December 3, 1977:

$H \subset l^2$ has the operator $L$

$H_2 = \text{Hilbert space of } u(t) \in l^2 \to \langle u(t) \sideset{\sum}{\wedge} \rangle = \frac{u(t+1) + u(t-1)}{2}$

We have produced an isomorphism

$L^2(S^1, 4\pi |\beta|^2 d\theta) \sim H_2$

$x \mapsto u(n, t) = \int e^{it} \phi(n, \lambda) \frac{\sin \theta}{\sin \Theta} d\Theta$

Moreover $x$ even $\Rightarrow \frac{x}{\sin \Theta}$ odd $\Rightarrow u(-t) = -u(t)$

and $x$ odd $\Rightarrow \frac{x}{\sin \Theta}$ even $\Rightarrow u(t) = u(-t)$

Recall $H_2$ is given the norm

$E(u) = \|u(t)\|^2 - \frac{1}{2} \langle u(t), u(t-1) \rangle - \frac{1}{2} \langle u(t+1), u(t) \rangle$

$= \|u(t)\|^2 + \|u(t+1)\|^2 - \langle Lu(t), u(t) \rangle - \langle Lu(t), u(t+1) \rangle$

$= \|u(t)\|^2 + \|u(t+1)\|^2 - \langle Lu(t), u(t) \rangle - \langle Lu(t), u(t+1) \rangle$

Let $H_2^{\text{ev}}$ consist of $u$ such that $u(-t) = u(t)$

and $H_2^{\text{odd}}$ consist of $u$ such that $u(-t) = -u(t)$. Since

$u(t) \mapsto u(-t)$ is a symmetry of $H_2$, it follows that

$H_2 = H_2^{\text{ev}} \oplus H_2^{\text{odd}}$

is an orthogonal direct sum.

Claim $H_2^{\text{odd}} \cong H_1$ via $u(t) \mapsto u(t)$. In
effect for \( u \in H_2 \) odd we have \( u(0) = 0 \), so for any \( u \in H_1 \) there is a unique \( u(t) \) such that \( u(0) = 0 \)

\( u(1) = v \). It follows that \( u(t) + u(-t) = (u(0) = 0 \) so \( u(-1) = -v \). Hence \( u(t) = -u(-t) \) since both coincide for \( t = 0, 1 \). Moreover we have

\[
E(u) = \|u(0)\|^2 - \frac{1}{2} (u(1), u(-1)) - \frac{1}{2} (u(-1), u(1))
\]

\[
= \|u(1)\|^2.
\]

Next let \( u(t) \in H_2^e \),

\[
\frac{u(t) - u(1)}{2} = 0
\]

\[
\frac{u(1) + u(-1)}{2} = Lu(0)
\]

\[
\implies u(1) = Lu(0)
\]

hence \( H_2^e \rightarrow H_1, u(t) \rightarrow u(0) \) is injective.

Given \( v \in H_1 \) let \( u(t) \in H_2 \) be given by

\[
u(0) = v
\]

\[
u(1) = Lv
\]

Then

\[
u(-1) + u(1) = Lv \longrightarrow \nu(-1) = \frac{Lv}{2} \rightarrow u(-1) = \frac{Lv}{2}
\]

and so \( u(-t) = Lu(t) \) as they coincide for \( t = 0, 1 \). Thus the map \( H_2^e \rightarrow H_1 \) is injective. We have

\[
E(u) = \|u(0)\|^2 - \frac{1}{2} (u(1), u(-1)) - \frac{1}{2} (u(-1), u(1))
\]

\[
= \|u(0)\|^2 - \|u(1)\|^2 = \|u(0)\|^2 - \|Lu(0)\|^2
\]

Thus we see that \( H_2 \) is the direct sum of \( H_1 \) and \( H_1 \) with the norm \( \|(1 - t^2) u, u\| \). Hence in general \( H_2 \) won’t be complete unless we require \( \|u\| < 1 \).
So we get
\[ u(t) \longrightarrow u(1) = \frac{u(1) - u(-1)}{2} \]
\[ L^2(S^1, 4\pi |B|^2 d\theta) \sim H_2 \sim H_1 \]
\[ \lambda \longmapsto \int e^{-t} \phi(n, \lambda) \frac{x}{2\pi \lambda} d\theta \longrightarrow \int \phi(n, \lambda) \frac{e^{-t} x}{2\pi \lambda} d\theta \]
\[ = \frac{i}{2\pi} \int \phi(n, \lambda) x d\theta \]

So inside \( L^2(S^1, 4\pi |B|^2 d\theta) \), there has to be an orthonormal basis corresponding to \( e_n \) under the isomorphism \( \lambda \longmapsto \int \phi(n, \lambda) x d\theta \).

\[
\left( \int \phi(n, \lambda) x d\theta, e_n \right) = \int \phi(n, \lambda) x d\theta = \int \frac{\phi(n, \lambda)}{4\pi |B|^2} \times 4\pi |B|^2 d\theta
\]

hence the element corresponds to \( e_n \) in the even function
\[
\frac{\phi(n, \lambda)}{4\pi |B|^2}
\]

As a check we recall
\[
\frac{d\mu(\lambda)}{2\pi |B|^2 \sqrt{1 - \lambda^2}} = \frac{1}{4\pi |B|^2} \int \phi(n, \lambda) \phi(m, \lambda) d\lambda = \delta_{nm}
\]

For \( n \) large we have
\[
\frac{\phi(n, \lambda)}{4\pi |B|^2} = \overline{\phi(n, \lambda)} = \frac{\overline{B(z) z^{-n} + B(z) z^n}}{4\pi |B|^2} = \frac{1}{4\pi} \left( \frac{z^n}{B} + \frac{z^{-n}}{B} \right)
\]
\[
= \frac{1}{2\pi} \Re \left( \frac{z^n}{A(z)} \right) = \frac{1}{4\pi} \left( \frac{z^n}{A(z)} + \frac{z^{-n}}{A(z^{-1})} \right)
\]
Let's now consider the other half of the Hilbert space which consists of odd \( \alpha \) in \( S^1 \) with the

\[ \text{norm from } 4\pi |B|^2 \sin^2 \theta \]

If I multiply \( \alpha \) by \( \sin \theta \) I get an even function.

So back to

\[ \alpha \mapsto \int z^{-t} \phi(n, \lambda) \, d\theta \]

\[ E\left( \int z^{-t} \phi(n, \lambda) \, d\theta \right) = \int |x|^2 \cdot 4\pi |B|^2 \sin^2 \theta \, d\theta \]

\[ E\left( \int z^{-t} \phi(n, \lambda) \, f \, d\theta \right) = \int \frac{|x|^2}{f} \cdot 4\pi |B|^2 \sin^2 \theta \, f \, d\theta \]

so if we choose \( f = \frac{1}{4\pi |B|^2 \sin^2 \theta} \) and put

\[ dv = f \, d\theta \]

we have

\[ E\left( \int z^{-t} \phi(n, \lambda) \, dv \right) = \int |x|^2 \, dv \]

Then we have

\[ L^2(S^1, dv) \xrightarrow{\sim} H^2 \]

\[ \alpha \mapsto \int z^{-t} \phi(n, \lambda) \, dv \]

and

\[ L^2(S^1, dv) \xrightarrow{\text{odd}} H^2 \xrightarrow{\sim} H^1 \]

\[ \alpha \mapsto \int z^{-t} \phi(n, \lambda) \, dv \mapsto \int \frac{z^{-1}z}{2} \phi(n, \lambda) \, dv \]

\[ u(t) \mapsto t_1(1) = \frac{u(1) - u(-1)}{2} \]
hence we get

\[ L^2(S^1, dv)_{\text{odd}} \xrightarrow{\sim} L^2, H^1 = L^2 \]

\[ x \mapsto \int \phi_x(\sin \theta) \, dv \]

Figure out what corresponds to \( e_n \):

\[ (\int \phi_x(\sin \theta) \, dv, e_n) = \int \phi(x, \lambda) \sin \theta \, dv \]

and so we see \( e_n \) corresponds to \( \phi(x, \lambda) \sin \theta \). Conclude that

\[ \phi(x, \lambda) \sin \theta \]

is an orthonormal basis for \( L^2(S^1, dv)_{\text{odd}} \).

December 4, 1977

Let \( dv \) be a measure on \( S^1 \) and \( p_0, p_1, \ldots \)

the sequence of poly obtained by orthonormalizing \( 1, z, z^2, \ldots \)

Define \( h_n, k_n \) for \( n \geq 0 \) by

1) \[ z p_n = k_n p_{n+1} - h_n z^m p_n^* \]

Then \( p_{n+1} \) orth to \( z^m p_n^* \) \( \Rightarrow \) \( 1 = |k_n|^2 + |h_n|^2 \). Let \( h_n \) = leading coefficient of \( p_n \). Then

\[ l_n = k_n k_{n+1} \]

so \( k_n > 0 \)

and

\[ k_n = \sqrt{1 - |h_n|^2} \]

and

\[ l_n = (k_{n-1} \cdots k_0)^{1/2} \left( \prod_{i=0}^{m-1} |1 - h_i|^2 \right)^{-1/2} l_0 \]

Also put \( z = 0 \) and \( \sin 1 \), gives

\[ 0 = k_n p_{n+1}(0) - h_n l_n \]
\[ p_{n+1}(0) = \frac{\ln l_n}{\ln l_{n+1}} \]

We get a basis for \( C[z, z^{-1}] \) using

\[ p_0, \quad z^{-1} p_2, \quad z^{-2} p_4, \quad z^2 p_2^*, \quad z^2 p_4^* \]

Better: Consider the filtration of \( C[z, z^{-1}] \) given by

\[ F_n C[z, z^{-1}] = \langle z^{-n}, \ldots, z^n \rangle \]

so that \( F_0 \subset F_1 \subset F_2 \subset \ldots \)

\[ \dim = 1 \quad 3 \quad 5 \]

In \( F_n \cap F_{n-1} \) we find \( z^{-n+1} p_{2n-1}, z^{-n} p_{2n} \)

and their *'s.

Unfortunately, \( z^n p_{2n}, z^n p_{2n}^* \) are not orthogonal.

In fact

\[ (z^{-n} p_{2n}, z^n p_{2n}^*) = (p_{2n}, z^{2n} p_{2n}^*) \]

and

\[ (p_n, z^n p_n^*) = (p_n(0), z^n p_n^*) = \frac{p_n(0)}{(z^n p_n^*)(0)} (z^n p_n^*, z^n p_n^*) \]

with \( z \ldots z^n \)

\[ = \frac{p_n(0)}{\ln \xi} \]
Check the formulas:
\[ z p_n = k_n p_{n+1} - h_n z^n p_n^* \]
\[ l_n = k_n l_{n+1} \quad \text{so that} \]
\[ l_{n+1} = \frac{1}{k_n} l_n = \frac{1}{k_n} \frac{1}{k_{n-1}} l_0 \]
\[ 0 = k_n p_{n+1}(0) - h_n l_n \]
\[ \therefore p_{n+1}(0) = \frac{h_n l_n}{k_n} = h_n l_{n+1} \]
so that
\[ \frac{p_{n+1}(0)}{l_{n+1}} = \frac{h_n}{l_n} \]
\[ (p_n, z^n p_n^*) = p_n(0) (1, z^n p_n^*) = \frac{p_n(0)}{l_n} (z^n p_n^*, z^n p_n^*) = \frac{p_n(0)}{l_n} \]
\[ = h_{n-1} \quad \text{for } n \geq 1. \]

Suppose now that \( d \nu \) is even: \( d \nu(-\theta) = d \nu(\theta) \). Then \((z^i, z^j) \in \mathbb{R}^2\) so that the \( p_n \) as obtained by Gram-Schmidt are real polyn, hence the \( h_n \) are real. Let
\[ q_0, q_1, q_2, \ldots, \]
\[ g_1, g_2, \ldots, \]
be the sequence of even (resp. odd) Laurent polyns obtained by orthonormalizing the sequences
\[ 1, z+z^{-1}, z^2+z^{-2}, \ldots, \]
\[ z^{-1}, z^{-2}, \ldots, \]
In general if \( F_n \subset \mathbb{C}[z^\pm 1] \) is the space of \( \sum a_i z^i \), then it is the space \( F_n \otimes F_{n-1} \) which I want a nice basis for. This space contains \( z^n p_{2n}, z^{-n+1} p_{2n-1} \) and
their stars, and the problem seems to be to choose an orthonormal basis for this space. First we should try to find the recursion formulas for the \( \{g_n\} \) and \( \{r_n\} \) in terms of the numbers \( \{h_n\} \)

\[
z^{-n}p_{2n} + z^n p^*_{2n} \text{ is proportional to } g_n \quad \text{and it has leading coefficient} \quad \left( \begin{align*} & n > 0 \\
\end{align*} \right)
\]

\[
l_{2n} + p_{2n}(0) = l_{2n}(1+h_{2n-1}) > 0
\]

Also

\[
\|z^{-n}p_{2n} + z^n p^*_{2n}\|^2 = 1 + 1 + 2 \operatorname{Re}(z^{-n}p_{2n} + z^n p^*_{2n})
\]

\[
= 2(1 + h_{2n-1})
\]

so

\[
g_n = \frac{1}{\sqrt{2(1+h_{2n-1})}} \left( z^{-n}p_{2n} + z^n p^*_{2n} \right) \quad \text{for } n > 1
\]

has the leading term

\[
\frac{l_{2n}}{\sqrt{2}} \sqrt{1 + h_{2n-1}} \cdot z^n
\]

\[
z^{-n}p_{2n} - z^n p^*_{2n} \text{ is proportional to } r_n \quad \text{and has the leading term}
\]

\[
(l_{2n} - p_{2n}(0)) z^n = l_{2n}(1-h_{2n-1}) z^n \quad \text{coff} > 0
\]

\[
\|z^{-n}p_{2n} + z^n p^*_{2n}\| = 2(1 - h_{2n-1})
\]

so

\[
r_n = \frac{1}{\sqrt{2(1-h_{2n-1})}} \left( z^{-n}p_{2n} - z^n p^*_{2n} \right)
\]

has the leading term

\[
\frac{l_{2n}}{\sqrt{2}} \sqrt{1 - h_{2n-1}} \cdot z^n
\]
\[ \| z^{-n+1} p_{2n-1} + z^{-n} p_{2n-1}^* \|^2 = 1 + 1 + 2 \text{Re}(z^{-n+1} p_{2n-1} z^{-n} p_{2n-1}^*) \]
\[ = 2 + 2 \text{Re}(z p_{2n-1} z^{-2n-1} p_{2n-1}^*) \]
\[ = 2 + 2 \text{Re}(h_{2n-1} p_{2n} - h_{2n-1} z^{2n-1} p_{2n-1}^* z^{2n-1} p_{2n-1}) \]
\[ = 2(1 - h_{2n-1}) \] (holds even if \( h \) not real)

\[ z^{-n+1} p_{2n-1} + z^{-n} p_{2n-1}^* \] is proportional to \( q_n \)

and has the leading term \( a_{2n-1} \), hence

\[ q_n = \frac{1}{\sqrt{2(1 - h_{2n-1})}} (z^{-n+1} p_{2n-1} + z^{-n} p_{2n-1}^*) \]

has leading term \( \frac{a_{2n-1}}{\sqrt{2(1 - h_{2n-1})}} z^n \)
Consider the Dirac system
\[
\begin{pmatrix}
1 & 0 \\
0 & i
\end{pmatrix}
\frac{du}{dx} + \begin{pmatrix}
0 & i \bar{p} \\
-i p & 0
\end{pmatrix} u = \lambda u
\]

on \(0 \leq x < \infty\) with \(\psi(x, \lambda) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\) as usual, so that \(\psi_2(x, \lambda)\) is a \(\delta\)-function for all \(x > 0\). Question: Can we make sense out of this system when \(p\) is a \(\delta\)-function?

Write the equation \(A \frac{du}{dx} + Bu = \lambda u\) where \(B = \beta \delta(x-x_0)\) where \(\beta\) is Hermitian, say to fix the ideas:
\[
\beta = \begin{pmatrix} 0 & i \beta \\ -i \beta & 0 \end{pmatrix}
\]
\(b \in \mathbb{C}\).

Then any solution should be of the form \(\begin{pmatrix} e^{ix_0}c_1 \\ e^{-ix_0}c_2 \end{pmatrix}\) for \(x < x_0\) and of a similar form for \(x > x_0\).

To see what goes on around \(x_0\) we integrate from \(x_0^-\) to \(x_0^+\):
\[
A(u(x_0^+) - u(x_0^-)) + \beta \int_{x_0^-}^{x_0^+} \delta(x-x) u(x) \, dx = \lambda \int_{x_0^-}^{x_0^+} u(x) \, dx
\]

The last term should be zero, but there is some ambiguity about the first integral since \(u\) is not continuous at \(x_0\).

The obvious choice is the average:
\[
A \left( u(x_0^+) - u(x_0^-) \right) + \beta \left( \frac{u(x_0^+) + u(x_0^-)}{2} \right) = 0
\]

or
\[
(A + \beta) u(x_0^+) = (A - \beta) u(x_0^-)
\]

or
\[
(I + \frac{A^{-1} \beta}{2}) u(x_0^+) = (I - \frac{A^{-1} \beta}{2}) u(x_0^-)
\]
Now

\[- A^{-1} \beta = \left( \begin{array}{cc} -i & 0 \\ 0 & i \end{array} \right) \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) = \left( \begin{array}{cc} 0 & \frac{\hat{b}}{2} \\ -\frac{\hat{b}}{2} & 0 \end{array} \right) \]

so

\[
(I - \frac{A^{-1} \beta}{2}) = \left( \begin{array}{cc} 1 & \frac{\hat{b}}{2} \\ \frac{\hat{b}}{2} & 1 \end{array} \right), \quad (I + \frac{A^{-1} \beta}{2}) = \left( \begin{array}{cc} 1 & -\frac{\hat{b}}{2} \\ \frac{\hat{b}}{2} & 1 \end{array} \right)
\]

Except for the fact that \( \frac{\hat{b}}{2} \) is not required to be of modulus < 1, it is clear that

\[
\left( I + \frac{A^{-1} \beta}{2} \right)^{-1} = \left( I - \frac{A^{-1} \beta}{2} \right)^{-1}
\]

will give an element of \( SU(2,1) \). Note that provided \( \left| \frac{\hat{b}}{2} \right| \neq 1 \) this is well defined and it has determinant

\[
\left( 1 - \frac{1}{4} \right)^{-1} \left( \frac{1}{4} \right)^{-1} = 1
\]

For \( \left| \frac{\hat{b}}{2} \right| < 1 \) it gives an element of \( SU(2,1) \), hence probably also for \( \left| \frac{\hat{b}}{2} \right| > 1 \). Maybe in general?

\[
\left( \begin{array}{cc} 1 & -\frac{\hat{b}}{2} \\ \frac{\hat{b}}{2} & 1 \end{array} \right) \left( \begin{array}{cc} 1 & \frac{\hat{b}}{2} \\ \frac{\hat{b}}{2} & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & \frac{\hat{b}}{2} \\ \frac{\hat{b}}{2} & 1 \end{array} \right) \frac{1}{1 - \frac{1}{4}}
\]

\[
= \left( \begin{array}{cc} 1 + \frac{1}{4} \frac{\hat{b}}{4} \\ \frac{\hat{b}}{4} \end{array} \right) \frac{1}{1 - \frac{1}{4}}
\]

This definitely doesn't work for \( |\hat{b}| = 2 \). This suggests that the averaging interpretation is probably not the good one.
I need to understand how to interpret
\[ A \frac{du}{dx} + \beta \delta(x-x_0) u = \lambda u \]

Simpler case
\[ \frac{du}{dx} = \delta(x-x_0) u \]

\[ \frac{du}{u} = \delta(x-x_0) \, dx \]

so \( u \) is right-continuous

\[ \ln u = \begin{cases} 1 & x > x_0 \\ 0 & x < x_0 \end{cases} + c \]

or
\[ u = \begin{cases} c & x < x_0 \\ ce^x & x > x_0 \end{cases} \]

Another interpretation: Choose a parameter \( t \) such that \( \delta(x-x_0) \, dx \) is absolutely continuous with respect to \( dt \). First, choose \( t \) such that \( dt = \delta(x-x_0) \, dx \), i.e.,

\[ \begin{cases} t = 1 & \text{for } x > x_0 \\ t = 0 & \text{for } x < x_0 \end{cases} \]

Then \( \frac{du}{dt} = u \), so \( u = ce^t \) as above.

2nd choice: Put \( t = x \) for \( x < x_0 \) and \( t = x+1 \) for \( x > x_0 \). Then

\[ dt = dx + \delta(x-x_0) \, dx \]

hence
\[ \delta(x-x_0) \, dx = dt - dx = f(t) \]

\[ f(t) = \begin{cases} 0 & \text{outside } x_0 \leq t \leq x_0 + 1 \\ 1 & \text{outside } x_0 \leq t \leq x_0 + 1 \end{cases} \]

Hence
\[ \frac{du}{dt} = f(t) u \]
Both of these choices agree with the idea of exponentiating the integral of \( v(x-x_0) \) dx. So now return to

\[
A \frac{du}{dx} + Bu = \lambda u
\]

where \( B = \beta \delta(x-x_0) \). The good interpretation of what happens as we pass through \( x_0 \) is now fairly clear. Let

\[
dt = dx + \delta(x-x_0) \cdot dx
\]

and write the equation

\[
A \frac{du}{dt} + B (\delta(x-x_0) \cdot du) = \lambda \frac{dx}{dt} u
\]

so we have

\[
A \frac{du}{dt} + \beta f u = \lambda (1-t) u
\]

so we have

\[
A \frac{du}{dt} = \lambda u \quad \text{on} \quad t \leq x_0. \quad \text{Then we have}
\]

\[
A \frac{du}{dt} + \beta u = 0 \quad \text{on} \quad x_0 \leq t \leq x_{0+1}
\]

and back for

\[
A \frac{du}{dx} = \lambda u \quad \text{on} \quad x_{0+1} \leq t.
\]

In the intermediate range, \( A^{-1} \beta \) is constant hence we have

\[
u(x_0^+) = e^{-A^{-1} \beta} u(x_0^-)
\]

\[
e^{(0 \ b) \cdot \ h} u(x_0^-)
\]

and indeed

\[
e^{(0 \ b) \cdot \ h} = \left( \begin{array}{c} 1 \\ \frac{h}{k} \end{array} \right) \frac{1}{\sqrt{1 + h^2}}
\]

for some \(|hl| < 1\).
Recall that a map \( g: S^1 \to \mathbb{C}^* \) which is holomorphic determines a holomorphic line bundle over \( P_1 \), denote it \( L_g \), whose sections are pairs \((f_0, f_\infty)\) where \( f_0 \) is holom. in \(|z|<1\) and \( f_\infty \) is holom. in \(|z|\geq 1\) such that \( f_\infty = f_0 \).

For example, if \( g = z^n \), then we get the sections \((z^i, z^{n-i})\).

Put \( D_0 = \text{span } 1, z, \ldots \text{ in } L^2(S^1), \) \( D_\infty = \text{span } 1, z^n, \ldots \).

Then \( L(S^1) \) can be identified with \( gD_\infty \cap D_0 \), i.e. holom. first on \(|z|<1\) when divided by \( g \) becomes holom. in \(|z|\geq 1\).

Suppose that \( g \) is in the form
\[
 g = \frac{z^n \delta^*(z)}{\delta(z)}
\]
where \( \delta \) is a poly of degree \( \leq n \) having all its roots in \(|z|>1\). Now
\[
gD_\infty \cap D_0 = \frac{z^n \delta^*(z)}{\delta(z)} D_\infty \cap D_0 \quad \leftrightarrow \quad z^n \delta^* D_\infty \cap \delta D_0
\]
\[
f_0 \quad \leftrightarrow \quad \delta f_0
\]
and \( \delta D_0 = D_\infty, \) \( \delta^* D_\infty = D_0, \) hence \( gD_\infty \cap D_0 \) is \((n+1)\)-dimensional with the basis \( \frac{z^i}{\delta^*}, 0 \leq i \leq n. \)

Moreover, if \( g: S^1 \to S^1 \) we can introduce an inner product on \( L(S^1) \) by putting
\[
 \| (f_0, f_\infty) \|^2 = \int |f_0|^2 \frac{d\theta}{2\pi} = \int |f_\infty|^2 \frac{d\theta}{2\pi}
\]
because \(|g|=1\). In the example \( g = \frac{z^n \delta^*}{\delta} \) this amounts to
using the inner product on polynomials of degree \(\leq n\)

\[
(z^i, z^j) = \int z^i \frac{d\theta}{2\pi|z|^2}.
\]

so it is more or less clear that I have found

some sort of discrete analogue of the de Branges spaces. For \(E\) take 

\[z^{-n/2} S(z)\]

for \(n \geq \deg(E)\). Then \(B(E)\) should consist of (say \(n\) even) all \(f\) holomorphic for \(0 < |z| < \infty\)

such that

\[
\frac{f(z)}{z^{-n/2} S(z)} \in D_0
\]

and

\[
\frac{f(z)}{z^{n/2} S(z)} \in D_\infty
\]

with the norm

\[
\int \left| \frac{f(z)}{z^{-n/2} S(z)} \right|^2 \frac{d\theta}{2\pi} = \int \frac{|f|^2 d\theta}{2\pi|z|^2}
\]

The first two conditions imply that \(f\) is a Laurent

poly of degree \(\leq n/2\).

Let's shift so that \(B\) consists of \(f\) holomorphic for \(0 < |z| < \infty\) with

\[
\frac{f}{z^{n/2} S(z)} \in D_\infty
\]

with the same norm. Then \(f\) has to be holomorphic at \(0\)

and \(f\) has to be holomorphic at \(\infty\), hence \(f\) is a poly in \(z\)
of degree \(\leq n\). Let's work out the point evaluations in \(B\)

using Cauchy's formula:

\[
\frac{1}{2\pi i} \int \frac{f(z)}{z-a} \frac{dz}{z-a} = \begin{cases} 
\frac{f(a)}{S(a)} & |a| < 1 \\
0 & |a| > 1
\end{cases}
\]
\[
\left. \frac{1}{2\pi i} \int \frac{f(z)}{z^a \delta(z)} \frac{dz}{z-a} \right|_{a} = \begin{cases} -\frac{f(a)}{a^a \delta^a(a)} & |a| > 1 \\ 0 & |a| < 0 \end{cases}
\]

So

\[
\int f(z) \left\{ \frac{\delta(a)}{\delta(z)} - \frac{a^a \delta^a(a)}{z^a \delta^a(z)} \right\} \frac{z}{z-a} \frac{dz}{2\pi i z} = f(a)
\]

Hence

\[
\int f(z) \left\{ \frac{\delta(a) \delta^a(z)}{1-a \overline{z}} - \frac{z^a a^a \delta^a(a) \delta^a(z)}{1-a \overline{z}} \right\} \frac{i}{1-a \overline{z}} \, d\theta
\]

So

\[
J_a(z) = \frac{\delta(a) \delta(z) - z^a a^a \delta^a(a) \delta^a(z)}{1-a \overline{z}}
\]

not quite.

The only problem is that this is a poly of degree \(n-1\). The problem is that (\#) isn't valid unless \(\deg f < n\) because \(\frac{dz}{z-a}\) is singular at \(\infty\).

\[
\frac{d}{\omega - a} = -\frac{1}{\omega - a} \frac{d\omega}{\omega - a} = -\frac{d\omega}{\omega(1-a\omega)}
\]

so you need to replace \(n\) by \(n+1\) in \(\#\). Thus the good formula is

\[
J_a(z) = \frac{\delta(a) \delta(z) - z^a a^a \delta^a(a) \delta^a(z)}{1-a \overline{z}} = \sum_{i=0}^{n} p_i(a) p_i(z)
\]
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Let's consider a Dirac system

\[ \frac{dx}{dt} = \left( \begin{array}{c} i \lambda \\ \rho - i \Delta \end{array} \right) x \]

where \( \rho \) is a \( \delta \)-function at \( \lambda = 1, 2, \ldots \). If \( \phi(x, \lambda) \) is the solution starting with \( \phi(0, \lambda) = (1) \), then we have

\[ \phi(x, \lambda) = \left( \begin{array}{c} e^{iux} \\ e^{-iux} \end{array} \right) \quad 0 < x < 1 \]

\[ \phi(1^+, \lambda) = R(h_1) \phi(1^-, \lambda) \]

\[ R(h) = \exp \left( i \int_{1^-}^{1^+} b dx \right) \]

etc. so that

\[ \phi(n^+, \lambda) = R(h_n) \left( \begin{array}{c} e^{iud} \\ e^{-iud} \end{array} \right) \ldots R(h_1) \left( \begin{array}{c} e^{iud} \\ e^{-iud} \end{array} \right) (1) \]

If we put \( z = e^{2iad} \), then this system parallels an orthogonal system of polynomials on \( S^1 \). I'd like to see if there is a sensible way of incorporating \( z^{1/2} \) and more generally \( z^n \), \( n \in \mathbb{Q} \) into the circle setup.

First consider the case where \( n \) is even and the circular de B pair is

\[ \left( \begin{array}{c} z^m \theta \\ z^{n-m} \theta \end{array} \right) \quad m = \frac{n}{2} \]

\( \delta \) has roots outside \( S^1 \) and degree \( 8 \leq 2m \). Thus

\[ \left( \begin{array}{c} z^{n/2} \theta \\ z^{n/2} \theta \end{array} \right) = R(h_n) \left( \begin{array}{c} z^{1/2} \theta \\ \theta z^{-1/2} \end{array} \right) \ldots R(h_1) \left( \begin{array}{c} z^{1/2} \\ \theta z^{-1/2} \end{array} \right) (1) \]

so that

\[ \phi_2(n^+, \lambda) = e^{-iud} \delta(e^{2iad}) \]

I propose to find the point evaluator for the space.
of Laurent polynomials of degree \( \leq m \) with the norm \( \int |f(z)|^2 \frac{d\theta}{2\pi} \). Start with Cauchy:

\[
\frac{1}{2\pi i} \int \frac{f(z)}{z^m \delta(z)} \frac{dz}{z-a} = \begin{cases} \frac{f(a)}{a^m \delta(a)} & |a| < 1 \\ 0 & |a| > 1 \end{cases}
\]

\[
\frac{1}{2\pi i} \int \frac{f(z)}{z^{m+1} \delta(z)} \frac{dz}{z-a} = \begin{cases} \frac{-f(a)}{a^{m+1} \delta(z)} & |a| > 1 \\ 0 & |a| < 1 \end{cases}
\]

\[
f(a) = \int f(z) \left\{ \frac{a^{-m} \delta(a)}{z^{-m}} \delta(z) - \frac{a^{m+1} \delta^*(a)}{z^{m+1}} \delta^*(z) \right\} \frac{z-a}{\delta^2} \frac{d\theta}{2\pi i^2}
\]

\[
= \int f(z) \left\{ a^{-m-\frac{1}{2}} \delta(a) - a^{m+\frac{1}{2}} \delta^*(a) \right\} \frac{z^{-\frac{1}{2}} a^{\frac{1}{2}}}{z-a} \frac{d\theta}{2\pi i^2}
\]

\[
hence \quad J(z) = \begin{bmatrix} \frac{z^m \delta^*(z)}{z^{m+1} \delta^*(z)} & a^{-m-\frac{1}{2}} \delta(a) \\ \frac{z^{-\frac{1}{2}} a^{\frac{1}{2}}}{z^m \delta(z)} & a^{m+\frac{1}{2}} \delta^*(a) \end{bmatrix}
\]

Now if I put

\[
E(\lambda) = \begin{bmatrix} z^{m+\frac{1}{2}} \delta^*(z) & z^{-\frac{1}{2}} \delta(z) \\ \frac{z^m \delta^*(z)}{z^{m+1} \delta^*(z)} & \frac{z^m \delta(z)}{z^{m+1} \delta(z)} \end{bmatrix} = e^{i(m+\frac{1}{2}) \lambda} \delta^*(z)
\]

\[
E(\lambda) = e^{-i(m+1)\lambda} \delta^*(e^{2i\lambda})
\]

and \( e^{2i\alpha} = a \). Then

\[
\begin{bmatrix} \frac{z^\frac{1}{2} a^{\frac{1}{2}}}{z^{-\frac{1}{2}} a^{-\frac{1}{2}}} \\ \frac{z^{-\frac{1}{2}} a^{\frac{1}{2}}}{z^\frac{1}{2} a^{-\frac{1}{2}}} \end{bmatrix} = e^{i\lambda} e^{-i\alpha} - e^{-i\lambda} e^{i\alpha} = 2i \sin(\lambda-\alpha).
\]
$J_{\alpha}(\xi) = \frac{|E_{\alpha}(\lambda) E_{\beta}(\xi)|}{|E(\lambda) E^{*}(\xi)|} = \frac{i}{2 \sin(\lambda - \xi)} \begin{vmatrix} E(\lambda) & E(\xi) \\ E^{*}(\lambda) & E^{*}(\xi) \end{vmatrix}$

somehow I have roughly the same Hilbert space except I am maybe replacing $L^2$ norms with the mean $\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T}$ on the real axis. No. The above formula makes sense only if $\frac{E^{*}(\lambda)}{E(\lambda)} = \frac{E^{*}(\lambda')}{E(\lambda')}$ when $\lambda - \xi \in \pi \mathbb{Z}$
Let $d\nu$ be a prob. measure on $S^1$ and $p_0 = 1, p_1, \ldots$ the associated sequence of orthogonal polynomials. Let $h_n, n \geq 1$ be defined by

$$z p_{n-1} = k_n p_n - h_n z^{n-1} p_{n-1}, \quad k_n = \sqrt{1 - h_n^2}$$

and let $l_n$ be the leading coefficient of $p_n$, so that

$$l_{n-1} = k_n l_n, \quad l_n = \frac{1}{k_n \cdots k_1}$$

Recall

$$p_n(z) z^n p^*_n = p_n(0) (1, z^n p^*_n) = \frac{p_n(0)}{l_n} \left( z^n p^*_n, z^n p^*_n \right) = \frac{p_n(0)}{l_n}$$

Starting with $d\nu$, I want to construct a nice orthonormal basis in $L^2(S^1, d\nu)$ which is adapted to the filtration $F_n C[z, z^{-1}] = \langle z^n, \ldots, z \rangle$. Let $W_n = F_n C[z, z^{-1}] \oplus F_n C[z, z^{-1}]$, so that $W_n$ is 2 dimensional for $n \geq 1$, one-dink for $n = 0$. Suppose $n \geq 1$. Note that $W_n$ is closed under the operation of conjugation $*$ which preserves norm. Hence $W_n$ is the complexification of a 2dink Euclidean space, so it has a real orth. basis $e_1, e_2$. Let $ae_1 + be_2$ be an element of $W_n$. It is orthogonal to its conjugate

$$(ae_1 + be_2, \bar{ae}_1 + \bar{be}_2) = a^2 + b^2 = 0$$

iff $b = \pm a$. Hence the vectors orthogonal to their conjugates form the union of the two lines spanned by $e_1, ie_2, e_1, -ie_2$. 

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so we can find an orthonormal basis for \( W \), consisting of \( (\phi, \phi^*) \) which is unique up to multiplying and interchanging \( \phi \) and \( \phi^* \) by a scalar of modulus 1. In fact we take:

\[
\phi = \frac{1}{\sqrt{2}}(e_1 + i e_2)
\]

\( W_n \) has the basis \( z^n p_{2n}, z^n p_{2n}^* \) so we try to find an element orthogonal to its conjugate in the form \( z^n p_{2n} + t z^n p_{2n}^* \):

\[
(z^n p_{2n} + t z^n p_{2n}^* , z^n p_{2n}^* + \overline{t} z^n p_{2n})
\]

\[
= (z^n p_{2n}, z^n p_{2n}^*) + t \| z^n p_{2n}^* \|^2 + \| z^n p_{2n} \|^2 t + t^2 (z^n p_{2n}^*, z^n p_{2n})
\]

\[
= h_{2n} + t + t + t^2 \overline{h_{2n}} = 0
\]

\[
t = \frac{-1 \pm \sqrt{1 - |h_{2n}|^2}}{|h_{2n}|} = \frac{-1 \pm k_{2n}}{|h_{2n}|}
\]

\[
\| z^n p_{2n} + t z^n p_{2n}^* \|^2 = (z^n p_{2n} + t z^n p_{2n}^*, z^n p_{2n} + t z^n p_{2n}^*)
\]

\[
= 1 + t \overline{h_{2n}} + \overline{t} h_{2n} + |t|^2
\]

\[
t = \frac{-1 - k_{2n}}{|h_{2n}|}
\]

\[
= 1 - 2(1 + k_{2n}) + \frac{1 + 2k_{2n} + k_{2n}^2}{|h_{2n}|^2}
\]
On $SU(1,1)$. This is the subgroup of $SL_2(\mathbb{C})$ consisting of matrices of the form \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]
i.e., fixed under flipping conjugation
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}
\]
followed by complex conjugation. The Lie algebra consists of all \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
with \(a+\bar{a}=0\), i.e., all
\[
\begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix}
\]
with \(\alpha \in \mathbb{R}\). Thus a good maximal compact $T$ in $G$: $SU(1,1)$ is the diagonal matrices. Let $G = SU(1,1)$ act on $|z| < 1$ in the obvious way, then $T$ fixes $0$ so we have an isomorphism
\[
\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \mapsto \exp\left(\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}\right)
\]
between $O$ and $|z| < 1$. We have
\[
\exp\left(\begin{pmatrix} 0 & it \\ -it & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh(t) & i\sinh(t) \\ -i\sinh(t) & \cosh(t) \end{pmatrix}
\]
\[
\exp\left\{\frac{it}{2}\begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}\right\}
\]
hence
\[
\exp\left(\begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}
\]
In general
\[
\begin{pmatrix}
0 & b \\
\bar{b} & 0
\end{pmatrix} =
\begin{pmatrix}
e^{i\theta} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
|b| & 0 \\
0 & |\bar{b}|
\end{pmatrix}
\begin{pmatrix}
e^{-i\theta} & 0 \\
0 & 1
\end{pmatrix}
\]

where \( \theta = \arg(b) \), so that
\[
\exp\left( \frac{a}{b} \right) =
\begin{pmatrix}
\cos |b| & e^{i\theta} \sin |b| \\
-e^{-i\theta} \sin |b| & \cos |b|
\end{pmatrix}
\]


Consider a 2-dimensional complex Hilbert space with involution \( \ast \) and suppose given \( u, u^* \) are independent and \( \|u\| = 1 \). Then I have seen that the set of vectors \( au + bu^* \) in \( W \) orthogonal to their stars is the union of two lines. Hence I can find an orthonormal basis of the form \( (\phi, \phi^*) \) which is unique up to multiplying \( \phi \) by an elt of \( S^1 \) and also interchanging \( \phi, \phi^* \).

\[
\phi = au + bu^* \\
\phi^* = \overline{bu} + \overline{au}^*
\]

It should be the case that
\[
\det\begin{pmatrix} a & b \\ \bar{b} & \overline{a} \end{pmatrix} = |a|^2 - |b|^2 \neq 0
\]
so that we can normalize our choices by requiring \( |a| > |b| \).

Work out formulas: Suppose \( (u, u^*) = h \)
\[
(u + tu^*, u^* + t\overline{u}) = h + t + t + t^2 \overline{h}
\]

\[
\Rightarrow t = \frac{-1 \pm \sqrt{1 - 4|\overline{h}|^2}}{2}
\]

Put \( k = \sqrt{1 - 4|\overline{h}|^2} \). Notice that
\[
|k| = |(u, u^*)| = \|u\| \|u^*\| = 1 \] with
equality if \( u, u^* \) are proportional, which is impossible so \( |h|<1 \). We want \( |t|<1 \) so we take
\[
t = \frac{1-k}{h} = \frac{-h}{1+k}
\]
Hence if \( \psi = (1+k)u - h u^* \), then \( (\psi, \psi^*) = 0 \).
\[
||\psi||^2 = (1+k)^2 - 2((1+k)h\bar{u} + h\bar{u})
\]
\[
= (1+k)\left[ 1 + \bar{k} - 2(1-k^2) + i(-k) \right] = 2k^2(1+k)
\]
Thus
\[
(\phi) = \frac{1}{\sqrt{2k^2(1+k)}} \begin{pmatrix} 1+k & -h \\ -h & 1+k \end{pmatrix} \begin{pmatrix} u \\ u^* \end{pmatrix}
\]
Denote this \( T(h) \) \( \det(T(h)) = \frac{1}{k} \)

to now given \( dv \) a prob. measure on \( S^1 \) let us define \( \phi_n \) by
\[
\begin{pmatrix} \phi_n \\ \phi_n^* \end{pmatrix} = T(h_n) \begin{pmatrix} z^{-n/2} p_n \\ z^{n/2} p_n^* \end{pmatrix}
\]
where \( h_n = (z^{-n/2} p_n, z^{n/2} p_n^*) = (p_n, z^n p_n^*) \). Then \( \phi_n \) is a poly of degree \( n \) in \( z \)

\[
\begin{pmatrix} \phi_n \\ \phi_n^* \end{pmatrix} = T(h_n) \begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix}
\]
We have the recursion formula.
\[
\begin{align*}
(2^n \phi_n^* ) &= T(h_n)(p_n^{* *}) = T(h_n) R(h_n) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} (p_{n-1}^{* *}) \\
&= T(h_n) R(h_n) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} T(h_{n-1})^{-1} \begin{pmatrix} \phi_{n-1}^* \\ z_{n-1}^* \phi_{n-1}^* \\ \end{pmatrix}
\end{align*}
\]

Observation: Except for a scalar factor $T(h)$, is the negative square root of $R(h)$:

\[
T(h)^2 = \frac{1}{2h^2(1+h^2 - 2(1+h)h) - 2h(1+h)} \begin{pmatrix} (1+h)^2 + 1h^2 - 2(1+h)h \\ -2h(1+h) \\ 1h^2 + (1+h)^2 \end{pmatrix} = \frac{1}{k^2} \begin{pmatrix} 1 - h \\ -h \end{pmatrix}
\]

\[
= \frac{1}{k} R(h)^{-1} \quad T(h)^2 R(h) = \frac{1}{k}
\]

Hence

\[
\begin{pmatrix} z_{n/2}^* \phi_n^* \\ z_{n/2}^* \phi_n^* \end{pmatrix} = \frac{1}{k_n} T(h_n)^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} T(h_{n-1})^{-1} \begin{pmatrix} \phi_{n-1}^* \\ z_{n-1/2}^* \phi_{n-1}^* \end{pmatrix}
\]

which is not such a bad transition formula.

Question: Suppose given $h_1, \ldots, h_n$ whence you get a matrix

\[
F(z) = R(h_n)(z_{n/2}^* 0) \cdots R(h_1)(z_{n/2}^* 0)
\]

which for $|z| = 1$ maps $S^1$ to $S^1$. Now fix boundary conditions at the ends, say $u_1 = u_2$ at $0$ and $u_1 = z_{n/2}^* e^{i\theta}$. Can these $z_{n/2}^*$ which are compatible with these boundary values:

\[
e^{i\theta} = F(z) \cdot 1
\]

be interpreted as eigenvalues of a unitary operator?
Here's how this is done for the Dirac D.E.
\[ \frac{du}{dx} = \left( \begin{array}{c} i \\ \bar{p} \end{array} \right) u \]
on \[0 \leq x \leq L\] with boundary conditions \( u_1 = u_2 \) at \( x = 0 \)
\( u_1 = e^{i\theta} u_2 \) at \( x = L \). The point is that we get a self-adjoint operator in the Hilbert space \( L^2(0, L)^2 \)
defined by the differential operator

\[ \left( \begin{array}{c} 0 \\ i \end{array} \right) \frac{d}{dx} + \left( \begin{array}{c} 0 \\ -i \end{array} \right) \bar{p} \]

together with the boundary conditions. Hence I get a 1-parameter unitary group in this Hilbert space.

Idea is to get a J-matrix picture for a unitary operator \( U \) plus cyclic unit vector \( e \) (= probability measure on \( S^1 \)) in the finite-dimensional case.

Suppose \( H \) is an \( n \)-dimensional Hilbert space with a unitary operator \( U \) and cyclic vector \( e \) (say \( \| e \| = 1 \)). By Gram-Schmidt we can construct an orthonormal basis \( e_0, e_1, \ldots , e_{n-1} \) for \( H \) from \( e, Ue, \ldots , U^{n-1} e \). I might as well suppose \( H = L^2(S^1, d\theta) \) where \( d\theta \) has support of card \( n \). Clearly \( p_i(U) = e_i \) for \( i = 0, \ldots , n-1 \). Moreover we get \( h_0, \ldots , h_{n-1} \) such that

\[ p_i = h_0 \bar{e}_{i-1} + h_i \bar{e}_{i-1}^* \]
where \( 0 \leq i \leq n-1 \) is equivalent to \( e_{i-1} \), \( \bar{e}_{i-1} \).
Let \( g \) be the unique monic poly of degree \( n \) such that \( g(z) = 0 \), i.e. \( \det (z - U) \). Then

\[
z p_{n-1}(z) = \ell_{n-1} f(z) + r(z)
\]

when \( \ell_{n-1} = \text{leading coeff of } p_{n-1} \) and deg \( r < n \). \( r \) is orthogonal to \( z^{n-1} \), hence \( r \) must be a multiple of \( z^{n-1} p_{n-1}^* \). Put \( n = -h_n z^{n-1} p_{n-1}^* \). Then

\[
z p_{n-1}(z) = -h_n z^{n-1} p_{n-1}^* \quad \text{in } L^2(S^1, d\nu)
\]

so taking norms: \( |h_n| = 1 \). Also

\[
z p_{n-1}(z) + h_n z^{n-1} p_{n-1}^* = \ell_{n-1} f(z)
\]

is a formula for the relation. Hence from (4, 4, c) we have managed to construct \( h_1, h_2, \ldots, h_{n-1} \) of modulus \( < 1 \) and \( h_n \) of modulus 1. So we get

\[
\begin{pmatrix}
\bar{z}^i p_i \\
p_i
\end{pmatrix} = R(h_i) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix}
\bar{z}^{i-1} p_{i-1}^* \\
p_{i-1}^*
\end{pmatrix} \quad i = 1, \ldots, n-1
\]

and finally

\[
\ell_{n-1} f(z) = \begin{pmatrix} 1 & h_n \end{pmatrix} \begin{pmatrix} \bar{z} & 0 \\ 0 & 1 \end{pmatrix} R(h_{n-1}) \begin{pmatrix} \bar{z} & 0 \\ 0 & 1 \end{pmatrix} \cdots R(h_1) \begin{pmatrix} \bar{z} & 0 \\ 0 & 1 \end{pmatrix} (1)
\]

In other words, we have shown that the Lee-Yang polynomial belong to a linear graph is essentially the characteristic poly of a unitary operator.