August 15, 1977

Go back to $\qquad \dfrac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$

If p is even, then we have seen that

$$+m^-(\lambda) = \frac{u_1(0,-\lambda)}{-u_2(-0,-\lambda)} = -m(\lambda) \qquad\qquad m = m^+$$

or $\qquad m^-(\lambda) = -m(-\lambda)$

so that the eigenvalue function is

$$\boxed{m(\lambda) - m^-(\lambda) = m(\lambda) + m(-\lambda)} \qquad \text{if } p \text{ even}$$

I was hoping this might be $\hat{\jmath}(\frac{1}{2}+i\lambda)$ for a suitable choice of p. But notice that $\hat{\jmath}$ is real [redacted] for λ real, and $|m(\lambda)| = 1$ so that one would have to have

$$m(-\lambda) = \overline{m(\lambda)} = \frac{1}{m(\lambda)}$$

[redacted] for λ real, hence $m(-\lambda) = \dfrac{1}{m(\lambda)}$ for all λ. Thus we would have

$$\hat{\jmath}\left(\tfrac{1}{2}+i\lambda\right) = m(\lambda) + \frac{1}{m(\lambda)}$$

and we would have the contradiction that

$$\hat{\jmath}\left(\tfrac{1}{2}+i\lambda\right) \in [-2,2] \implies \lambda \text{ real}$$

Does this [redacted] same situation take place when we use the real form of the system?

Notice that p real $\implies m(-\lambda) = \dfrac{1}{m(\lambda)}$.

Instead of having $\hat{f}(\frac{1}{2}+i\lambda) = m(\lambda) - m^-(\lambda) = m(\lambda)+m(-\lambda)$ one might have instead

$$\hat{f}\left(\frac{1}{2}+i\lambda\right) = \begin{vmatrix} u_1^+(x,\lambda) & u_1^-(x,\lambda) \\ u_2^+(x,\lambda) & u_2^-(x,\lambda) \end{vmatrix} = \begin{vmatrix} u_1^+(x,\lambda) & -u_1^+(-x,-\lambda) \\ u_2^+(x,\lambda) & u_2^+(-x,-\lambda) \end{vmatrix}$$

$$= u_1^+(0,\lambda)\,u_2^+(0,-\lambda) + u_1^+(0,-\lambda)\,u_2^+(0,\lambda)$$

This is evidently symmetric in λ. What does it mean for this to be real when λ is real? Note that when $\lambda \in \mathbb{R}$, the equation is symmetric under $x \mapsto x$, $\lambda \mapsto \lambda$ $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \overline{u_2} \\ \overline{u_1} \end{pmatrix}$, hence one has a constant $c(\lambda)$ $\ni$

$$\begin{pmatrix} u_1^+(x,\lambda) \\ u_2^+(x,\lambda) \end{pmatrix} = \begin{pmatrix} \overline{u_2^+(x,\lambda)} \\ \overline{u_1^+(x,\lambda)} \end{pmatrix} c(\lambda)$$

$$= \begin{pmatrix} \overline{c(\lambda)\,\overline{u_1^+(x,\lambda)}} \\ \overline{c(\lambda)\,\overline{u_2^+(x,\lambda)}} \end{pmatrix} c(\lambda) = c(\lambda)\,\overline{c(\lambda)} \begin{pmatrix} u_1^+(x,\lambda) \\ u_2^+(x,\lambda) \end{pmatrix}$$

So $|c(\lambda)| = 1$.

Now the solutions $u^+(x,\lambda)$ is only defined up to a holomorphic multiple, hence if I rewrite the above

$$\begin{pmatrix} f(\lambda)\,u_1^+(x,\lambda) \\ f(\lambda)\,u_2^+(x,\lambda) \end{pmatrix} = \begin{pmatrix} \overline{f(\lambda)\,u_2^+(x,\lambda)} \\ \overline{f(\lambda)\,u_1^+(x,\lambda)} \end{pmatrix} \frac{c(\lambda)\,f(\lambda)}{\overline{f(\lambda)}}$$

so that $c(\lambda)$ can be changed to $\frac{c(\lambda)\,f(\lambda)}{\overline{f(\lambda)}}$ if I want.

Suppose $c(\lambda) = 1$, i.e. $u_2(x,\lambda) = \overline{u_1(x,\lambda)}$ for λ real.
Then

$$\begin{vmatrix} u_1^+(0,\lambda) & -u_1^+(0,-\lambda) \\ u_2^+(0,\lambda) & u_2^+(0,-\lambda) \end{vmatrix} = \begin{vmatrix} u_1^+(0,\lambda) & -u_1^+(0,-\lambda) \\ \overline{u_1^+(0,\lambda)} & \overline{u_1^+(0,-\lambda)} \end{vmatrix}$$

gets conjugated when we substitute λ for $-\lambda$.

?

Suppose p even so that $\overline{m}(\lambda) = -m(-\lambda)$. The condition for an eigenvalue is $m(\lambda) + m(-\lambda) = 0$. The function $m(\lambda) + m(-\lambda)$ is symmetric, however, I don't think we want it to be $\hat{\rho}(\frac{1}{2} + i\lambda)$ because we've seen $m(\lambda) + m(-\lambda) \in \mathbb{R}$ with $\lambda \in \mathbb{R}$ forces $m(-\lambda) = \overline{m(\lambda)} = \frac{1}{m(\lambda)}$ so $m(\lambda) + m(-\lambda) = m(\lambda) + \frac{1}{m(\lambda)}$ $\in [-2,2]$ forces λ real.

So $\hat{\rho}(\frac{1}{2} + i\lambda)$ is not to be $m(\lambda) + m(-\lambda)$, and perhaps we can arrange for it to be the Wronskian

$$u_2^+(0,\lambda)\, u_2^+(0,-\lambda) \left[m(\lambda) + m(-\lambda) \right]$$

which is again symmetric. The question is whether $u_2^+(0,\lambda)$ can be chosen so that this is real for λ real.

Now we know that our equation has the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$ for λ real, so suppose $u_2^+(0,\lambda)$ can be chosen such that u^+ is fixed under this symmetry:

$$u_1^+(x,\lambda) = \overline{u_2^+(x,\lambda)}$$

If so, then the Wronskian is $\begin{vmatrix} u_1^+(0,\lambda) & -u_1^+(0,-\lambda) \\ \overline{u_1^+(0,\lambda)} & \overline{u_1^+(0,-\lambda)} \end{vmatrix}$

$$= u_1^+(0,\lambda)\, \overline{u_1^+(0,-\lambda)} + \overline{u_1^+(0,\lambda)}\, u_1^+(0,-\lambda)$$

which is obviously a real number. Now the condition $u_1^+(x,\lambda) = \overline{u_2^+(x,\lambda)}$ says that

$$m(\lambda) = \frac{u_1^+(0,\lambda)}{u_1^+(0,\lambda)} = \overline{\frac{u_2^+(0,\lambda)}{u_2^+(0,\lambda)}}$$

so all we have to do to achieve this desirable state is to take $m(\lambda)$ and factor it

$$m(\lambda) = \frac{\overline{f(\bar\lambda)}}{f(\lambda)}$$

where f is meromorphic. Note

$$f(\lambda)f(-\lambda)\left[m(\lambda)+m(-\lambda)\right] = f(-\lambda)\overline{f(\bar\lambda)} + f(\lambda)\overline{f(-\bar\lambda)}$$

is real for λ real.

August 16, 1977.

To understand theory of strings (book of Dym + McKean) due to Krein, which generalizes Stieltjes' theory of continued fractions + the moment problem.

The idea: I want to try to explain this theory as a sup of two examples: smooth strings and discrete strings.

Smooth strings: Suppose given a string on $0 \le x \le l$ with a variable density function $m'(x)$ which we supposed is smooth and ≥ 0. Then we can consider the S-L system

$$\frac{d^2u}{dx^2} = \lambda m'(x)u$$

on the interval. This SL equation is what one obtains by separating variables in the wave eqn. describing the motion of the string:

$$\frac{\partial^2 u}{\partial x^2} = m' \frac{\partial^2 u}{\partial t^2}$$

and λ can be interpreted as $-\omega^2$ where ω is the frequency of vibration. We have to give boundary conditions at the endpoints. At $x=0$ we take the boundary condition

$$\frac{du}{dx}(0) = 0$$

which means that this end of the string is free. At the other end we have to give a boundary condition which is of the form

$$u(\ell) + k\, u'(\ell) = 0$$

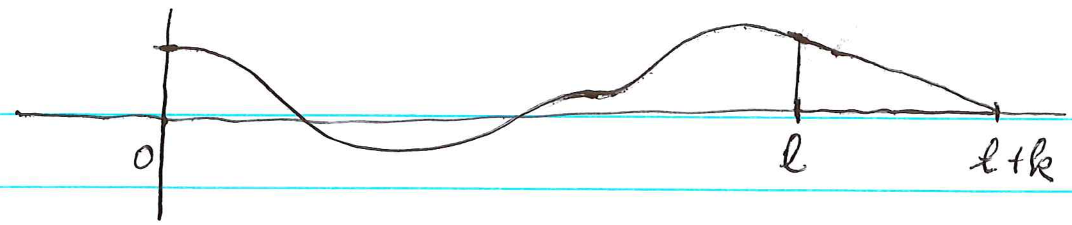
with k real to be self-adjoint. I think it turns out that one must have $0 \leq k \leq \infty$ for the eigenvalues $\lambda = -\omega^2$ to be ≤ 0. Suppose u is an eigenfunction. Thus

$$\int_0^\ell \lambda u \bar{u}\, m' dx = \int_0^\ell \frac{d^2 u}{dx^2} \bar{u}\, dx = \left[\frac{du}{dx} \bar{u} \right]_0^\ell - \int_0^\ell \left| \frac{du}{dx} \right|^2 dx$$

$$\lambda \int_0^\ell |u|^2 m' dx = -k |u(\ell)|^2 - \int_0^\ell \left| \frac{du}{dx} \right|^2 dx$$

Thus we want $0 \leq k \leq \infty$ so that λ has to be ≤ 0. ($k=\infty$ means $u'(\ell)=0$).

One can interpret this boundary condition as tying onto the end of the string at $x=\ell$ a weightless string tied down to the x-axis at $x = l+k$:



$$\frac{u(l)}{k} = -u'(l)$$

Speed of waves: From the characteristics of $\frac{\partial^2 u}{\partial x^2} = m' \frac{\partial^2 u}{\partial t^2}$ one knows that disturbances move along the string with speed $(m')^{-1/2}$. Better the characteristic in the forward direction is

$$dt = (m')^{1/2} dx$$

so that the time taken for a disturbance to propagate from x_0 to x_1 is

$$\int_{x_0}^{x_1} (m')^{1/2} dx.$$

Hence if $m' \doteq 0$ the speed is very fast. Also m' approximates a δ-function near x, i.e.

$$m' \doteq \frac{1}{\varepsilon} \quad \text{in a interval } I \text{ of length } \varepsilon$$

then $$\int_I (m')^{1/2} dx \doteq \varepsilon^{1/2}$$

which goes to zero as $\varepsilon \to 0$. Thus in the limiting case of a discrete string disturbances propogate instantaneously.

The spectral measure $d\mu$ for the string is defined as follows: First let $u(x,\lambda)$ denotes the solution of $\dfrac{d^2u}{dx^2} = \lambda m' u$ such that

$$\begin{cases} u'(x,0) = 0 \\ u(x,0) = 1. \end{cases}$$

Then the eigenvalues are those λ such that the boundary condition at $x=\ell$ is satisfied. Let $\lambda_1 > \lambda_2 > \cdots$ be the sequence of eigenvalues. We know that any nice function on $[0,\ell]$ can be expanded in terms of the eigenfunctions

$$f(x) = \sum a_j \, u(x,\lambda_j)$$

where

$$\int_0^\ell f(x)\, u(x,\lambda_j)\, m'(x)\,dx = a_j \int_0^\ell u(x,\lambda_j)^2\, m'(x)\,dx$$

(The reason we integrate with respect to $dm = m'(x)dx$ is to insure orthogonality of eigenfunctions:

$$L u_i = \lambda_i M u_i \qquad i = 1, 2.$$

$$\begin{aligned} (Lu_1, u_2) &= \lambda_1 (Mu_1, u_2) \\ \| \quad & \\ (u_1, Lu_2) &= \lambda_2 (u_1, Mu_2) \end{aligned} \Rightarrow (Mu_1, u_2) = 0 \quad \text{if } \lambda_1 \neq \lambda_2)$$

So now let $d\mu(\lambda) = \sum r_j^{-1} \delta(\lambda - \lambda_j)$

where $r_j = \int_0^\ell u(x,\lambda_j)^2\, dm$. One gets

$$f(x) = \int a(\lambda)\, u(x,\lambda)\, d\mu(\lambda) \quad \text{where} \quad a(\lambda) = \frac{1}{r(\lambda)} \int_0^\ell f(x)\, u(x,\lambda)\, dm$$

$$\lambda \in \{\lambda_j\}.$$

In other words one has an isometry

$$L^2(\mathbb{R}, d\mu) \xrightarrow{\sim} L^2([0,\ell], dm)$$

$$a(\lambda) \longmapsto \int_0^\ell a(\lambda)\, u(x,\lambda)\, d\mu(\lambda)$$

$$\frac{1}{h(\lambda)} \int_0^\ell f(x) u(x,\lambda)\, dm \longleftarrow f(x)$$

Let M denote multiplication by m' and let $L = \frac{d^2}{dx^2}$ so that the equation defining the eigenvalues is

$$Lu = \lambda M u$$

or

$$M^{-1} L u = \lambda u$$

Note that with respect to the inner product

$$(Mu, u) = \int |u|^2 m' dx$$

the operator $M^{-1}L$ is formally self-adjoint:

$$(MM^{-1}Lu, v) = (Lu, v) = (u, Lv) = (Mu, M^{-1}Lv)$$

So therefore we are really dealing with the operator $M^{-1}L = \frac{1}{m'}\frac{d^2}{dx^2} = \frac{d}{dm}\frac{d}{dx}$ and its eigenvalues.

The Green's function should be

$$(\lambda - M^{-1}L)^{-1} u = \sum_j \frac{u_j}{\lambda - \lambda_j} (Mu, u_j)$$

which is represented by the kernel

$$\sum_j \frac{u(x,\lambda_j)\, u(y,\lambda_j)}{\lambda - \lambda_j} = \int \frac{u(x,\hat\lambda)\, u(y,\hat\lambda)}{\lambda - \hat\lambda}\, d\mu(\hat\lambda)$$

However one also has the representation for the Green's function described in terms of the solutions $u(x,\lambda)$ and the solution $v(x,\lambda)$ satisfying the boundary condition at $x=l$ and normalized so the Wronskian

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = -1$$

i.e. $v'(0,\lambda) = -1$. This defines v except for the eigenvalues.

~~One has~~ $\cancel{f(y) = \int \delta(x-y) f(x)\,dx = \int \frac{\delta(x-y)}{m'(x)} f(x)\,dm(x)}$

~~and~~ $\cancel{\left(\lambda - \frac{1}{m'}\frac{d^2}{dx^2}\right) g(x,y) = \frac{\delta(x-y)}{m'(x)}}$

To find the kernel for the operator $(\lambda - M^{-1}K)^{-1}$, i.e. the function $g(x,y)$ such that

$$\left(\lambda - \frac{1}{m'}\frac{d^2}{dx^2}\right)\int g(x,y) f(y)\,dm(y) = f(x) = \int \frac{\delta(x-y)}{m'(y)} f(y)\,dm(y)$$

Thus we want $\left(\lambda - \frac{1}{m'(x)}\frac{d^2}{dx^2}\right)^{-1} g(x,y) = \frac{\delta(x-y)}{m'(x)}$ so

$g(x,y)$ is continuous in x satisfying both bdry conditions and $\frac{d}{dx}g(x,y)$ jumps by -1 at y.

$$g_\lambda(x,y) = \begin{cases} a\,u(x,\lambda) & x<y \\ b\,v(x,\lambda) & x>y \end{cases}$$

$$\begin{cases} a\,u(y,\lambda) - b\,v(y,\lambda) = 0 \\ a\,u'(y,\lambda) - b\,v'(y,\lambda) = +1 \end{cases}$$

Solution is $a = v(y,\lambda)$ $\quad b = u(y,\lambda)$. Thus

$$g_\lambda(x,y) = \begin{cases} u(x,\lambda)\, v(y,\lambda) & x<y \\ v(x,\lambda)\, u(y,\lambda) & x>y \end{cases}$$

$$= \int \frac{u(x,\hat\lambda)\, u(y,\hat\lambda)}{\lambda-\hat\lambda}\, d\mu(\hat\lambda)$$

($\exists$ problems with this if x,y are in same mass-free interval)

So we can recover the spectral measure by putting $x=y=0$, whence we get

$$v(0,\lambda) = \int \frac{d\mu(\hat\lambda)}{\lambda-\hat\lambda}$$

Example: $m'(x)=1$ on $0 \le x \le \pi$ with the boundary condition $u'(\pi)=0$. Then

$$u(x,\lambda) = \cos(\omega x) \qquad \lambda = -\omega^2$$

and $$v(x,\lambda) = \frac{\cos \omega(x-\pi)}{\omega \sin(\omega\pi)} = -\frac{\cos \omega(\pi-x)}{\omega\sin(\omega\pi)}$$

$$v'(x,\lambda) = \frac{+\sin(\omega x-\omega\pi)\,(+\omega)}{\omega\sin(\omega\pi)} = -1 \quad \text{at } x=0$$

so $$v(0,\lambda) = -\frac{\cos \pi\omega}{\omega\sin\pi\omega} = -\frac{1}{\omega}\cot(\pi\omega)$$

Now $$\pi\cot(\pi\omega) = \sum_{n\in\mathbb{Z}} \frac{1}{\omega-n} = \frac{1}{\omega} + \sum_{n=1}^{\infty} \frac{2\omega}{\omega^2-n^2}$$

so $$-\frac{1}{\omega}\cot(\pi\omega) = -\frac{1}{\pi\omega}\left(\pi\cot(\pi\omega)\right)$$

$$= \frac{+1}{\pi}\left[\frac{1}{-\omega^2} + \sum_{n=1}^{\infty} \frac{2}{-\omega^2+n^2}\right]$$

or $$v(0,\lambda) = \frac{1}{\pi}\left[\frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{2}{\lambda - n^2}\right]$$

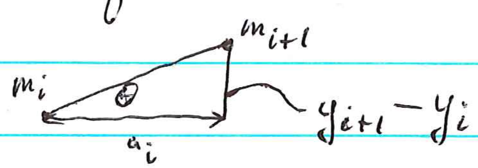
Now eigenvalues occur when $u'(\pi,\lambda) = -\omega \sin(\omega\pi) = 0$, hence when $\omega = n$, $n = 0, 1, 2, \ldots$.

$$\int_0^{\pi} \cos(nx)^2\, dx = \begin{cases} \pi & n = 0 \\ \frac{\pi}{2} & n = 1, 2, 3, \ldots \end{cases}$$

so $$\int \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda} = \frac{1}{\pi}\frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{2}{\pi}\frac{1}{\lambda + n^2}$$ so it works!

---

discrete string: One has masses $m_0, m_1, m_2, \ldots$ located at $x_0 = 0$, $x_1 = a_1$, $x_2 = a_1 + a_2$, ... Let y_n be the displacement of the n-th mass.



The string between the masses is considered weightless, and under a uniform tension $T = 1$. So the force on m_i due to the i-th segment is

$$= T\sin\theta \doteq T\,\frac{y_{i+1} - y_i}{a_i} = k_i(y_{i+1} - y_i)$$

where $k_i = \frac{1}{a_i}$. Equation of motion:

$$m_i \ddot{y}_i = k_i(y_{i+1} - y_i) + k_{i-1}(y_{i-1} - y_i)$$

which leads to the eigenvalue problem

$$\lambda m_i u_i = k_i u_{i+1} + (-k_i - k_{i-1}) u_i + k_{i-1} u_{i-1}$$

which I will write $\lambda M u = K u$. I work in the Hilbert space with inner product $(Mu, u) = \sum m_i u_i^2$ in which the operator $M^{-1}K$ is self-adjoint. Denote by $u(\lambda)$ the solution with the initial value

$$u(\lambda)_0 = 1.$$

Suppose there are only $n+1$ masses $m_0, \dots, m_n$ and m_{n+1} is tied down, i.e. the other boundary condition is $y_{n+1} = 0$. Then for λ not an eigenvalue, one can define $v(\lambda)$ to be the solution such that

$$\lambda m_0 v(\lambda)_0 = k_0 v(\lambda)_1 - k_0 v(\lambda)_0 + 1$$

$$v(\lambda)_{n+1} = 0.$$

(as if one had $k_{-1} v_{-1} = 1$). Rewrite the equations

$$\frac{k_{i-1} v_{i-1}}{v_i} = k_{i-1} + \lambda m_i + k_i - \frac{k_i^2}{\frac{k_i v_i}{v_{i+1}}}$$

Then

$$v(\lambda)_0 = \frac{1}{\lambda m_0 + k_0 -}\,\frac{k_0^2}{k_0 + \lambda m_1 + k_1 -}\cdots\frac{k_{n-1}^2}{k_{n-1} + \lambda m_n + k_n}$$

or

$$\boxed{v(\lambda)_0 = \frac{1}{\lambda m_0 +}\,\frac{1}{a_0 +}\,\frac{1}{\lambda m_1 +}\cdots\cdots\frac{1}{\lambda m_n +}\,\frac{1}{a_n}}$$

On the other hand one has $v = (\lambda M - K)^{-1} e_0$. The spectral measure is defined as follows. Let $\lambda_0 > \lambda_1 > \dots > \lambda_n$ be the eigenvalues. Then one can expand any vector y into eigenfunctions

$$y = \sum a_j u(\lambda_j) \quad \text{where} \quad (My, u(\lambda_j)) = a_j \underbrace{(Mu(\lambda_j), u(\lambda_j))}_{r_j}$$

so that if we put

$$d\mu(\lambda) = \sum_j \frac{1}{r_j}\,\delta(\lambda-\lambda_j)$$

then we get an isometry

$$L^2(\mathbb{R}, d\mu) \xrightarrow{\sim} \left\{ (y_i)_{0\le i\le n} \text{ with } (My,y) \text{ inner prod.} \right\}$$

$$a(\lambda) \longmapsto \int a(\lambda)\,u(\lambda)\,d\mu(\lambda)$$

Check: $\varphi = \int a(\lambda)\,u(\lambda)\,d\mu(\lambda) = \sum_j a(\lambda_j)\,u(\lambda_j)\,\frac{1}{r_j}$

$$(M\varphi,\varphi) = \sum \frac{a(\lambda_j)^2}{r_j^2}\big(Mu(\lambda_j),u(\lambda_j)\big) = \sum a(\lambda_j)^2\frac{1}{r_j}$$

$$= \int a(\lambda)^2\,d\mu(\lambda).$$

Suppose

$$e_0 = \sum a_j\,u(\lambda_j)$$

Then $m_0 = \big(Me_0, u(\lambda_j)\big) = a_j r_j$ so $a_j = \dfrac{m_0}{r_j}$ so

$$e_0 = \sum m_0\,\frac{u(\lambda_j)}{r_j} = m_0\int u(\hat\lambda)\,d\mu(\hat\lambda)$$

Hence

$$v(\lambda) = (\lambda M - K)^{-1}e_0 = (\lambda - M^{-1}K)^{-1}M^{-1}e_0 = \int \frac{u(\hat\lambda)}{\lambda-\hat\lambda}\,d\mu(\hat\lambda)$$

and so taking the component of degree 0, we get:

$$v(\lambda)_0 = \int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}} = \frac{1}{\lambda m_0 +}\;\frac{1}{a_0 +}\cdots\cdots\frac{1}{\lambda m_n +}\;\frac{1}{a_n}$$

Dual string: Let $y_i^+ = \dfrac{y_{i+1} - y_i}{a_i} = k_i(y_{i+1} - y_i)$ be the slope of the i-th segment. Then

$$\lambda m_i y_i = y_i^+ - y_{i-1}^+$$

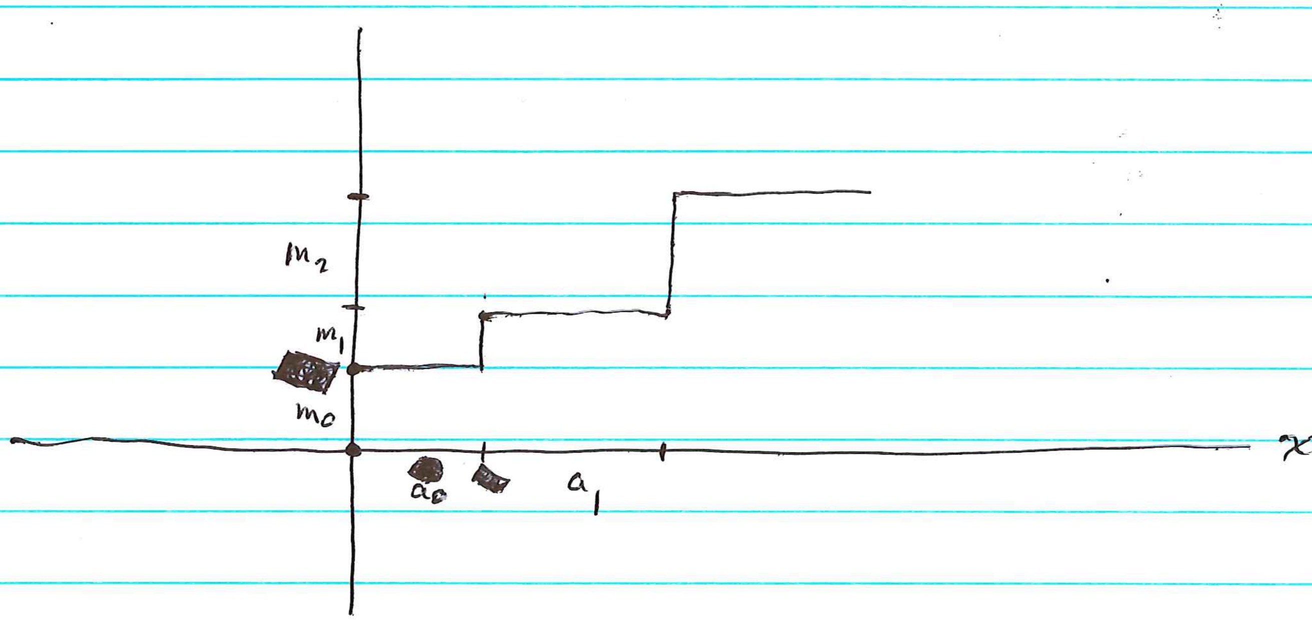
so

$$\lambda(y_{i+1} - y_i) = \frac{y_{i+1}^+ - y_i^+}{m_{i+1}} + \frac{y_{i-1}^+ - y_i^+}{m_i}$$

or

$$\lambda a_i\, y_i^+ = \frac{1}{m_{i+1}}(y_{i+1}^+ - y_i^+) + \frac{1}{m_i}(y_{i-1}^+ - y_i^+)$$

which is the equation for a string with the masses a_i and the separation m_{i+1} between a_i and a_{i+1}.



In the smooth string case the equation is

$$\frac{d}{dm}\frac{du}{dx} = \lambda u$$

which leads to

$$\frac{d}{dx}\frac{d}{dm}\left(\frac{du}{dx}\right) = \lambda\left(\frac{du}{dx}\right)$$

so you get a new string with x, m interchanged and u replaced by $\frac{du}{dx}$.

Now the boundary conditions

$$u'(0) = 0$$

$$u(\ell) + k u'(\ell) = 0$$

for the original string become for the dual string

$$w(0) = 0$$

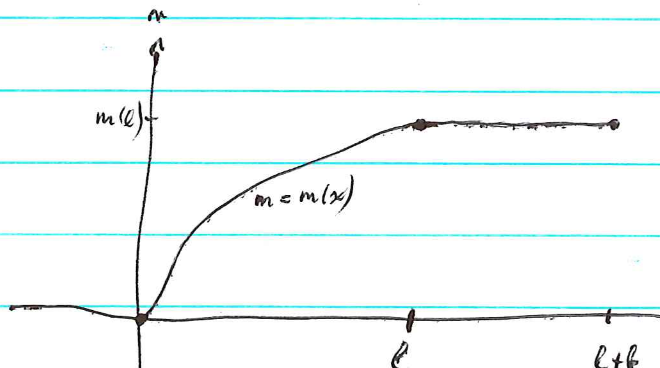
and $\quad k\, w(m(\ell)) + \frac{1}{\lambda}\frac{dw}{dm}(m(\ell)) = 0.\quad$ So at least

for $k = 0$ and $k = \infty$ these boundary conditions are nice:

$k = 0$: $\quad \begin{cases} u'(0) = 0 \\ u(\ell) = 0 \end{cases} \qquad \begin{cases} w(0) = 0 \\ \frac{dw}{dm}(m(\ell)) = 0 \end{cases}$

$k = \infty$ $\quad \begin{cases} u'(0) = 0 \\ u'(\ell) = 0 \end{cases} \qquad \begin{cases} w(0) = 0 \\ w(m(\ell)) = 0 \end{cases}$

But I have seen that k can be interpreted as putting on a weightless piece of length k from ℓ to $\ell + k$, so this should be the same as attaching to the dual string a point mass of mass k at the point $m(\ell)$:

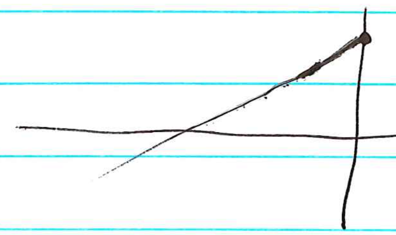


Interesting case is that of a smooth string of length with a point mass $m[\ell]$ at $x=\ell$. Then the equation of motion is still

$$\frac{\partial^2 y}{\partial x^2} = m' \frac{\partial^2 y}{\partial t^2} \qquad \text{in} \qquad 0 \leq x < \ell$$

but at ℓ we want

$$\frac{\partial y}{\partial x}(\ell) = -m[\ell]\frac{\partial^2 y}{\partial t^2}(\ell)$$



Hence we get the SL problem

$$\frac{d^2 u}{dx} = \lambda m'(x) u \qquad 0 \leq x < \ell$$

$$u'(0) = 0$$

$$u'(\ell) + \lambda m[\ell] u(\ell) = 0$$

since $\lambda = -\omega^2 < 0$ this means both u, u' have same sign.

where λ also appears in the boundary conditions. If u is an eigenfunction

$$\int_0^\ell \lambda u^2 m' dx = \int_0^\ell \frac{d^2 u}{dx^2} u \, dx = \left[\frac{du}{dx} u\right]_0^\ell - \int_0^\ell \left(\frac{du}{dx}\right)^2 dx$$

$$= -\lambda m[\ell] u(\ell)^2 - \int_0^\ell \left(\frac{du}{dx}\right)^2 dx$$

$$\therefore \quad \lambda\left(\int_0^\ell u^2 m' dx + m[\ell] u(\ell)^2\right) = -\int_0^\ell \left(\frac{du}{dx}\right)^2 dx \leq 0$$

so the eigenvalues are ≤ 0.

August 19, 1977

Definition: A de Branges function is an entire function $E(\lambda)$ such that

$$\operatorname{Im}(\lambda) > 0 \implies |E(\lambda)| > |E^{\#}(\lambda)|$$

where $E^{\#}(\lambda) = \overline{E(\bar{\lambda})}$.

Example: Let $u = \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix}$ be the solution of

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

starting with values

$$\begin{pmatrix} u_1(0,\lambda) \\ u_2(0,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha} \end{pmatrix}$$

where α does not depend on λ. Notice that the DE has the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$ $\lambda \mapsto \bar{\lambda}$, hence one has

$$\begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} = c \begin{pmatrix} \overline{u_2(x,\bar{\lambda})} \\ \overline{u_1(x,\bar{\lambda})} \end{pmatrix}$$

where c is a constant one can evaluate by setting $x=0$:

$$e^{i\alpha} = c\,\overline{e^{-i\alpha}} \quad \therefore \quad c = 1.$$

Hence $u_2(x,\lambda) = \overline{u_1(x,\bar{\lambda})}$. On the other hand

$$\frac{d}{dx}\left(|u_1|^2 - |u_2|^2\right) = (i\lambda u_1 + \cancel{\bar{p}u_2})\bar{u}_1 + (-i\bar{\lambda}\bar{u}_1 + \cancel{p\bar{u}_2})u_1$$
$$- (\cancel{pu_1} - i\lambda u_2)\bar{u}_2 - (\cancel{\bar{p}\bar{u}_1} + i\bar{\lambda}\bar{u}_2)u_2$$

$$= i(\lambda - \bar{\lambda})\left(|u_1|^2 + |u_2|^2\right) = -2(\operatorname{Im}\lambda)\left(|u_1|^2+|u_2|^2\right) < 0$$

if $\operatorname{Im}(\lambda) > 0$. Hence $|u_1|^2 - |u_2|^2$ decreases as x increases, so that $|u_1(x,\lambda)| < |u_2(x,\lambda)|$ for any $x > 0$. Therefore if we put

$$E(\lambda) = u_2(x,\lambda)$$

for some λ we have

$$|E(\lambda)| = |u_2(x,\lambda)| > |u_1(x,\lambda)| = \left|\overline{u_2(x,\bar{\lambda})}\right| = |E^{\#}(\lambda)|$$

hence $E(\lambda)$ is a de Branges function for any $x > 0$.

Notice also that if we had taken initial values:

$$\begin{pmatrix} u_1(0,\lambda) \\ u_2(0,\lambda) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

with $|c_1| = |c_2| > 0$, then we have

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c\begin{pmatrix} \bar{c}_2 \\ \bar{c}_1 \end{pmatrix} \qquad \text{hence} \qquad \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} = c\begin{pmatrix} \overline{u_2(x,\bar{\lambda})} \\ \overline{u_1(x,\bar{\lambda})} \end{pmatrix}$$

with $c = \dfrac{c_1}{\bar{c}_2} = \dfrac{c_1 c_2}{|c_2|^2} = \dfrac{c_1 c_2}{|c_1|^2} = \dfrac{c_2}{\bar{c}_1}$. Then again we have that $u_2(x,\lambda)$ is a de Branges function, since

$$|u_2(x,\lambda)| > |u_1(x,\lambda)| = |c|\left|\overline{u_2(x,\bar{\lambda})}\right| = \left|\overline{u_2(x,\bar{\lambda})}\right|$$

Reality condition for a de Branges function is:

$$E^{\#}(\lambda) = E(-\lambda)$$

i.e. $\overline{E(\lambda)} = E(-\lambda)$ if λ real. This means that

$$E(\lambda) = A(\lambda) - iB(\lambda)$$

where $A(\lambda)$ is even $B(\lambda)$ is odd and both are real for $\lambda \in \mathbb{R}$.

Example. Assume p is real-valued; then the system has the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$ and $\lambda \to -\lambda$. Hence if we have the initial condition

$$\begin{pmatrix} u_1(0,\lambda) \\ u_2(0,\lambda) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

stable under this symmetry: $\dfrac{c_1}{c_2} = \dfrac{c_2}{c_1}$ or $c_1 = \pm c_2$, then we have

$$(*) \qquad \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} = \tilde{c} \begin{pmatrix} u_2(x,-\lambda) \\ u_1(x,-\lambda) \end{pmatrix}$$

where

$$\tilde{c} = \frac{c_1}{c_2} = \frac{c_2}{c_1}$$

In general we have

$$\begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} = c \begin{pmatrix} u_2^{\#}(x,\lambda) \\ u_1^{\#}(x,\lambda) \end{pmatrix} \qquad c = \frac{c_1}{\bar{c}_2} = \frac{c_2}{\bar{c}_1} \in S^1$$

$$c_i = u_i(0,\lambda).$$

And $(*)$ above when $p = \bar{p}$. So to get the reality condition for the de Branges fn. $u_2(x,\lambda)$ we need to have

$$c = \tilde{c} \qquad \text{or} \qquad \frac{c_1}{c_2} = \frac{c_1}{\bar{c}_2}$$

hence c_1, c_2 are real and $c_2 = \pm c_1$.

August 21, 1977

Consider $\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$ on $0 \le x \le \ell$

and let $u(x,\lambda)$ be the solution starting with $u(0,\lambda) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $|c_1| = |c_2| = 1$. Then we have

$$\begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} = c \begin{pmatrix} \overline{u_2(x,\bar\lambda)} \\ \overline{u_1(x,\bar\lambda)} \end{pmatrix} \qquad c = \frac{c_1}{\bar c_2} = \frac{c_2}{\bar c_1}$$

and we have seen that $|u_2(x,\lambda)| > |u_2(x,\bar\lambda)|$ for $\operatorname{Im}\lambda > 0$ and any $x > 0$.

If we specify a boundary condition at $x = \ell$ of the type $u_1(\ell,\lambda) = e^{-i\alpha} u_2(\ell,\lambda)$ $\qquad \alpha \in \mathbb{R}$

then we get a self-adjoint bdry value problem, and hence a sequence of eigenvalues λ_j, and an expansion thm.

$$y(x) = \int u(x,\lambda)\,(y, u(\cdot,\lambda))\, d\mu(\lambda)$$

where $(u(\cdot,\lambda), y) = \int_0^\ell y^*(x)\, u(x,\lambda)\, dx$ is an entire function of λ such that

$$\int_{-\infty}^{\infty} |(u(\cdot,\lambda), y)|^2 \, d\mu(\lambda) = \int_0^\ell |y(x)|^2 dx$$

Thus we get an isomorphism of $(L^2([0,\ell]))^2$ with a Hilbert space consisting of entire functions whose norm depends on the values of these functions on the real line. Hopefully this will turn out to be a de Branges space. Note that this space doesn't depend on the bdry condition at $x = \ell$.

Example: $p = 0$; and $c_1 = c_2 = 1$. Then

$$u(x,\lambda) = \begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x} \end{pmatrix}$$

If the boundary condition at $x = l$ is $u_1 = u_2$, then the eigenvalues are determined by

$$e^{i\lambda l} = e^{-i\lambda l} \quad \text{or} \quad e^{2i\lambda l} = 1.$$

Suppose $l = \pi$, so that the eigenvalues are $\lambda = n \in \mathbb{Z}$.

Expansion: Given y_1, y_2 on $[0,\pi]$ you want:

$$y_1 = \sum a_n e^{inx}$$

$$y_2 = \sum a_n e^{-inx} = \sum a_{-n} e^{inx}$$

$$\cancel{\frac{y_1+y_2}{2} = \sum_{n\in\mathbb{Z}} (a_n + a_{-n}) \cos nx}$$

$$\cancel{\frac{y_1-y_2}{2i} = \sum_{n\in\mathbb{Z}} (a_n - a_{-n}) \sin nx}$$

Rewrite the second as $\quad y_2(-x) = \sum a_n e^{inx}$ for $-\pi \le x \le 0$.
Then it is clear that we get the **Fourier** series of the function $\tilde{y}$ which is $y_1(x)$ for $0 \le x \le \pi$ and $y_2(-x)$ for $-\pi \le x \le 0$:

$$2\pi a_n = \int_0^\pi \left(y_1 e^{-inx} + y_2 e^{inx} \right) dx = \int_{-\pi}^{\pi} \tilde{y}(x) e^{-inx}\, dx$$

In this case the entire function is

$$f(\lambda) = \int_{-\pi}^{\pi} \tilde{y}(x)\, e^{-i\lambda x}\, dx$$

which is of exponential type $\leq \pi$. By Paley-Wiener the space of these transforms consists of all entire functions of type $\leq \pi$ square-integrable on the real line.

Probability interpretation: Suppose one is given a measure $d\mu(\lambda)$ on $\mathbb{R}$. Then one gets a Hilbert space $Z = L^2(\mathbb{R}, d\mu)$ which comes equipped with a 1-parameter group of unitary operators, namely, U_t = multiplication by $e^{i\lambda t}$. (Recall this is the continuous version of having a Hilbert space + unitary operator U + cyclic vector. Here one gets the model $L^2(S^1, d\mu)$ with U = mult. by z and cyclic vector given by 1.)

Fix $T > 0$ and let Z^T be the span of the functions $e^{i\lambda t} = U_t \cdot 1$ for $|t| \leq T$. The problem is to "construct" the orthogonal projection of $e^{i\lambda T}$ on Z^T.

Suppose that $d\mu(\lambda)$ is the spectral measure belonging to a string which I will suppose to be smooth with the boundary condition $u'(l) = 0$. As usual $A(x, \lambda)$ denotes the solution of

$$\frac{d^2 y}{dx^2} = -\lambda^2 \rho y \qquad\qquad \rho = m'(x)$$

satisfying $A(0,\lambda) = 1$, $A'(0,\lambda) = 0$ and

$$B(x,\lambda) = -\frac{1}{\lambda} A'(x,\lambda)$$

Thus

$$\frac{dB}{dx} = -\frac{1}{\lambda}\frac{d^2 A}{dx^2} = \left(-\frac{1}{\lambda}\right)(-\lambda^2 \rho A) = \lambda \rho A$$

hence A, B satisfy the system:

$$\begin{cases} \lambda B = -\dfrac{dA}{dx} \\ \lambda A = \dfrac{dB}{dm} \end{cases} \qquad dm = \rho\, dx$$

Now it's known that any $f \in L^2(\mathbb{R}, d\mu)$ can be represented

$$f(\lambda) = \int_0^\ell A(x,\lambda)\alpha(x)\, dm(x) + \int_0^\ell B(x,\lambda)\beta(x)\, dx$$

where $\alpha \in L^2([0,\ell], dm)$ and $\beta \in L^2([0,\ell], dx)$ and the first summand is the even part $\frac{f(\lambda)+f(-\lambda)}{2}$ and the second is the odd part. We compute the effect of multiplying by λ on f.

$$\int_0^\ell \lambda A(x,\lambda)\alpha(x)\, dm(x) = \int_0^\ell \frac{dB}{dm}(x,\lambda)\alpha(x)\, dm(x) = \int_0^\ell \frac{dB}{dx}\alpha\, dx$$

$$= \Big[B\alpha\Big]_0^\ell - \int_0^\ell B\frac{d\alpha}{dx}\, dx$$

$$= \underbrace{B(\ell,\lambda)}_{=0}\alpha(\ell) - \int_0^\ell B(x,\lambda)\frac{d\alpha}{dx}\, dx$$

$$\int_0^\ell \lambda B(x,\lambda)\beta(x)\, dx = -\int_0^\ell \frac{dA}{dx}\beta\, dx = -\Big[A\beta\Big]_0^\ell + \int A\frac{d\beta}{dx}\, dx$$

$$= -\underbrace{A(\ell,\lambda)}_{=0}\beta(\ell) + \int_0^\ell A(x,\lambda)\frac{d\beta}{dm}\cdot dm$$

when λ is an eigenvalue

~~So if $\alpha(\ell)=\beta(\ell)=0$, then~~ Thus multiplication by λ corresponds to

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}\dfrac{d\beta}{dm}\\ -\dfrac{d\alpha}{dx}\end{pmatrix}$$

Suppose $l = \infty$. Then I have identified $L^2(\mathbb{R}, d\mu)$ with pairs $\binom{\alpha}{\beta}$ $\alpha \in L^2([0,\infty), dm)$, $\beta \in L^2([0,\infty), dx)$ in such a way that multiplication by λ corresponds to the operator $\binom{\alpha}{\beta} \longmapsto \begin{pmatrix} \frac{d\beta}{dm} \\ -\frac{d\alpha}{dx} \end{pmatrix}$

(*)

So what seems to be the case is that we have in the Hilbert space $L^2([0,\infty], dm) \times L^2([0,\infty), dx)$ a self-adjoint operator given by (*) and the boundary condition

$$\begin{cases} \alpha = 1 \\ \beta = 0 \end{cases} \quad \text{at} \quad x = 0$$

Also corresponding to the virtual element 1 of $L^2(d\mu)$ one has the virtual element $\alpha(x) = \frac{1}{m'(0)} \delta(x)$, $\beta = 0$. The 1-parameter family of multiplication by $e^{i\lambda t}$ corresponds to time evolution of the solutions of the wave equation

$$\frac{1}{i} \frac{\partial}{\partial t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{d\beta}{dm} \\ -\frac{d\alpha}{dx} \end{pmatrix}$$

(I notice now that in the prediction problem one assumes $\int d\mu < \infty$ so that $1 \in L^2(d\mu)$. This means the corresponding string has a point mass at $x=0$.)

We can mimic the above in the case of $\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$

Rewrite this equation in the form:

$$i\lambda u = \begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix} u$$

or
$$\lambda u = \begin{pmatrix} \frac{1}{i}\frac{d}{dx} & \left(\frac{p}{i}\right)^{-} \\ \frac{p}{i} & -\frac{1}{i}\frac{d}{dx} \end{pmatrix} u$$

and call the latter operator P. Provided boundary conditions at $x=0$, $x=\ell$ are given we get a self-adjoint operator $\hat{P}$ on $L^2([0,\ell])^2$. The eigenfunction expansion

$$y = \int_{-\infty}^{\infty} u(x,\lambda)\overline{(u(\ ,\lambda),y)}\, d\mu(\lambda)$$

provides an isomorphism between $L^2(\mathbb{R}, d\mu)$ and $L^2([0,\ell])^2$ such that multiplying by λ corresponds to the operator $\hat{P}$.

The next thing to understand is how the de Branges Hilbert spaces fit into this picture. For each $0 \le x \le \ell$ we consider the subspace of $L^2(d\mu)$ we get consisting of transforms $(u(\ ,\lambda), y) = \int_0^x y(x)^* u(x,\lambda)\, dx$ where y vanishes to the right of x. This gives one a subspace K^x which ought to be a de Branges space, i.e. a Hilbert space of entire functions satisfying de Branges axioms.

August 22, 1977

Return to $\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$ suppose boundary conditions at $x=0$ and $x=\ell$ are given. Let $u(x,\lambda)$ be the solution with initial conditions $u(0,\lambda) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $|c_1| = |c_2| = 1$. We let $v(x,\lambda)$ be the solution satisfying the boundary condition at $x=\ell$ and normalized so that

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = -1 \qquad \text{change to } +1$$

$v(x,\lambda)$ is defined when λ is not an eigenvalue. I want to construct the Green's matrix. Write the equations in the form

$$\lambda u = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix} u = Lu$$

Then the Green's matrix satisfies

$$(\lambda - L)\, g(x,y,\lambda) = I\,\delta(x-y)$$

and the boundary conditions (wrt x; y is fixed). Thus for $x<y$, one has

$$g(x,y,\lambda) = \begin{pmatrix} u_1(x)a_1 & u_1(x)a_2 \\ u_2(x)a_1 & u_2(x)a_2 \end{pmatrix} = u(x)\,a^t$$

and similarly $g(x,y,\lambda) = v(x)\,b^t$ for $x>y$. No matter what a,b are $(\lambda - L)g$ is a distribution supported at $x=y$, and the point is to choose a,b so that one gets $I\cdot\delta(x-y)$. (To be a bit more precise, we take $g(x,y,\lambda)$ to be the distribution defined by the function $u(x)a^t$ $x<y$ and $v(x)b^t$ for $x>y$, then try to rig a,b so that $(\lambda-L)g = I\delta(x-y)$)

Look at the first column of g. It should be clear that the second entry has to be continuous at y

$$u_2(y)a_1 = v_2(y)\,b_1$$

and that the first entry jumps by i $\quad(\text{by } -i)$

$$u_1(y)a_1 + i = v_1(y)b_1$$

$$u_1(y)\,a_1 - v_1(y)b_1 = -i$$

hence $\quad a_1 = i v_2(y) \qquad b_1 = i u_2(y)$

Similarly

$$u_1(y)a_2 = v_1(y)b_2$$

$$u_2(y)a_2 - i = v_2(y)\,b_2$$

or $\quad u_2(y)\,a_2 - v_2(y)\,b_2 = i$

hence $\quad a_2 = i\,v_1(y) \qquad b_2 = i\,u_1(y)$

Thus

$$\boxed{g(x,y,\lambda) = \begin{cases} i\begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda)\end{pmatrix}\begin{pmatrix} v_2(y,\lambda) & v_1(y,\lambda)\end{pmatrix} & x<y \\[2ex] i\begin{pmatrix} v_1(x,\lambda) \\ v_2(x,\lambda)\end{pmatrix}\begin{pmatrix} u_2(y,\lambda) & u_1(y,\lambda)\end{pmatrix} & x>y \end{cases}}$$

$\cancel{g(y,x,\lambda) = g(x,y,\lambda)^*}$ NO

Now one also has

$$g(x,y,\lambda) = \sum_{\lambda} \frac{u(x,\lambda_j)\;u(y,\lambda_j)^*}{\lambda - \lambda_j}\;\frac{1}{\|u(\cdot,\lambda_j)\|^2}$$

$$= \int_{-\infty}^{\infty} \frac{u(x,\hat\lambda)\, u(y,\hat\lambda)^*}{\lambda - \hat\lambda}\, d\mu(\hat\lambda)$$

In this formula put $x = 0$ and let $y \to 0$.

$$\lim_{y\to 0} g(0,y,\lambda) = i \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} v_2(0,\lambda) & v_1(0,\lambda) \end{pmatrix}$$

$$= \int_{-\infty}^{\infty} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} \bar c_1 & \bar c_2 \end{pmatrix} \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda}$$

hence

$$i c_1 v_2(0,\lambda) = c_1 \bar c_1 \int_{-\infty}^{\infty} \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda}$$

so it seems we get

$$i v_2(0,\lambda) = \bar c_1 \int_{-\infty}^{\infty} \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda}$$

$$i v_1(0,\lambda) = \bar c_2 \int_{-\infty}^{\infty} \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda}$$

which doesn't make sense. So obviously we've run into the discontinuity of g on the diagonal. However we ~~might~~ be able to argue this way for the components off the main diagonal which are continuous. e.g. for $x < y$

$$i\, u_2(x,\lambda)\, v_2(y,\lambda) = \int_{-\infty}^{\infty} \frac{u_2(x,\hat\lambda)\, \overline{u_1(y,\hat\lambda)}}{\lambda - \hat\lambda}\, d\mu(\hat\lambda)$$

$$i\, u_1(x,\lambda)\, v_1(y,\lambda) = \int_{-\infty}^{\infty} \frac{u_1(x,\hat\lambda)\, \overline{u_2(y,\hat\lambda)}}{\lambda - \hat\lambda}\, d\mu(\hat\lambda)$$

Same problem seems to arise. So we need an example to sort out the difficulty.

Let's begin with the operator

$$Lu = \frac{1}{i}\frac{du}{dx} = \lambda u$$

on the interval $0 \le x \le \pi$ with periodic boundary conditions: $u(0) = u(2\pi)$. The eigenfunctions are e^{inx}, $n \in \mathbb{Z}$

so

$$g(x,y,\lambda) = \sum \frac{e^{inx}\,\overline{e^{iny}}}{\lambda - n}\frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\sum_n \frac{e^{in(x-y)}}{\lambda - n}$$

On the other hand

$$g(x,y,\lambda) = \begin{cases} a\,e^{i\lambda x} & x<y \\ b\,e^{i\lambda x} & x>y \end{cases}$$

and one has

$$a = b e^{2\pi i\lambda} \qquad \text{boundary condition}$$

$$\left(\lambda - \frac{1}{i}\frac{d}{dx}\right) g(x,y,\lambda) = \delta(x-y)$$

hence

$$-\frac{1}{i}(b-a)\,e^{i\lambda y} = 1$$

$$a\left(e^{-2\pi i\lambda} - 1\right)e^{i\lambda y} = -i$$

or

$$a = \frac{i e^{-i\lambda y}}{-e^{-2\pi i\lambda} + 1}$$

Thus

$$g(x,y,\lambda) = \frac{i e^{i\lambda(x-y)}}{1 - e^{-2\pi i\lambda}} \qquad x<y$$

$$= \frac{i e^{i\lambda(x-y)}\,e^{-2\pi i\lambda}}{1 - e^{-2\pi i\lambda}} \qquad x>y$$

or finally
$$g(x,y,\lambda)=\begin{cases}\dfrac{i\,e^{-i\lambda(x-y)}}{1-e^{-2\pi i\lambda}} & x<y\\[2ex] \dfrac{i\,e^{i\lambda(x-y)}}{e^{2\pi i\lambda}-1} & x>y\end{cases}$$

By residues one gets the expansion
$$\frac{i\,e^{i\lambda(x-y)}}{1-e^{-2\pi i\lambda}}=\sum_n \frac{i}{2\pi i}\,\frac{e^{-in(x-y)}}{\lambda-n}\qquad x<y$$

and similarly for $x>y$.

Now consider $\dfrac{du}{dx}=\begin{pmatrix} i\lambda & 0\\ 0 & -i\lambda\end{pmatrix}u$ on $0\leq x\leq \pi$ with bdry conditions $u_1=u_2$ at both ends. [redacted] ~~also~~ One has $u(x,\lambda)=\begin{pmatrix} e^{i\lambda x} & \\ & e^{-i\lambda x}\end{pmatrix}$ and the eigenvalues are $\lambda=n\in\mathbb{Z}$. Hence

$$g(x,y,\lambda)=\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\,\frac{1}{\lambda-n}\begin{pmatrix} e^{inx}\\ e^{-inx}\end{pmatrix}\begin{pmatrix} e^{-iny} & e^{+iny}\end{pmatrix}$$

$$g(x,y,\lambda)=\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\,\frac{1}{\lambda-n}\begin{pmatrix} e^{in(x-y)} & e^{+in(x+y)}\\ e^{-in(x+y)} & e^{-in(x-y)}\end{pmatrix}$$

But we have
$$u(x,\lambda)=\begin{pmatrix} e^{i\lambda x}\\ e^{-i\lambda x}\end{pmatrix}$$
$$v(x,\lambda)=c\begin{pmatrix} e^{i\lambda(x-\pi)}\\ e^{-i\lambda(x-\pi)}\end{pmatrix}$$

$$c\begin{vmatrix} u_1 & v_1\\ u_2 & v_2\end{vmatrix}=c\begin{vmatrix} e^{i\lambda x} & e^{i\lambda x}e^{-\pi i\lambda}\\ e^{-i\lambda x} & e^{-i\lambda x}e^{+\pi i\lambda}\end{vmatrix}=c\left(e^{\pi i\lambda}-e^{-\pi i\lambda}\right)=+1$$

$$c = \frac{+1}{e^{\pi i\lambda} - e^{-\pi i\lambda}} \qquad \text{so}$$

$$g(x,y,\lambda) = \frac{+i}{e^{\pi i\lambda}-e^{-\pi i\lambda}} \begin{pmatrix} e^{i\lambda(x-y+\pi)} & e^{i\lambda(x+y-\pi)} \\ e^{i\lambda(-x-y+\pi)} & e^{i\lambda(-x+y-\pi)} \end{pmatrix} \qquad x<y$$

$$= \frac{i}{e^{\pi i\lambda}-e^{-\pi i\lambda}} \begin{pmatrix} e^{-i\lambda(x-y-\pi)} & e^{i\lambda(x+y-\pi)} \\ e^{i\lambda(-x-y+\pi)} & e^{-i\lambda(-x+y+\pi)} \end{pmatrix} \qquad x>y$$

So the problem seems to be this: In what sense is

$$(*) \qquad g(x,y,\lambda) = \sum_j \frac{u(x,\lambda_j)\, u(y,\lambda_j)^*}{\lambda-\lambda_j} \frac{1}{\|u(\,,\lambda_j)\|^2} \quad ?$$

What I wanted to do was to put $x=y=0$.

Example of $L = \frac{1}{i}\frac{d}{dx}$ on $0 \le x \le 2\pi$, periodic conditions
In what sense is

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{e^{in(x-y)}}{\lambda-n} = \begin{cases} \dfrac{i\, e^{i\lambda(x-y)}}{1-e^{-2\pi i\lambda}} & 0 \le x < y \\[2ex] \dfrac{i\, e^{i\lambda(x-y)}}{e^{2\pi i\lambda}-1} & y < x \le 2\pi \end{cases}$$

$$= \frac{1}{2}\cot(\pi\lambda) \qquad 0 \le x = y \le 2\pi \qquad ?$$

From Fourier series this should hold for x,y fixed, provided the sum $\sum_{n\in\mathbb{Z}}$ is taken $\sum_{|n|\le N}$ then $N\to\infty$.

Conjecture: The equality (*) holds as for Fourier series. Thus when $x=y$ the series on the right converges to the average of the diagonal limits.

Test this out in the example.

$$g(0,0+,\lambda) = \frac{i}{e^{\pi i\lambda}-e^{-\pi i\lambda}}\begin{pmatrix} e^{i\lambda\pi} & e^{-i\lambda\pi} \\ e^{+i\lambda\pi} & e^{-i\lambda\pi}\end{pmatrix}$$

$$g(0+,0,\lambda) = \frac{i}{e^{\pi i\lambda}-e^{-\pi i\lambda}}\begin{pmatrix} e^{-i\lambda\pi} & e^{-i\lambda\pi} \\ e^{i\lambda\pi} & e^{i\lambda\pi}\end{pmatrix}$$

$$\frac{g(0,0+;\lambda)+g(0+,0;\lambda)}{2} = \frac{1}{2}\begin{pmatrix} \cot(\pi\lambda) & \frac{i}{e^{2\pi i\lambda}-1} \\ \frac{i}{1-e^{-2\pi i\lambda}} & \cot(\pi\lambda)\end{pmatrix}$$

However the series for $g(x,y)$ converges for $x=y=0$ to

$$\frac{1}{2}\begin{pmatrix} \cot(\pi\lambda) & \cot(\pi\lambda) \\ \cot(\pi\lambda) & \cot(\pi\lambda)\end{pmatrix}$$

so the conjecture doesn't hold.

Perhaps I should ask for a formula relating $v(0,\lambda)$ to the spectral measure $d\mu(\lambda)$.

August 25, 1977

Consider $\dfrac{d^2u}{dx^2} + (\lambda - q)u = 0$ on $0 \le x \le \ell$
with the boundary condition

$$\frac{u(0)}{u'(0)} = \frac{\cos\alpha}{\sin\alpha}$$

and let $\varphi(x,\lambda)$, $\psi(x,\lambda)$ be the solutions with initial values

$$\begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}_{x=0} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Any solution not satisfying the boundary condition at $x=0$ is proportional to $m\psi + \varphi$ for some $m \in \mathbb{C}$. If the boundary condition at $x = \ell$:

$$\frac{u(\ell)}{u'(\ell)} = \cot\beta$$

is given then $\chi = m\psi + \varphi$ satisfies this iff

$$\begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}_{x=\ell} (m) = \cot\beta$$

or

$$m = \begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}_{x=\ell}^{-1} (\cot\beta) = \frac{\varphi'(\ell,\lambda)\cot\beta - \varphi(x,\ell)}{-\psi'(\ell,\lambda)\cot\beta + \psi(x,\ell)}$$

As $\cot(\beta)$ runs over $\mathbb{P}_1(\mathbb{R})$, m runs around a circle $C(\lambda)$ in $\mathbb{C}$. Put

$$\begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}_{x=\ell} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \qquad x = \cot\beta = \frac{Am+B}{Cm+D}$$

~~Then $m \in C(\lambda) \iff \frac{Am+B}{Cm+D} = \frac{\bar{A}\bar{m}+\bar{B}}{\bar{C}\bar{m}+\bar{D}}$ ... circle~~

(See April 1, 77)

The center of $C(\lambda)$ is the reflection through $C(\lambda)$ of $m=\infty$ which goes to $x = A/C$. Reflecting through $x \in \mathbb{R}$ gives $x = \frac{\bar{A}}{\bar{C}}$ which corresponds to the center. Thus

$$\text{center of } C(\lambda) = \frac{D\left(\frac{\bar{A}}{\bar{C}}\right) - B}{-C\left(\frac{\bar{A}}{\bar{C}}\right) + A} = \frac{\bar{A}D - B\bar{C}}{A\bar{C} - \bar{A}C}$$

$$\text{radius of } C(\lambda) = \left| \underbrace{\frac{\bar{A}D - B\bar{C}}{A\bar{C} - \bar{A}C}}_{\text{center}} - \underbrace{\left(\frac{D}{-C}\right)}_{\text{pt. corresp to } x=\infty} \right| = \left| \frac{\bar{A}CD - B\bar{C}C + A\bar{C}D - \bar{A}CD}{(A\bar{C} - \bar{A}C)C} \right|$$

$$= \left| \frac{1}{|A\bar{C} - \bar{A}C|} \right|$$

The points of $C(\lambda)$ are those m such that $\chi = m\psi + \varphi$ has real boundary values at $x = b$ i.e. such that

$$\begin{vmatrix} \chi & \bar{\chi} \\ \chi' & \bar{\chi}' \end{vmatrix}_{x=b} = 0$$

The coefficient of $m\bar{m}$ in this equation is $\begin{vmatrix} \psi & \bar{\psi} \\ \psi' & \bar{\psi}' \end{vmatrix}(b)$, so

$$\text{interior of } C(\lambda) \doteq \frac{\begin{vmatrix} \chi & \bar{\chi} \\ \chi' & \bar{\chi}' \end{vmatrix}}{\begin{vmatrix} \psi & \bar{\psi} \\ \psi' & \bar{\psi}' \end{vmatrix}}(b) < 0$$

Now since q is real if u is a soln. of the DE

$$\frac{d}{dx}\begin{vmatrix} u & \bar{u} \\ u' & \bar{u}' \end{vmatrix} = \begin{vmatrix} u & \bar{u} \\ (q-\lambda)u & (q-\bar{\lambda})\bar{u} \end{vmatrix} = (\lambda - \bar{\lambda})|u|^2 = 2i\,\mathrm{Im}(\lambda)\,|u|^2.$$

If $\mathrm{Im}(\lambda) > 0$, then

$$\mathrm{rad}(C(\lambda))^{-1} = \left\| \begin{matrix} \psi & \bar\psi \\ \psi' & \bar\psi' \end{matrix} \right\|_{x=\ell} = 2\,\mathrm{Im}(\lambda) \int_0^\ell |\psi|^2 dx$$

Also

$$\begin{vmatrix} \chi & \bar\chi \\ \chi' & \bar\chi' \end{vmatrix}(0) = \begin{vmatrix} m\psi + \varphi & \bar m\psi + \varphi \\ m\psi' + \varphi' & \bar m\psi' + \varphi' \end{vmatrix}_0 = \begin{vmatrix} (m-\bar m)\psi & \bar m\psi + \varphi \\ (m - \bar m)\psi' & \bar m\psi' + \varphi' \end{vmatrix}$$

$$= (m - \bar m)\begin{vmatrix} \psi & \varphi \\ \psi' & \varphi' \end{vmatrix} = m - \bar m = 2i\,\mathrm{Im}(m)$$

so

$$\frac{1}{2i}\begin{vmatrix} \chi & \bar\chi \\ \chi' & \bar\chi' \end{vmatrix}(\ell) = \mathrm{Im}(m) + \mathrm{Im}(\lambda)\int_0^\ell |\chi|^2 dx$$

so interior of $C(\lambda)$ is determined by

$$\frac{\frac{1}{2i}\begin{vmatrix} \chi & \bar\chi \\ \chi' & \bar\chi' \end{vmatrix}(\ell)}{\frac{1}{2i}\begin{vmatrix} \psi & \bar\psi \\ \psi' & \bar\psi' \end{vmatrix}(\ell)} = \frac{\mathrm{Im}(m) + \mathrm{Im}(\lambda)\int_0^\ell |\chi|^2 dx}{\mathrm{Im}(\lambda)\int_0^\ell |\psi|^2 dx} < 0$$

or simply

$$\int_0^\ell |\chi|^2 dx < -\frac{\mathrm{Im}(m)}{\mathrm{Im}(\lambda)}$$

for the interior of $C(\lambda)$.

As a check I note that for $\mathrm{Im}(\lambda) > 0$, one has

$$\frac{d}{dx}\,\frac{1}{2i}\begin{vmatrix} u & \bar u \\ u' & \bar u' \end{vmatrix} = \mathrm{Im}(\lambda)\,|u|^2 > 0$$

hence

$$\frac{1}{2i}\begin{vmatrix} \chi & \bar\chi \\ \chi' & \bar\chi' \end{vmatrix}(0) < 0 \qquad \text{if } \frac{\chi}{\chi'} \text{ real at } x = \ell$$

$$\overset{\shortparallel}{\frac{1}{2i}\left[\frac{\chi(0)}{\chi'(0)} - \frac{\bar\chi(0)}{\bar\chi'(0)}\right]|\chi'(0)|^2} = \mathrm{Im}(m(\lambda)) \cdot |\chi'(0)|^2 .$$

So the equation of $C_0(\lambda)$ is

$$\int_0^{l} |m\psi(x,\lambda)+\varphi(x,\lambda)|^2\,dx = -\frac{Im(m)}{Im(\lambda)}$$

Notice that $\chi(x,\lambda) = m\psi(x,\lambda)+\varphi(x,\lambda)$ is the solution satisfying the left boundary condition (here $m=m(\lambda)$ corresp. to the boundary condition given at $x=l$) whose Wronskian with ψ is 1:

$$\begin{vmatrix} \psi & m\psi+\varphi \\ \psi' & m\psi'+\varphi' \end{vmatrix} = \begin{vmatrix} \psi & \varphi \\ \psi' & \varphi' \end{vmatrix} = 1$$

Let's apply the Parseval thm. to $\chi(x,\lambda) = m(\lambda)\psi(x,\lambda)+\varphi(x,\lambda)$ where λ is fixed not an eigenvalue

$$\int_0^l |\chi(x,\lambda)|^2\,dx = \sum |a_j|^2 \frac{1}{r_j}$$

where
$$a_j = \int_0^l \chi(x,\lambda)\,\psi(x,\lambda_j)\,dx \qquad r_j = \int_0^l \psi(x,\lambda_j)^2\,dx$$

Now since $(L-\lambda)\chi(x,\lambda)=0$ one has

$$= \int_0^l L\chi(x,\lambda)\,\psi(x,\lambda_j)\,dx - \int_0^l \chi(x,\lambda)\,L\psi(x,\lambda_j)\,dx$$

$$= (\lambda-\lambda_j)\,a_j$$

$$= -\Big[\chi'(x,\lambda)\psi(x,\lambda_j) - \chi(x,\lambda)\psi'(x,\lambda_j)\Big]_0^l$$

$$= \begin{vmatrix} \chi & \psi \\ \chi' & \psi' \end{vmatrix}(0) = +1$$

$$\therefore \quad a_j = \frac{1}{\lambda-\lambda_j}$$

so that

$$\int_0^l |\chi(x,\lambda)|^2 dx = \sum{}' \frac{1}{|\lambda-\lambda_j|^2}\frac{1}{r_j} = \int \frac{d\mu(\hat{\lambda})}{|\lambda-\hat{\lambda}|^2}$$

Consider $L = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix}$ with boundary conditions given at $x=0$ and $x=l$. Let $u(x,\lambda)$ be the solution with the initial condition $u(0,\lambda) = \binom{c_1}{c_2}$ where recall: $|c_1| = |c_2| > 0$.) If λ is not an eigenvalue there is a solution $v(x,\lambda)$ independent of $u(x,\lambda)$ which satisfies the bdry condition at $x=l$ and we can choose it so that

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = i$$

for all x. Let's expand v in eigenfunctions

$$v = \sum{}' a_j u_j \qquad u_j = u(x,\lambda_j) \qquad r_j = \|u_j\|^2$$

$r_j a_j = (v, u_j)$. Now

$$(\lambda-\lambda_j)(v,u_j) = (Lv,u_j) - (v,Lu_j)$$

$$= \int_0^l \left[u_j^* Lv - (Lu_j)^* v\right] dx$$

$$= \left[u_j^* A v\right]_0^l \qquad A = \frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The boundary conditions at $x=l$ are of the form $\frac{v_1(l)}{v_2(l)} = e^{i\beta}$

Hence $(u_j^* A v)(\ell) = \frac{1}{i}\left[\overline{u_1^*(\ell)}\, v_1(\ell) - \overline{u_2(\ell)}\, v_2(\ell)\right]$

$$= \frac{1}{i}\left[\overline{u_2(\ell)}\, e^{-i\beta} e^{i\beta} v_2(\ell) - \overline{u_2(\ell)}\, v_2(\ell)\right] = 0$$

and
$$(u_j^* A v)(0) = \frac{1}{i}\left[\bar{c}_1 v_1(0) - \bar{c}_2 v_2(0)\right]$$

But by the choice of v we have

$$\begin{vmatrix} c_1 & v_1(0) \\ c_2 & v_2(0) \end{vmatrix} = c_1 v_2(0) - c_2 v_1(0) = i$$

Since $|c_1| = |c_2|$ we have $\frac{\bar{c}_2}{c_1} = \frac{\bar{c}_1}{c_2} = c$, $|c| = 1$
and up to a scalar we could have arranged that
$\binom{c_1}{c_2} = \binom{e^{i\alpha/2}}{e^{-i\alpha/2}}$ so that $c_1 = \bar{c}_2$. If this is the case then

$$(u_j^* A v)(0) = \frac{1}{i}(-i) = -1$$

so
$$(\lambda - \lambda_j)(v, u_j) = -u_j^* A v\,(0) = 1$$

and hence
$$(v, u_j) = \frac{1}{\lambda - \lambda_j}.$$

Thus in the ℓ^2-sense we have

$$v(x,\lambda) = \int \frac{u(x,\hat{\lambda})\, d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

<u>Example</u>: For $p = 0$ $\binom{c_1}{c_2} = \binom{1}{1}$ $\wedge$ $\ell = \pi$ we found ($v_1 = v_2$ at:)

$$u(x,\lambda) = \begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x} \end{pmatrix} \qquad v(x,\lambda) = \frac{i}{e^{\pi i\lambda} - e^{-\pi i \lambda}} \begin{pmatrix} e^{i\lambda x} e^{-i\lambda\pi} \\ e^{-i\lambda x} e^{i\lambda\pi} \end{pmatrix}$$

Now
$$\int \frac{u(x,\hat{\lambda})\, d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\lambda - n} \begin{pmatrix} e^{inx} \\ e^{-inx} \end{pmatrix}$$

and so where the problem encountered on p. 324 arises, namely this Fourier series for $x=0$ does not converge to $v(0,\lambda)$. For:

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{1}{\lambda-n}=\frac{1}{2}\cot(\pi\lambda)=\frac{i}{2}\,\frac{e^{\pi i\lambda}+e^{-\pi i\lambda}}{e^{\pi i\lambda}-e^{-\pi i\lambda}}$$

and
$$v(0,\lambda)=\frac{i}{e^{\pi i\lambda}-e^{-\pi i\lambda}}\begin{pmatrix}e^{-i\lambda\pi}\\ e^{i\lambda\pi}\end{pmatrix}$$

Question: What does the series $\displaystyle\int\frac{u(x,\tilde\lambda)\,d\mu(\tilde\lambda)}{\lambda-\tilde\lambda}$ actually converge to?

My idea here is that this series should make sense as a distribution of some sort. I should maybe think of C_0^∞ as being those C^∞ functions on $[0,\ell]$ which satisfy the boundary conditions. Such functions should have Fourier coefficients which decay rapidly. not quite - see below:

$$(f,u_j)=\frac{1}{\lambda_j}(f,Lu_j)\underset{\uparrow}{=}\frac{1}{\lambda_j}(Lf,u_j)$$

because f satisfies bdy conditions

We ought to get a tower of Hilbert spaces

$$\cdots\subset H_2\subset H_1\subset H_0\subset\cdots\cdots$$

where $H_0=L^2([0,\ell])^2$ and $H_n=G^nH_0$ where $G=(\lambda-L)^{-1}$ for some regular value λ, and we ought to have

$$C_0^\infty([0,\ell])=\bigcap H_n \qquad \bigcup H_n=\text{Distributions on }[0,\ell]$$

$\uparrow$ means the extension by 0 to $\mathbb{R}$ is C^∞. $\leftarrow$ not quite

Example: Take $L = \frac{d^2}{dx^2}$ on $[0, \pi]$ with bdry conditions $u(0) = u(\pi) = 0$. Eigenfunctions are $\sin(nx)$, $n = 1, 2, \ldots$. Suppose $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ converges rapidly enough so that we can differentiate term by term as many times as we please. Then we see that $\forall k \geq 0$

$$D^{2n+1} f \text{ vanishes at } 0 \text{ and } \pi$$

i.e. $L^k f$ satisfies the boundary conditions for every $k \geq 0$. The converse is obvious from:

$$\left(f, \sin(nx)\right) = \frac{1}{(-n^2)^k}\left(f, L^k \sin(nx)\right)$$
$$= \frac{1}{(-n^2)^k}\left(L^k, \sin(nx)\right)$$

which shows the coefficients go to zero faster than n^{-2k} for any $k \geq 0$. Hence the space H_∞ consists of all C^∞ odd periodic functions. $H_{-\infty}$ is a quotient of the space of distributions on $[0, \pi]$ where one kills the even derivative distributions supported at 0 and at π. $\blacksquare$

August 26, 1977

Let $u(x,\lambda)$ satisfy $\frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix} u = \lambda u$ with $u(0,\lambda) = \begin{pmatrix} c_1 \\ \bar{c}_1 \end{pmatrix}$. Choose $\tilde{c}_1$ $\tilde{c}_2$ so that $\begin{vmatrix} c_1 & \tilde{c}_1 \\ \bar{c}_1 & \tilde{c}_2 \end{vmatrix} = i$.

For example, suppose $c_1 = e^{i\alpha}$. Then

$$\begin{vmatrix} e^{i\alpha} & -\frac{1}{2} i e^{i\alpha} \\ e^{-i\alpha} & \frac{1}{2} i e^{-i\alpha} \end{vmatrix} = i$$

so I can arrange $\tilde{c}_2 = \bar{\tilde{c}}_1$ if I want. Let $\tilde{u}(x,\lambda)$ be the solution with $\tilde{u}(0,\lambda) = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}$. If $v = mu + \tilde{u}$ then we have

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = i$$

For λ not an eigenvalue we get a unique $m(\lambda)$ such that $v(x,\lambda)$ satisfies the bdry condition at $x = l$. Then $m(\lambda)$ is a meromorphic function with simple poles at the eigenvalues.

Calculate the eigenfunction expansion of v.

$$(\lambda - \lambda_j)(v, u_j) = (Lv, u_j) - (v, Lu_j) = \int_0^l \left[u_j^* Lv - (Lu_j)^* v\right] dx$$

$$= \left[u_j^* A v\right]_0^l \qquad A = \frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= -(u_j^* A v)(0) = -\frac{1}{i}\left[\bar{c}_1(mc_1 + \tilde{c}_1) - \bar{c}_2(mc_2 + \tilde{c}_2)\right]$$

$$= \frac{1}{i}\left[c_1 \tilde{c}_2 - c_2 \tilde{c}_1\right] = 1 \qquad c_2 = \bar{c}_1$$

Parseval's relation is $\int_0^l |v|^2 dx = \int_{-\infty}^{\infty} \frac{d\mu(\Lambda)}{|\lambda - \Lambda|^2}$

On the other hand, one has

$$\frac{d}{dx}(v^*Av) = v^*Lv - (Lv)^*v = (\lambda - \bar{\lambda})|v|^2$$

so

$$2i\,\mathrm{Im}(\lambda)\int_0^\ell |v|^2 dx = [v^*Av]_0^\ell = -(v^*Av)(0)$$

$$= -\frac{1}{i}\left(|mc_1 + \tilde{c}_1|^2 - |mc_2 + \tilde{c}_2|^2\right)$$

$$= -\frac{1}{i}\Big(|m|^2(|c_1|^2 - |c_2|^2) + m(c_1\bar{\tilde{c}}_1 - c_2\bar{\tilde{c}}_2) + \bar{m}(\bar{c}_1\tilde{c}_1 - \bar{c}_2\tilde{c}_2)$$

want $|\tilde{c}_1| = |\tilde{c}_2|$

$$+ |\tilde{c}_1|^2 - |\tilde{c}_2|^2\Big)$$

$$= -\frac{1}{i}\left(mi + \bar{m}(-i)\right) = -(m - \bar{m})$$

$$= -2i\,\mathrm{Im}(m).$$

So

$$-\frac{\mathrm{Im}(m)}{\mathrm{Im}(\lambda)} = \int_0^\ell |v|^2 dx = \int_{-\infty}^{\infty} \frac{d\mu(\hat{\lambda})}{|\lambda - \hat{\lambda}|^2}$$

Next notice that for $\hat{\lambda}$ real

[redacted]

$$\mathrm{Im}\left(\frac{1}{\lambda - \hat{\lambda}}\right) = \mathrm{Im}\left(\frac{|\lambda - \hat{\lambda}|}{\lambda - \hat{\lambda}}\right)\frac{1}{|\lambda - \hat{\lambda}|}$$

$$= -\mathrm{Im}\left(\frac{\lambda - \hat{\lambda}}{|\lambda - \hat{\lambda}|}\right)\frac{1}{|\lambda - \hat{\lambda}|} = -\mathrm{Im}(\lambda)\frac{1}{|\lambda - \hat{\lambda}|^2}$$

hence [redacted]

$$\int \mathrm{Im}\left(\frac{1}{\lambda - \hat{\lambda}}\right) d\mu(\hat{\lambda}) = -\mathrm{Im}(\lambda)\int \frac{d\mu(\hat{\lambda})}{|\lambda - \hat{\lambda}|^2} = \mathrm{Im}(m)$$

Now I want to prove that $m(\lambda) = \int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$. Now I know that $m(\lambda)$ is meromorphic with simple poles at the eigenvalues, so if the integral exists [redacted] in [redacted] a region of the complex plane, then the difference $m(\lambda) - \int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$

is an analytic function with zero imaginary part and hence it is a real constant. The trouble with this approach is that it involves estimating the growth of the eigenvalues. ~~However the Sturm comparison~~ see p. 338
~~Another approach would be to show that $m(\lambda)$~~

But it ought to be the case that for large λ, since we are working on a finite interval, the effect of p should be negligible. So it seems then that the integral $\int \frac{d\mu(\hat\lambda)}{\lambda-\hat\lambda}$ ought to be Eisenstein convergent and hence should <u>differ</u> from $m(\lambda)$ by a real constant.

Example again: $\begin{pmatrix} c_1 & \tilde c_1 \\ c_2 & \tilde c_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{i}{2} \\ 1 & \frac{i}{2} \end{pmatrix}$ $\quad u = \begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x} \end{pmatrix}$

$\tilde u = \begin{pmatrix} -\frac{i}{2} e^{-i\lambda x} \\ \frac{i}{2} e^{-i\lambda x} \end{pmatrix}$. So $mu + \tilde u = \begin{pmatrix} (m-\frac{i}{2}) e^{-i\lambda x} \\ (m+\frac{i}{2}) e^{-i\lambda x} \end{pmatrix}$ satisfies

the right-hand bdry condition $\iff$

$$\frac{(m-\frac{i}{2}) e^{i\lambda\pi}}{\cancel{(m+\frac{i}{2}) e^{-i\lambda\pi}}} = \left(m+\frac{i}{2}\right) e^{-i\lambda\pi}$$

$$m\left(e^{i\lambda\pi} - e^{-i\lambda\pi}\right) = \frac{i}{2}\left(e^{i\lambda\pi} + e^{-i\lambda\pi}\right)$$

$$m = \frac{i}{2}\,\frac{e^{i\lambda\pi} + e^{-i\lambda\pi}}{e^{i\lambda\pi} - e^{-i\lambda\pi}}$$

and the formula

$$m(\lambda) = \frac{1}{2\pi} \sum_n \frac{1}{\lambda - n}$$ holds.

We've used above that $c_2 = \overline{c_1}$ $\quad |\tilde c_2| = |\tilde c_1|$ $\quad \begin{vmatrix} c_1 & \tilde c_1 \\ c_2 & \tilde c_2 \end{vmatrix} = i$

Say $c_1 = e^{-i\alpha}$ $\tilde{c}_1 = re^{-i\beta}$, $\tilde{c}_2 = re^{i\gamma}$, $r>0$. Then

$$r\,e^{i(\alpha+\gamma)} - r\,e^{i(-\alpha+\beta)} = i$$

Now two points on the unit circle with difference a positive imaginary must be conjugate:

$$e^{i(\alpha+\gamma)} = e^{i(\alpha-\beta)}$$

so $e^{i\gamma} = e^{-i\beta}$, i.e. $\tilde{c}_2 = \overline{\tilde{c}_1}$. This latter condition persists if we add a real multiple of $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ to $\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}$. Therefore we see that changing $\tilde{u}$ by a real multiple of u doesn't affect any of the eigenvalues λ_j or their masses $\|u_j\|^2$, but it does change m by an additive real constant.

By choosing $\tilde{u}$ suitably, one can arrange for the additive constant to disappear. Question: Does the good choice for $\tilde{u}$ depend on the boundary condition at $x=l$?

Try a more invariant viewpoint: Idea is that we have a real projective line of possible boundary values at $x=0$ described by

$$(u^* A u)(0) = 0.$$

By describing v as $mu + \tilde{u}$ we coordinatize this real projective line so that the bdry value at $x=0$ corresponds to $m = \infty$.

Let $d\mu(\lambda)$ be a measure on the line such that

$$(*) \qquad \int_{|\lambda|\geq 1} \frac{d\mu(\lambda)}{\lambda^2} < \infty$$

I'd like to define an analytic function in the complement of $\mathrm{Supp}(d\mu)$ by

$$(1) \qquad f(z) = \int \frac{d\mu(\lambda)}{z-\lambda}$$

However this integral might not converge. Notice however that the imaginary part does converge absolutely

$$(2) \qquad \int \mathrm{Im}\left(\frac{1}{z-\lambda}\right) d\mu(\lambda) = -\mathrm{Im}(z) \int \frac{d\mu(\lambda)}{|z-\lambda|^2}$$

by the hypothesis $(*)$. Hence the problem is that the real part of (1) doesn't converge. Next notice that for any $z_0 \notin \mathrm{Supp}(d\mu)$ the integral

$$(3) \qquad \int \left[\frac{1}{z-\lambda} - \frac{1}{z_0-\lambda}\right] d\mu(\lambda)$$

converges absolutely since

$$\frac{1}{z-\lambda} - \frac{1}{z_0-\lambda} = \frac{z_0-z}{(z-\lambda)(z_0-\lambda)} \sim (z_0-z)\frac{1}{\lambda^2}$$

We can improve (3) in some sense so that it gives an analytic function with the imaginary part (2). We want to add to (3) the imaginary part of the correction term. So we want:

$$f(z) = \int \left[\frac{1}{z-\lambda} - \mathrm{Re}\left(\frac{1}{z_0-\lambda}\right)\right] d\mu(\lambda)$$

simplest choice for z_0 is i. Then

$$\mathrm{Re}\left(\frac{1}{i-\lambda}\right) = \frac{\mathrm{Re}(-i-\lambda)}{\lambda^2+1} = -\frac{\lambda}{1+\lambda^2}$$

and the final formula for the desired analytic fn. is

$$(4) \qquad f(z) = \int_{-\infty}^{\infty} \left[\frac{1}{z-\lambda} + \frac{\lambda}{1+\lambda^2}\right] d\mu(\lambda)$$

summary: (4) gives an analytic function on $\mathbb{C} - \mathrm{supp}\, d\mu$ having

$$\mathrm{Im}\, f(z) = -(\mathrm{Im}\,\lambda) \int_{-\infty}^{\infty} \frac{d\mu(\lambda)}{|z-\lambda|^2}$$

Suppose again we look at $Lu = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix} u = \lambda u$ on $0 \le x \le l$ with a boundary condition

$$u(0,\lambda) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad c_2 = \bar{c}_1 \qquad |c_1| = 1.$$

and denote the solution by $u_\lambda(x) = u(x,\lambda)$. If a self-adj. boundary condition is given at $x = l$, then we get a spectral measure $d\mu(\lambda)$.

Given $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2([0,l])^2$ we put

$$\hat{f}(\lambda) = (f, u_{\bar{\lambda}}) = \int_0^l u_{\bar{\lambda}}(x)^* f(x)\,dx$$

so that $\hat{f}(\lambda)$ is an entire function of λ and

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\mu(\lambda) = \int_0^l |f|^2 dx$$

by the Plancherel formula. Thus as f runs over

$L^2([0,\ell])^2$ we get a Hilbert space of analytic fns.
Fix λ_0 and try to represent the point evaluation:

$$\hat f(\lambda) \longmapsto \hat f(\lambda_0) \in \mathbb{C}.$$

$$\hat f(\lambda_0) = (f, u_{\bar\lambda_0}) = (\hat f, \hat u_{\bar\lambda_0})$$

So this linear functional is represented by $\hat u_{\bar\lambda_0}$:

$$\hat u_{\bar\lambda_0}(\lambda) = (u_{\bar\lambda_0}, u_{\bar\lambda})$$

But $(\bar\lambda_0 - \lambda)(u_{\bar\lambda_0}, u_{\bar\lambda}) = (L u_{\bar\lambda_0}, u_{\bar\lambda}) - (u_{\bar\lambda_0}, L u_{\bar\lambda})$

$\cancel{= \left[u^*_{\bar\lambda_0} A u_{\bar\lambda}\right]_0^\ell}$

$\cancel{= (u^*_{\bar\lambda_0} A u_{\bar\lambda})(\ell)}$ [redacted]

$\cancel{= \frac{1}{i}\left[u_1(\ell,\bar\lambda_0)\, u_1(\ell,\lambda) - u_2(\ell,\bar\lambda_0)\, u_2(\ell,\bar\lambda)\right]}$ [redacted]

$$= \left[u^*_{\bar\lambda} A u_{\bar\lambda_0}\right]_0^\ell$$

$$= u^*_{\bar\lambda} A u_{\bar\lambda_0}(\ell)$$

$$= \frac{1}{i}\left[\overline{u_1(\ell,\bar\lambda)}\, u_1(\ell,\bar\lambda_0) - \overline{u_2(\ell,\bar\lambda)}\, u_2(\ell,\bar\lambda_0)\right]$$

Thus the point evaluation $\hat f \mapsto \hat f(\lambda_0)$ is represented by

$\hat u_{\lambda_0} : \lambda \longmapsto$ [redacted] $i\,\dfrac{u_1^\#(\ell,\lambda)\, u_1(\ell,\bar\lambda_0) - u_2^\#(\ell,\lambda)\, u_2(\ell,\bar\lambda_0)}{\lambda - \bar\lambda_0}$

Notice that this doesn't depend upon the boundary condition at $x=\ell$.

: We know that $L^2([0,\ell])^2$ is isomorphic to $L^2(d\mu)$ by the transform $f \longleftrightarrow \hat{f}$. On the other hand we have seen how to extend each transform to an entire function of λ, and we have seen that the resulting Hilbert space structure on these entire functions doesn't depend on the boundary condition at $x=\ell$. Thus we seem to get lots of measures on $\mathbb{R}$ describing the inner product.

Recall the D.E. is symmetric under $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$ $\lambda \mapsto \bar{\lambda}$

hence $\begin{pmatrix} \bar{u}_2(x,\bar{\lambda}) \\ \bar{u}_1(x,\bar{\lambda}) \end{pmatrix} = c \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix}$

and one has $c=1$ if one assumes $c_2 = \bar{c}_1$. Then

$$u_2(x,\lambda) = u_1^{\#}(x,\lambda) = \overline{u_1(x,\bar{\lambda})}.$$

so

$$\hat{u}_{\bar{\lambda}_0}(\lambda) = (\hat{u}_{\bar{\lambda}_0}, \hat{u}_{\bar{\lambda}}) = i\, \frac{u_2(\ell,\lambda)\, u_1(\ell,\bar{\lambda}_0) - u_1(\ell,\lambda)\, u_2(\ell,\bar{\lambda}_0)}{\lambda - \bar{\lambda}_0}$$

$$= \frac{1}{i}\, \frac{1}{\lambda - \bar{\lambda}_0} \begin{vmatrix} u_1(\ell,\lambda) & u_1(\ell,\bar{\lambda}_0) \\ u_2(\ell,\lambda) & u_2(\ell,\bar{\lambda}_0) \end{vmatrix}$$

(Note that the formula for $\hat{u}_{\bar{\lambda}_0}$ on p. 340 implies

$$0 < \hat{u}_{\bar{\lambda}_0}(\lambda_0) = \frac{|u_1(\ell,\bar{\lambda}_0)|^2 - |u_2(\ell,\bar{\lambda}_0)|^2}{\operatorname{Im} \lambda_0}$$

hence $\operatorname{Im}(\lambda_0) > 0 \implies |u_2(\ell,\lambda_0)| = |u_1(\ell,\bar{\lambda}_0)| > |u_2(\ell,\bar{\lambda}_0)|$

showing $u_2(\ell,\bar{\lambda}_0)$ is a de Branges function.

August 27, 1977.

Def: A de Branges function is an entire fn. $E(\lambda)$ such that $\operatorname{Im}\lambda > 0 \Longrightarrow |E(\lambda)| > |E(\bar\lambda)|$. The de Branges space $B(E)$ based on E is the set of entire functions f such that

$$\|f\|^2 = \int_{\mathbb{R}} \left|\frac{f(\lambda)}{E(\lambda)}\right|^2 d\lambda < \infty$$

$$\left|\frac{f(\lambda)}{E(\lambda)}\right| \leq \frac{c(f)}{(\operatorname{Im}\lambda)^{1/2}} \qquad \operatorname{Im}\lambda > 0$$

$$\left|\frac{f(\lambda)}{E(\bar\lambda)}\right| \leq \frac{c(f)}{(\operatorname{Im}\bar\lambda)^{1/2}} \qquad \operatorname{Im}\lambda < 0.$$

Clearly $B(E)$ is a vector space over $\mathbb{C}$, and it is a pre-Hilbert space.

The condition on E implies that $E(\lambda) \neq 0$ for $\operatorname{Im}\lambda > 0$. If $E(\lambda) = 0$ with $\lambda \in \mathbb{R}$, then any f in $B(E)$ also vanishes at λ, otherwise $\|f\|$ would not be finite. Hence $\dfrac{f(\lambda)}{E(\lambda)}$ is analytic for $\operatorname{Im}\lambda \geq 0$. Since

$$\int_0^\pi \left|\frac{f}{E}(Re^{i\theta})\right| d\theta \leq c(f) \int_0^\pi \frac{1}{(R\sin\theta)^{1/2}}\, d\theta \to 0 \quad \text{as } R \to \infty$$

one gets by Cauchy's formula:

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} \dfrac{f(z)}{E(z)} & \operatorname{Im} z > 0 \\ 0 & \operatorname{Im} z < 0 \end{cases}$$

similarly

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(\lambda)}{E^\#(\lambda)}\,\frac{d\lambda}{\lambda - z} = \begin{cases} -\dfrac{f(z)}{E^\#(z)} & \operatorname{Im} z < 0 \\ 0 & \operatorname{Im} z > 0 \end{cases}$$

hence

$$\frac{1}{2\pi i}\int_{\mathbb{R}} f(\lambda)\left[\frac{E(z)}{E(\lambda)} - \frac{E^\#(z)}{E^\#(\lambda)}\right]\frac{d\lambda}{\lambda - z} = f(z) \qquad \operatorname{Im} z \neq 0.$$

Same formula has to hold for z real (can move the contour a bit to avoid the zero in deriving the formula and then use the fact that the integrand is defined at $z=\lambda$). So rewrite the above as

$$\int_{\mathbb{R}} f(\lambda)\,\frac{E(z)E^\#(\lambda) - E^\#(z)E(\lambda)}{2\pi i\,(\lambda - z)}\,\frac{d\lambda}{|E(\lambda)|^2} = \int f(\lambda)\,\overline{J_z(\lambda)}\,\frac{d\lambda}{|E(\lambda)|^2}$$

where

$$J_z(\lambda) = \frac{\overline{E(z)}E(\lambda) - E(\bar{z})\overline{E(\bar\lambda)}}{-2\pi i\,(\lambda - \bar{z})}$$

It remains to show J_z belongs to $B(E)$.

$$\frac{J_z(\lambda)}{E(\lambda)} = -\frac{1}{2\pi i(\lambda - \bar z)}\left(\overline{E(z)} - E(\bar z)\,\underbrace{\frac{E^\#(\lambda)}{E(\lambda)}}_{\text{bounded for } \operatorname{Im}(\lambda) \geq 0}\right)$$

Clear that $\displaystyle\int \left|\frac{J_z(\lambda)}{E(\lambda)}\right|^2 d\lambda \leq C\int \frac{d\lambda}{1+\lambda^2} < \infty.$

Also one has by Cauchy:

$$\frac{J_z(\omega)}{E(\omega)} = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{J_z(\lambda)}{E(\lambda)}\,\frac{d\lambda}{\lambda-\omega} \qquad \operatorname{Im}(\omega)>0.$$

so
$$\left|\frac{J_z(\omega)}{E(\omega)}\right|^2 \leq \text{const.} \int \left|\frac{J_z(\lambda)}{E(\lambda)}\right|^2 d\lambda \cdot \int_{\mathbb{R}} \frac{d\lambda}{|\lambda-\omega|^2}$$

$$\int_{\mathbb{R}} \frac{d\lambda}{|\lambda-a-bi|^2} = \int_{\mathbb{R}} \frac{d\lambda}{|\lambda-bi|^2} = \int \frac{b\,d\lambda}{|b\lambda-bi|^2} = \frac{1}{b}\overbrace{\int \frac{d\lambda}{1+\lambda^2}}^{\pi}$$

so one gets that $J_z \in B(E)$.

From $\quad f(\omega) = (f, J_\omega) \quad$ we get

$$|f(\omega)|^2 \leq \|f\|^2\,\|J_\omega\|^2 = \|f\|^2\, J_\omega(\omega) = \|f\|^2\,\frac{|E(\omega)|^2-|E^\#(\omega)|^2}{4\pi(\operatorname{Im}\omega)}.$$

Let's do things in reverse. Start with

$$Lu = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix} u = \lambda u$$

and let $u(x,\lambda)$ be the solution ~~with~~ initial value $\begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha}\end{pmatrix}$ so that one has $\quad u_2^\#(x,\lambda) = \overline{u_2(x,\bar\lambda)} = u_1(x,\lambda)$

Working on the interval $0 \leq x \leq l$ we can associate to each $f \in L^2([0,l])^2$ an entire function

$$\hat{f}(\lambda) = (f, u_{\bar\lambda}) = \int_0^l u(x,\bar\lambda)^* f(x)\,dx = \int_0^l \left(u_1^\#(x,\lambda)f_1(x) + u_2^\#(x,\lambda)f_2(x)\right)dx$$

If I select a boundary condition at $x=\ell$, then I get a sequence of eigenvalues and a spectral measure $d\mu(\lambda) = \sum_j \frac{\delta(\lambda-\lambda_j)}{\|u_{\lambda_j}\|^2}$ such that the Plancherel formula:

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \, d\mu(\lambda) = \int_0^{\ell} |f(x)|^2 \, dx$$

holds. Moreover we get an isomorphism between $L^2([0,\ell])^2$ and $L^2(d\mu)$, which means that in the Hilbert space of analytic functions $\hat{f}(\lambda)$ we get an orthonormal basis formed by the unique elements $\hat{f}_i$ such that

$$\hat{f}_i(\lambda_j) = \delta_{ij}\, r_j^{1/2} \qquad r_j = \|u_{\lambda_j}\|^2.$$

Suppose the boundary value at $x=\ell$ is $\frac{u_1(\ell,\lambda)}{u_2(\ell,\lambda)} = e^{i\beta}$ or $\frac{E^\#(\lambda)}{E(\lambda)} = e^{i\beta}$.

Compute the point evaluator:

$$\hat{f}(\lambda_0) = (f, u_{\bar{\lambda}_0}) = (\hat{f}, \hat{u}_{\bar{\lambda}_0}).$$

Calcute $\hat{u}_{\bar{\lambda}_0}(\lambda) = (u_{\bar{\lambda}_0}, u_{\bar{\lambda}})$ and we find:

$$\hat{u}_{\bar{\lambda}_0}(\lambda) = i\,\frac{E(\lambda)\overline{E(\lambda_0)} - E^\#(\lambda)\overline{E^\#(\lambda_0)}}{\lambda - \bar{\lambda}_0}$$

where $E(\lambda) = u_2(\ell,\lambda)$. Now if I write J_{λ_0} for the point evaluator at λ_0, i.e. $J_{\lambda_0} = \hat{u}_{\bar{\lambda}_0}$, then

$$J_{\lambda_0}(\lambda_0) = \|J_{\lambda_0}\|^2 = \frac{|E(\lambda_0)|^2 - |E^\#(\lambda_0)|^2}{2\,\mathrm{Im}\,\lambda_0}$$

This shows $|E(\lambda)| > |E^\#(\lambda)|$ if $\mathrm{Im}\,\lambda > 0$ since $u_\lambda \neq 0 \Rightarrow J_\lambda \neq 0$.

Take an $\hat{f}$:

$$\hat{f}(\lambda) = (f, u_{\bar{\lambda}}) \leq \|f\|^2 \|u_{\bar{\lambda}}\|^2 = \|f\|^2 \frac{|E(\lambda)|^2 - |E^{\#}(\lambda)|^2}{2\,\mathrm{Im}\,\lambda}$$

Hence we have an estimate

$$\left|\frac{\hat{f}(\lambda)}{E(\lambda)}\right| < \frac{c(f)}{\mathrm{Im}\,\lambda}$$

in the upper half plane. This allows us to establish the Cauchy formula

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\hat{f}(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} \dfrac{\hat{f}(z)}{E(z)} & \mathrm{Im}(z) > 0 \\ 0 & \mathrm{Im}(z) < 0 \end{cases}$$

and similarly

$$\frac{1}{2\pi i}\int \frac{\hat{f}(\lambda)}{E^{\#}(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} 0 & \mathrm{Im}(z) > 0 \\ -\dfrac{\hat{f}(z)}{E^{\#}(z)} & \mathrm{Im}(z) < 0 \end{cases}$$

hence

$$\frac{1}{2\pi i}\int \hat{f}(\lambda)\left[\frac{E(z)}{E(\lambda)} - \frac{E^{\#}(z)}{E^{\#}(\lambda)}\right]\frac{d\lambda}{\lambda - z} = \hat{f}(z)$$

$$\parallel$$

$$\int \hat{f}(\lambda)\left[\frac{E(z)\overline{E(\lambda)} - E^{\#}(z)\overline{E^{\#}(\lambda)}}{(2\pi i)(\lambda - z)}\right]\frac{d\lambda}{|E(\lambda)|^2}$$

Conclude the bracket represents the point evaluation at z

Still need to prove $\displaystyle\int\left|\frac{\hat{f}(\lambda)}{E(\lambda)}\right|^2 d\lambda < \infty$. However one should write by showing that is in $B(E)$.

What is the limit of $i \dfrac{\overline{E(z)}\,E(\lambda) - \overline{E^{\#}(z)}\,E^{\#}(\lambda)}{\lambda - \bar{z}} = (J_z, J_\lambda) = J_z(\lambda)$

when $z = \lambda$ is real?

$$(J_\lambda, J_\lambda) = \frac{|E(\lambda)|^2 - |E^{\#}(\lambda)|^2}{2\,\mathrm{Im}\,\lambda} = \frac{|E(a+ib)|^2 - |E(a-ib)|^2}{2b}$$

if $a+ib = \lambda$. Now let $b \to 0$, you get

$$\frac{d}{db}|E(a+ib)|^2\Big|_{b=0} = E'(a)\,i\,\overline{E(a)} + E(a)\,\overline{i\,E'(a)}$$

$$= |E(a)|^2\, i \left[\frac{E'(a)}{E(a)} - \overline{\frac{E'(a)}{E(a)}} \right]$$

$$= |E(a)|^2 (-2)\, \mathrm{Im}\, \frac{d}{da} \log E(a)$$

$$= -2|E(a)|^2 \frac{d}{da}\left(\arg E(a)\right)$$

Thus we have for λ real:

$$(J_\lambda, J_\lambda) = (u_\lambda, u_\lambda) = -2|E(\lambda)|^2 \frac{d}{d\lambda}\left(\arg E(\lambda)\right)$$