Using inequalities.

Let $X$ be a finite poset and $\mu$ a probability measure on $X$ given by a function: $\mu(\{x\}) = U(x)$. One wants to prove correlation inequalities:

$$\sum_{x \in X} f(x) g(x) U(x) \geq \left( \sum_{x \in X} f(x) U(x) \right) \left( \sum_{x \in X} g(x) U(x) \right)$$

when $f, g$ are monotone real-valued functions on $X$. (Note that one has equality when $f, g$ are independent, so this means monotone functions tend to behave non-independently.)

Note that any monotone function on $X$ is a non-negative linear combination of characteristic functions of subsets closed under specialization (I call them open); specifically, suppose the range of $f$ is

$$\{a_0 < \cdots < a_n\}.$$ Then

$$f(x) = a_0 + \sum_{i=1}^{n} (a_i - a_{i-1}) \cdot \chi_{ \{x \mid f(x) > a_i\} }.$$
Therefore (1) is equivalent to
\[ \mu(A \cap B) \geq \mu(A) \mu(B) \]
if \( A, B \) are open subsets of the poset \( X \).
(Recall two subsets \( A, B \) are independent if \( \mu(A \cap B) = \mu(A) \mu(B) \); this is the same as \( X_A \) and \( X_B \) being independent).

Other versions of (1): Consider the space of real functions on \( X \) with mean: \( \sum f(x) u(x) = 0 \). On this space one has the inner product \( (f, g) = \sum f(x) g(x) u(x) = E(fg) \).

Note that the monotone functions form a convex cone with non-empty interior. Condition (1) is equivalent to \( (f, g) \geq 0 \) if \( f, g \) are in this cone, i.e. the angle between two vectors in the cone is \( \leq 90^\circ \).

Observe that (2) holds if \( X \) is a chain because then either \( B \subseteq A \) or \( A \subseteq B \).

FKL theorem asserts (1) holds if \( X \) is a distributive lattice and \( u \) satisfies
\[ u(x \lor y) u(x \land y) \geq u(x) u(y) \]
for example \( u(x) = e^{-h(x)} \) with
\[ h(x \lor y) + h(x \land y) \leq h(x) + h(y). \]
Thm. (Simon's book p. 280)

Let \( \mathbb{R}^n \) be given the product order; let \( d\nu_1, \ldots, d\nu_n \) be measures on \( \mathbb{R} \) and \( U(x_1, \ldots, x_n) \) a strictly positive function >

(1) \[ U(xy) U(xy) \geq U(x) U(y) \]

\[ d\mu = U(x_1, \ldots, x_n) d\nu_1(x_1) \cdots d\nu_n(x_n) \]

Put \( \langle f \rangle = \int f \, d\mu / S d\mu \). If \( f, g \) are monotone then

(2) \[ \langle fg \rangle \geq \langle f \rangle \langle g \rangle. \]

Assume the \( d\nu_i \) have compact support; other cases can be handled by passing to the limit.

Proof by induction on \( n \). If \( n=1 \), (1) is trivial and (2) follows from

(3) \[ \int (f(x) - f(y))(g(x) - g(y)) \, d\mu(x) \geq 0 \]

\[ 2(\int fg \, d\mu)(\int d\mu) - 2(\int f \, d\mu)(\int g \, d\mu) \]

and the fact that the integrand in (3) is always \( \geq 0 \) when \( f, g \) are monotone.

Write \( x \in \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^n \) as \( (p, s) \) and

\[ \int (f(x) - f(y))(g(x) - g(y)) \, d\mu(x) d\nu(x) = \int \beta(s, t) \, d\nu_1(s) \, d\nu_n(t) \]

\[ \beta(s, t) = \int (f(p, s) - f(q, t))(g(p, s) - g(q, t)) \, U(p, s) U(q, t) \prod_{i=1}^{n-1} d\nu_i(p) d\nu_i(q). \]

It suffices to prove \( \beta(s, t) \geq 0 \) and since \( \beta(s, t) = \beta(t, s) \)
we can suppose \( s \leq t \). Put

\[
F(s) = \int f(p, s) \ U(p, s) \prod_{i=1}^{n-1} \, d\nu_i(p_i)
\]

\[
G(s) = \int g(p, s) \ U(p, s)
\]

\[
H(s) = \int f(p, s) g(p, s) \ U(p, s)
\]

\[
Z(s) = \int \ U(p)
\]

Then

\[
Z(s) Z(t) \beta(s, t) = Z(s) Z(t) \left[ H(s) Z(t) + Z(s) H(t) - F(s) G(t) - F(t) G(s) \right]
\]

\[
= Z(s)^2 \left[ Z(t) H(t) - F(t) G(t) \right]
\]

\[
+ Z(t)^2 \left[ Z(s) H(s) - F(s) G(s) \right]
\]

\[
+ \left[ Z(s) F(t) - Z(t) F(s) \right] \left[ Z(s) G(t) - Z(t) G(s) \right]
\]

Now

\[
Z(t) H(t) \geq F(t) G(t)
\]

namely apply induction to \( f(\cdot, t) \ g(\cdot, t) \ U(\cdot, t) \)

and \( d\nu_1 \ldots d\nu_{n-1} \). Similarly the second term is \( \geq 0 \).

Next

\[
p \mapsto \frac{U(p, s)}{U(p, t)}
\]

is increasing. Hence by induction

\[
F(s) Z(t) = \int f(s, q) U(s, q) \prod_{i=1}^{n-1} \, d\nu_i(q) \int \frac{U(t, s)}{U(s, t)} U(t) \prod_{i=1}^{n-1} \, d\nu_i(t)
\]

\[
\leq \int U(s) \int f(s, s) U_t \leq \int U(s) \int f(s, s) U_t = Z(s) F(t)
\]
because \( f_s \leq f_t \). Similarly, \( G(s)Z(t) \leq G(t)Z(s) \), so the proof is complete.

Another version of the proof in the special case \( X = S \times Y \) where \( S = \{0, 1\} \) and \( X \) is finite. Put

\[
F(s) = \sum_{y \in Y} f(s, y) U(s, y)
\]

and define \( G(s), H(s), Z(s) \) similarly. We want to prove:

\[
\sum_{s} H(s) \sum_{s} Z(s) \geq \sum_{s} F(s) \sum_{s} G(s)
\]

By the induction assumption \( \frac{F(s)}{Z(s)} \geq \frac{G(s)}{Z(s)} \) are increasing. Note that if \( f, g \) are increasing and \( s \leq t \)

\[
(*) \quad f(s)g(s) + f(t)g(t) \geq f(o)g(t) + f(t)g(s)
\]

because the difference is \((f(s) - f(t))(g(s) - g(t)) \geq 0\).

Thus

\[
(H(0) + H(1))(Z(0) + Z(1)) \geq \left( \frac{F(0)G(0)}{Z(0)} + \frac{F(1)G(1)}{Z(1)} \right)(Z(0) + Z(1))
\]

\[
\geq F(0)G(0) + F(1)G(1) + \frac{F(0)G(0)}{Z(0)} Z(0) Z(1) + \frac{F(1)G(1)}{Z(1)} Z(0) Z(1)
\]

\[
\geq F(0)G(0) + F(1)G(1) + \left( \frac{F(1)G(0)}{Z(0)} + \frac{F(0)G(1)}{Z(1)} \right) Z(0) Z(1)
\]

\[
= (F(0) + F(1))(G(0) + G(1))
\]
Better proof: Assume \( H(s)Z(s) \geq F(s)G(s) \) with \( Z(s) > 0 \) and that \( \frac{F(s)}{Z(s)} \), \( \frac{G(s)}{Z(s)} \) are increasing.

Then
\[
\sum H(s)Z(s) \geq \sum \frac{F(s)}{Z(s)} \frac{G(s)}{Z(s)} Z(s) \sum Z(s)
\]
\[
\geq \sum \frac{F(s)}{Z(s)} Z(s) \cdot \sum \frac{G(s)}{Z(s)} Z(s)
\]
\[
= \sum F(s) \sum G(s)
\]
because of what we know for monotone functions on a chain.

Note the same formula will hold \( \circ \) under the weaker assumption that \( Z(s) > 0 \) but that \( Z(s) = 0 \Rightarrow F(s) = G(s) = H(s) = 0 \); namely you delete these from \( S \).

Generalize: Suppose I have a map of posets \( p: X \rightarrow S \) where \( S \) is a chain. If \( s \leq t \) in \( S \), then
\[
F(t)Z(s) = \sum_{x \in X_t} f(x)U(x) \sum_{y \in X_s} U(y) \geq \sum \frac{f(y)x}{U(y)} U(x) \sum \frac{U(y)}{y \in X_s} U(y)
\]
\[
= \sum \frac{f(y)}{y \in X_s} \sum_{x \in X_t} U(x) \cdot \sum \frac{U(y)}{y \in X_s} U(y)
\]
\[
\geq \sum \frac{f(y)U(y)}{y \in X_s} \cdot \sum \frac{U(x)}{x \in X_t} = F(s)Z(t)
\]
Here \( \varphi : X_t \to X_s \) is some sort of pull-back map satisfying:

\[
\begin{align*}
\text{a) } & \quad \varphi(x) \leq x \\
\text{b) } & \quad y \leq y' \implies \frac{\sum_{x \in \varphi^{-1}(y')} U(x)}{U(y)} \leq \frac{\sum_{x \in \varphi^{-1}(y')} U(x)}{U(y')} 
\end{align*}
\]

Note b) holds if one has a map \( \varphi : X_t \to X_t y' \) embedding \( \varphi^{-1}(y) \) in \( \varphi^{-1}(y') \) and if

\[ U(x \cup y') U(y) \geq U(y') U(x). \]

So now if \( H(s) = \sum_{x \in X_s} f(x) g(x) U(x) \), then assuming \( H(s) Z(s) \geq F(s) G(s) \) on each fibre, we get

\[
\begin{align*}
\sum_{s} H(s) \sum_{s} Z(s) & \geq \sum_{s} F(s) G(s) Z(s) \sum_{s} Z(s) \\
& \geq \sum_{s} F(s) Z(s) \sum_{s} G(s) Z(s) \\
& = \sum_{s} F(s) \sum_{s} G(s).
\end{align*}
\]
Next I want to allow \( U \) to be zero sometimes. To simplify return to the case

\[
X = \{ s \times y \}.
\]

For \( s \leq t \)

\[
F(t) Z(s) = \sum_y f(t, y) U(t, y) \sum_y u(s, y).
\]

\[
\geq \sum_y f(s, y) \frac{U(t, y)}{u(s, y)} U(s, y) \sum_y u(s, y).
\]

Note that the function \( y \mapsto \frac{U(t, y)}{u(s, y)} \) defined on the set of \( y \) such that \( U(s, y) > 0 \) is increasing.

\[
U(s, y) U(t, y') \geq U(s, y') U(t, y)
\]

and \( U(s, y), U(s, y') > 0 \)

\[
\Rightarrow \frac{U(t, y')}{U(s, y')} \geq \frac{U(t, y)}{U(s, y)}
\]

But \( \{ y \mid U(s, y) > 0 \} \) is a sublattice.

Suppose next that \( L \) is a finite distributive lattice. Let \( J \) be the set of irreducibles in \( L \), so that \( L \) is isomorphic to the lattice of closed subset of \( J \). Let \( U : L \to \mathbb{R}^\geq \) satisfy \( U(x \cup y) U(x \cap y) > U(x) U(y) \), and let \( f, g \) be monotone functions on \( L \). To prove

\[
\sum' f(x) g(x) U(x) \sum' u(x) \geq \sum' f(x) U(x) \sum g(x) U(x)
\]
I want to prove this by induction on \( \text{card}(L) \).

First note that we can enlarge \( L \) to the lattice of all subsets of \( T \). In effect we extend \( U \) to \( 2^T \) by zero outside of \( L \). The inequality \( U(x \cup y) U(x \cap y) \geq U(x)U(y) \) still holds because if either \( x, y \notin L \) then the right side is zero. (Notice that the support of \( U \) is a sublattice of \( L \)). Next one can extend \( f \) from \( L \) to \( 2^T \) by defining \( f(x) = f(\overline{x}) \); \( x \leq x' \Rightarrow \overline{x} \leq \overline{x'} \), etc. So it suffices to prove the theorem when \( L = 2^T \), but where \( U \) is allowed to have the value zero.

Then I would try induction on \( \text{card}(T) \). So write \( L = S \times Y \) and put

\[
F(s) = \sum_{y \in Y} f(s, y) U(s, y) = \sum_{y \in L_s} f(s, y) U(s, y)
\]

where \( L_s = \{ y \in Y \mid U(s, y) > 0 \} \). Note \( L_s \) is a sublattice of \( Y = 2^{T'} \), \( T' = T - \text{some } S \). If \( S \subseteq T \), then the function

\[
\frac{U(s, y)}{U(s, y)}
\]

defined on \( L_s \) is monotone, hence it can be extended to all of \( Y \) to be a monotone function (its value at \( y \) is the value at the smallest element of \( L_s \geq y \)). Thus we can argue...
\[ F(t) \mathcal{Z}(s) = \sum_y \frac{h(t, y)}{u(s, y)} u(s, y) \] \[ \sum \quad ? ? \]

I seem to run into trouble if \( L_0 \) doesn't contain the largest element of \( \mathcal{Y} \).

So let's try selecting an irreducible \( p \in \mathcal{T} \) and considering the map \( L \rightarrow \{0, 1\} \) \( x \mapsto 0, 1 \) according as \( p \not\preceq x \) or \( p \preceq x \). Assume \( p \) maximal. Then \( L_0 = \{ x \in L \mid p \not\preceq x \} \) = all closed subsets of \( \mathcal{T} \) not containing \( p \). Clearly \( L_0 \) contains the largest element of \( \mathcal{Y} \) in this case, and the same is true for \( L_1 \).

Try again this time using induction on card \( \mathcal{T} \).

Suppose then \( L \) is the lattice of closed subsets of the finite poset \( \mathcal{T} \) and that \( U : L \rightarrow \mathbb{R}_{>0} \) satisfies \( U(x \cup y) U(x \cap y) \geq U(x) U(y) \) and that \( f, g \) are monotone functions on \( L \). Then pick a maximal element \( p \) of \( \mathcal{T} \). We then have

\[
\sum_{x \in L} F(x) U(x) = \sum_{p \not\preceq x} F(x) U(x) + \sum_{p \preceq x} F(x) U(x)
\]

Put \( L_0 = \{ x \in L \mid p \not\preceq x \} \) = closed subsets of \( \mathcal{T} - \{ p \} \) and \( L_1 = \{ x \in L \mid p \preceq x \} \) = closed subsets of \( \mathcal{T} - \mathcal{T} - \{ p \} \). Thus

\[
\sum_{x \in L} F(x) U(x) = F(0) + F(1) \quad F(d) = \sum_{x \in L} F(x) U(x)
\]
As before put \( Z(a) = \sum_{x \in L_0} u(x) \)
Then
\[
F(1) = \sum_{y \in L_1} f(y) u(y)
= \sum_{x \in L_0} f(x \cup \{p\}) \frac{u(x \cup \{p\})}{u(x)} u(x)
\]
where by convention \( f(x \cup \{p\}) = f(\overline{x \cup \{p\}}) \) and \( u(x \cup \{p\}) = 0 \) if \( x \cup \{p\} \) is not closed. Now if \( x \leq x' \) and \( x \cup \{p\} \) is not closed, then \( x \cup \{p\} \) is not closed (for \( x' \cup \{p\} = (x \cup \{p\}) \cup x' \)), hence we have
\[
\frac{u(x \cup \{p\})}{u(x)} \leq \frac{u(x' \cup \{p\})}{u(x')}
\]
in all cases. Thus
\[
F(1) Z(0) = \sum_{x \in L_0} f(x \cup \{p\}) \frac{u(x \cup \{p\})}{u(x)} u(x) \sum_{x \in L_0} u(x)
\geq \sum_{x \in L_0} f(x \cup \{p\}) u(x) \sum_{x \in L_0} u(x \cup \{p\})
\geq F(0) Z(1)
\]
so
\[
\sum_{a} H(a) \sum_{a} Z(a) \geq \sum_{a} \frac{F(a)}{Z(a)} G(a) Z(a) \sum_{a} Z(a)
\geq \sum_{a} F(a) \sum_{a} G(a)
\]
as before.
Let $L$ be a finite distributive lattice. I want to understand the different kinds of functions $U: L \to R_{\geq}$ such that

1. $U(x \lor y) \geq U(x) U(y)$

Put $H(x) = -\log U(x)$; then (1) becomes

2. $H(x \lor y) + H(x \land y) \leq H(x) + H(y)$.

Call a function satisfying (2) semi-modular.

Begin by classifying semi-modular functions on $L \times \{0,1\}$. Think of $L$ as the lattice of closed subsets of a poset $J$; then $L \times \{0,1\}$ is the lattice of closed subsets of $J \cup \{\emptyset\}$. First note any semi-modular $H$ on $L \times \{0,1\}$ gives semi-modular functions $H_i$ on $L_i = L \times \{i\}$ for $i = 1, 2$ such that

3. $x \preceq x' \in L_i \implies H_i(x' \lor \{p\}) - H_i(x \lor \{p\}) \leq H_i(x) - H_i(x)$

Conversely a pair $H_i$ on $L_i$, $i = 1, 2$ satisfying (3) gives an $H$ on $L \times \{0,1\}$. To verify (3) one can suppose $x \in L_0$, $y \in L_1$, say $y = x \lor \{p\}$. Then

\[ H_0(x) + H_1(x, x_0, \{p\}) - H_1(x_0 x, \{x_1\}, \{p\}) - H_0(x_0 x_1) \]

\[ \geq H_0(x_0 x_1) - H_0(x_1) - (H_1(x_0 x, \{p\}, \{p\}) - H_1(x, \{p\})) \]

\[ \geq 0 \quad \text{by (3) applied to } x_1 \leq x_0 x_1. \]
Suppose now that $L$ is the lattice of closed sets of a poset $J = J' \cup \{p\}$ where $p$ is maximal in $J$. Then

$$L = L_0 \cup L_1,$$

where $L_0$ = closed subset of $J'$ and $L_1 = \{ y \in L \mid p \in y \}$. By sending $y \in L_1$ to $y - \{p\} \in L_0$ we get an isomorphism

$$L_1 \xrightarrow{\sim} (L_0) \supseteq \omega \quad \omega = J \times p$$

Any semi-modular $H$ on $L$ determines $H_i$ on $L_i$ such that

$$(4) \quad w \leq x \leq x' \text{ in } L_0 \implies H_1(x' \cup \{p\}) - H_1(x \cup \{p\}) \leq H_0(x') - H_0(x)$$

Conversely given $H_i$ on $L_i$, $i = 1, 2$ satisfying $(4)$ I claim we get an $H$ on $L$. To verify $(2)$ one can suppose $x \in L_0$ and $y \in L_1$, say $y = x_1 \cup \{p\}$ with $w \leq x_1 \in L_0$. Then

$$H_0(x) + H_1(x_1 \cup \{p\}) - H_1(x \cup x_1 \cup \{p\}) - H_0(x \land x_1)$$

$$\geq H_0(x \cup x_1) - H_0(x_1) - (H_1(x \cup x_1 \cup \{p\}) - H_1(x_1 \cup \{p\}))$$

$$\geq 0.$$
Simplification of (2): Suppose we choose a maximal chain from \( x \land y \) to \( x \) and one from \( x \land y \) to \( y \).

Then

\[
H(y) - H(x \land y) = H(y_o) - H(y_0) = \sum_{i=1}^{p} H(y_i) - H(y_{i-1})
\]

note

\[
y_{i+1}(x \land y_i) = y_{i-1}
\]

\[
\geq \sum_{i=1}^{p} H(x \land y_i) - H(x \land y_{i-1}) = H(x \land y) - H(x)
\]

Hence if we know (2) holds when \( y \) covers \( x \land y \), then it holds in general. Similarly we see that (2) when \( x, y \) cover \( x \land y \) simply it works in general.

Problem: What is the height of an element of a finite distributive lattice \( L \)? Clearly, the card of the number of irreducibles \( \preceq x \), i.e., the cardinality of \( x \) as a subset of \( T \).

How to describe all semi-modular functions \( H \) on \( L \):

Construct them inductively with respect to height. First select \( H(f) \), then \( H(p) \) for all \( p \in T \). Then for each \( x \) of height 2 look at the two elements it covers; if \( x = \{p, q, \} \) with \( \{p, q, \} \subseteq \{q\} \) closed, then we have a condition

\[
H(x) \leq H(p) + H(q).
\]
if on the other hand \( x = \{ p, q \} \) with \( p < q \), then there is no condition on \( H(x) \). In general \( H(x) \) for \( x \) irreducible is completely arbitrary.

Note that the cone of semi-modular functions on \( L \) contains the subspace of "modular" functions, i.e. such that \( H(x \circ y) + H(x \circ y) = H(x) + H(y) \). Such an \( H \) can be identified with a function on \( J \) and a constant \( c \):

\[
H(x) = \sum_{p \in x} f(p) + c \quad \text{where } f(p) = H(J \leq p) - H(J < p).
\]

So we can normalize things by requiring that \( H(x) = 0 \) if \( x = \emptyset \) or if \( x \) is irreducible. For each reducible \( x \) one consider those elements it covers, i.e. \( x - \{ p \} \) for each \( p \) a generic point of \( x \). For each pair \( p, q \) of generic points of \( x \) with \( p \neq q \) one has a condition

\[
H(x) \leq H(x - \{ p \}) + H(x - \{ q \}) - H(x - \{ p, q \})
\]

so the possible choices for \( H(x) \) form a half-line \( \mathbb{R}_{\geq} \).