March 5, 1977

Let's go back to the one-sided $J$-matrix problem. Suppose $\bar{J}$ is a two-sided $J$-matrix and $\overline{J_+}$ is the associated one-sided matrix. We fix $\lambda$. The corresponding $\overline{J_+}$ eigenfunction is given by

$$(y_n, y_{n+1}) = (y_{n-1}, y_n) \begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ \frac{\lambda - b_n}{a_n} & 1 \end{pmatrix}$$

starting from $$(y_0, y_1) = (0, 1)$$. Thus

$$(y_n, y_{n+1}) = \begin{pmatrix} 0 & -\frac{c_0}{a_1} \\ \frac{\lambda - b_1}{a_1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ \frac{\lambda - b_n}{a_n} & 1 \end{pmatrix}$$

$$\overline{\Phi_n}(\lambda)$$

How is this related to the continued fraction?

If we put

$$\overline{J_n} = \begin{pmatrix} b_1 & a_1 & & & \\ c_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1} & b_n & a_n \\ e_{1} & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$B_n = \det(\lambda I - \overline{J_n})$$, then we get

$$B_n = -(-c_{n-1})(-a_{n-1})B_{n-2} + (\lambda - b_n)B_{n-1}$$

$$B_n = -c_{n-1}a_{n-1}B_{n-2} + (\lambda - b_n)B_{n-1}$$
Consequently

\[ y_{n+1}(\lambda) = \det(A - J_n) \]

This is also clear because we know that both sides are polynomials of degree \( n \) with the same leading terms and the same number of distinct roots. Note \( J_n \) has simple eigenvalues, for there is a cyclic vector.

When is this function \( y_n \) bounded in \( n \)?

We have fractional linear transformation \( \Phi_n(A) \)

carrying \( \text{Im}(z) > 0 \) strictly into itself if \( \text{Im}(\lambda) > 0 \),

\[ \lim_{n \to \infty} \Phi_n(A)^{2} \{ \text{Im} z > 0 \} \]

is either a limit circle or a limit point. In the latter case

\[ \frac{y_n(\lambda)}{y_{n+1}(\lambda)} \to \infty \]

Suppose \( J \) periodic with period \( \tau \). There are \((\tau - 1)\) distinct \( \lambda \) such that

\[ y_{2}(\lambda) = \frac{\det(A - J_{n-1} - J_{1})}{a_{1} \cdots a_{n-1}} = 0. \]
For each of these $F(\lambda)$ has the eigenvector $(0, 1)$ which is real, hence the eigenvalue $\varepsilon$ must be real. This implies that $\lambda$ is outside the interior of the bands.

**Problem:** For each $\lambda$ such that $y_n(\lambda) = 0$ determine whether $|y_{n+1}(\lambda)|$ is $> 1$ or $< 1$.

If $|y_{n+1}(\lambda)| = 1$, then $y_{n+1}(\lambda) = \pm 1$ and so we are in the situation where $\varepsilon = 1$ or $-1$, hence there is a periodic or half-periodic solution. So it is not in $L^2$.

**Isospectral deformation.** Suppose $L(t)$ is a one-parameter family of operators of the form

$$L(t) = U(t) L_0 U(t)^{-1}$$

Then

$$L_t = U_t L_0 U^{-1} - U L_0 U^{-1} U_t U^{-1}$$

where $L_t = \frac{\partial L}{\partial t}$ etc. So

$$L_t = B L - L B = [B, L]$$

where $B = U_t U^{-1}$. If $U(t)$ is unitary, then

$$U U^* = 1 \quad \Rightarrow \quad U_t U^* + U U_t^* = 0$$

$$\Rightarrow \quad B + B^* = 0 \quad \Rightarrow \quad B \text{ skew-adjoint}$$
Lax applies this to \( L = \partial^2 + g \) (\( \partial = \frac{\partial}{\partial x} \) \( g \) a function of \( x, t \)) when \( \dot{L} = \partial_t^* \). He tries to construct a \( \mathcal{B} \) which works which is a differential operator.

If \( \mathcal{B} = a \partial + b \) is skew-adjoint, then

\[
\mathcal{B}^* = -a \partial - bx + b = -a \partial - b
\]

\[
\Rightarrow \quad a_x = 2b \quad \text{or} \quad b = \frac{1}{2} a_x
\]

So if \( \mathcal{B} = \partial^3 + a \partial + \frac{1}{2} a x \), then

\[
[B, L] = \begin{bmatrix} \partial^3 + a \partial + \frac{1}{2} a_x, \partial^2 + g \end{bmatrix} = 3g_x \partial^2 + 3g_{xx} \partial + g_{xxx} -2a_x \partial^2 - a_{xx} \partial + a_{xx} -a_{xx} \partial - \frac{1}{2} a_{xxx}
\]

\[
= (3g_x - 2a_x) \partial^2 + (3g_{xx} - 2a_{xx}) \partial + (g_{xxx} - \frac{1}{2} a_{xxx} + a_{xx})
\]

So if \( a = \frac{3}{2} g \), we have.

\[
[B, L] = \frac{1}{4} g_{xxx} + \frac{3}{2} g_{xx}
\]

This should be equal to \( g_t \). Thus if

\[
\frac{1}{4} g_{xxx} + \frac{3}{2} g_{xx} = g_t
\]

one has at least in some formal sense \( g \) that as \( t \) varies the operators \( L = \partial^2 + g \) are all conjugate and hence have the same spectra.
More lax:

Let $M$ be the (infinite-diml) manifold of functions $u(x)$ say for $u \in A$. Consider functions $F : M \to \mathbb{R}$ say for example

$$F(u) = \int_0^1 L(x, u, u_x) \, dx$$

Then one defines its derivative in the direction $v$

$$F(u + \varepsilon v) = \int_0^1 L(x, u + \varepsilon v, u_x + \varepsilon v_x) \, dx$$

$$F(u + \varepsilon v) - F(u) = \varepsilon \int_0^1 \left( \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x \right) \, dx + O(\varepsilon^2)$$

$$\frac{d}{d\varepsilon} F(u + \varepsilon v) \bigg|_{\varepsilon=0} = \int_0^1 \left( \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x \right) \, dx$$

and one expresses this as an inner product

$$\frac{d}{d\varepsilon} F(u + \varepsilon v) \bigg|_{\varepsilon=0} = \left( G_F(u), v \right)$$

where $G_F$ is the gradient of $F$. In this example

$$G_F = \frac{\partial L}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial u_x} \right)$$

Now what?
March 6, 1977.

Let us relate unimodular transforms shrinking the unit circle $|z| \leq 1$, with those shrinking the unit disk $|z| < 1$.

Let $M$ be the monoid of unimodular transformations carrying $\text{Im} \ w > 0$ into itself. Then

\[ M = \text{PSL}_2(\mathbb{R}) \]

in fact, $\text{PSL}_2(\mathbb{R})$ is the set of invertible elements of $M$. Also $M$ contains the translations $w \mapsto w + t$ with $\text{Im} \ t \geq 0$.

Consider the set of circles in $\text{Im} \ w > 0$; do not include lines. $\text{Im}(w) = \text{constant}$ for they meet $\text{Im}(w) = 0$ at $\infty$. $\text{PSL}_2(\mathbb{R})$ acts on $\mathcal{C}$. One can successively reflect to obtain a sequence of circles

\[ C_0 = \{ \Re u \geq 0 \}, \quad C_1 = C, \quad C_2 = \text{reflection of } C_0 \text{ in } C_1, \quad \text{etc.} \]

which converge to a point in the UHP. To see this, draw the circles orthogonal to $C_0$ and $C_1$. So it is now clear that if we move this limit point to $w = i$, then the rotations around $i$ leave $C$ fixed. The
for now let $\Theta \in PSL(2, \mathbb{C})$ induce the unit disk $D$

in its induced $\Theta(1,0)(D) = \Theta(D)$ with $0<1<2$, hence

$\Theta^{-1}(1,0)(D) = \Theta^{-1}(D)$ is concentric with $D$

and $\Theta(0,1)(D)$ is concentric with $\overline{D}$

$\Theta^{-1}(0,1)(D)$ is concentric with $\overline{D}$ and $\Theta(0,1)(D)$ is concentric with $D$

$\Theta^{-1}(0,1)(D)$ is concentric with $\overline{D}$ and $\Theta(0,1)(D)$ is concentric with $D$

and there is a homothety $\Theta$ with $\Theta(0,1)(D)$ coincides with an infinite transformation $\Theta(D)$ such that

$\Theta^{-1}(0,1)(D)$ is concentric with $\overline{D}$ and the

$\Theta^{-1}(0,1)(D)$ is concentric with $\overline{D}$ and $\Theta(0,1)(D)$ is concentric with $D$

and the stabilizers of any circle $S^1$.

and the invariant of $\mathbb{C}$ in its radius, i.e. the main block of

and the stabilizers of any circle $S^1$. Conclude
Note that $\phi_1$ is unique up to a rotation, hence if $H$ is the subgroup preserving $2D$, then one has

$$H \times S^1 \{ x \mid |x|<1 \} \times S^1$$

for the monoid $\tilde{M} = \{ \Theta \in \text{PSL}_2(\mathbb{C}) \mid \Theta(D) \subset \text{Int}(D) \}$. Observe that one gets the right dimension for $M$:

$$3 + 1 + (3-1) = 6.$$
such that
\[
f(\text{Im } z > 0) \subset \overline{M}
\]
\[
f(\text{Im } z < 0) \subset (\overline{M})^{-1}
\]
such \( f \) are determined by the restriction to \( R \) which
as a map \( f: R \to \Gamma' \cong \text{PSL}_2(R) \).

First problem is to get a 2D related to \( R \).
So I want to convert the matrices occurring in Lee-Yang
to the ones occurring in the Jacobi problem. Any isom
\[
|y| < 1 \iff \text{Im}(\omega) > 0
\]
will be unique up to an element of \( \text{PSL}_2(R) \). The
simplest is given by
\[
y : i \iff 0 : \omega \quad \quad \quad y(\omega i + 1) = +i \omega + 1
\]
\[-1 \iff \infty \quad \quad \quad -(y + i) \omega = 1 - y
\]
\[
\begin{cases}
\frac{1 + i \omega + 1}{-i \omega + 1} \\
\frac{-y - 1}{i \frac{y - 1}{y + 1}}
\end{cases}
\]
\[
\omega = \frac{1 - y}{i(1 + y)} = -i \frac{y - 1}{y + 1}
\]
\[
y = 0 \iff \omega = i
\]

Thus multiplication by \( i^2 \) in the \( \gamma \) plane
\[
\frac{1}{2\gamma} (i \, i) (\lambda^{-1} \, i) (i \, i) = (i \, i) (-i) (i \, i) = \frac{1}{2\gamma} (i \, i \, i \, i) (i \, i \, i \, i)
\]
If \((a \ b) \in \text{SL}_2(\mathbb{R})\) has \(i\) as fixed point:

\[
\frac{ai+b}{ci+d} = i
\]

\[
\Rightarrow ai+b = -c+di \Rightarrow a=d, \ b=-c
\]

and

\[
ad-bc = a^2+b^2 = 1.
\]

Thus the rotation matrices \((\cos \theta \ \sin \theta \ -\sin \theta \ \cos \theta)\) leave \(i\) fixed. Observe that for \(\theta = \pi\) this gives \(-I\) which acts as identity in UHP. Thus:

\[y \mapsto e^{i\theta} y\]

corresponds to

\[
\begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-(\sin \frac{\theta}{2}) & (\cos \frac{\theta}{2})
\end{pmatrix}
\]

Thus the transformations leading to Lee-Yang polyno:

\[
\begin{pmatrix}
z^{\frac{1}{2}} & 0 \\
0 & z^{-\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
\]

\[z = e^{i\theta}\]

\[z = e^{i\theta}\]

can be more generally any elt. of \(\Gamma\) will correspond to the transformations

\[
\begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-(\sin \frac{\theta}{2}) & (\cos \frac{\theta}{2})
\end{pmatrix}
\begin{pmatrix}
1-a & 0 \\
0 & 1+a
\end{pmatrix}
\]

\[w \mapsto \frac{(1-a)w}{1+a}\]

no need \(-1 < a < 1\).

can be more generally any elt. of \(\text{SL}_2(\mathbb{R})\).
I am interested in all matrices of the form
\[
\Phi(\theta) = (R_{\theta/2} A_1) (R_{\theta/2} A_2) \cdots (R_{\theta/2} A_n)
\]
where \( A_1, \ldots, A_n \in \mathbb{PSL}_2(\mathbb{R}) \) and \( R_{\theta/2} = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \).

Putting
\[
B_i = A_i A_{i+1} \cdots A_n
\]
we have
\[
A_i = B_i B_{i+1}^{-1} \quad \text{for } i = 1, \ldots, n-1 \quad \text{and} \quad A_n = B_n = B_n B_1^{-1} \quad \text{if } B_1 = A_1, \ldots, A_n = 1.
\]

\[
\Phi(\theta) = R_{\theta/2} B_1 (B_2^{-1} R_{\theta/2} B_2) \cdots (B_n^{-1} R_{\theta/2} B_n)
\]
Consequently, we see that \( \Phi(\theta) \) is \( B_1 \) times conjugates of the basic loop \( \Theta \rightarrow R_{\theta/2} \) in \( \mathbb{PSL}_2(\mathbb{R}) \).

Let \( M = \{ \Theta \in \mathbb{PSL}_2(\mathbb{R}) \mid \Theta H = \text{Int } \overline{H} \} \)
and let \( \overline{M} = \text{closure of } M \). To describe elements of \( \partial M - \mathbb{PSL}_2(\mathbb{R}) \), there are \( \Theta \) carrying a "generalized" circle in \( \overline{H} \), tangent to \( \partial H \) at one point.

Changing \( \Theta \) to \( \Phi' \Theta \) with \( \Phi \in \Gamma \), we can assume \( \Phi' \Theta(\partial H) \) tangent to \( \partial H \) at \( \infty \), hence \( \Phi^{-1} \Theta(\partial H) = \{ w \mid \text{Im}(w) = c > 0 \} \). Thus we see that if \( \tau_c(\omega) = \omega + c \),
then \( \Phi^{-1} \Theta(\partial H) = \tau_c(\partial H) \).
So \( \Theta = \Phi^{-1} \tau_c \Phi', \) some \( \Phi', \in \Gamma \).
Put \( B = \{(a, b) \in \Gamma^2 \mid a > 0, \ b \in \mathcal{R} \} \). Then we get
the description
\[
\Gamma \times B \begin{cases} 
(\frac{a}{a^{-1}},) \mid \text{Im} \tau > 0 \end{cases} \times B \Gamma
\]
for those \( \Theta \in G = \text{PSL}_d(\mathcal{C}) \) such that \( \Theta \in \overline{\mathcal{M}} - \mathcal{M} - \Gamma \).
Now
\[
\Gamma \backslash \Gamma / B \cong B \mid \mathcal{P}(R) = 2 \cdot \text{pts}.
\]

Given a linear Ising chain with periodic conditions does there exist an eigenvalue problem whose characteristic roots are the roots of the partition function?
Thus the partition function is the trace of a certain 2x2 matrix which one can adjust to have determinant +1.
For the trace to vanish means then that it has the eigenvalues +1, -1.

Look at the linear Ising chain with the same constants
\[
P(z) = \text{tr} \left[ \begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & a \\ \frac{\alpha}{V_1 - \alpha z} & \frac{1}{V_1} \end{pmatrix} \right]^n = \lambda_+^n + \lambda_-^n.
\]

Make this matrix of determinant +1
\[
P(z) = \text{tr} \left( \begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix} \frac{1}{V_1 - \alpha z} \right)^n = \lambda_+^n + \lambda_-^n
\]
\( \lambda_+ \lambda_- = 1 \)

has trace \( \frac{z^{\frac{1}{2}} + z^{-\frac{1}{2}}}{V_1 - \alpha z} \)
\[ P(z) = 0 \implies (\lambda_+/\lambda_-)^n = -1, \quad \lambda_+/\lambda_- = e^{\frac{2i(2d+1)}{n} \pi i} \]

\[ \lambda_\pm = e^{\pm \frac{2i(2d+1)}{2n} \pi i} \]

so the roots of \( P(z) \) are given by \( z = e^{i\varphi} \) where

\[
\frac{\cos(\varphi/2)}{\sqrt{1-a^2}} = \cos\left(\frac{2d+1}{n} \pi \right)
\]

\[
\cos(\varphi/2) = \sqrt{1-a^2} \cos\left(\frac{2d+1}{2n} \pi \right)
\]

\[
\cos \varphi = 1 \cos^2\left(\frac{\varphi}{2}\right) - 1 = 2(1-a^2) \cos\cos\left(\frac{2d+1}{2n} \pi \right) - 1
\]

\[
= (1-a^2) \left( 1 + \cos\left(\frac{2d+1}{n} \pi \right) \right) - 1
\]

\[
\cos \varphi_j = -a^2 + (1-a^2) \cos\left(\frac{2j+1}{n} \pi \right)
\]

Visualize this as homotopy with parameter \( t = a^2 \) moving the root pair \( \exp(\pm i\left(\frac{2d+1}{n} \pi \right)) \) to \(-1\).

If \( n \) is even

\[
P(z) = z^{\frac{-n}{2}} \prod_{j=1}^{n/2} \left( z^2 - 2\left(-a^2 + (1-a^2) \cos\left(\frac{2j+1}{n} \pi \right) \right)^2 + 1 \right)
\]

and if
\[ P(z) = z^{-n} (z+1)^{[n/2]} \prod_{j=1}^{r} \left( z^2 - 2 \left( -a^2 + (1-a^2) \cos \left( \frac{2j+1}{n} \pi \right) \right) z + 1 \right) \]

What is the Heilman-Lieb dimer limit of this partition function? Here \( a = e^{-J \beta} \) and \( \beta \to 0 \) whence \( a \to 1 \). Then all the roots tend to \( z = -1 \).

\[
\left[ z^4 - 2 (1 - a^2)(1 + \cos(\frac{2j+1}{n} \pi)) - 1 \right] + z^{-1}
\]

\[
= (z + 2 + z^{-1}) - 2 (1 - a^2)(1 + \cos(\frac{2j+1}{n} \pi))
\]

\[ 1 - a^2 = -e^{-2J \beta} = 2J \beta - 2J^2 \beta^2 + \ldots \]

So to have an interesting limit one wants

\[ \frac{z+2+z^{-1}}{\beta^{1/2}} = \frac{(\frac{1}{2} + \frac{1}{2})^2}{\beta} \]

\[
\frac{z + 2 + z^{-1}}{1/2} = \frac{(\frac{1}{2} + \frac{1}{2})^2}{1/2} \]

So if one puts

\[ \frac{z^{1/2} + z^{-1/2}}{\beta^{1/2}} = 2x \]

\[
\cos \frac{\phi}{2} = \beta^{1/2} x \quad \sin \frac{\phi}{2} = \sqrt{1 - \beta x}
\]

\[
\begin{cases}
  z^{1/2} = \beta^{1/2} x + i \sqrt{1 - \beta x} \\
  z^{-1/2} = \beta^{1/2} x - i \sqrt{1 - \beta x}
\end{cases}
\]

\[ n = 2m+\varepsilon \quad \varepsilon = 0, 1 \]
\[ P(z) = \left( z^{\frac{1}{2}} + z^{-\frac{1}{2}} \right)^{n} \prod_{j=1}^{m} \left( z + 2 + z^{-1} - 2 (2J) \left( 1 + \cos \left( \frac{2j + 1}{n} \pi \right) \right) \right) \]

\[ = 2 \beta^{n/2} x^{2} \prod_{j=1}^{m} \left( x^{2} - 2J \left( 1 + \cos \left( \frac{2j + 1}{n} \right) \right) \right) \]

\[ = 2 \beta^{n/2} \prod_{j=1}^{m} \left( x - \sqrt{2J \cos \left( \frac{2j + 1}{n} \pi \right)} \right) \]

which is essentially the Chebyshev polynomial \( \cos \left( \frac{\pi \cos^{-1}(\frac{x}{\sqrt{2J}})}{n} \right) \).

Is it generally true that the dimer polynomial is obtained from an eigenvalue problem?

Review limit procedure before

\[ z = e^{i\eta} \quad \cos \frac{\eta j}{2} = \sqrt{1 - \frac{a^{2}}{n^{2}}} \cos \left( \frac{2j + 1}{n} \pi \right) \]

Now we put \( z^{\frac{1}{2}} = e^{i\eta/n} \), i.e. \( \eta_{j} = \frac{\eta j}{n} \) and let \( a \rightarrow \frac{\eta}{n} \)

\[ \cos \left( \frac{\eta_{j}}{2} \right) = \left( 1 - \frac{a^{2}}{n^{2}} \right)^{\frac{1}{2}} \cos \left( \frac{2j + 1}{n} \frac{\pi}{2} \right) \]

\[ J \left( 1 - \frac{1}{2} \left( \frac{\eta_{j}^{2}}{n^{2}} \right) \right) = \left( 1 - \frac{a^{2}}{2n^{2}} \right) \left( 1 - \frac{1}{2} \left( \frac{2j + 1}{n} \pi \right)^{2} \frac{1}{n^{2}} \right) \]

\[ u_{j}^{2} = a^{2} + \left( \frac{(2j + 1)\pi}{n} \right)^{2} \]

Limiting Roots are \( u_{j} = \pm \sqrt{a^{2} + \left( \frac{(2j + 1)\pi}{n} \right)^{2}} \).
An important point to notice is that letting \( a \to 0 \) corresponds to \( \beta \to 0 \), that is, to the low-temperature limit. Now if we fix the particular Ising system under consideration then the low-temp limit is \( 1 + z^n \).

Recall \( M = \{ \Theta \in SL_2(\mathbb{C}) \mid \Theta(\text{Im}(w)) > 0 \} \subset \text{Im}(w) > 0 \} \).

Consider a 1-parameter subgroup \( e^{tA} \) in \( SL_2(\mathbb{C}) \). It induces a flow in \( P_1(\mathbb{C}) \) which sends \( w \) to \( e^{tA}w \) at time \( t \). Thus we get a vector field on \( P_1(\mathbb{C}) \) and we can ask when the flow carries the UHP \( H \) into itself.

If \( \Theta = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \) then

\[
e^{tA} \cdot w = \frac{a(t)w + b(t)}{c(t)w + d(t)}
\]

\[
\frac{d}{dt} A(t) w \bigg|_{t=0} = \left( \frac{(c(t)w + d(t))(a'(t)w + b'(t)) - (a(t)w + b(t))(c'(t)w + d'(t))}{(c(t)w + d(t))^2} \right)_{t=0}
\]

\[
= \frac{1}{2} \left( a'(0)w + b'(0) - w(c'(0)w + d'(0)) \right)
\]

\[
= -c'(0)w^2 + (a'(0) - d'(0))w + b'(0)
\]

Now\( \begin{pmatrix} a'(0) & b'(0) \\ c'(0) & d'(0) \end{pmatrix} = \frac{d}{dt} e^{tA} \bigg|_{t=0} = A \)

So if \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) we want

\( w \in \mathbb{R} \Rightarrow \text{Im} \left( -\gamma w^2 + (\alpha - \delta)w + \beta \right) > 0 \)

\[-w^2(\text{Im} \gamma) + w(\text{Im} \alpha - \text{Im} \delta) + \text{Im} \beta > 0 \]
which is the case iff
\[(\text{Im} \alpha - \text{Im} \delta)^2 \leq -4(\text{Im} \theta)(\text{Im} \beta)\] \[\text{Im} \delta < 0 \quad \text{Im} \beta > 0\]

If \(A\) has trace zero, then \(\text{Im} \delta = -\text{Im} \alpha\), so this becomes
\[(\text{Im} \alpha)^2 \leq (-\text{Im} \delta)(\text{Im} \beta) \quad \text{and} \quad \text{Im} \delta < 0, \quad \text{Im} \beta > 0,\]

Thus if \(A = A_0 + iB_0\) with \(A_0, B_0\) real, then \(A_0\) can be arbitrary while \(B_0 = \begin{pmatrix} r & q \\ -q & r \end{pmatrix}\) with \(r \geq 0, g > 0\) and \(\det(B_0) = -r^2 + qr > 0\).

March 9, 1977.

To classify all \(u \mapsto A(u)\) holomorphic such that
\[\text{tr} A(u) = 0, \quad u \in \mathbb{R} \Rightarrow A(u) \text{ real, } \text{Im}(u) > 0 \Rightarrow A \text{ satisfies } \text{(x)} \text{ above, and } \text{Im}(u) < 0 \Rightarrow A(u) \text{ satisfies the opposite of } \text{(x)}.\]

\[\overline{A(u)} = A(u) \Rightarrow \text{Im} A(\bar{u}) = -\text{Im} A(u).\]

\[A(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & -a(u) \end{pmatrix}, \text{ then }\]
\[\begin{array}{c}
\text{Im}(u) > 0 \Rightarrow \text{Im}(b(u)) > 0 \\
\text{Im}(u) = 0 \Rightarrow \text{Im}(b(u)) = 0 \\
\text{Im}(u) < 0 \Rightarrow \text{Im}(b(u)) < 0
\end{array}\]

The open mapping thm. \(\Rightarrow\) if \(b \neq 0\), then \(\text{Im}(u) > 0\).
\[ \Rightarrow \text{Im}(\text{blu}) > 0. \]  Thus \( b^{-1}(1) = \text{IR}. \)  The map \( b: \text{IR} \to \text{IR} \) is etale: suppose \( b'(x) = 0; \) translate \( b \) so that \( x = 0 \) and \( b = 0; \) then \( b(z) = z^n(a_0 + a_1 z + \ldots) \) around the origin looks like \( z \to z^n, \) which does not preserve \( \text{Im} z > 0 \) for \( n > 1. \)  By Picard an entire function misses at most one value, hence \( b: \text{IR} \to \text{IR} \) must be an isomorphism.  Similarly by symmetry of \( b(z) \) under conjugation, one sees that \( b(c) = c. \)  \( b \) has a single zero, so
\[ b(z) = z e^{h(z)} \]
with \( h(z) \) entire, translating so that the zero is at zero.
Scaling we can assume \( h(0) = 1. \)  Note that \( h \) is real: \( \overline{h(z)} = h(\overline{z}). \)

Look at the critical values of the map \( b: \text{C} \to \text{C}. \)  Note that the tangent space map is either 0 or an isomorphism, and is never of rank 1.  Thus the inverse image of a smooth embedded curve in \( \text{C} \) under \( b \) which avoids the critical points is a smooth codim. 1 submanifold of \( \text{C} \) etale over the given curve.

Wait: the big Picard theorem says that in a nbhd of essential singularity at most one value is omitted, I think.  This means that if \( b \) is not a polynomial, then for some \( a \in \text{IR} \) \( b(z) - a \) has infinitely many zeroes.

So it seems that the only possibility for \( b(z) \) is
\[ b(z) = g_1 z + h_1 \quad h \in \mathbb{R}, \ g_1 > 0. \quad \text{Similarly} \]
\[ c(z) = -g_2 z + h_2 \quad h_2 \in \mathbb{R}, \ g_2 \geq 0. \]

\[ \text{Im}(a(w))^2 \leq g_1 g_2 \text{Im}(u)^2 \]
\[ |\text{Im}(a(w))| \leq \sqrt{g_1 g_2} |\text{Im}(u)|. \]

\[ |e^{i a(w)}| = e^{\text{Re}(i a(w))} = e^{-\text{Im}(a(w))} \leq e \frac{|\text{Im}(a(w))|}{e} \leq e^{\text{Im}(a(w))} \]

so \( e^{i a(w)} \) is an entire function of exp. type without zeroes hence is \( e^{P(z)} \), deg \( P(z) \leq 1 \). \( \Rightarrow \ a(w) \) linear in \( u \).

Therefore it seems that the only possibilities for a holom. function

\[ u \mapsto \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} \]

of the good type is linear

\[ \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} + u \begin{pmatrix} P & 0 \\ -k & -p \end{pmatrix} \]

arbitrary \( g, n \geq 0, \ g n - p^2 \geq 0. \)

In the past we’ve seen the following examples for \( (P, g) \):

\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -p \end{pmatrix} \]
But note
\[
\begin{pmatrix}
d - b \\
-c & a
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
+ b & d \\
- a & - c
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
\[
= \begin{pmatrix}
ab + cd & b^2 + d^2 \\
-a^2 - c^2 & -ab - cd
\end{pmatrix}
\]

Observe that any \( \begin{pmatrix}
p & q \\
r & -p
\end{pmatrix} \) with \(-p^2 + qr = 1\)
and \(q > 0, r > 0\) is in this form. In fact, take vector \((a, c)\) of length \(\sqrt{r^2}\), \((b, d)\) of length \(\sqrt{q}\) and with angle \(\theta\) between them given by
\[
\cos \theta = \frac{p}{\sqrt{qr}}
\]
so that \((a, c) \cdot (b, d) = \frac{p}{\sqrt{qr}} \cdot \sqrt{q} \cdot \sqrt{r} = p\). Then from the identity
\[
(a + cd)^2 + (ad - bc)^2 = a^2b^2 + c^2d^2 + a^2d^2 + b^2c^2 = (a^2 + c^2)(b^2 + d^2)
\]
one gets \(ad - bc = \pm 1\), and you can make the sign +1 by interchanging \((a, c), (b, d)\).

Next
\[
\begin{pmatrix}
d - b \\
-c & a
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
0 & + d \\
0 & - c
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
\[
= \begin{pmatrix}
cd & d^2 \\
-c^2 & -cd
\end{pmatrix}
\]
\[
\begin{pmatrix}
\begin{array}{ccc}
   d & -b & \ 0 & 0 \\
   -c & a & -1 & 0 \\
   0 & 0 & c & d
\end{array}
\end{pmatrix}
\begin{pmatrix}
   a & b \\
   c & d
\end{pmatrix}
= 
\begin{pmatrix}
   a & b \\
   -c & a
\end{pmatrix}
\begin{pmatrix}
   b & 0 \\
   0 & b^2
\end{pmatrix}
= 
\begin{pmatrix}
   +ab & b^2 \\
   -c^2 & -ab
\end{pmatrix}
\]

Now given \( \begin{pmatrix}
   P & Q \\
   -r & -p
\end{pmatrix} \) with \( g, r \geq 0 \) and \( p^2 = gr \).

Put \( d = \sqrt{g} \) \( c = \pm \sqrt{r} \) \( cd = p \).

At least if not both \( g, c \) are zero, then not both \( c, d \) are zero so we can find \( a, b \) with \( ad - bc = 1 \).

**Summary.**

**Proposition:** Let \( A \in SL_2(C) \). Then the flow on \( P_1(C) \) induced by \( e^{tA} \) carries \( H \) into itself for \( t \geq 0 \) iff \( \text{Im}(A) \) has the form

\[
\begin{cases}
   \text{Im}(A) = \begin{pmatrix}
   P & Q \\
   -r & -p
\end{pmatrix} \\
   q \geq 0, r \geq 0 \text{ and } p^2 = gr.
\end{cases}
\]

The group \( SL_2(R) \) acts on the matrices of the form (*) and a cross-section for the orbits is the set

\[
\begin{pmatrix}
   0 & 1 \\
   -r & 0
\end{pmatrix} \quad r \geq 0.
\]