March 19, 1978:

Let A be closed symmetric with deficiency indices $(1,1)$, let $u_i \in ((A+i)\mathcal{D}_A)^\perp$ $u_{-i} \in ((A-i)\mathcal{D}_A)^\perp$ be unit vectors and let U be the unitary extension of $V = (A-i)(A+i)^{-1}$ such that $U(u_i) = u_{-i}$. Suppose $\mathcal{H}$ is generated by u_{-i} under U. Then we get an isomorphism

$$L^2(S^1, d\nu) \xrightarrow{\sim} \mathcal{H}$$

such that U corresponds to mult. by ζ and

$$1 \longmapsto u_{-i} \qquad\qquad \int_{S^1} d\nu = 1.$$
$$\zeta^{-1} \longmapsto u_i$$

Let p denote the mass at 1 of $d\nu$; we decompose S^1 into $\zeta = 1$ and its complement which are parameterized using $\zeta = \frac{x-i}{x+i}$.

$$L^2(S^1, d\nu) \xrightarrow{\sim} L^2(\mathbb{R}, d\mu) \oplus \mathbb{C}$$

$$f(\zeta) \longmapsto \left(f\left(\frac{x-i}{x+i}\right)\frac{1}{x+i} \ , \ f(1) \right)$$

Here $d\mu$ is defined by

$$\int |f(\zeta)|^2 d\nu = \int \left| f\left(\tfrac{x-i}{x+i}\right) \tfrac{1}{x+i} \right|^2 d\mu + p|f(1)|^2$$

e.g. if $d\nu = \rho(\zeta)\frac{d\theta}{2\pi}$, then since

$$\frac{d\theta}{2\pi} = \frac{d\zeta}{2\pi i \zeta} = \frac{1}{2\pi i\left(\frac{x-i}{x+i}\right)} \frac{(x+i)-(x-i)}{(x+i)^2} = \frac{1}{\pi}\frac{dx}{x^2+1}$$

we have $\displaystyle\int |f(\zeta)|^2 d\nu = \int |f(\zeta)|^2 \rho(\zeta)\frac{d\theta}{2\pi} = \int \left|f\left(\tfrac{x-i}{x+i}\right)\right|^2 \rho\left(\tfrac{x-i}{x+i}\right)\frac{1}{\pi}\frac{dx}{x^2+1}$

so $$d\mu = \frac{1}{\pi}\,\rho\left(\frac{x-i}{x+i}\right)dx$$

It follows that in this new model

$$u_{-i} = \left(\frac{1}{x+i},\ 1\right)$$

$$u_i = \left(\frac{1}{x-i},\ 1\right)$$

so that $$\int \frac{d\mu}{x^2+1} + p = 1.$$

In this new model what is V? It is defined on (f,a) perpendicular to u_i:

$$\left((f,a),\ u_i\right) = \left(\begin{pmatrix} f \\ a \end{pmatrix}, \begin{pmatrix} \frac{1}{x-i} \\ 1 \end{pmatrix}\right) = \int \frac{f\,d\mu}{x+i} + a = 0$$

and $$V\begin{pmatrix} f \\ a \end{pmatrix} = \begin{pmatrix} \frac{x-i}{x+i} f \\ a \end{pmatrix}.$$ $\mathcal{D}_A = \operatorname{Im}(I-V)$ should consist of

$$\begin{pmatrix} f \\ a \end{pmatrix} - \begin{pmatrix} \frac{x-i}{x+i} f \\ a \end{pmatrix} = \begin{pmatrix} \frac{2i}{x+i} f \\ 0 \end{pmatrix} \qquad \text{with } \int \frac{f\,d\mu}{x+i} + a = 0$$

Thus $\mathcal{D}_A$ consists of all $\begin{pmatrix} f \\ 0 \end{pmatrix}$ with $f \in L^2(d\mu)$, $xf \in L^2(d\mu)$.
and

$$A\begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} xf \\ -\frac{1}{p}\int f\,d\mu \end{pmatrix}.$$

What is u_λ? It seems to be $\begin{pmatrix} \frac{1}{x-\lambda} \\ 1 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{x-\lambda} \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{x}{x^2+1} \\ 1 \end{pmatrix} - m(\lambda)\begin{pmatrix} \frac{1}{x^2+1} \\ 0 \end{pmatrix} \in \mathcal{D}_A$$

This is OKAY but it doesn't specify $m(\lambda)$

You need

$$\begin{pmatrix} \begin{pmatrix} \frac{1}{x-\lambda} \\ 1 \end{pmatrix} \\ \lambda \begin{pmatrix} \frac{1}{x-\lambda} \\ 1 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} \frac{x}{x^2+1} \\ 1 \end{pmatrix} \\ \begin{pmatrix} -\frac{1}{x^2+1} \\ 0 \end{pmatrix} \end{pmatrix} - m(\lambda) \begin{pmatrix} \begin{pmatrix} \frac{1}{x^2+1} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{x}{x^2+1} \\ 1 \end{pmatrix} \end{pmatrix} = \overset{\in \Gamma_A}{\begin{pmatrix} \begin{pmatrix} \frac{1}{x-\lambda} - \frac{x}{x^2+1} - \frac{m(\lambda)}{x^2+1} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{x}{x-\lambda} - \frac{x^2}{x^2+1} - \frac{m(\lambda)x}{x^2+1} \\ -\frac{1}{p}\int \left[\frac{1}{x-\lambda} - \frac{x+m(\lambda)}{x^2+1}\right] d\mu \end{pmatrix} \end{pmatrix}}$$

i.e. $$\lambda - m(\lambda) = -\frac{1}{p}\int \left[\frac{1}{x-\lambda} - \frac{x}{x^2+1}\right] d\mu + \frac{m(\lambda)}{p}$$

or $$p\lambda + \int \left[\frac{1}{x-\lambda} - \frac{x}{x^2+1}\right] d\mu = \underbrace{(1-p)}_{?}\, m(\lambda)$$

for some reason this 1 shouldn't be here. NO it's OKAY. The correction is

$$\lambda - m(\lambda) = -\frac{1}{p}\int \left[\frac{1}{x-\lambda} - \frac{x+m(\lambda)}{x^2+1}\right] d\mu$$

$$= -\frac{1}{p}\int \left[\frac{1}{x-\lambda} - \frac{x}{x^2+1}\right] d\mu + \frac{1}{p} m(\lambda) \int \frac{d\mu}{x^2+1}$$

$$\lambda + \frac{1}{p}\int \left[\frac{1}{x-\lambda} - \frac{x}{x^2+1}\right] d\mu = m(\lambda) + \frac{1}{p} m(\lambda) \int \frac{d\mu}{x^2+1}$$

$$p\lambda + \int \left[\frac{1}{x-\lambda} - \frac{x}{x^2+1}\right] d\mu = m\left\{p + \int \frac{d\mu}{x^2+1}\right\} = m(\lambda)$$

so it all works.

$\mathcal{H}$ Hilbert space, A closed symmetric operator with deficiency indices $(1,1)$. Choose unit vectors $u_i \in \mathcal{H} \ominus (A+i)\mathcal{D}_A$, $u_{-i} \in \mathcal{H} \ominus (A-i)\mathcal{D}_A$, and let U be the unitary operator on $\mathcal{H}$ which extends the Cayley transform $V = (A-i)(A+i)^{-1}$ and is such that $U(u_i) = u_{-i}$. We have an orthogonal decomposition

$$(P\Gamma_A)^{\perp} = \Gamma_A \oplus \mathbb{C}\cdot\begin{pmatrix} u_i \\ iu_i \end{pmatrix} \oplus \mathbb{C}\cdot\begin{pmatrix} u_{-i} \\ -iu_{-i} \end{pmatrix}$$

The choice of U determines a subspace $\tilde{\Gamma}$ between Γ_A and $(P\Gamma_A)^{\perp}$ (which $= \Gamma_{A^*}$ when A is densely-defined) as follows:

First from $V = (A-i)(A+i)^{-1}$ one has

$$\begin{tikzcd} \mathcal{D}_V & \mathcal{D}_A \arrow[l, "A+i"', "\sim"] \arrow[r, "A-i", "\sim"'] \arrow[d, "\wr"] & \mathcal{R}_V \\ & \Gamma_A & \end{tikzcd}$$

$$y \longmapsto \begin{pmatrix} (A+i)^{-1}y \\ A(A+i)^{-1}y \end{pmatrix}$$

$$Vy = (A-i)(A+i)^{-1}y$$
$$y = (A+i)(A+i)^{-1}y$$

$$\therefore \quad \frac{I-V}{2i}y = (A+i)^{-1}y \qquad\qquad \frac{I+V}{2}y = A(A+i)^{-1}y$$

so that

$$\Gamma_A = \left\{ \begin{pmatrix} \frac{I-V}{2i}y \\ \frac{I+V}{2}y \end{pmatrix} \,\middle|\, y \in \mathcal{D}_V \right\}$$

So we put $$\tilde{\Gamma} = \left\{ \begin{pmatrix} \left(\frac{I-u}{2i}\right)y \\ \left(\frac{I+u}{2}\right)y \end{pmatrix} \middle| \, y \in \mathcal{H} \right\}$$

and since $\mathcal{H} = \mathcal{D}_V \oplus \mathbb{C}u_i$, we have

$$\tilde{\Gamma} = \Gamma_A \oplus \mathbb{C} \cdot \begin{pmatrix} \frac{u_i - u_{-i}}{2i} \\ \frac{u_i + u_{-i}}{2} \end{pmatrix}.$$

$\tilde{\Gamma}$ is isotropic for P and when A is densely-defined $\tilde{\Gamma} = \Gamma_{\tilde{A}}$ where $\tilde{A}$ is self-adjoint ext. of A.

Put $\mathcal{N}_{\bar{\lambda}} = \mathcal{H} \ominus [(A-\lambda)\mathcal{D}_A]$; this is one-dimensional for λ non-real. Moreover if $u \in \mathcal{N}_\lambda$, then

$$\left(P\begin{pmatrix} u \\ \lambda u \end{pmatrix}, \begin{pmatrix} u \\ \lambda u \end{pmatrix}\right) = \left(\begin{pmatrix} \lambda u \\ -u \end{pmatrix}, \begin{pmatrix} u \\ \lambda u \end{pmatrix}\right) = (\lambda - \bar{\lambda})|u|^2$$

and hence for λ non-real we know that $\begin{pmatrix} u \\ \lambda u \end{pmatrix} \notin \tilde{\Gamma}$. Therefore there exists a unique $u_\lambda \in \mathcal{N}_\lambda$ and a number $m(\lambda)$ such that

$$\begin{pmatrix} u_\lambda \\ \lambda u_\lambda \end{pmatrix} \equiv \begin{pmatrix} \frac{u_i + u_{-i}}{2} \\ -\frac{u_i - u_{-i}}{2i} \end{pmatrix} + m(\lambda) \begin{pmatrix} \frac{u_i - u_{-i}}{2i} \\ \frac{u_i + u_{-i}}{2} \end{pmatrix} \qquad \text{mod } \Gamma_A$$

Because $\mathcal{N}_\lambda$ varies holomorphically in λ, u_λ and $m(\lambda)$ vary holomorphically in λ.

$$\left(P\begin{pmatrix} u_\lambda \\ \lambda u_\lambda \end{pmatrix}, \begin{pmatrix} u_z \\ z u_z \end{pmatrix}\right) = (\lambda - \bar{z})(u_\lambda, u_z) = m(\lambda) - \overline{m(z)}$$

which shows that $\overline{m(\bar{\lambda})} = m(\lambda)$ and that

$$(u_\lambda, u_z) = \frac{m(\lambda) - m(\bar{z})}{\lambda - \bar{z}}$$

$$(u_{\bar{z}}, u_{\bar{\lambda}}) = \frac{m(\lambda) - m(\bar{z})}{\lambda - \bar{z}}$$

So m is analytic off $\mathbb{R}$ has $\frac{\operatorname{Im}(m(\lambda))}{\operatorname{Im}(\lambda)} > 0$.

But now you define the transform $\hat{f}$ of any elt. f in $\mathcal{H}$ by

$$\hat{f}(\lambda) = (f, u_{\bar{\lambda}})$$

This will be an [redacted] analytic function of λ off $\mathbb{R}$. Now by general non-sense one knows that because of the positivity of the form

$$(u_{\bar{z}}, u_{\bar{\lambda}}) = \frac{m(\lambda) - m(\bar{z})}{\lambda - \bar{z}}$$

there is a unique Hilbert space $\mathcal{L}(m)$ of analytic functions on $\mathbb{C} - \mathbb{R}$ with point evaluators $J_z(\lambda) = (u_{\bar{z}}, u_{\bar{\lambda}})$ [redacted]. It also follows that the transform defined a Hilbert projection

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{L}(m) \\ f & \longmapsto & \hat{f} \\ u_{\bar{z}} & \longleftarrow\!\shortmid & J_z \end{array}$$

gives the adjoint.

The next step would be to describe this quotient space $\mathcal{L}(m)$ a bit [redacted] differently. It seems the simplest approach is to follow through from the unitary point of view.

Thus we decompose $\mathcal{H}$ into the cyclic subspace $\mathcal{H}_0$ for U spanned by u_{-i} and its orthogonal complement which we denote $\mathcal{H}_1$. [redacted] Since $\mathcal{H}_1 \subset \mathcal{D}_V$ = orth comp. of u_i and [redacted] $U = V|\mathcal{D}_V$, it follows that V is the direct sum $V = V_0 \oplus V_1$, where V_1 is a unitary operator on $\mathcal{H}_1$, and V_0 is the restriction of U to $\mathcal{D}_V = \mathcal{H}_0 = \mathcal{H}_0 - \mathbb{C}u_i$. So it is clear that we might as well concern ourselves with the [redacted] case $\mathcal{H}_0 = \mathcal{H}$.

But there we know by the spectral theorem that there is an isomorphism

$$\mathcal{H} \xrightarrow{\sim} L^2(S^1, d\nu)$$
$$U \longleftrightarrow \text{mult. by } \zeta$$
$$u_i \longleftrightarrow \zeta^{-1}$$
$$u_{-i} \longleftrightarrow 1$$

where $d\nu$ is a probability measure on S^1. Next we decompose S^1 into $\{1\}$ and $S^1 - \{1\}$ which we parameterize via the map $x \mapsto \frac{x-i}{x+i}$. This ~~[illegible]~~ provides us with an isomorphism

$$L^2(S^1, d\nu) \longrightarrow L^2(\mathbb{R}, d\mu) \oplus \mathbb{C} \qquad \text{if } \quad p = \nu\{1\} > 0.$$
$$f \longmapsto \left[f\left(\tfrac{x-i}{x+i}\right) \frac{1}{x+i} \,,\ f(1) \right]$$

where $d\mu$ is the measure on $\mathbb{R}$ such that

$$\int_{S^1} |f(\zeta)|^2 d\nu = \int_{\mathbb{R}} \left| f\left(\frac{x-i}{x+i}\right)\right|^2 \frac{d\mu}{x^2+1} + |f(1)|^2 p$$

and the factor $\mathbb{C}$ is equipped with the inner product $\|[0, a]\|^2 = p\,|a|^2$.

To simplify the follows I suppose $p = \nu\{1\} = 0$. which means that U doesn't have $\zeta = 1$ as a discrete eigenvalue. Suppose U has 1 as a discrete eigenvalue. Then if v is the eigenvector we have

$$v = (A+i)y + c u_i$$
$$v = Uv = (A-i)y + c u_{-i}$$

so

$$0 = 2iy + c(u_i - u_{-i})$$

or $$-y = \frac{u_i - u_{-i}}{2i} \in \mathcal{D}_A$$

Better: $$\begin{pmatrix} \frac{1-U}{2i} v \\ \frac{1+U}{2} v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} \in \tilde{\Gamma}$$ hence

$$\left(P\begin{pmatrix} x \\ Ax \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix} \right) = (-x, v) = 0 \quad \text{for all } x \in \mathcal{D}_A$$

so that $\mathcal{D}_A$ is not dense.

In any case we see that

$\mathcal{D}_A$ dense $\Longrightarrow$ U doesn't have discrete eigenvalue 1

$\Longrightarrow$ $\tilde{A} = i\,\frac{1+U}{1-U}$ is a self-adjoint extension of A.

So consider this case. Then you get

$$\mathcal{H} \simeq L^2(S^1, d\mu) \simeq L^2(\mathbb{R}, d\mu)$$

$$u_i \longmapsto \frac{1}{x-i}$$

$$u_{-i} \longmapsto \frac{1}{x+i}$$

$\tilde{A} \longmapsto$ mult. by x

$A \longmapsto$ mult. by x on $\{ f \in \mathcal{D}_A \mid \int f\,d\mu = 0 \}$

March 23, 1978: (yesterday Carl turned 13).

$-u'' + qu = \omega^2 u$. Trubowitz claims that

$$E(\omega) = \phi'(x_0, \omega^2) - i\omega\phi(x_0, \omega^2) \qquad x_0 \text{ fixed } > 0$$

is a deBranges function, where ϕ is the solution with $\phi(0)=0$ $\phi'(0) = 1$. If this is true, then $\phi'(x_0, \omega^2) = 0 \implies$

$$\overline{E(\bar\omega)} = \phi'(x_0, \omega^2) + i\omega\phi(x_0, \omega^2) = i\omega\phi(x_0, \omega^2)$$

has the same modulus as $E(\omega)$, so ω has to be real. Therefore the eigenvalues for the SL problem

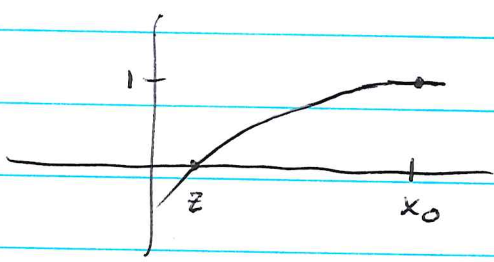
$$(*) \quad \begin{cases} -u'' + qu = \lambda u \\ u(0) = 0 \\ u'(x_0) = 0 \end{cases}$$

are ≥ 0. In fact the eigenvalues will be > 0 provided $\omega = 0$ is not an eigenvalue, i.e. provided $\phi'(x_0, 0) \neq 0$.

Suppose now that all the eigenvalues for $(*)$ are > 0. Let $v(x, \lambda)$ be the solution of

$$-v'' + qv = \lambda v$$
$$v(x_0) = 1$$
$$v'(x_0) = 0$$

By Sturm comparison it follows that $v(x, 0)$ doesn't vanish for $0 \leq x \leq x_0$, since otherwise decreasing λ would move a zero z in $(0, x_0)$ back to zero.



Hence the function $p(x) = \dfrac{v'(x,0)}{v(x,0)}$ is well-defined on

$[0, x_0]$ and it satisfies

$$p' + p^2 = q$$

so that we can factor the SL operator

$$-u'' + qu = -\left(\frac{d}{dx} + p\right)\left(\frac{d}{dx} - p\right)u = \omega^2 u$$

and replace this by

$$\begin{cases} \left(\frac{d}{dx} - p\right)u_1 = \omega u_2 \\ \left(\frac{d}{dx} + p\right)u_2 = -\omega u_1 \end{cases} \qquad \text{where} \qquad \begin{aligned} u &= u_1 \\ u_2 &= \frac{1}{\omega}\left(\frac{d}{dx} - p\right)u \end{aligned}$$

The latter system is in the form

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} 0 & -p \\ -p & 0 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \omega\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and hence we know that for any fixed $x > 0$ that

$$\operatorname{Im}\omega > 0 \implies \operatorname{Im}\frac{u_1(x,\omega)}{u_2(x,\omega)} > 0 \quad \text{if} \quad \frac{u_1(0,\omega)}{u_2(0,\omega)} \in \mathbb{R}$$

so

$$\text{''} \implies \operatorname{Im}\frac{\phi(x_0,\omega^2)}{\frac{1}{\omega}\left(\frac{d}{dx} - p\right)\phi(x_0,\omega^2)} > 0$$

By construction $p(x_0) = 0$, so we have

$$\operatorname{Im}\left\{\frac{\omega\phi(x_0,\omega^2)}{\phi'(x_0,\omega^2)}\right\} \begin{array}{ll} > 0 & \text{in the UHP} \\ < 0 & \text{in the LHP} \end{array}$$

which implies that

$$i\phi'(x_0,\omega^2) + \omega\phi(x_0,\omega^2), \quad \phi'(x_0,\omega^2) - i\omega\phi(x_0,\omega^2)$$

are de B functions.

Finally suppose the eigenvalues for (*) are ≥ 0. Then if x_0 is replaced by $x_1 = x_0 - \varepsilon$ the eigenvalues become > 0 and [redacted] hence we know that $\phi'(x_1, \omega^2) - i\omega\phi(x_1, \omega^2)$ is a dB function. We want to let $x_1 \uparrow x_0$. In the limit we get an entire fn. $E(\omega)$ with

$$|E(\omega)| \geq |\overline{E(\bar\omega)}|$$

for ω in the UHP. [redacted] By Hurwitz $E(\omega)$ can only vanish on the real axis, so then by maximum modulus applied to $\dfrac{\overline{E(\bar\omega)}}{E(\omega)}$ one sees it has to be < 1 in the UHP. So $E(\omega)$ [redacted] $= \phi'(x_0, \omega^2) - i\omega\phi(x_0, \omega^2)$ has to be a dB function. This proves:

<u>Proposition</u>: Let $\phi(x, \lambda)$ be the solution of $-u'' + qu = \lambda u$ with
$$\phi(0,\lambda) = [redacted]\ 0$$
$$\phi'(0,\lambda) = [redacted]\ 1$$
[redacted]

and let x_0 be fixed > 0. Then $E(\omega) = \phi'(x_0, \omega^2) - i\omega\phi(x_0, \omega^2)$ is a deBranges function iff all the eigenvalues for the SL problem

$$\begin{cases} -u'' + qu = \lambda u \\ [redacted]\ u(0)\ [redacted] = 0 \\ u'(x_0) = 0 \end{cases}$$

[redacted] are ≥ 0.

Curiousity: The obvious way to show $\phi'(x_0, \omega^2) - i\omega\phi(x_0, \omega^2)$ is a de B function would be to use the system

$$\frac{d}{dx}\begin{pmatrix} \omega u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ \frac{q}{\omega} - \omega & 0 \end{pmatrix}\begin{pmatrix} \omega u \\ u' \end{pmatrix}$$

The matrix $\begin{pmatrix} 0 & \omega \\ \frac{q}{\omega}-\omega & 0 \end{pmatrix}$ when exponentiated has the UHP shrinking property for $\operatorname{Im}\omega > 0$ provided $q \geq 0$. Certainly $q \geq 0 \Rightarrow$ the solution of $u'' = qu$, $u(x_0)=1$, $u'(x_0)=0$ is concave upwards, so the SL problem in the prop. has no solution with $\lambda \leq 0$. But one doesn't obtain best possible results this way.

March 25, 1978:

Suppose we consider $L^2(\mathbb{R}, d\mu)$ with $\int \frac{d\mu}{x^2+1} < \infty$ with the symmetric operator A of type $(1,1)$ given by multiplication by x on $\mathcal{D}_A = \{ f \in L^2 \mid xf \in L^2 \text{ and } \int f d\mu = 0 \}$.

According to de B there is a canonical filtration of this gadget by de B spaces.

To fix the ideas suppose $d\mu$ is a measure [redacted] supported on ∞ points. Then we obtain a decreasing filtration:

$$\mathcal{H} \supset \mathcal{D}_A \supset \mathcal{D}_{A^2} \supset \cdots\cdots$$

where
$$\mathcal{D}_{A^n} = \{ f \in L^2 \mid \int f d\mu = \int xf\, d\mu = \cdots = \int x^{n-1} f\, d\mu = 0 \}$$
$$= \mathcal{H} \ominus F_{n-1}\mathbb{C}[x]$$

This shows that the filtration by degree on polynomials is intrinsically determined from $\mathcal{H}$ and A.

More generally suppose given an $\mathcal{H}$, A symm. of type $(1,1)$ and suppose $\mathcal{D}_A$ is closed in $\mathcal{H}$. This means that A is bounded and would be a bounded self-adjoint operator except that its domain is of codimension 1 in $\mathcal{H}$. [redacted] If $\tilde{A}$ is [redacted] a self-adjoint extension, then $\tilde{A}$ is a bounded self-adjoint operator. Let e be a unit vector $\perp \mathcal{D}_A$. If (H, A) minimal, then e has to be acyclic for $\tilde{A}$, so we end up with a model for A like the above where

$$\mathcal{D}_{A^n} = \mathcal{H} \ominus F_{n-1}\mathbb{C}[x]$$

so that the canonical deBranges filtration is by $0 \subset (\mathcal{D}_A)^{\perp} \subset (\mathcal{D}_{A^2})^{\perp} \subset \cdots$.

The above example gives a new viewpoint.

Consider on $0 \leq x \leq b$ a Schroedinger DE

$$-y'' + qy = \lambda y$$

together with a real boundary condition at $x = b$. We get in $\mathcal{H} = L^2(0,b)$ a symmetric operator A of type (1,1) by closing up $L = -\frac{d^2}{dx^2} + q$ on smooth functions vanishing near 0 and satisfying the boundary condition at $x = b$. [redacted]

We get an isomorphism of $\mathcal{D}_{A^*}/\mathcal{D}_A$ with $\mathbb{C}^2$ by sending u to $\begin{pmatrix} u'(0) \\ u(0) \end{pmatrix}$. By Green's formula

$$(A^* u, v) - (u, A^* v) = \int_0^b \left[(Lu)\bar{v} - u\overline{Lv} \right] dx$$

$$= \left[-u'\bar{v} + u\bar{v}' \right]_0^b = \begin{vmatrix} u'(0) & \overline{v'(0)} \\ u(0) & \overline{v(0)} \end{vmatrix}$$

$$= \left(P\begin{pmatrix} u'(0) \\ u(0) \end{pmatrix}, \begin{pmatrix} v'(0) \\ v(0) \end{pmatrix} \right)$$

hence if ψ_λ [redacted] $\in \mathcal{N}_\lambda$, then

$$(\lambda - \bar{\lambda}) |\psi_\lambda|^2 = \left(P\begin{pmatrix} \psi_\lambda'(0) \\ \psi_\lambda(0) \end{pmatrix}, \begin{pmatrix} \psi_\lambda'(0) \\ \psi_\lambda(0) \end{pmatrix} \right)$$

$$2i \operatorname{Im}\lambda \, |\psi_\lambda|^2 = |\psi_\lambda(0)|^2 \, 2i \operatorname{Im}\left(\frac{\psi_\lambda'(0)}{\psi_\lambda(0)} \right).$$

so [redacted] $\operatorname{Im}\left(\frac{\psi_\lambda'(0)}{\psi_\lambda(0)} \right) = \operatorname{Im}\lambda \cdot |\psi_\lambda|^2 / |\psi_\lambda(0)|^2$

has the same sign as $\operatorname{Im}\lambda$.

The above calculates the power form on $\mathcal{D}_{A^*}/\mathcal{D}_A \simeq \mathbb{C}^2$. There is also an inner product which we could compute in terms of the boundary values for ψ_i and ψ_{-i}. We would need this calculation to compute the $m(\lambda)$ belonging to a ~~boundary~~ boundary condition at $x=0$ via the fmla

$$u_\lambda = \frac{u_i + u_{-i}}{2} + m(\lambda)\,\frac{u_i - u_{-i}}{2i}$$

However if we are not concerned about having $m(\lambda)$ normalized so that $m(i) = i$ we can proceed [redacted] as follows. Let the boundary condition at $x=0$ be given by

$$\begin{pmatrix} u'(0) \\ u(0) \end{pmatrix} \sim \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^2 - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ [redacted]

and choose a complementary boundary condition: $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ such that

$$\begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix} \in SL_2(\mathbb{R}).$$

Then one [redacted] can define $m(\lambda)$ by

$$\begin{pmatrix} \psi'(0,\lambda) \\ \psi(0,\lambda) \end{pmatrix} \sim \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + m(\lambda) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

i.e.

$$\frac{\psi'(0,\lambda)}{\psi(0,\lambda)} = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix} (m(\lambda))$$

or

$$m(\lambda) = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}^{-1} \left(\frac{\psi'(0,\lambda)}{\psi(0,\lambda)} \right)$$

which shows Im [redacted] $(m(\lambda))$ has the same sign as $\mathrm{Im}(\lambda)$.

Note: Recognize: $\psi = m\phi + \tilde{\phi}$ where $W(\phi,\psi) = W(\phi,\tilde{\phi}) = \pm 1$.

March 26, 1978

General theory: Let A be closed symmetric densely-defined of type $(1,1)$ on $\mathcal{H}$ with no self-adjoint components. If we choose a symplectic basis for $\mathcal{D}_{A^*}/\mathcal{D}_A$ (a basis h,k with $P(h,h)=P(k,k)=0$, $P(h,k)=1$), then we get an $m(\lambda)$ defined by

$$u_\lambda = m(\lambda)h + k \quad \in \mathcal{N}_\lambda \qquad \text{mod } \mathcal{D}_A$$

and we get an isomorphism of $\mathcal{H}$ with $L^2(\mathbb{R}, d\mu) \xrightarrow{\sim} L(m)$ determined by $u \mapsto \hat{u}$ where

$$(u, u_{\bar\lambda}) = \int \frac{\hat{u}(x)}{x-\lambda}\, d\mu .$$

Q: Is there a formula for $\hat{u}(x)$?

Specific example: Take A to be the op. on $\mathcal{H} = L^2(0,b)$ defined by $L = -\frac{d^2}{dx^2} + q$ together with a boundary condition at $x=b$. Yesterday I saw that $\mathcal{D}_{A^*}/\mathcal{D}_A$ could be identified with $\mathbb{C}^2$ via $u \mapsto \begin{pmatrix} u'(0) \\ u(0)\end{pmatrix}$ and that with this description the power form becomes

$$P(u,v) = \begin{vmatrix} u'(0) & \overline{v'(0)} \\ u(0) & \overline{v(0)} \end{vmatrix}$$

Suppose $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ $k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ are real, so $\begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix} \in SL_2(\mathbb{R})$. $m(\lambda)$, ψ_λ are defined by

$$\psi_\lambda = m(\lambda)\phi_\lambda + \tilde{\phi}_\lambda$$

where $\phi_\lambda, \tilde{\phi}_\lambda$ are eigenfns. with boundary values h, k. By the general theory if $d\mu(\lambda)$ is the measure belonging to m we get an isomorphism

$$L^2(0,b) \xrightarrow{\sim} L^2(\mathbb{R}, d\mu)$$
$$\psi_\lambda \longleftarrow \frac{1}{x-\lambda}$$
$$u \longmapsto \hat{u}$$

where all I know at the moment is that

$$(u, \psi_{\bar\lambda}) = \int \frac{\hat{u}(x)}{x-\lambda}\, d\mu(x).$$

I want a formula for $\hat{u}$. It should be the case that

$$\hat{u}(\lambda) = (u, \phi_\lambda).$$

Review the Sz.-Nagy theorem about contractions. Let T be a contraction operator on $\mathcal{H}$. Then there is unique triple $(\tilde{\mathcal{H}}, U, i)$ with U unitary on $\tilde{\mathcal{H}}$, and $i : \mathcal{H} \to \tilde{\mathcal{H}}$ with

1) $$T^n = i^* U^n i \qquad \text{for all } n \geq 0.$$

and such that $\tilde{\mathcal{H}}$ is generated by U and $i\mathcal{H}$. Note that 1) for $n=0 \Rightarrow i^*i = I$ so i is an isometric embedding. Also

$$(T^*)^n = (i^* U^n i)^* = i^* U^{-n} i \qquad \text{for } n \geq 0.$$

Construction of $\tilde{\mathcal{H}}$. Try to define an inner product on $\bigoplus_{n\in\mathbb{Z}} z^n \mathcal{H}$ by requiring

$$(z^n\alpha, z^m\beta) = \begin{cases} (T^{n-m}\alpha, \beta)_{\mathcal{H}} & n \geq m \\ ((T^*)^{m-n}\alpha, \beta)_{\mathcal{H}} & n \leq m \end{cases}$$

If this inner product is ≥ 0, then completion gives $\tilde{\mathcal{H}}$ and

multiplication by z gives the unitary operator U, and i is obvious. To show positivity we want to see that
$$\left(\sum_n z^n \alpha_n, \sum_m z^m \alpha_m\right) = \sum_{n,m} \left(\gamma_{n-m}\alpha_n, \alpha_m\right)_{\mathcal{H}} \geq 0$$
where $\gamma_n = \begin{cases} T^n & n \geq 0 \\ (T^*)^{-n} & n \leq 0 \end{cases}$. We write this inner product as an integral over S^1

since
$$\int \left(\left(\sum_p z^{-p}\gamma_p\right)\left(\sum_n z^n \alpha_n\right), \sum_m z^m \alpha_m\right)_{\mathcal{H}} \frac{d\theta}{2\pi}$$
$$= \int \sum_{n,m,p} \left(\gamma_p \alpha_n, \alpha_m\right) z^{-p+n-m} \frac{d\theta}{2\pi} = \sum_{n,m} \left(\gamma_{n-m}\alpha_n, \alpha_m\right)$$
we only have to show $\sum_p z^{-p}\gamma_p \geq 0$ for each $z \in S^1$. (Note that replacing T by rT and letting $r \uparrow 1$ we can assume $\|T\| < 1$ so this series converges.
$$\sum_p z^{-p}\gamma_p = \sum_{n \geq 0} z^{-n}T^n + \sum_{n<0} z^{-n}(T^*)^{-n}$$
$$= (1-z^{-1}T)^{-1} + (zT^*)(1-zT^*)^{-1}$$
$$= (1-z^{-1}T)^{-1}\left(1 - zT^* + (1-z^{-1}T)zT^*\right)(1-zT^*)^{-1}$$
$$= (1-z^{-1}T)^{-1}(1-TT^*)(1-zT^*)^{-1}$$
which is clearly > 0. In fact we see that $\|T\| < 1$
$$\Longrightarrow \quad \tilde{\mathcal{H}} \xleftarrow{\sim} \bigoplus_{n \geq 0}^{\ell^2} z^n \mathcal{H}.$$

March 27, 1978.

T contraction on $\mathcal{H}$. [redacted] To obtain $\tilde{\mathcal{H}}$ we complete $\bigoplus z^n \mathcal{H}$ = space of Laurent polys with coeffs in $\mathcal{H}$ with respect to the inner product

$$(p, q)_{\tilde{\mathcal{H}}} = \int (\gamma(z)p(z), q(z))_{\mathcal{H}} \frac{d\theta}{2\pi}$$

where

$$\gamma(z) = \sum_{p \gtrless 0} z^{-p} \begin{cases} T^p & p \geq 0 \\ (T^*)^{-p} & p \leq 0 \end{cases} = \sum_{p\geq 0} z^{-p} T^p + \sum_{p>0} z^{-p} (T^*)^{-p}$$

$$= (1 - z^{-1}T)^{-1} + (1 - zT^*)^{-1} zT^*$$

$$= (1 - zT^*)^{-1} \left[1 - zT^* + zT^*(1 - z^{-1}T) \right] (1 - z^{-1}T)^{-1}$$

$$= (1 - zT^*)^{-1} \left[1 - T^*T \right] (1 - z^{-1}T)^{-1}$$

Yesterday we found also

$$\gamma(z) = (1 - z^{-1}T)^{-1} \left[1 - TT^* \right] (1 - zT^*)^{-1}.$$

Thus $\gamma(z) \frac{d\theta}{2\pi}$ is an operator measure like the dE_λ that occurs in the spectral thm. [redacted] ?

Over S^1 we take the trivial bundle with fibre $\mathcal{H}$ and we put the metric given by $\gamma(z)$ in the fibre over z. Then integration gives us a metric on sections.

Assume $\|T\| < 1$ so the above calculation make sense. Put

$$\varphi(z) = (1 - TT^*)^{1/2} (1 - zT^*)^{-1}$$

so that for $z \in S^1$ we have

$$\gamma(z) = \varphi(z)^* \varphi(z).$$

It follows that

$$\|p\|^2_{\tilde{\mathcal{H}}} = \int (\gamma(z)p(z), p(z))_{\mathcal{H}} \frac{d\theta}{2\pi}$$

$$= \int \|\varphi(z)p(z)\|^2_{\mathcal{H}} \frac{d\theta}{2\pi}$$

so that we have an isomorphism

$$L^2(S^1;\mathcal{H}) \xleftarrow{\ \sim\ } \tilde{\mathcal{H}}$$

$$\varphi p \longleftarrow\!\shortmid\ p$$

Hence if we associate to $v \in \mathcal{H}$ the element

$$\varphi(z)^{-1}v = (1-zT^*)(1-TT^*)^{-1/2}v$$

we have

$$(z^n\varphi(z)^{-1}v, \varphi(z)^{-1}w)_{\tilde{\mathcal{H}}} = \int (\varphi(z)^*\varphi(z)\, z^n\varphi(z)^{-1}v, \varphi(z)^{-1}w)_{\mathcal{H}} \frac{d\theta}{2\pi}$$

$$= \begin{cases} 0 & n \neq 0 \\ (v,w) & n=0 \end{cases}$$

What has this got to do with scattering? In scattering you have a unitary operator U on space $\mathcal{V}$ and 2 outgoing subspaces $\mathcal{D}_0$ and $\mathcal{D}_1$. We then get an isomorphism

$$L^2(S^1, \mathcal{D}_0 \ominus U\mathcal{D}_0) \xrightarrow{\sim} \mathcal{V} \xleftarrow{\sim} L^2(S^1, \mathcal{D}_1 \ominus U\mathcal{D}_1)$$

which is given by a function $S(z)$ on S^1 with values in unitary maps from $\mathcal{D}_0 \ominus z\mathcal{D}_0$ to $\mathcal{D}_1 \ominus z\mathcal{D}_1$. $S(z)$ is the scattering operator.

When $\mathcal{D}_0 \supset \mathcal{D}_1$, then U induces on $\mathcal{D}_0 \ominus \mathcal{D}_1$ a contraction operator T.

Q: Given T on $\mathcal{H}$ form $\tilde{\mathcal{H}}$ and put $\mathcal{D}_0 =$ span of

$\mathcal{H}, z\mathcal{H}, \dots\dots\dots$ Is $\mathcal{D}_0$ an outgoing subspace as well as $\mathcal{D}_1 = \mathcal{D} \ominus \mathcal{H}$?

This is clear when $\|T\| < 1$?? What is $S(z)$ in this case?

To a measurable map $S: S^1 \longrightarrow \mathcal{U}(\mathcal{H})$ belongs an outgoing subspace $S\mathcal{D}_0$ [redacted] of $L^2(S^1;\mathcal{H})$, where $\mathcal{D}_0 = H^2(S^1;\mathcal{H})$. If S has a holomorphic extension to $|z|<1$, then $S\mathcal{D}_0 \subset \mathcal{D}_0$ and conversely. In this case we can associate [redacted] to S the contraction operator T on $\mathcal{H} = \mathcal{D}_0/S\mathcal{D}_0$ induced by multiplication by z. [redacted] Invariant subspaces of $\mathcal{H}$ correspond to outgoing subspaces $\mathcal{D}_1$ between $\mathcal{D}_0$ and $S\mathcal{D}_0$ and hence correspond to factorizations [redacted]

$$S = S_2 S_1 \qquad\qquad \mathcal{D}_1 = S_1\mathcal{D}_0$$

of the operator function S into factors of the same type. Notice that $\mathcal{D}$ [redacted] determines S up to right multiplication by a constant map, so there is some non-uniqueness which one would like to remove by normalizing S so that $S(1) = 1$.

Problem: Relate $\mathcal{N}$ and $\mathcal{H}$. $\mathcal{N}$ has to do with the multiplicity of the spectrum of U on $\tilde{\mathcal{H}}$ and [redacted] somehow involves completing $\mathcal{H}$ with the norm defined by $1-T^*T$. For example since $1-T^*T \geq 0$ one has

$$(1-T^*T)x = 0 \iff 0 = ((1-T^*T)x, x) = \|x\|^2 - \|Tx\|^2 \iff Ux = Tx$$

Similarly $(1-TT^*)x = 0 \iff u^{-1}x = T^*x$.

More precisely suppose that the spectrum of T is contained in $|\lambda|<1$, whence for $|z|=1$ the operators $(1-zT^*)$, $1-z^{-1}T$ are invertible. For example, this happens if $\|T^k\|<1$ for some k. Then from the formula for $\gamma(z)$

$$\gamma(z) = \begin{cases} (1-zT^*)^{-1}[1-T^*T](1-z^{-1}T)^{-1} & \text{or} \\ (1-z^{-1}T)^{-1}[1-TT^*](1-zT^*)^{-1} & \end{cases}$$

it follows that $\tilde{\mathcal{H}}$ is isomorphic to the space of functions on S^1 with values in $\mathcal{H}$ with the norm $((1-T^*T)x,x)$.

At the moment I do not understand the following: The scattering matrices connected with contraction operators compose in a natural way, but those obtained as n-ports don't compose. Is there any relation between the two? For example, you have de Branges functions connected with orth. polys. on S^1.

$$\begin{array}{ccc} \operatorname{Ker}(1-T^*T) & \longrightarrow & \operatorname{Ker}(1-TT^*) \\ x & \longmapsto & Tx \\ T^*y & \longleftarrow\!\shortmid & y \end{array}$$

obviously inverse, so these kernels are canonically isomorphic as Hilbert spaces.

Example: Suppose $T : \mathcal{H} \longrightarrow \mathcal{H}$ is an isometric embedding i.e. $T^*T = 1$. Then $1 - T^*T = 0$ so one sees from this example that one has to be careful about the formulas for $\gamma(z)$ when $\|T\| = 1$. In this case we know what $\tilde{\mathcal{H}}$ looks like, namely,

$$\tilde{\mathcal{H}} = L^2(S^1; \mathcal{N}) \qquad \mathcal{N} = \mathcal{H} \ominus T\mathcal{H} = \operatorname{Ker} TT^*$$
$$\mathcal{H} = H^2(S^1; \mathcal{N}) \qquad = \operatorname{Im}(1 - TT^*).$$

In this example $\mathcal{D}_0 = \mathcal{H}$ so there is no candidate for $\mathcal{D}_1$ hence no scattering matrix S belonging to such a contraction operator.

$$\begin{array}{c} \mathcal{H} = \operatorname{Ker}(1 - T^*T) \oplus \mathcal{N} \\ T\Big\downarrow\Big\uparrow T^* \\ \mathcal{H} = \operatorname{Ker}(1 - TT^*) \oplus \mathcal{N}_1 \end{array}$$

Define $\mathcal{N}, \mathcal{N}_1$ to be the indicated orthogonal complements. Since T carries $\mathcal{K} = \operatorname{Ker}(1 - T^*T)$ to $\mathcal{K}_1 = \operatorname{Ker}(1 - TT^*)$, T^* carries $\mathcal{N}_1$ to $\mathcal{N}$:

$$(T^*\mathcal{N}_1, \mathcal{K}) = (\mathcal{N}_1, T\mathcal{K}) \subset (\mathcal{N}_1, \mathcal{K}_1) = 0$$

Similarly $T(\mathcal{N}) \subset \mathcal{N}_1$. $\blacksquare$

Suppose $\mathcal{N}, \mathcal{N}_1$ are 1-dimensional. Can you describe $\tilde{\mathcal{H}}$? Does there exist a scattering matrix.

Recall that if A is symm (1,1) operator on $\mathcal{H}_1$, without self-ad component then there is associated a 2-dim'l complex vector space

$$V = (J\Gamma_A)^{\perp}/\Gamma_A$$

with a hermitian form P of type (1,1) and a holomorphic map $m : \lambda \mapsto N_\lambda$ from $\mathbb{C} - \mathbb{R}$ to $\mathbb{P}V$ such that m carries UHP (resp LHP) into the positive (negative) disk for the power form P and such that

$$m(\bar{\lambda}) = m(\lambda)^*$$

where $*$ denotes reflection through the zero power circle in $\mathbb{P}V$. Notice also however that V has an inner product, which might mean that the triple (V, P, m) does not determine $(\mathcal{H}, A)$ up to canonical isomorphism. NO: One knows what N_i, N_{-i} are inside of V, and one knows the metrics on N_i, N_{-i} because the power form coincides with the metric up to a scalar factor on these subspaces.