Review: Consider a D-system \( Lu = P \frac{du}{dx} + Qu = \lambda u \) with \( P = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \).

\( \frac{1}{i} (P, u, u) = -|u_1|^2 + |u_2|^2 > 0 \) describes the disk \( \frac{|u_1|}{|u_2|} < 1 \) in \( \mathbb{R}^2 \).

Since

\[
\frac{d}{dx} (P, u, u) = (Lu, u) - (u, Lu) = (\lambda - \bar{\lambda}) |u|^2
\]

we have

\[
\frac{1}{i} (P, u, u)(b) = \frac{1}{i} (P, u, u)(0) + 2 \text{Im} \lambda \int_0^b |u|^2 dx
\]

hence propagation from 0 to \( b > 0 \) shrinks the disk for \( \text{Im}(\lambda) > 0 \).

Suppose given the boundary condition \( u_1 = u_2 \) at 0
\( u_1 = e^{i\theta} u_2 \) at \( b \). Let \( \varphi(x, \lambda), \psi(x, \lambda) \) denote the solutions with
\[
\varphi(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi(b, \lambda) = \begin{pmatrix} e^{i\theta/2} \\ 0 \end{pmatrix}
\]

Calculation leads to the following formula for the Green's matrix

\[
G(x, y, \lambda) = \begin{cases} \frac{i}{W}(\varphi_1(x), \psi_2(y), \varphi_1(y)) & x < y \\ \frac{i}{W}(\psi_1(x), \varphi_2(y), \psi_1(y)) & x > y \end{cases}
\]

\( W = W(\varphi, \psi) \)

\( (\lambda - \bar{\lambda}) G = 0 \) or \( -PG \bigg|_{x = y}^{x = y+} = I \) on \( G(y+) - G(y-) = -\rho \rho^{-1} \rho = \rho \).
Since \( \psi_2(x, \lambda) = \overline{\psi_1(x, \lambda)} \) this can be written

\[
G(x, y, \lambda) = \begin{cases} \frac{i}{W} \psi(x, \lambda) \psi(y, \lambda)^* & x < y \\
\frac{i}{W} \psi(x, \lambda) \psi(y, \lambda)^* & x > y 
\end{cases}
\]

Now

\[
W = W(\varphi(\lambda), \psi(\lambda)) = \begin{vmatrix} \psi_1(x, \lambda) & \psi_2(x, \lambda) \\
\psi_2(x, \lambda) & \psi_2(x, \lambda) \end{vmatrix} = \frac{1}{i} \psi(x, \lambda)^* \mathcal{P} \psi(x, \lambda) = \frac{1}{i} (\mathcal{P} \varphi(\lambda), \psi(\lambda))(x)
\]

also \( W = i (\mathcal{P} \varphi(\lambda), \psi(\lambda))(x) \)

is independent of \( x \) and it is an entire function of \( \lambda \). Its zeroes are the eigenvalues of the self-adjoint problem defined by \( L \) and the given boundary values.

Suppose \( \lambda_0 \) is an eigenvalue. \( \lambda_0 = \overline{\lambda_0} \)

\[
W = W(\varphi(\lambda), \psi(\lambda)) = W(\varphi(\lambda), \psi(\lambda))(b) = W(\varphi(\lambda), \psi(\lambda))(b) = i (\mathcal{P} \varphi(\lambda), \psi(\lambda))(b)
\]

\[
= i (\mathcal{P} \varphi(\lambda), \psi(\overline{\lambda_0}))(b)
\]

\[
= i (\mathcal{P} \varphi(\lambda), \psi(\overline{\lambda_0}))(b) + i(\lambda - \lambda_0) \int_0^b (\varphi(\lambda), \psi(\overline{\lambda_0})) \, dx
\]

\[
\rightarrow \lim_{\lambda \to \lambda_0} \frac{W}{\lambda - \lambda_0} = i \int_0^b (\varphi(\lambda_0), \psi(\overline{\lambda_0})) \, dx
\]

Hence

\[
\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0) G(x, y, \lambda) = \frac{\psi(x, \lambda_0) \psi(y, \lambda)^*}{(\varphi(\lambda_0), \psi(\lambda_0))_{[0,b]}} \quad x < y
\]
\[ \frac{\phi(x, \lambda_0) \phi(y, \lambda_0)^*}{\| y(0) \|_{L^2([0, b])}^2} \]

and similarly for \( x > y \).

Project: Rewrite the above using only the solution matrix of the Dirac system.

The de Branges’ idea: Construct a Hilbert space of pairs of entire functions belonging to the solution matrix over the interval \( 0 \leq x \leq b \).

Check details of the following: Suppose given a matrix \( K(\lambda, z) \), \((\lambda, z) \in \mathbb{R} \times \mathbb{R} \), where \( \mathbb{R} \) is a set. Assume this matrix is Hermitian and \( \geq 0 \); let \( \mathcal{H} \) be the associated Hilbert space, that is, the Hilbert space with generators \( J_z \) for \( z \in \mathbb{R} \) such that
\[ (J_z, J_{\lambda}) = K(\lambda, z). \]

For each \( h \in \mathcal{H} \) we can define a function on \( \mathbb{R} \) by
\[ h(z) = (h, J_z) \]

Note that \( h(z) = 0 \) for all \( z \in \mathbb{R} \) \( \Rightarrow \) \( h = 0 \).

We have
\[ |h(z)| \leq \| h \| \cdot \| J_z \| \quad \text{where} \quad \| J_z \|^2 = K(z, z) \]

Let \( S \) be a subset of \( \mathbb{R} \) on which the function \( z \mapsto K(z, z) \)
is bounded. Given \( h \in H \) and \( \varepsilon > 0 \) we know that because the \( J_\lambda \) are dense in \( H \) for \( \varepsilon \in \mathbb{R} \), there exists a finite linear combination \( \sum_{i=1}^{n} c_i J_{\lambda_i} \) within \( \varepsilon \) of \( h \).

Hence

\[
\left| h(\lambda) - \sum_{i=1}^{n} c_i J_{\lambda_i}(\lambda) \right| < \varepsilon \cdot K(\lambda, \lambda)
\]

which means that the function \( h(\lambda) \) can be uniformly approximated on \( \mathbb{R} \) by linear combinations of the functions \( J_\lambda(\lambda) = K(\lambda, \lambda) \).

So now suppose \( \Omega \) is an open subset of \( \mathbb{C} \) and that for each \( \varepsilon \) the function \( J_\lambda(\lambda) = K(\lambda, \lambda) \) is holomorphic on \( \Omega \). Provided \( K(\varepsilon, \varepsilon) \) is bounded on compact sets, for example, if it is continuous, then the above shows the function \( \lambda \mapsto h(\lambda) \) can be uniformly approximated on compact subsets of \( \Omega \) by holomorphic functions, hence \( \lambda \mapsto h(\lambda) \) is holomorphic.

Thus each element of \( H \) is associated as holomorphic function on \( \Omega \) which determines it. Hence we can identify \( H \) with a space of holomorphic functions on \( \Omega \).

Example: Consider \( L u = (-\frac{d^2}{dx^2} + q) u = \lambda u \) note: \( \lambda \)
on \[0 \leq x \leq l\] with boundary condition given at \( x = 0 \).

In \( L^2(0, l) \) we consider the elements \( \varphi_\lambda \) for \( \varphi \in \mathbb{C} \). Then

\[
(\varphi_\lambda, \varphi_\lambda)_{[0, l]} = \int_0^l \varphi(x, \varphi) \varphi(x, \lambda) \, dx = (\varphi_\lambda, \varphi_\lambda)_{[0, l]}
\]

\[
(\lambda - \bar{\lambda}) (\varphi_\lambda, \varphi_\lambda)_{[0, l]} = \left\{ (L \varphi_\lambda, \varphi_\lambda) - (\varphi_\lambda, L \varphi_\lambda) \right\}_{[0, l]}
\]

\[
= \int_0^l \left\{ -\frac{d^2}{dx^2} \varphi_\lambda \cdot \varphi_\lambda + \varphi_\lambda \cdot \frac{d^2}{dx^2} \varphi_\lambda \right\} \, dx = \left[ -\frac{d}{dx} \varphi_\lambda \cdot \varphi_\lambda + \varphi_\lambda \frac{d}{dx} \varphi_\lambda \right]_0^l.
\]
\[
\begin{vmatrix}
\varphi_1 & \varphi_2 \\
\varphi_1' & \varphi_2' \\
\end{vmatrix}
(l)
\]

Thus,
\[
(\varphi_1, \varphi_2)_{[0, \varepsilon]} = \frac{W(\varphi_1, \varphi_2)(l)}{\lambda - \varepsilon}
\]

For example if \( \varphi = 0 \) and \( \varphi(x, \lambda) = \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}} \) then
\[
(\varphi_1, \varphi_2)_{[0, \varepsilon]} = \frac{1}{\lambda - \varepsilon} \begin{vmatrix}
\sin \sqrt{\lambda} l & \sin \sqrt{\varepsilon} l \\
\cos \sqrt{\lambda} l & \cos \sqrt{\varepsilon} l \\
\end{vmatrix}
\]

Recall the formula
\[
J_{\varepsilon}(\lambda) = \frac{1}{\lambda - \varepsilon} \begin{vmatrix}
A(\lambda) & A(\varepsilon) \\
B(\lambda) & B(\varepsilon) \\
\end{vmatrix}
\]

for the de Branges space based on \( E = A - iB \). Hence we get the de Branges space based on
\[
E(\lambda) = \cos \sqrt{\lambda} l - i \frac{\sin \sqrt{\lambda} l}{\sqrt{\lambda}}
\]

If on the other hand \( \varphi(x, \lambda) = \cos \sqrt{\lambda} x \), then we get
\[
E(\lambda) = \cos(\sqrt{\lambda} l) - i \sqrt{\lambda} \sin(\sqrt{\lambda} l)
\]

Back in March 9, 1977 we looked at fractional linear transformations which shrink the upper half-plane. Especially we looked at systems
\[
\frac{d\nu}{dx} = A(\lambda) \nu \quad A(\lambda) = \alpha + i \beta
\]

where the solution matrix preserves, expands, shrinks the UHP according as \( \text{Im} \lambda = 0, < 0, > 0 \) respectively.
The basic calculation amounted to looking at matrices of the form

\[ I + \varepsilon A \beta \]

with this property. So if \( x \in \mathbb{R} \)

\[ (I + \varepsilon A \beta)(x) = \frac{(1 + \varepsilon A \beta_{11})x + \varepsilon A \beta_{12}}{\varepsilon A \beta_{21} x + (1 + \varepsilon A \beta_{22})} \]

\[ = \left[ x + \varepsilon A (\beta_{11}x + \beta_{12}) \right] \left[ 1 - \varepsilon A (\beta_{21}x + \beta_{22}) \right] \]

\[ = x + \varepsilon A \left[ \beta_{11}x + \beta_{12} - \beta_{21}x^2 - \beta_{22}x \right] \]

If \( \text{Im} \lambda > 0 \) we want this to point into the UHP. Hence for any \( x \) real we want

\[ \beta_{12} + (\beta_{11} - \beta_{22})x - \beta_{21}x^2 \geq 0 \]

i.e.

\[ \beta_{12} > 0, \quad \beta_{21} \leq 0, \quad \beta_{11} \leq -\beta_{21}\beta_{12}. \]

Thus \( \beta \) has the form

\[ \beta = \begin{pmatrix} p & q \\ -q & -p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ p & q \end{pmatrix} \]

where \( p > 0, q \geq 0, \quad p^2 \leq q^2, \) so the last matrix is \( \geq 0. \)

Notice also that a real matrix of trace 0 is of the form

\[ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} + x_{11} \end{pmatrix} \]

arbitrary real symmetric.
hence a system
\[
\frac{du}{dx} = (x + \lambda \beta) u
\]

with requisite shrinking property is a self-adjoint system
\[
P \frac{du}{dx} + Qu = \lambda R u
\]

where
\[
P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q, R \text{ are real + symmetric,}
\]
so
\[
\text{tr}(P^{-1}Q) = \text{tr}(P^{-1}R) = 0
\]
hence the solution matrix is unimodular, and finally \( R \geq 0 \).

So in the class of Nevanlinna matrices is included solution matrices for SL systems
\[(x) \quad (-\frac{d^2}{dx^2} + \beta) u = \lambda u\]
Rewrite this as
\[
\frac{d}{dx} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q^{-1} & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}
\]
or
\[
L(u) = \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u \\ u' \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}
\]
Hence if \( \phi, \phi' \) are solutions of \((x)\) corresponding to different boundary conditions at \(0\), then \((6)\) and \((9)\) resp.
\[
S(\lambda) = \begin{pmatrix} \phi(x, \lambda) & \phi'(x, \lambda) \\ \phi''(x, \lambda) & \phi''(x, \lambda) \end{pmatrix}
\]
is a Nevanlinna matrix all \( \lambda \geq 0 \).

To fix the ideas consider a 5-l system on \( 0 \leq x \leq L \).

1) \[
Lu = -\frac{d^2 u}{dx^2} + gu = \lambda u
\]

I can write it in the form

2) \[
\begin{pmatrix}
\frac{d}{dx}(u')
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-g & 0
\end{pmatrix}
\begin{pmatrix}
u'
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
u
\end{pmatrix}
\]
or
\[
p \frac{d}{dx} u + Q u = \lambda R u
\]
where \( R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

Notice that the Hilbert space associated to the matrix measure \( Rdx \) is just \( L^2((0,\ell)) \). Let \( M(\ell,\lambda) \) be the solution matrix for the system 2) on \( 0 \leq x \leq \ell \). For each \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2 \) we have the solution \( \phi_a(x,\lambda) = M(\ell,\lambda)a \) and we can form the transform
\[
\hat{f}_a(\lambda) = \int_0^\ell (Rf, \phi_a(\lambda)) \, dx = \text{inner product of } f, \phi_a(\lambda) \text{ in } L^2((0,\ell), Rdx).
\]

Then \( \hat{f} : a \mapsto \hat{f}_a \) is a homomorphism of \( \mathbb{R}^2 \) into entire functions associated to each \( f \) in \( L^2((0,\ell), Rdx) \) which we identify with \( L^2(0,\ell) \). So in this way we get a Hilbert space space consisting vector entire functions. The goal now is to understand the point-evaluator which characterizes this Hilbert space.

It's natural to work with entire functions with values in a vector-space. Hence we should think of \( \hat{f} \) as being an entire function with values in the space of conjugation linear functionals on the space of initial values at \( x = 0 \). Since \( \mathbb{C}^2 \) comes equipped with an inner product...
we can think of \( \hat{f}(\lambda) \) as an entire function with values in the initial value space of the system. Thus \( \hat{f}(\lambda) \) is the column vector of entire functions

\[
\hat{f}(\lambda) = \int_0^\lambda \begin{pmatrix} \phi_1(x, \lambda) \\
\phi_2(x, \lambda) \end{pmatrix} f(x) \, dx \quad \text{where} \quad M(x, \lambda) = \begin{pmatrix} \phi_1' & \phi_2' \\
\phi_1 & \phi_2 \end{pmatrix}
\]

so we know what the space of transforms consists of. We next have to describe the inner product. It should suffice to give the point evaluator. Represent the functional \( \hat{f} \mapsto \hat{f}_a(\lambda) = \langle \hat{f}, M(\lambda) a \rangle \) by \( \langle \hat{f}, \hat{M}(\lambda) a \rangle \) and \( \hat{M}(\lambda) a = (RM(\lambda)a, M(\lambda)b)_{[0, \varepsilon]} \)

Recall that \( (\lambda - \lambda)(RM(\lambda)a, M(\lambda)b)_{[0, \varepsilon]} \)

\[
= \left\{ (LM(\lambda)a, M(\lambda)b) - (LM(\lambda)b, LM(\lambda)a) \right\}_{[0, \varepsilon]}
\]

\[
= \left( PM(\lambda)a, M(\lambda)b \right)_0^\varepsilon
\]

\[
= \left( b^\ast M(\lambda)^\ast PM(\lambda)a \right)_0^\varepsilon
\]

Better approach: The Hilbert space of transforms consists of entire functions \( \hat{f} : \mathbb{C} \rightarrow \mathbb{C}^2 \). For each \( a \in \mathbb{C}^2 \) and \( \lambda \in \mathbb{C} \) we have a linear functional

\[
\hat{f} \mapsto \langle \hat{f}(\lambda), a \rangle
\]

which is representable by \( I_{\lambda, a} \). Since
\[
(f, J_{z,a}) = (\hat{f}(z)a) = (Rf, M(z)a)_{[a, e]}
\]

it follows that

\[
\hat{J}_{z,a} = \hat{M}(z)a
\]

hence

\[
(\hat{J}_{z,a}, J_{\lambda, b}) = (\hat{J}_{z,a}(\lambda), b) = (\hat{M}(z)a(\lambda), b)
\]

\[
= (RM(z)a, M(\lambda)b)_{[a, e]}
\]

\[
= [b^* M(\lambda)^* P M(z)a]_0 / (\lambda - \bar{z})
\]

The formulas might be prettier if instead one wants the basic map to be

\[
(z, a) \mapsto \gamma_a(z) \hat{a} = M(z)a
\]

for then the inner products are

\[
<M(\lambda)a, M(z)b> = [b^* M(\lambda)^* P M(z)a]_0 / (\lambda - \bar{z})
\]

\[
\langle M(\lambda)a, M(z)b \rangle = b^* \frac{M(\lambda)^* P M(z)a}{\lambda - \bar{z}} - P a
\]

Question: Given a Nevanlinna matrix \( M(\lambda) \) does one always get a Hilbert space of entire functions defined in this way?

Interesting point related to the fact already noted that a Toeplitz matrix \( (e^{i\pi j}) \) is positive
$\text{semi-definite} \iff g(z) = c_0 + 2 \sum_{n \geq 1} c_n z^n$ have a positive real part in the disk.

Take $a = (1_0)$ and $q = q_a = (q_{1}(x, \lambda))$. Drop the $a_2$ we have

$$M(x, \lambda) a = \begin{pmatrix} q_1(x, \lambda) \\ q_1'(x, \lambda) \end{pmatrix}$$

where $q_1(x, \lambda)$ is the solution of $Lu = -u'' + g u = \lambda u$ with initial values $(1)$. We know that in this case we get a Hilbert space of entire functions with inner product

$$(J_z, J_\lambda) = \int q_1(x, \lambda) \overline{q_1(x, \bar{\lambda})} \, dx$$

$$= \frac{1}{\lambda - \bar{\lambda}} \left( \langle P \varphi(\lambda), \varphi(z) \rangle (\zeta) \right)$$

$$= \frac{1}{\lambda - \bar{\lambda}} \left| \begin{array}{cc} q_1(\lambda) & q_1'(\zeta) \\ q_1'(\lambda) & q_1'\zeta) \end{array} \right| (\zeta)$$

Conversely suppose we are given entire functions $A(\lambda), B(\lambda)$ real on the real axis such that for all $\Im \lambda \neq 0$

$$\left| \begin{array}{cc} A(\lambda) & A(\lambda) \\ B(\lambda) & B(\lambda) \end{array} \right| > 0$$

or equivalently

$$\frac{A(\lambda)}{B(\lambda)} = \frac{A(\bar{\lambda})}{B(\bar{\lambda})} \frac{|B(\lambda)|^2}{|\lambda - \bar{\lambda}|} = \frac{\Im \left( \frac{A(\lambda)}{B(\lambda)} \right)}{\frac{\Im (A(\lambda))}{|B(\lambda)|^2}}$$

for $\Im \lambda \neq 0$. Notice that $A(\lambda) = 0 \Rightarrow 0 = A(\bar{\lambda}) = A(\lambda)$ so that $A, B$ have only real zeroes. Thus 1) amounts
A, B having only real zeroes and

\[ \frac{\text{Im}(\frac{A(\lambda)}{B(\lambda)})}{\text{Im} \lambda} > 0 \quad \text{for } \text{Im} \lambda \neq 0. \]

But then if we put \( E(\lambda) = A(\lambda) + iB(\lambda) \), then it should follow \( E \) is a de Branges function.

\[ \frac{E^*}{E} = \frac{(1 - i)(A)}{(1 + i)(B)} \]

\[ \frac{1}{2i(\lambda - \bar{\lambda})} \begin{vmatrix} E(\lambda) & E(\bar{\lambda}) \\ \overline{E(\lambda)} & \overline{E(\bar{\lambda})} \end{vmatrix} = 2i \begin{vmatrix} A(\lambda) & A(\bar{\lambda}) \\ \overline{B(\lambda)} & \overline{B(\bar{\lambda})} \end{vmatrix} \quad \text{etc.} \]

So by de Branges' theory we know that the matrix

\[ K(\lambda, \bar{\lambda}) = \frac{1}{\lambda - \bar{\lambda}} \begin{vmatrix} A(\lambda) & A(\bar{\lambda}) \\ B(\lambda) & B(\bar{\lambda}) \end{vmatrix} \]

has to be \( > 0 \).

**Remark:** If \( \alpha \in \text{SL}_2(\mathbb{R}) \), then \( \alpha^t \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \) because \( \text{SL}_2(\mathbb{R}) = \text{Sp}_2(\mathbb{R}) \). In fact

\[ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & -ad + bc \\ ad - bc & 0 \end{pmatrix} \]

so that \( \alpha^t \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \iff \det \alpha = 1 \).

Consequently, if \( M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \) is a matrix...
of entire functions which are real (i.e. real on \( \mathbb{R} \)) and det \( M(\lambda) = 1 \), then
\[
M(z)^* PM(\lambda) - P = M(z)^t PM(\lambda) - P
\]
vanishes for \( z = \lambda \) and hence
\[
K(\lambda, z) = \frac{M(z)^* PM(\lambda) - P}{\lambda - z}
\]
is an entire function of \( \lambda \) for any value of \( z \).

**Question:** Is \( K(\lambda, \lambda) \geq 0 \) when \( M \) is a Nevanlinna matrix?

To show \( K(\lambda, \lambda) \geq 0 \) we must prove
\[
(K(\lambda, \lambda) a, a) \geq 0
\]
for all \( a \in \mathbb{C}^n \). If \( a \) is real, then \( (Pa, a) = 0 \), so
\[
(K(\lambda, \lambda) a, a) = \frac{i}{2} \frac{(PM(\lambda) a, M(\lambda) a)}{2 \text{Im} \lambda} \geq 0
\]
because \( M \) Nevanlinna \( \Rightarrow \text{Im} (M(\lambda) a) \geq 0 \) for \( \text{Im} \lambda > 0 \).

If \( a \) is complex, say \( \alpha + i \beta \), then
\[
(PM(\lambda) \alpha, M(\lambda) \alpha) + (PM(\lambda) \alpha, i M(\lambda) \beta) + (i PM(\lambda) \beta, M(\lambda) \alpha) + (PM(\lambda) \beta, M(\lambda) \beta) + (Pa, \alpha) + (Pa, i \beta) + (i Pb, \alpha) + (i Pb, \beta)
\]
January 3, 1978

Lee Yang theorem: Let \( P(z_1, ..., z_n) = \sum a_I z^I \) where

I runs over subset of \( \{1, ..., n\} \) and

\[
a_I = \prod_{j \in I} c_{ij} \quad I' = \{1, ..., n\} - I
\]

and \( c_{ij} \) are numbers of modulus \( \leq 1 \) given for \( i \neq j \) such that \( c_{ij} = \overline{c_{ji}} \). The theorem asserts that if \( |z_1|, ..., |z_n| < 1 \)

then \( P(z_1, ..., z_n) \neq 0 \).

**Question:** Does \( P(z_1, ..., z_n) = \det \left( (\begin{array}{c} \mathbf{z_1} \\ \vdots \\ \mathbf{z_n} \end{array}) + T \right) \) for some unitary matrix \( T \) of determinant \( =1 \)?

**Example:** If \( n=2 \) \( P(z_1, z_2) = 1 + a z_1 + \overline{a} z_2 + z_1 z_2 \quad a = \bar{c}_{12} \)

and

\[
\det \left( \begin{array}{cc} x & a \\ -\overline{b} & y \end{array} \right) = \frac{x y + \overline{a} \overline{y} + |a|^2 + |b|^2}{1}
\]

**First reduction:** Look at the effect of the substitution

\( z_i \rightarrow \epsilon_i z_i \) where \( |\epsilon_i| = 1 \). I claim by such a substitution I can make \( P(\epsilon_1 z_1, ..., \epsilon_n z_n) \) a Lee-Yang polygon with \( 0 < \epsilon_i < 1 \).

In effect, choose \( \theta_{ij} \):

\[
c_{ij} = |c_{ij}| e^{i \theta_{ij}} \quad \text{with} \quad \theta_{ij} = -\theta_{ji}
\]

put \( \theta_{ii} = 0 \)

Then

\[
\prod_{i \in I} c_{ij} = \prod_{i \in I} |c_{ij}| e^{i \sum_{j \in I'} \theta_{ij}}
\]

\[
\sum_{i \in I} \theta_{ij} = \sum_{i \in I} \theta_{ij} + \sum_{j \in I'} \theta_{ij} = \sum_{i \in I} \left( \sum_{j=1}^n \theta_{ij} \right)
\]

by anti-symmetry.
So if I put \( \varepsilon_i = \exp \left( \sqrt{-1} \sum_{j=1}^{n} \theta_{ij} \right) \) I have

\[
a_I = \prod_{i \in I} c_{ii} = \prod_{i \in I} |c_{ii}| \cdot \varepsilon_I
\]

and so the claim is clear. Note \( \varepsilon_1 \ldots \varepsilon_n = 1 \).

On the other side

\[
\det \left( \begin{bmatrix} \varepsilon_1 z_1 & \cdots & \varepsilon_n z_n \\ \vdots & \ddots & \vdots \\ z_m & \cdots & z_n \end{bmatrix} + T \right) = \det \left( \begin{bmatrix} \varepsilon_1 z_1 & \cdots & \varepsilon_n z_n \\ \vdots & \ddots & \vdots \\ \varepsilon_1^{-1} z_1 & \cdots & \varepsilon_n^{-1} z_n \end{bmatrix} \right) \cdot \prod_{i \in I} \varepsilon_i
\]

consequently modulo this substitution the poly depends only on the class of \( T \) in \( SU_n / \text{diag tor.} \approx \text{flag manifold.} \) In fact if we use

\[
\det \left( \begin{bmatrix} \varepsilon_1 z_1 & \cdots & \varepsilon_n z_n \\ \vdots & \ddots & \vdots \\ \varepsilon_1^{-1} z_1 & \cdots & \varepsilon_n^{-1} z_n \end{bmatrix} + T \right) = \det \left( \begin{bmatrix} \varepsilon_1 z_1 & \cdots & \varepsilon_n z_n \\ \vdots & \ddots & \vdots \\ \varepsilon_1^{-1} z_1 & \cdots & \varepsilon_n^{-1} z_n \end{bmatrix} \right)
\]

we see the poly depends only on the image of \( T \) in \( \text{diag tor.} \backslash \text{SU}_n / \text{diag tor.} \approx \text{flag manifold.} \)

The latter has real dimension \( (n^2-n)-(n-1) = n^2-2n+1 - (n-1)^2 \). The set of \( c_{ij} \) has dimension \( n(n-1) \) so this makes one suspect that the possible \( T \)-polys are richer than Lee-Yang polynomials.

Symmetry of Lee-Yang poly:

\[
\overline{a_I} = \prod_{i \in I} c_{ii} = \prod_{i \in I} c_{ii} = a_{I'}
\]

hence

\[
\overline{P(z)} = \sum_{I} a_I \overline{z}^I = \prod_{i=1}^{n} \sum_{I} a_I (z^{I'})^{-1} = \left( \prod_{i=1}^{n} z_i \right) P(\overline{z}^{-1})
\]
\[
\prod_{i=1}^{n} z_i \cdot \prod_{i=1}^{n} \det(\overline{z_i}^* + T) = \prod_{i=1}^{n} z_i \cdot \det(z^{-1} + T)
\]

\[
= \det(z) \cdot \det(z^{-1} + T^*)
\]

\[T^* = \overline{T}^* = T^{-1}\]

\[
= \det(I + zT^{-1})
\]

\[
= \det(T + z) \quad \text{since} \quad \det(T) = 1
\]

Gradually, I am coming to the viewpoint that the right framework for Lee-Yang is polynomials of the form

\[P(z_1, \ldots, z_n) = \det(I - (z_1, \ldots, z_n)T)\]

where \(T\) is unitary. Notice that if \(|z_i| < 1\) then \(z_i T\) is a contraction operator so all eigenvalues are inside the disk and so \(P(z) \neq 0\). But more is true: suppose that \(|z_i| < 1\) and at least one \(z_i\) has modulus < 1. Then \(\|z T\| \leq 1\). What does it mean for \(z T\) to have the eigenvalue 1?

Write \(V = V_1 \oplus V_2\) where \(|z| < 1\) on \(V_1\) and \(|z| = 1\) on \(V_2\). Then \(\|z v\| = \|v\| \iff v \in V_2\). Hence \(\|z T v\| = \|v\| \iff \|z T v\| = \|v\| \iff v \in V_2\). Hence \(z T v = v\) \(\implies v, T v \in V_2\).

So it's clear that if \(z T\) has the eigenvalue 1, then this eigenspace is a subspace of \(V_2\) stable under \(T\). Hence it's clear that...
The above argument is what should explain statements such as if all \( |c_{ij}| < 1 \), then \( P(\zeta) \neq 0 \) if all \( |\zeta_i| < 1 \) and some \( |\zeta_i| < 1 \).

Let's start with a Ly poly \( P(\zeta) = P(\zeta_1, \ldots, \zeta_n) \) and try to write it in the form \( \det(I + \zeta T) \) where \( T \) is unitary. Suppose the \( c_{ij} > 0 \). Write \( P \) as a linear polynomial in \( \zeta_1 \):

\[
P(\zeta) = \sum_{j \in J} \prod_{i \in I} c_{i1} \prod_{j \in J'} c_{ij} \zeta_1^j + \sum_{j \in J'} \prod_{i \in I} c_{ij} \prod_{k \in J} c_{jk} \zeta_1 \zeta_j^k
\]

\[
= Q(c_{11} \zeta_1, \ldots, c_{1n} \zeta_n) + \prod_{j=2}^n Q\left( \frac{z_j}{c_{1j}}, \ldots, \frac{z_n}{c_{1n}} \right) \zeta_1
\]

Suppose \( T \) orthogonal, to simplify. Then

\[
P_1(\zeta) = \det(1 + \zeta T) = \zeta_1 \cdots \zeta_n \det(\zeta T^{-1} + I) = \zeta_1 \cdots \zeta_n \det(I + \frac{1}{\zeta_1} T)\]

\[
\text{since } T^t = T^{-1}
\]

It follows that if

\[
P_1(\zeta) = A(\zeta_1, \zeta_2, \ldots, \zeta_n) + B(\zeta_2, \ldots, \zeta_n) \zeta_1
\]

then

\[
A\left( \frac{1}{\zeta_1} \right) + B\left( \frac{1}{\zeta_1} \right) \frac{1}{\zeta_1} = \frac{1}{\zeta_1 \zeta'} (A(\zeta') + B(\zeta') \zeta_1)
\]

so

\[
B(\zeta') = \zeta'A\left( \frac{1}{\zeta_1} \right) \quad \text{And} \quad P_1(\zeta) = A(\zeta') + A\left( \frac{1}{\zeta_1} \right) \bar{\zeta} \zeta_1
\]
\[ P_1(z) = \begin{vmatrix} 1 + z_1 t_{11} & z_1 t_{12} & \cdots \\ z_2 t_{21} & 1 + z_2 t_{22} & \cdots \end{vmatrix} \]

Set \( z_1 = 0 \) to find \( A(z') \).

You get
\[ A(z') = \det (1 + z'S) \]

where
\[ S = \begin{pmatrix} t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \vdots \\ t_{n2} & \cdots & t_{nn} \end{pmatrix} \]

The idea is to find \( T \) so that \( P(z) = P_1(z) \), using by induction the fact that
\[ Q(z') = \det (1 + z'T') \]

for some orthogonal \( T' \) of \( \det 1 \). If this is to work we must have
\[ Q(c'z') = \det (1 + c'z'T') = \det (1 + z'S) \]

where \( c' = \begin{pmatrix} c_{21} & \cdots \\ \vdots & \ddots \\ c_{n1} \end{pmatrix} \). So the problem becomes this:

Given \( c' \) diagonal \( 0 < c' < I \) and \( T' \) orthogonal of determinant 1, can you find an \( T \) with
\[ \det (1 + c'z'T') = \det (1 + z'S) \]

This equation is probably equivalent to \( c'T' \) and \( S \) being conjugate by diagonal matrices. First possibility to try is
\[ S = (c')^{-1/2} T (c')^{-1/2} \]

This is a contraction operator. If there is an orthogonal matrix \( T \) extending it, then we get
\[ \det (1 + zT) = \det (1 + z'S) + \det (1 + \frac{1}{2} z') z' z_1 = \det (c'z') + \det (c'z_1) = P(z) \]

which is what we want.
Let \( T \) be a unitary \( n \times n \) matrix and \( S \) its lower \([2,n] \times [2,n]\) block.

\[
\begin{pmatrix}
t_{11} \\
t_{21} \\
t_{n1}
\end{pmatrix}
\]

The rows of \( T \) are orthonormal vectors. Hence if \( \mathbf{v}_i = (t_{i1}, \ldots, t_{in}) \) one has
\[
t_{i1} \overline{t}_{j1} + \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} \quad 2 \leq i, j \leq n
\]
i.e.
\[
\delta_{ij} - \langle \mathbf{v}_i, \mathbf{v}_j \rangle = (t_{i1})\begin{pmatrix} \overline{t}_{21} & \cdots & \overline{t}_{2n} \\ \vdots & \ddots & \vdots \\ \overline{t}_{n1} & \cdots & \overline{t}_{nn} \end{pmatrix}
\]

Hence we see that
\[
I - SS^* = \mathbf{v}\mathbf{v}^* \quad \mathbf{v} = \begin{pmatrix} t_{21} \\ \vdots \\ t_{n1} \end{pmatrix}
\]

Thus a necessary condition that a matrix \( S \) come from a unitary \( T \) in the above way is that \( I - SS^* \) be of rank 1 and \( \geq 0 \).

Conversely, given \( S, \mathbf{v} \) with \( I - SS^* = \mathbf{v}\mathbf{v}^* \) we reverse the above procedure to get a \((n-1) \times n\) matrix \((\mathbf{v}, S)\) whose rows are orthogonal unit vectors. Then there is a unique unit vector up to a scalar of modulus 1 orthogonal to these rows, so we get a unique choice for \( T \) having prescribed determinant.
So the question is whether given a unitary \( (n-1) \times (n-1) \) matrix \( U \) and a diagonal matrix \( 0 < c < 1 \) can we find a diagonal invertible matrix \( d \) such that
\[
dc Ud^{-1} = S
\]
extends in the above way to a unitary matrix \( UT \). So we want
\[
I - SS^* = I - dc Ud^{-1}(d^*)^{-1}U^{-1}cd^*
\]
to be of the form \( v v^* \). Put \( d^{-1} = S \). Then we want
\[
SS^* = (cU) SS^* (U^*c)
\]
to be \( \geq 0 \) of rank 1. Thus we might as well assume the diagonal entries of \( S \) are \( > 0 \). We want
\[
c^{-2}(S^2) = U(S^2)U^*
\]
to be \( \geq 0 \) of rank 1. Not clear why \( S^2 \) should have this property.