
W. Hahn: "Über Orthogonalpolynome, die q-Differenzen-

Suppose $L$ is a linear operator on polynomials decreasing degree by 1 such that one has the commutation formula with $x$ as follows:

$$L(xf) = f + (gx+w)Lf$$

Then

$$L(x^n) = x^{n-1} + (gx+w)\left\{x^{n-2} + (gx+w)Lx^{n-2}\right\}$$

$$= x^{n-1} + (gx+w)x^{n-2} + \ldots + (gx+w)^{n-1}$$

$$= \sum_{k=0}^{n-1} \frac{(gx+w)^k - x^k}{(gx+w) - x}$$

So in general

$$(Lf)(x) = \frac{f(gx+w) - f(x)}{(gx+w) - x}$$

(and when $gx+w = x$ this is to be interpreted as $\frac{df}{dx}$).

If we change variables $x = az + b$, then

$$a(\tilde{z} + b) = gx + w = g(az + b) + w$$

or

$$\tilde{z} = g\tilde{z} + \frac{w+b-b}{a}$$
Hence if \( q \neq 1 \) we can pick \( a = 1 \) \( \omega + (q-1) b = 0 \), and so arrange that \( \omega = 0 \).

Therefore we see that there are essentially three cases of \( L \) to study:

i) \( L = \frac{d}{dx} \) or \( q x + \omega = x \), whence you have differential equations.

ii) \( L = \Delta \) or \( q x + \omega = x + 1 \), whence you have difference equations.

iii) \( L = \frac{df}{dx} = \frac{f(qx) - f(x)}{(q-1)x} \), whence you have what Hahn calls \( q \)-difference equations.

Example: Put

\[ y_a(x) = \sum_{n=0}^{\infty} \frac{a(a+1) \ldots (a+n-1)}{n!} x^n \]

Then

\[ y_{a+1}(x) - y_a(x) = x y_{a+1}(x) \] or

\[ (1-x) y_{a+1}(x) = y_a(x) \]

Also

\[ \frac{d}{dx} y_a(x) = a y_{a+1}(x) \] so

\[ (1-x) \frac{d}{dx} y_a(x) = a y_a(x) \]

\[ \frac{y_a'}{y_a} = \frac{a}{1-x} = \frac{-a}{x-1} \]

\[ \log y_a = -a \log(1-x) + \text{const} \]

\[ y_a(x) = \frac{\text{const}}{(1-x)^a} \]
The $q$-analogue goes as follows. Put $[a] = \frac{q^a - 1}{q - 1}$.

\[ y_a(x) = \sum_{n \geq 0} \frac{[a] \ldots [a+n-1]}{[1] \ldots [n]} x^n. \]

Then $[a+n] - [a] = \frac{q^{a+n} - q^a}{q - 1} = q^a[n]$, so

\[ y_{a+1}(x) - y_a(x) = q^{a} y_{a+1}(x). \]

Hence

\[ (1 - q^{a} x) y_{a+1}(x) = y_a(x). \]

\[ (\Theta y_a)(x) = \sum_{n \geq 0} \frac{[a] \ldots [a+n-1]}{[1] \ldots [n]} \frac{q^n - 1}{q - 1} x^{n-1} = [a] y_{a+1}(x). \]

So

\[ (1 - q^{a} x) \frac{y_a(q^a x) - y_a(x)}{(q - 1) x} = \frac{q^{a} - 1}{q - 1} y_a(x). \]

\[ \frac{y_a(q^a x)}{y_a(x)} = 1 + \frac{(q^{a} - 1) x}{1 - q^{a} x} = \frac{1 - x}{1 - q^{a} x}. \]

So

\[ y_a(x) = \frac{1 - q^{a} x}{1 - x} y_a(q^a x) = \frac{(1 - q^{a} x)(1 - q^{a+1} x)}{(1 - x)(1 - q^{a} x)} y_a(q^{2a} x). \]

Assuming $|q| < 1$ (think of $q$ as $e^{2\pi i t}$ with $\text{Im} t > 0$), one has $y_a(q^{4a} x) \to 1$ as $n \to \infty$, so

\[ y_a(x) = \prod_{j=0}^{\infty} \left( \frac{1 - q^{a+j} x}{1 - q^{4a} x} \right). \]

Amazingly enough this approaches $(1 - x)^{-a}$ as $q \to 1$. 

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What is the $\Delta$-analogue of the above calculation?

Let us regard as basic the difference equations with constant coefficients constructed from the operator $L$. In other words, $L$ is the basic gadget; its eigenfunctions are the analogues of exponential functions, perhaps first order $L$-equations with rational coefficients. We will explain what are the analogues of $\Gamma$-functions.

Again, let $(L \phi)(x) = \frac{f(gx+\omega) - f(x)}{(gx+\omega) - x}$, Put

\[ p_n(x) = x(x-[1] \omega) \ldots (x-[n-1] \omega) \]

and note that

\[ (gx+\omega-[i] \omega) = (gx+\omega-(1+\ldots+g^{i-1}) \omega) = g \left( x - [i-1] \omega \right) \]

Hence

\[ L p_n(x) = \frac{\left( g(x+\omega) g^{n-1} - x \right) x \ldots (x+[n-2] \omega)}{(g-1) x + \omega} \]

\[ = \frac{(g^{n-1} x + [n] \omega)}{(g-1) x + \omega} x \ldots (x+[n-2] \omega) \]

\[ (L p_n)(x) = [n] p_{n-1}(x) \]

It follows that

\[ e(\lambda, x) = \sum_{n \geq 0} \frac{x(x-[1] \omega) \ldots (x-[n-1] \omega)}{[1][2] \ldots [n]} \lambda^n \]
\[ L e(\lambda, x) = \lambda e(\lambda, x) \]

**Example:** \( q = 1 \), whence \( [i^k] = i \). Then

\[
e(\lambda, x) = \sum_{n=0}^\infty \frac{x(x\omega - (x-n\omega + \omega))}{n!} \]

\[
= \sum_{n=0}^\infty \left( \frac{x}{\omega} \right) \left( \frac{x-1}{\omega} \right) \ldots \left( \frac{x-n+1}{\omega} \right) (\omega \lambda)^n
\]

\[
e(\lambda, x) = \left( \frac{x}{\omega} \right) \quad \rightarrow \quad e^{\lambda x} \quad \text{as} \quad \omega \to 0
\]

**Example:** \( \omega = 0 \). Put \( f(x) = e(\lambda, x) \). Then

\[
f(qx) - f(x) \quad \frac{f(qx)}{(q-1)x} \quad = \lambda f(x)
\]

\[
f(qx) - f(x) \quad 1 = \lambda (q-1)x
\]

\[
f(x) = \frac{1}{1 + \lambda (q-1)x}
\]

\[ f(x) = \frac{1}{1 - \lambda (1-q)x} \frac{1}{1 - \lambda (1-q)^2x} \ldots = \frac{1}{1 - \lambda (1-q)^nx}
\]

Hence

\[ e(\lambda, x) = \left( \prod_{j=0}^{\infty} (1 - \lambda (1-q)x) \right)^{-1}
\]

For this to converge one needs \( |q| < 1 \).

In the general case one has
\[
\frac{f(q^x + \omega) - f(x)}{(q-1)x + \omega} = \lambda f(x)
\]

\[
\frac{f(q^x + \omega)}{f(x)} = 1 + \lambda(q-1)x + \lambda \omega
\]

Sequence \( x, q^x + \omega, q^{2x} + 2\omega, \ldots, q^{nx} + [n]\omega \rightarrow \frac{\omega}{1-q} \)

\[
\frac{f(\frac{\omega}{1-q})}{f(x)} = \prod_{i=0}^{n} \left( 1 + \lambda \left( \frac{q^{i}x + [i]\omega}{1-q} \right) \right)
\]

So what is \( f(\frac{\omega}{1-q}) \)?

\[
\frac{\omega}{1-q} \sum_{i=1}^{\infty} \frac{q^{i}}{[1][2]\cdots[n]} \left( \frac{\omega}{1-q} \right)^{n}
\]

not useful. Instead put \( x=0 \) above. Note

\[
(q-1)(q^x + [j] \omega) + \omega = q^j (q-1)x + (q^j - 1) \omega + \omega
\]

\[
= q^j \left( (q-1)x + \omega \right).
\]

Hence

\[
f(\frac{\omega}{1-q}) = \prod_{j \geq 0} \left( 1 + \lambda q^j \left( \frac{\omega}{1-q} \right) \right)
\]
\[ c(\lambda, x) = \frac{\prod_{j=0}^{\infty} (1 + \lambda \delta_j \omega)}{\prod_{j=0}^{\infty} (1 + \lambda \delta_j ((q-1)x + \omega))} \]

\[(\Delta u)(x) = \frac{u(x+\omega) - u(x)}{\omega} = \lambda u(x)\]

If \( u(x) = e^{\alpha x} \), then
\[ \frac{e^{\alpha \omega} - 1}{\omega} = \lambda \quad \text{or} \quad \alpha = \frac{1}{\omega} \log(1 + \lambda \omega) \]

Thus \( \alpha \) is determined up to \( \frac{2 \pi i n}{\omega} \), \( n \in \mathbb{Z} \), hence we can alter \( e^{\alpha x} \) by any of the \( \omega \)-periodic functions \( e^{\frac{2 \pi i n x}{\omega}} \), \( n \in \mathbb{Z} \).

This is important to note perhaps, because the series solution
\[ (1 + \omega \lambda)^\frac{x}{\omega} = \sum_{n \geq 0} \frac{x(x-\omega) \cdots (x-n\omega + \omega)}{n!} \lambda^n \]

converges only for \( |\lambda| < 1 \), hence as we analytically continue the solution for different \( \lambda \) we get different solutions of the original difference equation.

Consider the equation
\[ (\Delta f)(x) = \lambda x f(x) \]
Try a series \( \sum a_n x^n = f(x) \). Then

\[(\theta f)(x) = \sum a_n \frac{d^n}{dx^n} x = \sum \lambda a_n x^{n+1} \]

leads to the recursion formula

\[a_n \frac{d^n}{dx^n} = \lambda a_{n-2} \quad \text{for } n \geq 2 \]

\[a_1 = 0 \]

\[a_0 \text{ arbitrary.} \]

So

\[ f(x) = \sum_{n \geq 0} \frac{\lambda^n x^{2n}}{2^n (n)!} \]

As \( q \to 1 \) this approaches \( \sum_{n \geq 0} \frac{\lambda^n x^{2n}}{2^n n!} \) which is the solution of the differential equation \( \frac{dx}{dx} f = \lambda x f \). On the other hand the original difference equation can be written

\[ \frac{f(qx) - f(x)}{(q-1)x^2} = \lambda x f(x) \]

So

\[ \frac{f(qx)}{f(x)} - 1 = \lambda (q-1) x^2 \]

So

\[ f(x) = \frac{1}{1 + \lambda (q-1) x^2} f(qx) \]

So

\[ f(x) = \left( \prod_{j \geq 0} \left( 1 + \lambda (q-1) q^{2j} x^2 \right) \right)^{-1} \]

So if we put \( \lambda = \frac{1}{1-q} \) we get the identity

\[ \frac{1}{\prod_{j \geq 0} (1-q^{2j} x^2)} = \sum_{n \geq 0} \frac{x^{2n}}{(1-q^2)(1-q^4) \cdots (1-q^{2n})} \]}
which is essentially the formula for the \( g \)-exponential. 

\[
\sum \frac{x^n}{[1][1][n]} = \prod_{j \geq 0} (1 + \lambda(g(j)g^jx)^{-1})^{-1}
\]

obtained previously.

Gauss identity. Put

\[
f_m(x) = \frac{x^n}{(1+x)(1+g^2x) \ldots (1+g^{m-1}x)}
\]

Then

\[
(1+x)f_m(gx) = f_{m+1}(x) = f_m(x)(1+g^m x)
\]

so if we put \( f_m(x) = \sum a_n x^n \), then

\[
(1+x)\sum a_n g^n x^n = \sum a_n x^n (1+g^m x)
\]

\[
a_n g^n + a_{n-1} g^{n-1} = a_n + a_{n-1} g^m
\]

\[
a_n (1+g^n) = a_{n-1} (g^m - g^{n-1})
\]

so

\[
a_n = \frac{g^{m-n} a_{n-1}}{1+g^n}
\]

\[
\sum_{n=0}^{\infty} \frac{2}{n(n-1)} g^{\frac{n(n-1)}{2}} \frac{(1-g^{m-n+1}) \ldots (1-g^m)}{(1-g) \ldots (1-g^n)}
\]

\[
(1+x)(1+g^2 x) \ldots (1+g^{m-1} x) = \sum_{n=0}^{m} \frac{g^{\frac{n(n-1)}{2}} [m] \ldots [m-n+1]}{[1] \ldots [n]} x^n
\]

\[
\prod_{j \geq 0} (1+g^jx) = \sum_{n=0}^{\infty} \frac{g^{\frac{n(n-1)}{2}}}{(1-g) \ldots (1-g^n)} x^n
\]
First order difference equations with rational coefficients:

\[ \frac{\partial f(x)}{\partial x} = R(x) f(x) \]

can be written \[ \frac{f(gx)}{f(x)} = 1 + (g-1)xR(x). \]

To solve this one can factor the rational function on the right into linear factors. A factor \((1 + \lambda x)\) leads to \[ \prod_{j=0}^{n} (1 + \lambda_j x)^{-1} \] as a factor of \(f\).

A singular case occurs, possibly if \(x\) occurs in the denominator of \(R\). I should consider first the case

\[ \frac{d}{dx} f(x) = R(x) f(x) \]

which has the solution

\[ f(x) = e^{\int R(x) dx} \]

This gives a good power series solution at \(x = 0\) provided \(R\) is regular at \(x = 0\). If \(R(x)\) has a simple pole at \(x = 0\), say \(R(x) = \frac{x}{x} + \text{reg} \), then \(x = 0\) is a regular singular pt, and the solution is of the form

\[ f(x) = x^a (1 + a_1 x + a_2 x^2 + ...) \]

so one expects the analogous thing with \(a_1\).
The good case is when $R$ is regular at $0$ and we get nice power series solutions. The next good case is when $R$ has a simple pole. Important special case is $R = \frac{\alpha}{x}$ which leads to the equation

\[
\frac{f(gx)}{f(x)} = 1 + (g-1)x = \lambda
\]

Try a solution $f(x) = x^\mu$. Then

\[
\frac{g^\mu x^\mu}{x^\mu} = \lambda \quad \Rightarrow \quad g^\mu = \lambda
\]

This determines $\mu$ up to an additive constant $\frac{2\pi i n}{\log g}$, but these differences in the choices for $\mu$ change $f$ by a $g$-periodic function.

In the above we supposed that $\lambda \neq 0$. If the equation

\[
\frac{f(gx)}{f(x)} = x^m
\]

has a Laurent series solution $f(x) = \sum a_n x^n$, then

\[
\sum a_n g^n x^n = \sum a_n x^{n+m} = \sum a_{n-m} x^n
\]

so

\[
a_n g^n = a_{n-m}. \quad \text{say} \quad m = -1. \quad \text{Then}
\]

\[
a_n = g^{n+1} a_{n-1} = g^{(n+1)+\ldots+1+0} a_0 = g^{\frac{m(n-1)}{2}} a_0
\]

so

\[
f(x) = \sum_{n \in \mathbb{Z}} g^{\frac{n(n-1)}{2}} x^n
\]

which is essentially a $\Theta$-function. If $m = +1$, ...
then we find that

\[ a_n q^n = a_{n-1} \]

or

\[ a_n = q^{-n} a_{n-1} = q^{-n} q^{-n+1} a_{n-2} = q^{-n} q^{-n+1} q^{-n+2} a_{n-3} \]

\[ = q^{-n} q^{-n+1} \cdots q^{-1} a_0 = q^{-\frac{n(n+1)}{2}} a_0 \]

so

\[ f(x) = \sum_{n=1}^{\infty} q^{-\frac{n(n+1)}{2}} x^n \]

except for the fact that for \(|q|<1\) this doesn't converge. Curious:

Put

\[ f(x) = \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} x^n \]

so that

\[ f(qx) = \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2} + n} x^n = \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} x^n \]

\[ = \sum_{n=1}^{\infty} q^{\frac{(n-1)n}{2}} x^{n-1} = x^{-1} f(x) \]

Now \( f(x) \) is defined and analytic for all \( x \neq 0 \).

Look at \( g(x) = f(x)^{-1} \). Then

\[ \frac{g(qx)}{g(x)} = \frac{f(x)}{f(qx)} = x \]

g(x) doesn't have a laurent series expansion. This means I guess that \( f(x) \) has lots of zeroes.

Look at the zeroes of \( f(x) \); if \( \lambda \) is a zero then so is \( q\lambda \) and \( q^{-1}\lambda \). So let us look at the function with the zeroes \( q^n \lambda \) for \( n \geq 1 \)

\[ h_1(x) = \prod_{n=1}^{\infty} \left( 1 - \frac{q^n \lambda}{x} \right) \]

and the function with the zeroes \( q^{-n} \lambda \) for \( n > 0 \) which
\( h_2(x) = \prod_{n=0}^{\infty} \left(1 - \frac{g^n x}{\lambda} \right) \)

Then

\[ h_1(gx) = (1 - \frac{A}{x}) h_1(x) \]

\[ h_2(gx) = \left(1 - \frac{x}{\lambda} \right)^{-1} h_2(x) \]

so

\[ h_1(gx) h_2(gx) = \frac{A(x-1)}{x} \frac{h_1(x) h_2(x)}{(1 - \frac{x}{\lambda})} = \left(\frac{A}{x}\right) h_1(x) h_2(x) \]

Thus if we take \( \lambda = -1 \), \( h_1 h_2 \) will satisfy the same functional equation as \( f(x) \). So we have

\[ f(x) = \sum_{n \in \mathbb{Z}} g^{\frac{n(n-1)}{2}} x^n = g(x) \frac{\prod_{n=0}^{\infty} (1 + g^n x)}{\prod_{n=1}^{\infty} (1 + g^n x)^2} \]

where \( g(x) \) satisfies:

\[ g(gx) = g(x) \]

Next note that \( \frac{n(n-1)}{2} \) is symmetric under \( n \rightarrow 1-n \), hence

\[ x f(x^{-1}) = \sum_{n=1}^{\infty} g^{\frac{n(n-1)}{2}} x^{1-n} = f(x) \]

which shows that \( f(-1) = 0 \). Thus \( g(x) \) is an entire function which is bounded, so \( g(x) \) is a constant in \( x \), given by

\[ g = \sum_{n \geq 0} g^{\frac{n(n+1)}{2}} \frac{\prod_{n=0}^{\infty} (1 + g^n x)^2}{\prod_{n=1}^{\infty} (1 + g^n x)^2} \]

so \( g \) is some power series in \( g \) with leading term 1, whose exact determination as an infinite product is possible.
Review: I've been considering δ-difference equations

\[
\frac{f(\delta x)}{f(x)} = \tilde{R}(x)
\]

where \( \tilde{R}(x) \) is a rational function of \( x \). I am interested in solutions with nice analytic properties as functions of \( x \). For example, if \( \tilde{R}(x) = 1 \), then a solution meromorphic on \( \mathbb{C} - \{0\} \) is the same thing as a doubly-periodic function on \( \mathbb{C} \) with periods 1 and \( \tau \) where \( e^{2\pi i \tau} = 8 \). This means that the only solutions holomorphic off zero are constants.

Thus the solutions of the homogeneous equation \( \frac{f(\delta x)}{f(x)} = 1 \) can be understood in terms of elliptic functions.

The next point is that if we split \( \tilde{R} \) into factors of degree 1, then it suffices to solve the separate equations — this is the principle used in finding particular solutions. So we end up with the following cases:

\[
\tilde{R}(x) = \text{constant, } \frac{1}{x}, \frac{1}{x}, (1 + \lambda x)^{-1}
\]

Consider the latter cases. If we assume a soln. of the form

\[
f(x) = \sum c_n x^n
\]
then
\[ f(gx) = \sum a_n g^n x^n = (1+\lambda x) \sum a_n x^n = \sum (a_n + \lambda a_{n-1}) x^n \]

leads to the recursion formula

\[ a_n g^n = a_n + \lambda a_{n-1} \]

\[ a_n (g^n - 1) = \lambda a_{n-1} . \]

Hence \( a_{-1} = 0 \), \( a_n = 0 \) for \( n \leq -1 \). If \( a_0 = 1 \) we get

\[ a_n = \frac{\lambda}{g^n - 1} a_{n-1} = \ldots = \frac{\lambda^n}{(g^n - 1)(g^{n-1} - 1)} . \]

So

\[ f(x) = \sum_{n \geq 0} \frac{(-\lambda)^n}{(1-g)(1-g^2) \ldots (1-g^n)} x^n = \prod_{j \geq 0} \frac{1}{1-\lambda g^j x} \]

On the other hand if \( \frac{f(gx)}{f(x)} = (1+\lambda x)^{-1} \) then

\[ (1+\lambda x) f(gx) = (1+\lambda x) \sum a_n g^n x^n = \sum (a_n g^n + \lambda a_{n-1} g^{n-1}) x^n \]

so

\[ a_n g^n + \lambda a_{n-1} g^{n-1} = a_n \]

\[ \lambda a_{n-1} g^{n-1} = a_n (1-g^n) \]

Again \( a_{-1} = a_2 = \ldots = 0 \) and if \( a_0 = 1 \), then

\[ a_n = \frac{\lambda g^{n-1}}{1-g^n} a_{n-1} = \ldots = \frac{\lambda^n g^{n(n-1)}}{(1-g)(1-g^2) \ldots (1-g^n)} \]

so

\[ f(x) = \sum_{n \geq 0} \frac{\lambda^n}{(1-g)(1-g^2) \ldots (1-g^n)} x^n = \prod_{j \geq 0} (1+\lambda g^j x) \]
Next thing to look at is those equations giving rise to simple recursion formulas for the coefficients. Thus consider

\[ \frac{f(\tilde{q}x)}{f(x)} = \frac{1 + \lambda x}{1 + \mu x} \]

\[ a_n \tilde{q}^n + \mu a_{n-1} \tilde{q}^{n-1} = a_n + \lambda a_{n-1} \]

\[ a_n (\mu \tilde{q}^{n-1} - \lambda) = a_n (1 - \tilde{q}^n) \]

As before these relation allow us to set \( q_0 = 1, a_1 = a_2 = \ldots = 0 \) and

\[ a_n = \frac{(\mu \tilde{q}^{n-1} - \lambda) \cdots (\mu - \lambda)}{(1 - \tilde{q}^n) \cdots (1 - \tilde{q})} \]

\[ = \frac{(1 - \frac{\mu}{\lambda}) \cdots (1 - \frac{\mu}{\lambda} \tilde{q}^{n-1}) (\lambda)^n}{(1 - \tilde{q}^n) \cdots (1 - \tilde{q})} \]

\[ = \frac{\prod_{k=0}^{n} [\alpha] \cdots [\alpha+n-1]}{[1] \cdots [n]} (-\lambda)^n \]

where \( \sqrt{q} = \frac{\mu}{\lambda} \) so we get the "binomial-type series"

\[ f(x) = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n} [\alpha] \cdots [\alpha+n-1]}{[1] \cdots [n]} (-\lambda)^n x^n \]

\[ = \prod_{j \geq 0} \left( \frac{1 + \mu \tilde{q}^j x}{1 + \mu x} \right) \]

Notice that if \( \lambda = \tilde{q}^m \) for some integer \( m \), then we can put \( \mu a_m = 1 \), \( a_{m+1} = a_{m+2} = \ldots = 0 \) and grind out a series solution holomorphic at \( \infty \).
Recall that if \( y_a(x) = \sum_{n \geq 0} \frac{[a] \ldots [a+n-1]}{[1] \ldots [n]} x^n \), then

\[ g y_a = [a] y_{a+1} \quad \text{and} \quad y_{a+1} - y_a = g^a x y_{a+1} \]

so that

\[ \frac{y_a(gx)}{y_a(x)} = \frac{1-x}{1-g^a x} \]

and so we get the first of the following formulas

\[ \prod_{j \geq 0} \frac{1 - g^j x}{1 - g^j x} = \sum_{n \geq 0} \frac{[a] \ldots [a+n-1]}{[1] \ldots [n]} x^n \]

\[ \prod_{j \geq 0} \frac{1}{1 - g^j x} = \sum_{n \geq 0} \frac{x^n}{(1-g) \ldots (1-g^n)} \]

\[ \prod_{j \geq 0} (1 + g^j x) = \sum_{n \geq 0} \frac{g^{n(n-1)/2} x^n}{(1-g) \ldots (1-g^n)} \]

The second is obtained by letting \( a \to +\infty \) in the first, and the third by substituting \( x \to -x g^{-a} \) in the first and letting \( a \to -\infty \).

\[ \prod_{j \geq 0} \frac{1 - g^j x}{1 - g^{-a} x} = \sum_{n \geq 0} \frac{(g^{-a-1}) \ldots (g^{-a} - g^{n-1})}{(1-g) \ldots (1-g^n)} x^n \]

\[ \prod_{j \geq 0} \frac{1 - g^j x}{1 - g^{j+a} x} = \sum_{n \geq 0} \frac{[a] \ldots [a-n+1]}{[1] \ldots [n]} g^{n(n-1)/2} (-x)^n \]

Hahn denotes the last function by \((1-x)_a\) since as \( g \to 1 \) it converges to \((1-x)^a\)
Consider a 2nd order DE with a regular singular point at \( x = 0 \):

\[
\left( x^2 p(x) \frac{d^2}{dx^2} + x g(x) \frac{d}{dx} + r(x) \right) y = 0
\]

where \( p, g, r \) are analytic at 0 and \( p(0) = 1 \). I want to assume that the recursion relation for the coefficients of a series solution is of the two term type. This will be the case if \( p, g, r \) are linear in \( x \) in which case the DE has the form

\[
\left( x^2 \frac{d^2}{dx^2} + c_1 x \frac{d}{dx} + c_2 \right) y = \left( c_3 x^3 \frac{d^2}{dx^2} + c_4 x^2 \frac{d}{dx} + c_5 x \right) y
\]

If we have a solution \( y = \sum_{n=0}^{\infty} a_n x^n \), then

\[
\sum_{n=0}^{\infty} a_n [\mu(n)(\mu+n-1) + c_1(\mu+n) + c_2] x^n
\]

\[
= \sum_{n=0}^{\infty} a_n \left[ \frac{\mu(n)(\mu+n-1) + c_4(\mu+n) + c_5}{\mu(n)(\mu+n-1) + c_1(\mu+n) + c_2} \right] x^{n+1}
\]

giving the indicial equation

\[ p(\mu n) + c_1\mu + c_2 = 0 \]

and the recurrence formula

\[
a_n = \frac{c_3(\mu+n-1)(\mu+n-2) + c_4(\mu+n-1) + c_5}{(\mu+n)(\mu+n-1) + c_1(\mu+n) + c_2} a_{n-1}
\]

Let me simplify things by requiring that \( \mu = 0 \) be a root of the indicial equation so that \( c_2 = 0 \). Then

\[
a_n = \frac{c_3(n-1)(n-2) + c_4(n-1) + c_5}{(c_1 + n-1)} n
\]
Case 1: \( c_3 = c_4 = 0, c_5 \neq 0 \)  

By scaling \( x \), we can suppose \( c_5 = 1 \). Then we get the D.E.

\[
\left( x \frac{d^2}{dx^2} + c_1 \frac{d}{dx} - 1 \right) y = 0
\]

which is essentially Bessel's DE (start with Bessel's DE)

\[
\left( \left( \frac{d}{dx} \right)^2 + \frac{z^2 - n^2}{4} \right) u = 0
\]

and put \( x = \left( \frac{z}{2} \right)^2 \). \( (dx = \frac{z}{2} dz \) \  \( x \frac{d}{dx} = \frac{z}{2} \frac{d}{dz} \)

\[
\left( \frac{d}{dz} \right)^2 = 4 \left( x \frac{d}{dx} \right)^2, \quad z^2 = 4x \)

and you get the DE

\[
\left( \left( \frac{d}{dx} \right)^2 + x - \frac{n^2}{4} \right) u = 0.
\]

Now put \( u = x^{n/2} v \) and you get

\[
\left( \left( \frac{d}{dx} + \frac{n}{2} \right)^2 - x - \frac{n^2}{4} \right) v = 0
\]

or

\[
\left( \left( x \frac{d}{dx} \right)^2 + nx \frac{d}{dx} + \frac{n^2}{4} - x - \frac{n^2}{4} \right) v = 0
\]

Case 2: \( c_3 = 0, c_4 \neq 0 \); by scaling can suppose \( c_4 = 1 \)

whence we have the confluent hypergeometric DE

\[
\left( x \frac{d^2}{dx^2} + (c_1 - x) \frac{d}{dx} - c_5 \right) y = 0
\]

Case 3: \( c_3 \neq 0 \); whence by scaling we can arrange \( c_3 = 1 \)

and we get the hypergeometric DE.

\[
\left( x(1-x) \frac{d^2}{dx^2} + (c_1-c_4 x) \frac{d}{dx} - c_5 \right) y = 0
\]
Now \[ c + [n \n] = c + \frac{g^n}{g-1} = -\frac{g^{-\gamma}}{g-1} + \frac{g^n}{g-1} \]
\[ = \frac{g^n - g^{-\gamma}}{g-1} = g^{-\gamma} [y+n] \]

provided \( c = -\frac{g^{-\gamma}}{g-1} = \frac{g^{-\gamma}}{1-g} = g^{-\gamma} [y] \). \( Y \) exists

provided \( c \neq \frac{1}{g-1} \). Now the numerator can be written

\[ c_q [n][n-1] + c_s [n] + c_0 \]
\[ = \alpha \cdot g^{2n} + \beta \cdot g^n + \gamma \]
\[ = \alpha_0 (g^n - r_1)(g^n - r_2) \]
\[ = \alpha_0 (g^n - g^{-\alpha})(g^n - g^{-\beta}) \]
\[ = \text{const.} [\alpha+n][\beta+n] \]

provided \( c_q \neq 0 \) and \( r_1, r_2 \neq 0 \). Hence the series we obtain is a \( \text{Heine hypergeometric series:} \)

\[ \sum \frac{[\alpha][\alpha+1]...[\alpha+n-1][\beta][\beta+1]...[\beta+n-1]}{[g][g+1]...[g+n-1]} \frac{(\lambda x)^n}{[n]} \]
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We saw that DE's of the form with $c_1 \neq 0$

\[(1) \quad \left( c_1 x^2 \frac{d^2}{dx^2} + c_2 x \frac{d}{dx} + c_3 \right) y = \left( c_4 x^2 \frac{d^2}{dx^2} + c_5 x \frac{d}{dx} + c_6 x \right) y \]

have nice series solutions around $x=0$. If we suppose $c_3 = 0$ (replace $y$ by $x^\mu y$ where $\mu$ is a root of the indicial equation), then we get DE's of the form

\[(2) \quad \left( a_1 x^2 + a_2 x + a_3 \right) \frac{d^2}{dx^2} + \left( a_4 x + a_5 \right) \frac{d}{dx} + a_6 \right) y = 0 \]

with $a_3 = 0$, $a_2 \neq 0$. Conversely suppose given a DE of the second type, we look at the quadratic factor $(a_1 x^2 + a_2 x + a_3)$. By translating and scaling we can bring it into one of the forms:

\[
\begin{align*}
&1 \\
&x \\
&x^2 \\
&x(1-x) \\
&x(1-x)^2 \\
&x(1-x)^3 \\
&x(1-x)^4 \\
&x(1-x)^5
\end{align*}
\]

The cases $x(1-x)$ and $x(1-x)^2$ belong to the type (1). Let's look at the other types:

Take $x^2$. Then we have to have $a_5 = 0$ or else $x = 0$ is an irregular singular point. So we have an equation of the form

\[
\left( \left( x^2 \frac{d}{dx} \right)^2 + b_1 \left( x \frac{d}{dx} \right) + b_2 \right) y = 0
\]

which has solutions $y = x^\mu$.

Take the case of 1. By an affine transformation we can make $(a_4 x + a_5) = 2x$. (Assume $a_4 \neq 0$; otherwise the DE has constant coefficients, hence solutions $y = e^{ax}$).
so we get the DE
\[
\left( \frac{d^2}{dx^2} + 2x \frac{d}{dx} + b \right)y = 0
\]

If \( y = e^{-x^2/2} u \), then
\[
e^{-x^2/2} \left( \frac{d}{dx} + 2x \right) \left( \frac{d}{dx} \right) e^{-x^2/2} u = \left( \frac{d}{dx} - x \right) \left( \frac{d}{dx} + x \right) = \frac{d^2}{dx^2} - x^2 + 1
\]
so we end up with the Hermite DE.

Hahn considers q-difference equations of the form
\[
\left( a_1 x^2 + a_2 x + a_3 \right) q^2 y + \left( a_4 x + a_5 \right) q y + a_6 y = 0
\]
but the analysis is quite different because the origin, as the unique fixed point of \( x \mapsto q x \), must remain fixed under any \( x \) change. Possibilities for the leading factor are:

\[
\begin{align*}
\text{deg } q & : 0, 1 \\
\text{deg 1:} & \quad x, \quad x - 1 \\
\text{deg 2:} & \quad x^2, \quad x(1-x), \quad (1-x)(1-a x), \quad (1-x)^2
\end{align*}
\]

He also considers solutions expanded in series about \( x = 1 \).

First consider series about \( x = 0 \) with 2-term recursion relations. Thus I consider
\[
\left( c_1 x^2 q^2 + c_2 x q + c_3 \right) y = \left( c_4 x^2 q^2 + c_5 x q + c_6 x \right) y
\]
with \( c_3 = 0 \), \( c_1 = 1 \), \( c_2 = c \). This leads to a recursion formula
\[
a_n = \frac{c_4 \left[ n \right] \left[ n-1 \right] + c_5 \left[ n-1 \right] + c_6 \left[ n \right]}{\left[ n \right] \left[ n-1 \right] + c \left[ n \right]} a_{n-1}
\]