July 3, 1997:

The question is whether I should be looking at difference equations instead of differential equations.

Possible approaches:

1) Do there exist difference eqn. analogues of the special DE's such as the Weber D.E, etc?

2) Jacobi matrices (one-sided) can be directly related to orthogonal polynomials and 2nd order difference equations. So we must have a 2nd order difference equation associated to the Weber D.E.

3) Take a linear Ising model with vertices the integers. This should be the same thing as a sequence of transition matrices of the type

$$\begin{pmatrix} e^{ia\lambda} & 0 \\ 0 & e^{-ia\lambda} \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}$$

$a>0$

shrinks unit disk for $\mathrm{Im}(\lambda)>0$

any matrix in $SL_2(\mathbb{C})$ preserving unit disk.

July 4, 1975

Review J-matrices:

Let n be an integer ≥ 1. Then we have an equivalence between the following gadgets:

1) A probability measure $d\mu$ on $\mathbb{R}$ whose support has cardinality n.

2) Iso classes of triples (V, A, v) where V is a Hilbert space of dimension n, A is a self-adjoint operator on V, and v is a unit vector cyclic with respect to A, V.

3) $n \times n$ Jacobi matrices $J = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix}$

where a_i, b_i are real and all $a_i > 0$.

The transitions between these gadgets are as follows:

$1) \to 2)$: Given $d\mu$ put $V = L^2(\mathbb{R}, d\mu)$ with $A =$ multiplication by x and $v =$ class of the const. function 1.

$2) \to 1)$. Given (V, A, v), let $\lambda_1 < \cdots < \lambda_n$ be the eigenvalues of A (multiplicities $= 1$, since $\exists$ cyclic vector), and let $v = v_{\lambda_1} + \cdots + v_{\lambda_n}$ be the eigenvector decomposition of v. Then

$$d\mu = \sum \|v_{\lambda_i}\|^2 \delta_{\lambda_i}$$

Another way to get $d\mu$ is to use the formula

$$\left((\lambda - A)^{-1} v, v\right) = \sum \frac{\|v_{\lambda_i}\|^2}{\lambda - \lambda_i} = \int \frac{1}{\lambda - x} d\mu(x).$$

$3) \to 2)$ obvious: $V = \mathbb{C}^n$, $v = e_1$, $A = J$. v is cyclic because all $a_i > 0$.

2) $\to$ 3). Given (V, A, v) you construct from the basis $v, Av, \ldots, A^{n-1}v$ an orthormal basis $v_1, \ldots, v_n$ by Gram-Schmidt and let J be the matrix of A relative to this basis.

1) $\to$ 3). Construct from $1, x, \ldots, x^{n-1}$ an orthormal sequence of polynomials $\phi_0, \ldots, \phi_{n-1}$ wrt the measure $d\mu$, then J is the matrix of multiplication by x:

$$x\phi_{i-1} = a_{i-1}\phi_{i-2} + b_i \phi_{i-1} + a_i \phi_i \qquad 1 \le i \le n$$

(This maybe looks better if you put $\phi_{i-1} = v_i$.)

$$\phi_i = \frac{x^i}{a_1 \cdots a_i} + \text{lower terms}$$

Here we can take a_n to be arbitrary >0 and

$$\phi_n = \frac{1}{a_1 \cdots a_n} \prod_{i=1}^{n} (x - \lambda_i) = \frac{1}{a_1 \cdots a_n} \det(xI_n - J).$$

3) $\to$ 1) This is interesting because it uses continued fractions: Put

$$g_i(x) = \det\left(xI_{n-i+1} - \begin{pmatrix} b_i & a_i & & \\ a_i & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix} \right) \qquad \begin{array}{l} g_n(x) = x - b_n \\ g_{n+1}(x) = 1 \end{array}$$

The minor expansion shows

$$g_i(x) = (x - b_i)\, g_{i+1}(x) - a_i^2\, g_{i+2}(x)$$

or

$$\frac{g_{i+1}}{g_i} = \cfrac{1}{x - b_i - a_i^2 \cfrac{g_{i+2}}{g_{i+1}}}$$

Now Cramer's rule gives that $\frac{g_2}{g_1} = ((xI-J)^{-1}e_1, e_1)$, hence we have

$$\int \frac{d\mu(\lambda)}{x-\lambda} = \cfrac{1}{x-b_1-\cfrac{a_1^2}{x-b_2-\cfrac{a_2^2}{\ddots \, \cfrac{a_{n-2}^2}{x-b_{n-1}-\cfrac{a_{n-1}^2}{x-b_n}}}}}$$

$$= \frac{1}{x-b_1-}\,\frac{a_1^2}{x-b_2-}\cdots\frac{a_{n-2}^2}{x-b_{n-1}-}\,\frac{a_{n-1}^2}{x-b_n}$$

Additional facts:

$$\phi_i(x) = \frac{1}{a_1\cdots a_i}\det\Big(xI_i - \underbrace{\begin{pmatrix} b_1 & a_1 & & 0 \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{i-1} \\ 0 & & a_{i-1} & b_i \end{pmatrix}}_{J_i}\Big)$$

One can see this by showing both sides satisfy the same recursion relations, or by noting that if $\phi_i(\lambda)=0$, then $(\phi_0(\lambda), \ldots, \phi_{i-1}(\lambda))$ is an eigenvector for J_i, so the roots of ϕ_i are the eigenvalues of J_i.

Also $d\mu$ is even $\Longleftrightarrow$ $b_1 = \cdots = b_n = 0$.

Also I could add business of isospectral flows on the space of these J-matrices. (see p. 17 March 20)

Here's another way of going from a J-matrix to a measure besides continued fractions. For any eigenvalue λ we let $u_\lambda(i)$ be defined by the recursion formulas

$$a_{i-1} u_\lambda(i-1) + b_i u_\lambda(i) + a_i u_\lambda(i+1) = \lambda u_\lambda(i)$$

$$u_\lambda(0) = 0 \qquad u_\lambda(1) = 1$$

(Pick some a_n)

for $i = 1, \dots, n$. Thus

$u_\lambda(i)$ is a polynomial of degree $i-1$, in fact, it is the $(i-1)$-th orthonormal polynomial ϕ_{i-1}, or also

$$u_\lambda(i) = \frac{1}{a_1 \cdots a_n} \det\left(\lambda I - \begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & & \\ & & & b_{i-1} \end{pmatrix}\right)$$

Now let u_λ be the vector $(u_\lambda(1), \dots, u_\lambda(n))$. One has

$$J u_\lambda - \lambda u_\lambda = -a_n u_\lambda(n+1) e_n$$

so u_λ is an eigenvector when λ is an eigenvalue. Now we can expand e_1 in terms of the eigenvectors

$$e_1 = \sum_{j=1}^{n} r_{\lambda_j} u_{\lambda_j}$$

and since $(u_\lambda, e_1) = 1$ by definition, one has

$$r_{\lambda_j} = \frac{1}{\|u_{\lambda_j}\|^2} \qquad \left((z-J)^{-1} e_1, e_1\right) = \sum \frac{r_{\lambda_j}}{z - \lambda_j}$$

Therefore

$$d\mu = \sum \frac{1}{\|u_{\lambda_j}\|^2} \delta_{\lambda_j}$$

<u>Example</u>: $b_i = 0$, $a_i = \frac{1}{2}$. The recursion equations are

$$\frac{1}{2}(y_{i-1} + y_{i+1}) = \lambda y_i$$

Since this has constant coefficients one tries solns of the form $y_\nu = e^{-i\nu\theta}$. This will be a solution if

$$\cos\theta = \frac{1}{2}(e^{-i\theta} + e^{i\theta}) = \lambda$$

The general solution is $y_\nu = c_1 e^{i\nu\theta} + c_2 e^{-i\nu\theta}$ as long as $e^{i\theta} \neq e^{-i\theta}$, i.e. $\lambda \neq \pm 1$. Initial conditions give

$$u_\lambda(\nu) = \frac{\sin \nu\theta}{\sin\theta}$$

The eigenvalues on the interval $1 \leq \nu \leq n$ are then roots of

$$u_\lambda(n+1) = \frac{\sin(n+1)\theta}{\sin\theta} = 0$$

i.e. $\theta = \frac{\pi j}{n+1}$, $j = 1, \ldots, n$; corresponding eigenvalues are $\lambda_j = \cos\left(\frac{\pi j}{n+1}\right)$.

$$\|u_{\lambda_j}\|^2 = \sum_{\nu=1}^{n} \frac{\left(\sin \nu\frac{\pi j}{n+1}\right)^2}{\left(\sin\frac{\pi j}{n+1}\right)^2}$$

Now
$$\sum_{\nu=1}^{n}\left(\sin\nu\frac{\pi j}{n+1}\right)^2 = \sum_{\nu=1}^{n}\left(-\frac{e^{2\pi i j\nu/n+1}}{4} + \frac{1}{2} - \frac{e^{-2\pi i j \nu/n+1}}{4}\right)$$

$$= \frac{n}{2} - \frac{1}{2}\left(\sum_{\nu=0}^{n} e^{2\pi i j\nu/n+1} - 1\right) = \frac{n+1}{2}$$

Thus
$$d\mu_n = \sum_{j=1}^{n} \frac{2}{n+1}\left(\sin\frac{\pi j}{n+1}\right)^2 \delta_{\cos\left(\frac{\pi j}{n+1}\right)}$$

and

$$\int \frac{d\mu_n(x)}{z-x} = \sum_{j=1}^{n} \frac{2}{n+1} \sin^2\left(\frac{\pi j}{n+1}\right) \frac{1}{z-\cos\frac{\pi j}{n+1}}$$

$$\longrightarrow 2\int_0^1 \frac{\sin^2(\pi y)}{z-\cos(\pi y)}\,dy \qquad \begin{array}{l} x = \cos\pi y \\ dx = -\pi\sin\pi y\,dy \end{array}$$

$$= \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{z-x}\,dx$$

Hence $$\lim_{n\to\infty} d\mu_n(x) = \frac{2}{\pi}\sqrt{1-x^2}\,dx .$$

Hermite polys. from this point of view.

$$h_s(z) = \frac{e^{-z^2/2}}{\Gamma(s)} \int_0^\infty e^{-t^2-2zt}\, t^s \frac{dt}{t} \quad \text{satisfies}$$

$$\left(\frac{d}{dz}+z\right)h_s(z) = -2\,\frac{\Gamma(s+1)}{\Gamma(s)}\,\frac{e^{-z^2/2}}{\Gamma(s+1)}\int_0^\infty e^{-t^2-2zt}\,t^{s+1}\frac{dt}{t} = -2s\,h_{s+1}(z)$$

$$\left(\frac{d}{dz}-z\right)h_{s+1}(z) = \frac{e^{-z^2/2}}{\Gamma(s)}\int_0^\infty e^{-t^2-2zt}(-2t-2z)\,t^{s+1}\frac{dt}{t}$$

$$= \frac{e^{-z^2/2}}{\Gamma(s+1)}\int_0^\infty e^{-t^2-2zt}\left(-\frac{d}{dt}\right)t^{s}\,dt = -h_s(z)$$

Put $$H_n(z) = e^{z^2/2}h_{-n}(z) = \frac{\Gamma(1+n)}{2\pi i\, e^{-i\pi n}}\oint e^{-t^2-2tz}\,t^s\frac{dt}{t}$$

$$= \frac{n!}{2\pi i}\oint e^{-t^2+2tz}\,t^s\frac{dt}{t}$$

so that $$e^{-t^2+2tz} = \sum_{n\geq 0} H_n(z)\frac{t^n}{n!}$$

One has
$$\begin{cases} \dfrac{d}{dz} H_n(z) = 2n\, H_{n-1}(z) \\ \left(\dfrac{d}{dz} - 2z\right) H_n(z) = -H_{n+1}(z) \end{cases}$$

hence
$$2n\, H_{n-1}(z) + H_{n+1}(z) = 2z\, H_n(z)$$

July 5, 1977:

So let's now consider the difference equation

$$y_{n+1} - 2z y_n + 2n y_{n-1} = 0$$

and try to figure out the different ways of solving it. Example:

$$y_n = \int_a^b t^n \phi(t)\,dt$$

$$n y_{n-1} = \int_a^b n t^{n-1} \phi(t)\,dt = \left[t^n \phi(t)\right]_a^b - \int_a^b t^n \phi'(t)\,dt$$

So we get a solution if $t^n \phi(t)$ vanishes at the endpts of the path of integration and if

$$t\phi(t) - 2z\phi(t) - 2\phi'(t) = 0$$

or
$$\phi'(t) = \left(\frac{t}{2} - z\right)\phi(t)$$

or
$$\phi(t) = e^{t^2/4 - zt}$$

So we see that

$$y_n = \int_{P} e^{t^2/4 - zt}\, t^n \, dt$$

satisfies the difference equation provided P is a path ending at $+i\infty$ or $-i\infty$. Also if $n > 0$ we could have 0 as an endpoint.

Another possibility would be to replace y_n by

$$u_n = \frac{y_n}{\Gamma(n+1)}$$

whence the difference equation becomes

$$(n+1)\,\frac{\Gamma(n+2)}{\Gamma(n+1)}\,\frac{y_{n+1}}{\Gamma(n+2)} - 2z\,\frac{y_n}{\Gamma(n+1)} + 2n\,\frac{y_{n-1}}{\cancel{\Gamma(n+1)}\;\Gamma(n)} = 0$$

or

$$n u_n - 2z u_{n-1} + 2u_{n-2} = 0$$

Now you try

$$u_n = \int t^{n-1}\phi(t)\,dt$$

$$n u_n = \int n t^{n-1}\phi(t)\,dt = \left[t^n\phi(t)\right]_a^b - \int t^n \phi'(t)\,dt$$

$$-t^2\phi'(t) - 2z\,\phi(t) + \frac{2}{t}\phi(t) = 0$$

$$\phi'(t) = \left[-\frac{2z}{t^2} + \frac{2}{t^3}\right]\phi(t)$$

$$\phi(t) = e^{\frac{2z}{t} - \frac{1}{t^2}}$$

So a better choice of integral would be

$$u_n = \int \phi(t)\, t^{-n} \frac{dt}{t}$$

and you would get $\quad u_n = \int e^{-t^2 + 2zt}\, t^{-n} \frac{dt}{t}$

So the two possible integral forms for the solution are:

$$y_n = \int e^{t^2/4 - zt}\, t^n\, dt$$

$$y_n = \Gamma(n+1) \int e^{-t^2 + 2zt}\, t^{-n} \frac{dt}{t}$$

July 7, 1977

It seems desirable to collect a zoo of examples of difference equations. For example what is the J-matrix belonging to the Poisson distribution: $\sum e^{-\lambda}\frac{\lambda^k}{k!}\delta(x-k)$?

Answer given in the book of G. Szegö, Orthogonal Polys. AMS Colloquium Pub 33 (1939).

First let's establish the orthogonality of Hermite polys using the generating function

$$\sum H_n(x)\frac{t^n}{n!} = e^{-t^2+2xt}$$

[redacted] $$\sum_{n,m\geq 0}\left(\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx\right)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$= \int e^{-t^2+2xt}e^{-u^2+2xu}e^{-x^2}dx$$

$$= \int e^{-x^2+2x(t+u)-(t^2+2tu+u^2)+2tu}dx = e^{2tu}\int e^{-(x-t-u)^2}dx$$

$$= \sqrt{\pi}\,e^{2tu} = \sum_{n\geq 0}\frac{\sqrt{\pi}\,2^n}{n!}t^n u^n$$

Hence

$$\int H_n(x)H_m(x)e^{-x^2}dx = \delta_{nm}\sqrt{\pi}\,2^n n!$$

This example shows how one can deduce orthogonality from the generating function. The generating function

is a function of two variables $\psi(x,t)$ such that
$$\int \psi(x,t)\,\psi(x,u)\,d\mu(x)$$
is a function of tu. I don't see a simple procedure for finding ψ

Poisson distribution

$$\int \psi(x,t)\psi(x,u)\,d\mu(x) = \sum_{k\geq 0} \psi(k,t)\psi(k,u)\frac{\lambda^{+k}}{k!}e^{-\lambda}$$

Try $\psi(x,t) = e^{a(t)+x\,b(t)}$ in order to be able to sum this.

[redacted]

$$e^{-\lambda + a(t) + a(u)} \sum_{k\geq 0} \frac{e^{k(b(t)+b(u))}\lambda^{+k}}{k!}$$

$$= e^{-\lambda + a(t) + a(u)}\, e^{+\lambda e^{b(t)+b(u)}}$$

Anyway what works is $e^{b(t)} = 1+t$ [redacted] and $a(t) = -\lambda t$ because

$$-\lambda - \lambda t - \lambda u + \lambda(1+t)(1+u) = \lambda tu$$

So we have
$$\psi(x,t) = e^{-\lambda t}(1+t)^x$$

and
$$\sum_{k\geq 0} \psi(x,t)\psi(x,u)\frac{\lambda^k}{k!}e^{-\lambda} = e^{\lambda tu}$$

Another possibility is $\psi(x,t) = e^{-\lambda\varepsilon t}(1+\varepsilon t)^x$ which would lead to $e^{\lambda tu}$ being replaced by $e^{\lambda\varepsilon^2 tu}$.

With $\psi(x;t) = e^{-\lambda t}(1+t)^x$ one gets the polys:

$$\sum p_n(x)\frac{t^n}{n!} = \sum \frac{(-\lambda)^n t^n}{n!} \sum \binom{x}{n} t^n$$

or

$$p_n(x) = n! \sum_{j=0}^{n} \binom{x}{j} \frac{(-\lambda)^{n-j}}{(n-j)!} = x^n + \cdots\cdots$$

Recursion formulas: Diff wrt t:

$$\sum p_n(x)\frac{t^n}{n!} = e^{-\lambda t}(1+t)^x$$

$$\sum_{n\geq 1} p_n(x)\frac{t^{n-1}}{(n-1)!} = -\lambda \sum p_n(x)\frac{t^n}{n!} + x\sum p_n(x-1)\frac{t^n}{n!}$$

$$\boxed{p_{n+1}(x) = -\lambda\, p_n(x) + x\, p_n(x-1)}$$

$$\sum p_n(x)\frac{t^n}{n!} = (1+t)\sum_{n\geq 0} p_n(x-1)\frac{t^n}{n!}$$

$$= \sum_{n\geq 0} p_n(x-1)\frac{t^n}{n!} + \underbrace{\sum_{n\geq 0} p_n(x-1)\frac{t^{n+1}}{(n+1)!}(n+1)}_{\sum_{n\geq 0} p_{n-1}(x-1)\frac{t^n}{n!}\, n}$$

$$\boxed{p_n(x) = p_n(x-1) + n\, p_{n-1}(x-1)}$$

$$p_n(x-1) = \frac{1}{x}\left(p_{n+1}(x) + \lambda p_n(x)\right)$$

$$p_n(x) = \frac{1}{x}\left(p_{n+1}(x) + \lambda p_n(x)\right) + n\frac{1}{x}\left(p_n(x) + \lambda p_{n-1}(x)\right)$$

$$\boxed{x p_n(x) = p_{n+1}(x) + (\lambda + n) p_n(x) + \lambda n p_{n-1}(x)}$$

Do the same thing for the Bernoulli distribution

$$\sum \binom{n}{k} p^k (1-p)^{n-k} \, \delta(x-k)$$

This time we want

$$\sum_{k=0}^{n} \psi(k,t)\psi(k,u)\binom{n}{k} p^k (1-p)^{n-k} = \text{function of } tu$$

so we try $\psi(x,t) = a(t)\, b(t)^x$. Then

$$\sum a(t)\, a(u) \binom{n}{k} \left(b(t) b(u) p\right)^k (1-p)^{n-k}$$

$$= a(t)\, a(u) \left((1-p) + p\, b(t)\, b(u)\right)^n$$

Try $b(t) = \dfrac{1+(1-p)t}{1-pt}$. Then

$$1-p+p\,\frac{1+(1-p)t}{1-pt}\,\frac{1+(1-p)u}{1-pu} = \frac{q[1-pt-pu+p^2tu] + p[1+qt+qu+q^2tu]}{(1-pt)(1-pu)}$$

$$= \frac{q+p+(p+q)pq\,tu}{(1-pt)(1-pu)} = \frac{1+pq\,tu}{(1-pt)(1-pu)}$$

where $q = 1-p$.

So if we take $a(t) = (1-pt)^n$, then

$$\psi(x,t) = (1-pt)^n \left(\frac{1+qt}{1-pt}\right)^x = (1-pt)^{n-x}(1+qt)^x$$

and we find

$$\sum \psi(k,t)\psi(k,u)\binom{n}{k} p^k(1-p)^{n-k} = \sum \binom{n}{k}\big(p(1+qt)(1+qu)\big)^k\big(q(1-pt)(1-pu)\big)^{n-k}$$

$$= \big(p(1+qt+qu+q^2tu) + q(1-pt-pu+p^2tu)\big)^n$$

$$= (1+pq\,tu)^n$$

as it should be.

Laguerre polys: Here $d\mu = e^{-x}dx$, so orthogonality becomes

$$\int_0^\infty \psi(x,t)\,\psi(x,u)\,e^{-x}dx = \text{fn of } tu$$

So try $\psi(x,t) = a(t)\,e^{-x\,b(t)}$ because then

$$\int_0^\infty a(t)a(u)\,e^{-x(1+b(t)+b(u))}dx = \frac{a(t)\,a(u)}{1+b(t)+b(u)} = f(tu)$$

Now we expect $a(t) = 1+O(t)$, $b(t) = t + O(t^2)$ so that the coefficient of t^n in $\psi(x,t)$ is a monic poly of degree n.) times $\frac{(-1)^n}{n!}$ Putting $u=0$ we get

$$\frac{a(t)}{1+b(t)} = 1$$

so

$$\frac{(1+b(t))(1+b(u))}{1+b(t)+b(u)} = 1 + \frac{b(t)\,b(u)}{1+b(t)+b(u)} = f(tu)$$

$$\frac{1+b(t)+b(u)}{b(t)b(u)} = g(tu)$$

$$\left(\frac{1+b(t)}{b(t)}\right)\left(\frac{1+b(u)}{b(u)}\right) = 1+g(tu)$$

(★) $$\left(\frac{t}{b(t)}+t\right)\left(\frac{u}{b(u)}+u\right) = \text{function of } tu$$

But $$\frac{t}{b(t)} = 1 + h_1 t + h_2 t^2 + \cdots$$

$$\left(1+(h_1+1)t+\cdots\right)\left(1+(h_2+1)u+\cdots\right) = \text{function of } tu$$

Better put $u=0$ in and you find $\frac{t}{b(t)}+t = \text{const.}$
So

$$\frac{t}{b(t)}+t = 1 \qquad \frac{1}{b(t)} = \frac{1}{t}-1 \qquad b(t) = \frac{t}{1-t}$$

and $$a(t) = 1+\frac{t}{1-t} = \frac{1}{1-t}\text{, so}$$

$$\psi(x,t) = \frac{1}{1-t}e^{-x(t/1-t)}$$

Multiple derivative method for orthogonal polys on the interval $0 \leq x < \infty$. Let $d\mu = p(x)\,dx$. Then assuming things behave at the ends

$$\int_0^\infty x^k \left(p^{-1}\frac{d^n}{dx^n}g(x)\right) p\,dx = \int_0^\infty \left(-\frac{d}{dx}\right)^n (x^k)\, g\,dx = 0$$

if $k<n$, hence if $p^{-1}\frac{d^n}{dx^n}g$ is a poly of degree n it is the orthogonal poly of degree n we seek. For example we take $g(x) = p(x)x^n$, then

$$p^{-1}\frac{d^n}{dx^n}(px^n) = \left(\frac{d}{dx}-\frac{p'}{p}\right)^n x^n$$

will be a poly of degree n when $\frac{p'}{p} = a + \frac{b}{x}$

i.e. $p = e^{ax}x^{b}$. Suppose $a = -1$ and let $b = \alpha$ > -1. Then we get the Laguerre poly except for sign

$$L_n^{\alpha}(x) = \cancel{\frac{(-1)^n}{n!}} \left(e^{-x}x^{\alpha}\right)^{-1} \frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$$

$$= \cancel{\frac{(-1)^n}{n!}} \left(\frac{d}{dx} - 1 + \frac{\alpha}{x}\right)^n x^n$$

$$= \cancel{\frac{(-1)^n}{n!}}\, x \left(\frac{d}{dx} - 1 + \frac{\alpha+1}{x}\right)^n x^{n-1}$$

$$= \cancel{\frac{(-1)^n}{n!}}\, x^n \left(\frac{d}{dx} - 1 + \frac{\alpha+n}{x}\right)^n 1$$

To establish recursion formulas use

$$\frac{d^n}{dx^n} \cdot x = x\frac{d^n}{dx^n} + n\frac{d^{n-1}}{dx^{n-1}}$$

because $\left[\frac{d}{dx}, x\right] = 1$ which commutes. So

$$L_{n+1}^{\alpha}(x) = \cancel{\frac{(-1)^{n+1}}{(n+1)!}} \left[x\left(\frac{d}{dx} - 1 + \frac{\alpha}{x}\right)^{n+1} x^n + (n+1)\left(\frac{d}{dx} - 1 + \frac{\alpha}{x}\right)^n x^n \right]$$

~~$= \frac{-1}{n+1} x\left(\frac{d}{dx} - 1 + \frac{\alpha}{x}\right) L_n^{\alpha}(x) + (-1) L_n^{\alpha}(x)$~~

July 8, 1977

To find recursion formula for Laguerre polys.

$$L_n(x) = e^x \frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \left(\frac{d}{dx}-1\right)^n x^n$$

$$L_{n+1}(x) = \left(\frac{d}{dx}-1\right)^{n+1} x \cdot x^{n+1} = x\left(\frac{d}{dx}-1\right)^{n+1} x^n + (n+1)\left(\frac{d}{dx}-1\right)^n x^n$$

$$= x\left(\frac{d}{dx}-1\right)^n \left(n x^{n-1} \cancel{}\right) + (n+1)L_n - xL_n$$

$$= n\left[\left(\frac{d}{dx}-1\right)^n x^n - n\left(\frac{d}{dx}-1\right)^{n-1} x^{n-1}\right] + (n+1-x)L_n$$

$$= (2n+1-x)L_n - n^2 L_{n-1}$$

So the recursion formula is

$$L_{n+1} - (2n+1-x)L_n + n^2 L_{n-1} = 0$$

or

$$L_{n+1} - (2n+1)L_n + n^2 L_{n-1} = -xL_n$$

so if I put $y_n = (-1)^n L_n$ so as to make $(-1)^n L_n(x)$ monic I get the recursion relation

$$\boxed{y_{n+1} + (2n+1)y_n + n^2 y_{n-1} = x y_n}$$

or

$$\boxed{(n+1)\left(\frac{y_{n+1}}{(n+1)!}\right) + (2n+1)\left(\frac{y_n}{n!}\right) + n\,\frac{y_{n-1}}{(n-1)!} = x\,\frac{y_n}{(n)!}}$$

which belongs to the J-matrix

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & 3 & 2 & & \\ & 2 & 5 & & \\ & & & \ddots & n \\ & & & n & 2n+1 \end{pmatrix}$$

$$L_n^\alpha = e^x x^{-\alpha} \frac{d^n}{dx^n}\left(e^{-x} x^{\alpha+n}\right) = \left(\frac{d}{dx} + \frac{\alpha}{x} - 1\right)^n x^n$$

$$L_{n+1}^\alpha = \left(\frac{d}{dx}+\frac{\alpha}{x}-1\right)^{n+1} x^{n+1} = x\left(\frac{d}{dx}+\frac{\alpha}{x}-1\right)^{n+1} x^n + (n+1)\left(\frac{d}{dx}+\frac{\alpha}{x}-1\right)^n x^n$$

$$= x\left(\frac{d}{dx}+\frac{\alpha}{x}-1\right)^n \left((n+\alpha)\, x^{n-1}\right) + (n+1)L_n^\alpha - xL_n^\alpha$$

$$= (n+\alpha)\left[\left(\frac{d}{dx}+\frac{\alpha}{x}-1\right)^n x^n - n\left(\frac{d}{dx}+\frac{\alpha}{x}-1\right)^{n-1} x^{n-1}\right] + (n+1-x)L_n^\alpha$$

$$= (2n+1+\alpha-x)L_n^\alpha - n(n+\alpha)L_{n-1}^\alpha$$

So the recursion formula is

$$L_{n+1}^\alpha - (2n+1+\alpha)L_n^\alpha + n(n+\alpha)L_{n-1}^\alpha = -xL_n^\alpha$$

In monic poly form it is

$$\boxed{y_{n+1} + (2n+1+\alpha)y_n + n(n+\alpha)y_{n-1} = xy_n}$$

Putting $\tilde{y}_n = \dfrac{y_n}{n!}$ we get

$$(n+1)\tilde{y}_{n+1} + (2n+1+\alpha)\tilde{y}_n + (n+\alpha)\tilde{y}_{n-1} = x\tilde{y}_n$$

and if we put $u_n = \dfrac{y_n}{\Gamma(n+\alpha+1)}$, then

$$\frac{(n+\alpha+1)\, y_{n+1}}{(n+\alpha+1)\Gamma(n+\alpha+1)} + (2n+1+\alpha)\frac{y_n}{\Gamma(n+\alpha+1)} + n(n+\alpha)\frac{y_{n-1}}{(n+\alpha)\Gamma(n+\alpha)} = x\frac{y_n}{\Gamma(n+\alpha+1)}$$

or

$$\boxed{(n+\alpha+1)u_{n+1} + (2n+1+\alpha)u_n + nu_{n-1} = xu_n}$$

Return to Poisson's measure: $\sum e^{-\lambda} \frac{\lambda^k}{k!} \delta_k$.

More generally consider a measure supported on the integer with masses $p(k)$. Then for $m<n$

$$0 = \sum_k \Delta^n(x^m)\, q(k) = \sum_k k^m \left((\Delta^*)^n q\right)(k)$$

$$= \sum_k k^m \left(p^{-1}(\Delta^*)^n q\right)(k)\, p(k).$$

Hence if $p^{-1}(\Delta^*)^n q$ is a poly of degree n, it is the orthogonal poly. For example if $p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, then

$$p^{-1}(\Delta^*)^n p(k) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{p(k-j)}{p(k)}$$

$$= \sum_{j=0}^{n} \frac{n!}{(n-j)!\, j!} (-1)^j \frac{k!}{(k-j)!} \lambda^{-j}$$

$$= \frac{n!}{(-\lambda)^n} \sum_{j=0}^{n} \frac{(-\lambda)^{n-j}}{(n-j)!} \binom{k}{j}$$

which is what we found on page 143.

Note for purposes of generalization that

$$\Delta = e^D - 1 \qquad D = \frac{d}{dx}$$

and in general any power series with 0 constant term

$$c_1 D + c_2 D^2 + \cdots\cdots$$

will give an operator on polynomials which decreases degree by 1, hence is nilpotent.

Start looking at difference equations first order & linear.

$$u(x+1) = p(x)\,u(x)$$

$$\log u(x+1) - \log u(x) = \log(p(x))$$

is solved by some sort of summation process to $\log(p(x))$. Specific examples:

$$u(x+1) = x\,u(x)$$

satisfied by $\Gamma(x)g(x)$ where g is an arbitrary periodic function. Notice that

$$\Gamma(x)\ \frac{\sin \pi x}{\pi} = \frac{1}{\Gamma(1-x)}$$

also is a solution which decays as $x \to -\infty$. Notice that that if we remove the disks $|x+n| < \varepsilon$, $n \in \mathbb{Z}$ then $\Gamma(x)$ also decays as $x \to +\infty$. Thus provided we stay away from the poles, we see that any solution of this difference equation decays at $x \to -\infty$ and blows up as $x \to +\infty$.

Put $x = \frac{1}{2} + it$, so $1 - x = \frac{1}{2} - it = \overline{\frac{1}{2} + it}$ and hence $\Gamma(1-x) = \overline{\Gamma(x)}$ when t is real. Thus

$$\left|\Gamma\left(\tfrac{1}{2}+it\right)\right|^2 = \frac{\pi}{\sin \frac{\pi}{2}(1+2it)} = \frac{\pi}{\cos 2\frac{\pi}{2}it} = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}$$

decays exponentially at $t \to \infty$

$$\Gamma\left(\tfrac{1}{2}+it\right) \sim \sqrt{2\pi}\, e^{-\frac{\pi}{2}t} \qquad t \to +\infty$$

So $\dfrac{1}{\Gamma\left(\frac{1}{2}+it\right)}$ looks like a facinating candidate for p, except for the fact it isn't even.

Actually we have for $a > 0$

$$\Gamma(a+i\lambda) = \int_0^\infty e^{-t} t^{a+i\lambda} \frac{dt}{t} = \int_{-\infty}^{\infty} \underbrace{e^{-e^u} e^{au}} e^{i\lambda u}\, du$$

The function $e^{-e^u} e^{au}$ is C^∞ and decays very fast as $u \to +\infty$ and like e^{au} as $u \to -\infty$, so it is rapidly-decreasing. It follows that $\Gamma(a+i\lambda)$ is a rapidly decreasing function of λ, which means it decays faster than λ^{-n} for any n. I conjecture it is asymptotic to $e^{-\pi a \lambda}$. NO $e^{-\lambda\frac{\pi}{2}}$ for all a.

Other observations:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt = \int_0^1 \sum_0^\infty \frac{(-t)^n}{n!} t^s \frac{dt}{t} + \underbrace{\int_1^\infty e^{-t} t^{s-1} dt}_{\text{entire function}}$$

$$= \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{1}{s+n} + \text{entire fn.}$$

gives the analytic continuation + residues at the poles instantly.

Stirling's approx. $\log \Gamma(z) \sim (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + O(\frac{1}{z})$

gives $\log \Gamma(a+i\lambda) \sim (a+i\lambda-\frac{1}{2}) \log(a+i\lambda) - (a+i\lambda) + O(1)$

$$\sim -\lambda \frac{\pi}{2}$$

so $\Gamma(a+i\lambda) \sim e^{-|\lambda|\frac{\pi}{2}} \cdot \lambda^{a-\frac{1}{2}} \cdot$ other smaller junk

as $|\lambda| \to +\infty$.

July 9, 1977:

Consider 2nd order difference equations

$$a(s+1)\, u_{s+1} + b(s)\, u_s + c(s-1) u_{s-1} = 0$$

where a, b, c are linear functions of s, say $a(s) = a_0 s + a_1$ etc. The Wronskian of two solutions satisfies

$$a(s+1) W(s) = \begin{vmatrix} u_s & v_s \\ a(s+1) u_{s+1} & a(s+1) v_{s+1} \end{vmatrix} = \begin{vmatrix} u_s & v_s \\ -c(s-1) u_{s-1} & -c(s-1) v_{s-1} \end{vmatrix}$$

$$= c(s-1)\, W(s-1)$$

hence $W(s) = \dfrac{c(s-1)}{a(s+1)} W(s-1)$. This means that up to a periodic function, W can be expressed in terms of Γ-functions. If the diff. eq is in the J-form i.e. $c(s-1) = a(s)$, then

$$a(s+1) W(s) = a(s) W(s-1)$$

is a periodic function.

Solve the ΔE by Laplace (or Mellin) transform:

$$u_s = \int \phi(t)\, t^{s-1}\, dt$$

shifting corresponding to multiplying ϕ by t

$$u_{s+1} = \int t\phi(t)\ t^{s-1} dt$$

and multiplying by s corresponds to $-t\frac{d}{dt}$

$$s u_s = \int \phi(t) \frac{d}{dt}(t^s) dt = -\int t \frac{d\phi}{dt}\, t^{s-1} dt$$

So the transform of

$$(a_0(s+1)+a_1)\,u_{s+1} + (b_0 s+b_1)\,u_s + (c_0(s-1)+c_1)\,u_{s-1} = 0$$

is

$$t\left(-a_0 t\frac{d}{dt}+a_1\right)\phi + \left(-b_0 t\frac{d}{dt}+b_1\right)\phi + t^{-1}\left(-c_0 t\frac{d}{dt}+c_1\right)\phi = 0$$

or

$$(a_0 t^2+b_0 t+c_0)\frac{d\phi}{dt} = (a_1 t+b_1+c_1 t^{-1})\,\phi$$

or

$$\frac{1}{\phi}\frac{d\phi}{dt} = \frac{1}{t}\frac{(a_1 t^2+b_1 t+c_1)}{(a_0 t^2+b_0 t+c_0)}$$

Now the actual form of ϕ will depend on the partial fraction decomposition of the rational function on the right. The generic case occurs when the roots of $a_0 t^2+b_0 t+c_0$ are distinct and $\neq 0$.

Notice that making a change of the form $u_s = h^s y_s$ where h is a constant gives the equation

$$h\,a(s+1)\,y_{s+1} + b(s)\,y_s + h^{-1}c(s-1)\,y_{s-1} = 0.$$

Hence we can arrange $a_0 = c_0 = 1$ when these are both $\neq 0$. Now if we translate s we can suppose that $c_1 = 0$, so

$$\frac{1}{\phi}\frac{d\phi}{dt} = \frac{a_1 t+b_1}{t^2+b_0 t+1} = \frac{\beta_1}{t-\alpha_1} + \frac{\beta_2}{t-\alpha_2}$$

$$\phi = (t-\alpha_1)^{\beta_1}(t-\alpha_2)^{\beta_2}$$

so

$$u_s = \int (t-\alpha_1)^{\beta_1}(t-\alpha_2)^{\beta_2}\,t^{s-1}\,dt.$$

This isn't so nice in the Jacobi-matrix-with-λ case because then the exponents β_1, β_2 depend on λ. Instead let us shift s so as to cancel one of the other poles in the denominator. Not possible because shift s by h changes $a_1t^2 + b_1t + c_1$ to $a_1t^2 + b_1t + c_1 + h(a_0t^2 + b_0t + c_0)$ which doesn't change the values at the roots of the denominator.

$$F(a,b,c;x) = 1 + \frac{ab}{c \cdot 1!}x + \frac{a(a+1)\,b(b+1)}{c(c+1)\;2!}x^2 + \cdots$$

$$\Gamma(c-a)\frac{\Gamma(a)}{\Gamma(c)}F(a,b,c;x) = \sum_{n\geq 0} \frac{\Gamma(c-a)\Gamma(a+n)}{\Gamma(c+n)} \; \frac{b(b+1)\cdots(b+n-1)}{n!}x^n$$

$$= \sum_{n\geq 0} \int_0^1 t^{c-a-1}(1-t)^{a+n-1}\,dt \; \frac{(-b)(-b-1)\cdots(-b-n+1)}{n!}(-x)^n$$

$$= \int_0^1 t^{c-a-1}(1-t)^{a-1} \sum_{n\geq 0} \binom{-b}{n} \left((1-t)(-x)\right)^n$$

$$= \int_0^1 t^{c-a-1}(1-t)^{a-1}\left(1 - x(1-t)\right)^{-b}\,dt$$

Hence

$$F(a,b,c;1) = \frac{\Gamma(c)}{\Gamma(c-a)\,\Gamma(a)} \int_0^1 t^{c-a-b-1}(1-t)^{a-1}\,dt \;\; \frac{\Gamma(c-a-b)\Gamma(a)}{\Gamma(c-b)}$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\;\Gamma(c-b)}$$

July 11, 1977

Review the way $\mathrm{Todd}(x) = \frac{x}{1-e^{-x}}$ occurs in topology.

Suppose h is a cohomology theory (multiplicative) such that complex vector bundles are oriented with respect to h, whence one gets Gysin maps $f: h(X) \to h(Y)$ when f is proper & complex-oriented map of manifolds. One has a formal group law over $h(pt)$ describing behavior of Euler classes of line bundles under $\otimes$

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2))$$

If $f: \mathbb{P}E \to X$ is a projective bundle where $E = L_1 + \cdots + L_n$ and $\xi = e(\mathcal{O}(1)) \in h(\mathbb{P}E)$, $x_i = e(L_i) \in h(X)$, one has

$$f_*(a(\xi)) = \operatorname{res} \frac{a(T)\,\omega}{\prod_{1 \le i \le n} F(x_i, T)} \qquad a(T) \in h(X)[[T]]$$

where ω is the invariant differential for F, i.e.

$$\omega = d\ell(T)$$

where ℓ is the logarithm: $\ell(F(X,Y)) = \ell(X) + \ell(Y)$, $\ell(X) = X + \cdots$. This residue formula applied to the case of $\mathbb{CP}^{n-1} \to pt$, shows that

$$[\mathbb{CP}^{n-1}] = \operatorname{res} \frac{\omega}{T^n}$$

hence
$$\omega = \sum_{n \ge 0} [\mathbb{CP}^n] T^n dT$$

$$\ell(T) = \sum_{n\geq 0} [\mathbb{C}P^n] \frac{T^{n+1}}{n+1}$$

For K-theory if $i: Y \longrightarrow X$ is the divisor associated to a transversal section of a line bundle L over X, then

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

so $\qquad i_* 1 = 1 - L^{-1}$

so $\qquad e(i^*L) = i^* i_* 1 = 1 - (i^*L)^{-1}$

Thus in general $\quad e(L) = 1 - L^{-1} \quad$ for line bundles.

So

$$1 - e(L_1 \otimes L_2) = (1 - e(L_1))(1 - e(L_2))$$

and $\qquad 1 - F(X,Y) = (1-X)(1-Y) \qquad$ or

$$F(X,Y) = X + Y - XY$$

The logarithm is $\quad \ell(X) = -\log(1-X) = \sum_{n\geq 0} \frac{X^{n+1}}{n+1}$

hence $[\mathbb{C}P^n] = 1$ in K-theory for all n.

RR: $\quad ch: K(X) \longrightarrow H(X) \quad$ not compatible with Gysin, so it is necessary to twist the Gysin f_* on $H(X)$:

$$f_!(\alpha) = f_*(\varphi(\nu_f)\alpha)$$

where φ is an exponential class with values in H. φ is to be chosen so as to be compatible with $e(L)$:

$$\begin{array}{c} ch(1 - L^{-1}) = \varphi(L) c_1(L) \\ \| \\ 1 - e^{-c_1(L)} \end{array}$$

so
$$\varphi(L) = \frac{1-e^{-c_1(L)}}{c_1(L)}$$

Thus
$$\varphi(\nu_f) = \mathrm{Todd}(\tau_f)\ , \text{ where } \mathrm{Todd}(x) = \frac{x}{1-e^{-x}}$$

For $\mathbb{CP}^n$:

$$1 = \operatorname{res} \frac{d\ell(T)}{T^{n+1}} = \operatorname{res} \frac{dT}{(\ell^{-1}(T))^{n+1}}$$

$$= \operatorname{res} \frac{dT}{(1-e^{-T})^{n+1}} = \operatorname{res}\left\{\left(\frac{T}{1-e^{-T}}\right)^{n+1} \frac{dT}{T^{n+1}}\right\}$$

The same formula results if we use
$$\mathrm{Todd}(\tau_{\mathbb{CP}^n}) = \mathrm{Todd}\left((n+1)\mathcal{O}(1)/\mathcal{O}\right) = \left(\frac{\xi}{1-e^{-\xi}}\right)^{n+1} \qquad \xi = c_1(\mathcal{O}(1))$$

Thus $\frac{x}{1-e^{-x}}$ is the unique series such that when raised to the $(n+1)$-th power, the nth coeff. is 1, for all n.

Signature homomorphism: $U^*(pt) \xrightarrow{\sigma} \mathbb{Z}$ sends $\mathbb{CP}^n$ to 1 n even, 0 if n odd. The group law belonging to this hom. is given by
$$d\ell(T) = \sum_{n\geq 0} T^{2n} = \frac{1}{1-T^2} = \frac{1}{2}\left(\frac{1}{1-T} + \frac{1}{1+T}\right)$$

$$\ell(T) = \frac{1}{2}\log\left(\frac{1+T}{1-T}\right) = u$$

$$\frac{1+T}{1-T} = e^{2u} \qquad 1+T = e^{2u} - (e^{2u})T$$

$$\ell^{-1}(u) = T = \frac{e^{2u}-1}{e^{2u}+1} = \frac{e^{u}-e^{-u}}{e^{u}+e^{-u}} = \tanh(u)$$

so

$$\text{signature}(\mathbb{CP}^n) = \text{res}\, \frac{d\ell(T)}{T^{n+1}} = \text{res}\, \frac{dT}{(\ell^{-1}(T))^{n+1}}$$

$$= \text{res} \left(\frac{e^T + e^{-T}}{e^T - e^{-T}}\, T \right)^{n+1} \frac{dT}{T^{n+1}}$$

which gives the Hirzebruch signature thm with $\mathcal{L}$-genus

$$\mathcal{L}(T) = \frac{e^T + e^{-T}}{e^T - e^{-T}}\, T = T \coth(T).$$

Next topic should be the ~~[illegible]~~ order of $\operatorname{Im} J$, the conjectured value for the torsion in $K_*(\mathbb{Z})$.

Question: Do there exist interesting discrete analogues of the system

$$\frac{du}{dx} = \begin{pmatrix} i(\lambda+b) & \rho \\ \rho & -i(\lambda+b) \end{pmatrix} u$$

where b, ρ are real functions of x. I might try

$$u(x+1) = \begin{pmatrix} e^{i(a(x)\lambda + b(x))} & 0 \\ 0 & e^{-i(a(x)\lambda + b(x))} \end{pmatrix} \begin{pmatrix} \cosh \rho(x) & \sinh \rho(x) \\ \sinh \rho(x) & \cosh \rho(x) \end{pmatrix} u(x)$$

where a, b, ρ are functions of x with $a(x) \geq 0$. A reason for looking at this is the formula:

$$\exp \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} = \begin{pmatrix} \cosh \rho & \sinh \rho \\ \sinh \rho & \cosh \rho \end{pmatrix}$$

(evident from power series).

Put [redacted] $\psi_c(x) = 1 + \frac{x}{c\cdot 1} + \frac{x^2}{c(c+1)\,2!} + \cdots$

$$= \sum \frac{x^n}{c(c+1)\cdots(c+n-1)\,n!}$$

$$\frac{d}{dx}\psi_c(x) = \sum \frac{x^{n-1}}{(n-1)!}\,\frac{1}{c(c+1)\cdots\underset{c+1+n-1-1}{\underset{\shortparallel}{(c+n-1)}}}$$

$$= \frac{1}{c}\psi_{c+1}(x)$$

$$\frac{d}{dx}\left(x^{c-1}\psi_c(x)\right) = \sum \frac{1}{c(c+1)\cdots(c+n-1)\,n!}\,\frac{d}{dx}\left(x^{c+n-1}\right)$$

$$= (c-1)\,x^{c-2} + \sum_{n\geq 1} \frac{x^{c+n-2}}{c(c+1)\cdots(c+n-2)\,n!}$$

$$= (c-1)\,x^{c-2}\left(1 + \sum_{n\geq 1} \frac{x^n}{(c-1)c\cdots\underset{c-1+n-1}{\underset{\shortparallel}{(c+n-2)}}\,n!}\right)$$

$$= (c-1)\,x^{c-2}\,\psi_{c-1}$$

So

$$\left(\frac{d}{dx} + \frac{c-1}{x}\right)\psi_c = \text{[redacted]}\ \frac{c-1}{x}\psi_{c-1}$$

and

$$\frac{1}{c}\psi_{c+1} + \frac{c-1}{x}\psi_c = \text{[redacted]}\ \frac{c-1}{x}\psi_{c-1}$$

$$\frac{x}{(c-1)c}\psi_{c+1} + \psi_c = \psi_{c-1}$$

so [redacted]

$$\frac{\psi_c}{\psi_{c+1}} = 1 + \cfrac{\frac{x}{c(c+1)}}{1 + \cfrac{\frac{x}{(c+1)(c+2)}}{1 + \cdots}}$$

See Perron Kettenbrüche p. 313. Incidentally ψ_c is essentially $J_c(x)$ up to $\frac{x^2}{2}$ nonsense.

July 12, 1977

Let $$A_n(\lambda) = \begin{pmatrix} e^{i(a_n\lambda+b_n)} & 0 \\ 0 & e^{-i(a_n\lambda+b_n)} \end{pmatrix} \begin{pmatrix} \cosh c_n & \sinh c_n \\ \sinh c_n & \cosh c_n \end{pmatrix}$$

where a_n, b_n, c_n are real, $a_n > 0$. (The matrix ↑ is an arbitrary real symmetric matrix of determinant 1 with pos. first entry – changing b_n by π we can allways arrange the first entry positive.) The matrix $A_n(\lambda)$ shrinks $|z| \leq 1$ when $\mathrm{Im}(\lambda) > 0$.

Consider the system

$$y_{n+1} = A_n y_n \qquad y_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

I hope that if $c_n \to \infty$ as $n \to +\infty$, then I can find a solution $y_n^+(\lambda)$ holom. in λ, non-vanishing, which decays as $n \to \infty$. One should have

$$\frac{u_0^+(\lambda)}{v_0^+(\lambda)} = \lim_{n\to\infty} A_0^{-1} \cdots A_{n-1}^{-1}(w)$$

where w is "generic". If $|w| = 1$, then $\frac{u_0^+}{v_0^+}(\lambda)$ is outside S^1 for $\mathrm{Im}(\lambda) > 0$ and inside for $\mathrm{Im}(\lambda) < 0$, so the equation

$$\frac{u_0^+}{v_0^+}(\lambda) = e^{i\theta} \qquad \theta \in \mathbb{R}$$

has only real roots.

If also $c_n \to \infty$ as $n \to -\infty$, there should be a solution $y_n^-(\lambda)$ decaying as $n \to -\infty$ given by

$$\frac{u_0^-}{v_0^-}(\lambda) = \lim_{n\to\infty} A_{-1} \cdots A_{-n}(w^-)$$

and for $|w| = 1$, $\frac{u_0^-}{v_0^-}(\lambda)$ is inside S^1 for $\mathrm{Im}(\lambda) > 0$. Hence the equation

$$\frac{u_0^-}{v_0^-}(\lambda) = \frac{u_0^+}{v_0^+}(\lambda)$$

has only real roots. But this equation is equivalent to the vanishing of the Wronskian

$$\begin{vmatrix} u_n^+(\lambda) & v_n^-(\lambda) \\ u_n^-(\lambda) & v_n^-(\lambda) \end{vmatrix}$$

which is independent of n.

The above suggests the matching condition

$$\frac{u_1^+}{u_2^+}(0,\lambda) = e^{i\theta} \frac{u_1^-(0,\lambda)}{u_2^-(0,\lambda)}$$

for solutions of $\frac{d}{dx} u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$. Recall that when p is odd, we showed that $\lambda = 0$ is an eigenvalue because the system is invariant under $x \mapsto -x$, $\lambda \mapsto -\lambda$, $u \mapsto u$, hence $u^+(-x,-\lambda) = u^-(x,\lambda)$, so $u^+(0,0)$ and $u^-(0,0)$ are linearly dependent. However if we use the matching condition

$$\begin{pmatrix} u_1^+(0,\lambda) \\ u_2^+(0,\lambda) \end{pmatrix} = \text{const} \begin{pmatrix} u_1^-(0,+\lambda) \\ -u_2^-(0,+\lambda) \end{pmatrix}$$

then this doesn't work. Instead it shows for p even that $\lambda = 0$ is an eigenvalue.

I want next to understand the recursion relations satisfied by hypergeometric series. So return to

$$y_c(x) = \sum{}' \frac{x^n}{c(c+1)\cdots(c+n-1)\, n!}$$

and note that

$$\frac{1}{c(c+1)\cdots(c+n-1)} - \frac{1}{(c+1)\cdots(c+n)} = \frac{1}{(c+1)\cdots(c+n-1)}\left(\frac{1}{c} - \frac{1}{c+n}\right)$$

$$= \frac{n}{c(c+1)\cdots(c+n-1)(c+n)}$$

hence

$$y_c(x) - y_{c+1}(x) = \frac{x}{c(c+1)} \sum{}' \frac{x^{n-1}}{(c+2)\cdots\underbrace{(c+n)}_{c+2+n-1-1}\,(n-1)!} = \frac{x}{c(c+1)}\, y_{c+2}(x)$$

Also if

$$u_a(x) = \sum_{n\geq 0} \frac{a(a+1)\cdots(a+n-1)}{n!} x^n$$

then

$$a(a+1)\cdots(a+n-1) - (a+1)\cdots(a+n) = \{(a+1)\cdots(a+n-1)\}(a-a-n)$$

so

$$u_a(x) - u_{a+1}(x) = -x \sum{}' \frac{(a+1)\cdots(a+n-1)}{(n-1)!} x^{n-1} = -x\, u_{a+1}(x)$$

so

$$u_a(x) = (1-x)\, u_{a+1}(x)$$

which agrees with

$$u_a(x) = \sum_{n\geq 0} \frac{(-a)(-a-1)\cdots(-a-n+1)}{n!}(-x)^n = (1-x)^{-a}$$

Next example:

$$F(a,c,x) = \sum \frac{a(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n-1)} \frac{x^n}{n!}$$

$$F(a,c,x) - F(a,c+1,x) = \sum \frac{a(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n)} \frac{x^n}{(n-1)!}$$

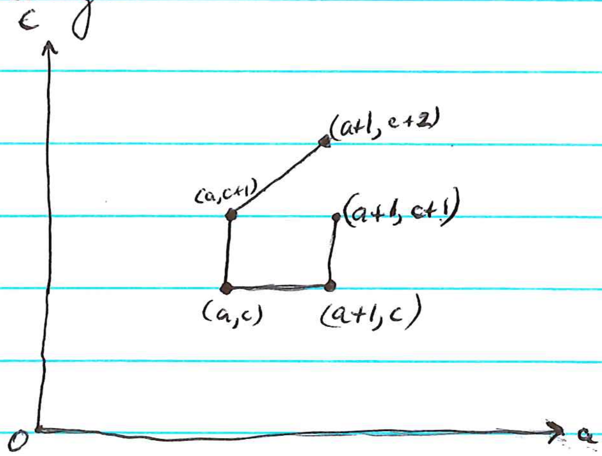
$$= \frac{ax}{c(c+1)} F(a+1, c+2, x)$$

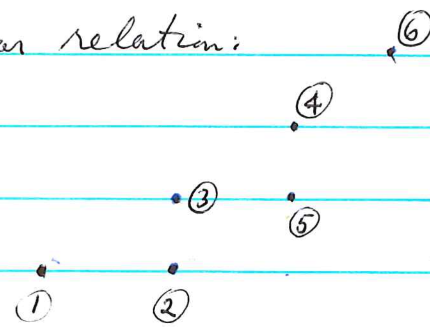
$$F(a,c,x) - F(a+1,c,x) = -\sum \frac{(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n-1)} \frac{x^n}{(n-1)!}$$

$$= -\frac{x}{c} F(a+1, c+1, x)$$

As a check put $x \mapsto \frac{x}{a}$ in the first and let $a \to \infty$; one gets the relation for y_c. Similarly put $x \mapsto cx$ and let $c \to \infty$; you get the relation for u_a.

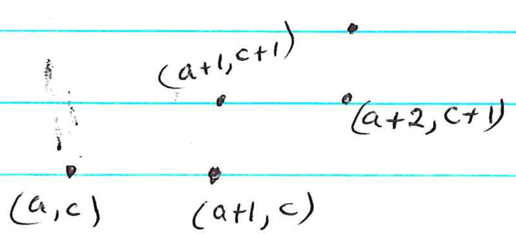
Visualize: lines indicate a linear relation:





So it's clear the only possible recurrence relation moves diagonally, i.e. it involves adjacent diagonal lines

NO.



I want to find $(a+1,c)$ in terms of (a,c) and $(a-1,c)$.

$$F(a,c-1;x) - F(a,c,x) = \frac{ax}{(c-1)c} F(a+1,c+1;x)$$

$$\frac{a}{c-1}\left\{ F(a,c,x) - F(a+1,c,x) = -\frac{x}{c} F(a+1,c+1;x) \right\}$$

$$F(a,c-1;x) + F(a,c;x)\left(\frac{a}{c-1} - 1\right) - \frac{a}{c-1} F(a+1,c;x) = 0$$

$$(c-1)F(a,c-1;x) + (a-c+1)F(a,c;x) = aF(a+1,c;x)$$

$$\boxed{\begin{aligned} F(a,c;x) - F(a+1,c;x) &= \left(-\frac{x}{c}\right) F(a+1,c+1;x) \\ F(a,c;x) + \left(\frac{a-c}{c}\right) F(a,c+1;x) &= \frac{a}{c} F(a+1,c+1;x) \end{aligned}}$$

Hence

$$\begin{pmatrix} F(a,c;x) \\ F(a,c+1;x) \end{pmatrix} = \begin{pmatrix} 1-\frac{a}{c} & \frac{a}{c} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F(a,c+1;x \\ F(a+1,c+1;x) \end{pmatrix}$$

$$\begin{pmatrix} F(a,c+1;x) \\ F(a+1,c+1,x) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{x}{c+1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F(a+1,c+1;x) \\ F(a+1,c+2;x) \end{pmatrix}$$

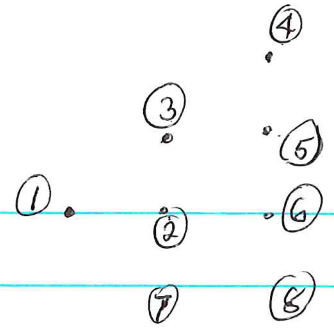
which isn't very enlightening. Find relation between (a,c) $(a+1,c+1)$ $(a+2,c+2)$:

$$F(a,c,x) - F(a+1,c,x) + \frac{x}{c} F(a+1,c+1,x) = 0$$

$$F(a+1,c,x) - F(a+1,c+1,x) = \frac{(a+1)x}{c(c+1)} F(a+2,c+2,x)$$

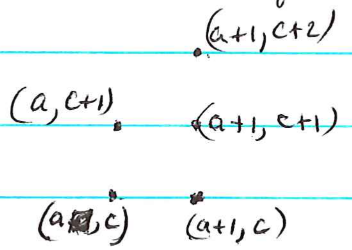
$$\boxed{F(a,c,x) + \left(\frac{x}{c} - 1\right) F(a+1,c+1,x) = \frac{(a+1)x}{c(c+1)} F(a+2,c+2,x)}$$

Actually we can get anywhere on the lattice $a+\mathbb{Z}$, $c+\mathbb{Z}$ by linear relations, e.g.:



Since there is a linear relation expressing $F(a+2,c)$ in terms of $F(a,c)$ and $F(a+1,c)$ we can also move to the left. So it should be possible to find recursion relations for a fixed and for c fixed.

First do for a fixed:



$$F(a,c) - F(a,c+1) = \frac{ax}{c(c+1)} F(a+1,c+2)$$

$$-F(a,c) + F(a+1,c) = +\frac{x}{c} F(a+1,c+1)$$

$$F(a,c+1) - F(a+1,c+1) = -\frac{x}{c+1} F(a+1,c+2)$$

add:

$$F(a+1,c) - F(a+1,c+1) = \frac{x}{c} F(a+1,c+1) + \frac{x}{c+1}\left(\frac{a}{c}-1\right) F(a+1,c+2)$$

$$\boxed{F(a+1,c) - \left(1+\frac{x}{c}\right) F(a+1,c+1) + \frac{(c-a)x}{c(c+1)} F(a+1,c+2) = 0}$$

$$F(a,c) - F(a,c+1) = \frac{ax}{c(c+1)} F(a+1,c+2)$$

$$\frac{a}{c}\Big(F(a,c+1) - F(a+1,c+1)\Big) = -\frac{x}{c+1} F(a+1,c+2)\frac{a}{c}$$

$$F(a,c) + \left(\frac{a}{c}-1\right)F(a,c+1) - \frac{a}{c}F(a+1,c+1) = 0$$

$$-\left(F(a+1,c) + \left(\frac{a+1}{c}-1\right)F(a+1,c+1) - \frac{a+1}{c}F(a+2,c+1) = 0\right)$$

$$-\left(F(a,c) - F(a+1,c) + \frac{x}{c}F(a+1,c+1) = 0\right)$$

$$\left(\frac{a}{c}-1\right)F(a,c+1) + \left[-\frac{a}{c} - \frac{a+1}{c} + 1 - \frac{x}{c}\right]F(a+1,c+1) + \frac{a+1}{c}F(a+2,c+1) = 0$$

$$(a-c)\,F(a,c+1) + [-2a-1+c-x]\,F(a+1,c+1) + (a+1)\,F(a+2,c+1) = 0$$

$$a\,F(a+1,c) + [-2a+2-1+c-1-x]\,F(a,c) + (a-c)\,F(a-1,c) = 0$$

or

$$\boxed{a\,F(a+1,c) + [c-2a-x]\,F(a,c) + (a-c)\,F(a-1,c) = 0}$$

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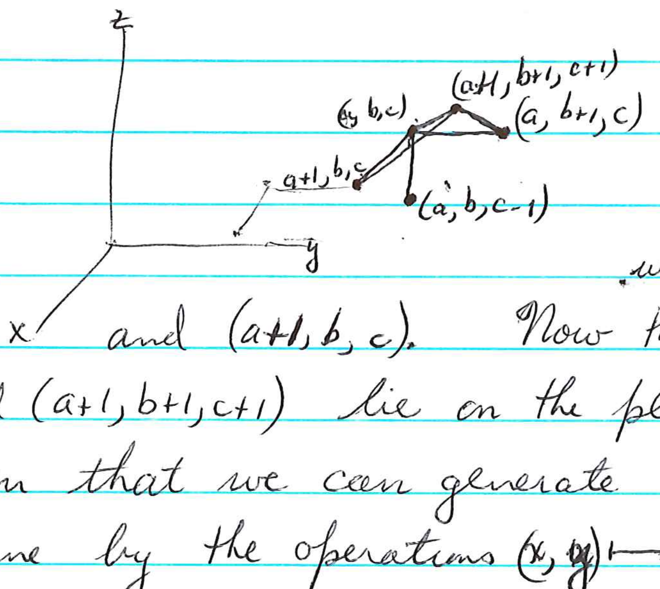
$$F(a,b,c;x) = \sum \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)\,1\cdots n}\,x^n$$

$$F(a,b,c;x) - F(a+1,b,c;x) = -\frac{bx}{c}\,F(a+1,b+1,c+1,x)$$

$$F(a,b,c;x) - F(a,b+1,c,x) = -\frac{bx}{c}\,F(a+1,b+1,c+1,x)$$

$$F(a,b,c;x) - F(a,b,c+1,x) = \frac{abx}{c(c+1)}\,F(a+1,b+1,c+2,x)$$

From the first two we link $(a+1,b,c)$, (a,b,c), $(a,b+1,c)$



From the bottom link $(a,b,c-1)$, (a,b,c), $(a+1,b+1,c+1)$. Thus from $(a+1,b+1,c+1)$ and $(a,b,c-1)$ we get (a,b,c), hence also $(a,b+1,c)$ and $(a+1,b,c)$. Now the points $(a+1,b,c)$, $(a,b,c-1)$, $(a,b+1,c)$ and $(a+1,b+1,c+1)$ lie on the plane $z-x-y=c-a-b-1$. It's clear then that we can generate all the lattice points on this plane by the operations $(x,y)\mapsto(x+1,y)$ and $(x,y)\mapsto(x,y+1)$. Since from points of $z-x-y=c-a-b-1$ we can get to (a,b,c) we can get all lattice points of $z-x-y=c-a-b$. Thus it should be possible to construct recursion relations in