April 27, 1977

On April 1, I considered the general D.E.

\[ \frac{dx}{dt} = AX \]

\[ A = A_0(t) + A_1(t) \lambda \]

where \( A_1(t) \) is of the form \( \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \) \( -\lambda^2 + \gamma^2 \geq 0 \).

I considered changing variables: \( X = UY \) where \( U = U(t) \) is in \( SL_2(\mathbb{R}) \).

Under this change the matrix \( A \) is replaced by

\[ U^{-1}AU - U^{-1}U. \]

Assuming \( \det A_1(t) = -\lambda^2 + \gamma^2 > 0 \) we can rescale, i.e. change it so as to make this determinant 1. Then we can choose \( U \) so as to make \( A_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Then I can further alter \( U \) to make \( A_0 \) symmetric, whence I saw that the DE was equivalent (by the change from UHP to unit disk) to the system:

\[ \frac{1}{i} \begin{pmatrix} \frac{d}{dx} \\ -\vec{\gamma} \end{pmatrix} Y = \lambda Y \]

with \( \vec{\gamma} \) complex.

I want to note now that the solution matrices obtained do not seem to exhaust the class of Ising model limits. I recall that a linear Ising model gives a matrix function of the form

\[ A_1 \begin{pmatrix} \cosh h_1 & \sinh h_1 \\ -\sinh h_1 & \cosh h_1 \end{pmatrix} A_2 \begin{pmatrix} \cosh h_2 & \sinh h_2 \\ -\sinh h_2 & \cosh h_2 \end{pmatrix} \ldots \ldots A_n \begin{pmatrix} \cosh h_n & \sinh h_n \\ -\sinh h_n & \cosh h_n \end{pmatrix} \]

where the \( A_i \in SL_2(\mathbb{R}) \) and the \( h_i > 0 \).
April 28, 1977

Consider the classical motion associated to the potential $e^{2x}$:

$$\left(\frac{dx}{dt}\right)^2 + e^{2x} = \lambda^2$$

Solution is

$$x = \log\left(\frac{\lambda}{\cosh \lambda t}\right) \quad \Rightarrow \quad x = e^x = \frac{\lambda}{\cosh \lambda t}$$

where one can replace $t$ by $t - t_0$. Check

$$\frac{dx}{dt} = -\frac{d}{dt} \log(\cosh \lambda t) = -\frac{\sinh \lambda t}{\cosh \lambda t}$$

$$\left(\frac{dx}{dt}\right)^2 + e^{2x} = \lambda^2 \frac{\sinh \lambda t}{\cosh^2 \lambda t} + \lambda^2 \frac{\lambda^2}{\cosh^2 \lambda t} = \lambda^2$$

An interesting question is how to relate the motion $x = \frac{\lambda}{\cosh (\lambda t)}$ for all different $\lambda$ with the basic wave motion $u(x, t) = e^{-r \cosh (\lambda t)}$. What we would like is some approximate representation of $u$ as a superposition of classically moving wave packets.
April 29, 1977

\[ E = h \nu \]

\[ E \text{ measured in eV} = g e \text{ cm}^2 \text{ sec}^{-1} \Rightarrow h \text{ in g cm}^2 \text{ sec} \]

\[ H = \frac{p^2}{2m} + V. \] As an operator \( p = \frac{\hbar}{i} \frac{\partial}{\partial x} \)

(momentum = \( \frac{\hbar}{i} \frac{\partial}{\partial x} ) \). Thus Schrödinger's eqn. is

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi \]

The time dependent value function is \[ u(x,t) = \psi(x) e^{-\frac{i}{\hbar}Et} \]

(Note \( E = h \nu \) means we should have \( \frac{E}{h} \) cycles per second, i.e.

\( \text{an angle of} \quad 2\pi \frac{E}{h} = \frac{E}{h} \text{ radians in one second.} \)

Time dep. Schrödinger equation is

\[ \frac{\hbar}{i} \frac{\partial u}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial x^2} - V(x) u \]

To understand the classical approximation put

\[ u(x,t) = e^{\frac{i}{\hbar}S(x,t)} \]

Then

\[-\frac{\hbar}{i} \frac{\partial u}{\partial t} = -\frac{\partial S}{\partial t} \cdot u \]

\[ (\frac{\hbar}{i} \frac{\partial}{\partial x})^2 u = (\frac{\hbar}{i} \frac{\partial}{\partial x})(S_x u) = (S_x^2 + \frac{\hbar}{i} S_{xx}) u \]

i.e.

\[ \frac{\partial S}{\partial t} + \frac{1}{2m} (S_x)^2 + V(x) + \frac{\hbar}{i} S_{xx} = 0 \]

If we let \( \hbar \to 0 \) then we get the Hamilton-Jacobi equation for the classical motion.
I notice now that my use of \( u, \psi \) is opposite to the physicist's convention. So we change. From now on \( \psi = \psi(x,t) \) and \( u = u(x,E) \) and we use the expansion formula

\[
\psi(x,t) = \int e^{-iEt} u(x,E) dE
\]

when \( \hbar = 1 \) and

\[
\psi(x,t) = \int e^{-iEt/\hbar} u(x,E) dE
\]

in general. \( \psi(x,t) \) is the state of the system at time \( t \) and evolves according to

\[
\psi = e^{-iHt/\hbar} \psi_0 \quad \psi_0 = \psi(.,0)
\]

The average value of an operator \( A \) when the system is in the state \( \psi \) is

\[
\langle A \psi, \psi \rangle = \int \psi^* A \psi \, dx.
\]

\[
\frac{d}{dt} \langle A \psi, \psi \rangle = \frac{d}{dt} \langle e^{iHt/\hbar} A e^{-iHt/\hbar} \psi_0, \psi_0 \rangle
\]

\[
= \left\langle \frac{i}{\hbar} [H, A] \psi, \psi \right\rangle
\]

Applying this to position and momentum:

\[
\frac{d}{dt} \left[ \frac{\hbar^2}{2m} \frac{d^2}{dx^2} x \right] = \frac{i}{\hbar} \left[ H, \frac{\hbar}{m} \frac{d}{dx} \right] = \frac{d}{dt} \left[ V, \frac{\hbar}{m} \frac{d}{dx} \right] = -\frac{P}{m}
\]
\[
\frac{i}{\hbar} [H, \psi] = -\frac{i}{\hbar} \frac{\hbar^2}{2m} \left[ \frac{d^2}{dx^2} \psi \right] = \frac{i}{\hbar} \frac{\hbar}{m} \frac{d}{dx} \psi = -\frac{p}{m}
\]

we get

\[
m \frac{d^2}{dt^2} \langle xy, \psi \rangle = \frac{d}{dt} \langle p \psi, \psi \rangle = -\langle \frac{dV}{dx} \psi, \psi \rangle
\]

Thus if \( \psi \) is supported in a small neighborhood around \( x = \langle xy, \psi \rangle \) one gets the classical motion:

\[
m \frac{d^2x}{dt^2} = -V(x).
\]

April 30, 1977:

Conservation of energy: From

\[
\frac{d}{dt} \langle A \psi, \psi \rangle = \langle \frac{d}{\hbar} [H, A] \psi, \psi \rangle
\]

one sees

\[
\langle H \psi, \psi \rangle = \left\langle -\frac{\hbar^2}{2m} \psi_{xx} + V \psi, \psi \right\rangle
\]

\[
= \frac{\hbar^2}{2m} \|\psi_x\|^2 + \left\langle V \psi, \psi \right\rangle
\]

\[
= \int \left( \frac{\hbar^2}{2m} \left| \frac{\partial \psi}{\partial x} \right|^2 + V \left| \psi \right|^2 \right) dx
\]

is constant in time.
Find fund. solution of heat equation

\[ u(x,t) = \frac{1}{2\pi i} \int e^{i \xi x} \hat{u}(\xi,t) \, d\xi \]

\[ u(x,0) = \delta(x) \Rightarrow \hat{u}(\xi,0) = 1 \]

\[ \frac{\partial u}{\partial t} = -\xi^2 u \quad \Rightarrow \quad \hat{u} = e^{-t \xi^2} \]

\[ u(x,t) = \frac{1}{2\pi} \int e^{-t \xi^2 + i \xi x} \, d\xi \cdot e^{-\frac{x^2}{4t}} \]

\[ = \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int e^{-t \xi^2} \frac{d\xi}{\sqrt{t}} = \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \sqrt{\frac{\pi}{t}} \]

\[ u = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \]

Consider now heat conditon on Fourier's ring of period \(2\pi\). The eigenfunctions are \(e^{-n^2 t} \cos nx\) so the solution with initial value \(\delta_{2\pi n}(x)\) is

\[ \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-n^2 t} e^{inx} = \sum_{n \in \mathbb{Z}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-\frac{2\pi n}{t})^2}{4t}} \]

Put \(x \mapsto 2\pi x\), \(t \mapsto \pi t^2\)

\[ \frac{1}{2\pi} \sum e^{-n^2 \pi t^2} e^{in 2\pi x} = \sum \frac{1}{2\pi t} e^{-\frac{(2\pi)^2 (x-n)^2}{4\pi t^2}} \]

\[ \sum e^{-\pi n^2 t^2} e^{2\pi i nx} = \frac{1}{t} \sum e^{-\pi \left[ \frac{x^2}{t^2} - \frac{2x n}{t^2} + \frac{n^2}{t^2} \right]} \]

Put \(x \mapsto xt\)

\[ \sum e^{-\pi n^2 t^2} e^{2\pi i nx} = \frac{1}{t} \sum e^{-\frac{\pi n^2}{t^2} + \frac{2\pi n x}{t}} e^{-\pi x^2} \]
Therefore if we put

\[ u(x,t) = \sum_{n} e^{-\pi n^2 t + 2\pi ntx - \frac{\pi x^2}{2t}} \]

\[ = \left( e^{\frac{\pi x^2}{t}} \right)^{\frac{1}{2}} \sum_{n} e^{-\pi (nt - x)^2} \]

we have the relation \( u(t,ix) = u\left(\frac{t}{x},x\right) \) hence \( u(t,x) = u\left(\frac{t}{x},\frac{x}{t}\right) = u(t,-x) \)

---

May 1, 1977  Start with the basic identity:

\[ \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} e^{2\pi inx} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi}{t} (x-n)^2} \]

\[ = e^{-\frac{\pi x^2}{t}} \sum_{n} e^{-\pi n^2 \frac{1}{t} + 2\pi nx \frac{1}{t}} \]

i.e.,

\[ \Theta(t,x) = \frac{e^{-\frac{\pi x^2}{t}}}{\sqrt{t}} \Theta\left(\frac{1}{t},\frac{x}{it}\right) \]

\[ \Theta(t,x) \text{ is periodic in } x \text{ of period } 1 \Rightarrow \Theta\left(\frac{1}{t},\frac{x}{it}\right) \]

is periodic in \( x \) of period \( it \). Thus if \( t \to \infty \), \( x = a + ib \) a fixed, \( b \to \pm \infty \), then \( \Theta\left(\frac{1}{t},\frac{x}{it}\right) \) should be bounded, so \( \Theta(t,x) \) grows like

\[ e^{-\frac{\pi}{t} (a+ib)^2} \sim e^{\frac{\pi}{t} b^2} \]
Asymptotic expansions.

(1) \[-\frac{d^2 u}{dx^2} + gu = \lambda^2 u\]

Put \[u = e^{iS(x, \lambda)}\].

\[-\frac{d}{dx} (e^{iS} i\frac{d}{dx} S) = -e^{iS} (-S_x^2 + iS_{xx}) = (\lambda^2 - g) e^{iS}\]

(2) \[S_x^2 - (\lambda^2 - g) - iS_{xx} = 0\]

I claim there is a unique asymptotic expansion

\[S = \lambda x + u_0(x) + u_1(x) \lambda^{-1} + \ldots\]

such that \[S(0, \lambda) = 0\]. To see this write

\[S = \lambda x + T\]

whence (2) becomes

\[(\lambda + T_x)^2 - \lambda^2 + g - iT_{xx} = 0\]

\[2\lambda T_x + T_x^2 - \lambda - iT_{xx} = 0\]

and it's clear one can successively solve for the coefficients of

\[T_x = u_0' + u_1' \lambda^{-1} + u_2' \lambda^{-2} + \ldots\]

Note \[u_0' = 0\].

Through terms up to \(\lambda^{-1}\) one gets

\[S_x = \lambda - \frac{g}{2} \lambda^{-1} - i \frac{g'}{4} \lambda^{-2}\]
which agrees up to factors independent of $x$ with

$$(x^2 - 8)^{-1/4} \int_0^1 (x^2 - 8)^{1/2} \mathrm{d}y$$

One obtains another asymptotic solution by replacing $\lambda$ by $-\lambda$.

**Question:** What have these asymptotic solutions to do with real solutions?

Rewrite (1) in the form

$$\frac{1}{x^2} \frac{d^2 u}{dx^2} + \left(1 - \frac{g(x)}{x^2}\right) u = 0$$

and put $y = \lambda x$, whence we get

$$(1') \quad \frac{d^2 u}{dy^2} + \left(1 - \frac{1}{x^2} g\left(\frac{y}{x}\right)\right) u = 0$$

Assuming $g$ analytic near 0, then $\frac{1}{x^2} g\left(\frac{y}{x}\right)$ is analytic around $\frac{1}{x} = 0$, so any solution of (1') with initial values independent of $\lambda$ should be holomorphic in $\frac{1}{x}$.

The problem is that if $\Phi(x, y)$ is the solution matrix for (1) starting at $y = 0$ is $\Phi(x, y)$ holomorphic in $\frac{1}{x}$, aside from the $e^{\pm i\lambda x}$ parts.
May 2, 1977

Consider

\[ \frac{d^2 u}{dy^2} + \left[ 1 - \left( 8 \left( \frac{4}{\lambda^2} \right) \right) \right] u = 0 \]

Put \( \varepsilon = \frac{1}{\lambda} \) and \( u = e^{iy} v \).

\[
\begin{align*}
\left( \frac{d^2 u}{dy^2} + u \right) &= e^{iy} \frac{d^2 v}{dy^2} + 2ie^{iy} \frac{dv}{dy} + (e^{iy}) v + e^{iy} v \\
&= \left( \frac{1}{\varepsilon} + \varepsilon^2 g(\varepsilon y) \right) e^{iy} v
\end{align*}
\]

or

\[
\frac{d^2 v}{dy^2} + 2i \frac{dv}{dy} = \varepsilon^2 g(\varepsilon y) v
\]

Look for a series solution \( v = a_0(y) + a_1(y) \varepsilon + \ldots \). If

\[ g(\varepsilon y) = g_0 + g_1 \varepsilon y + g_2 \varepsilon^2 y^2 + \ldots \]

then comparing coefficients of powers of \( \varepsilon \) we get

\[
\frac{d^2 a_i}{dy^2} + 2i \frac{da_i}{dy} = g_{i-2} y^{-i-2} a_0 + \ldots + g_1 y a_{i-3} + g_0 a_{i-2}
\]

Assuming inductively that \( a_j \) is a poly in \( y \) of degree \( \leq j-1 \), and using that \( \frac{d}{dy} + 2i \) acts as an automorphism on the poly of degree \( \leq n \) for any \( n \), one sees there is a unique choice for \( a_j \) as a poly in \( y \) and its degree is \( \leq i-1 \), up to a constant. So we can make \( a_i \) unique by requiring it to vanish at \( 0 \).

It remains to decide whether the resulting series for \( V(y, \varepsilon) \) is convergent. Can you find a formula for \( (\frac{d}{dy} + 2i)^{-1} \) on polynomials?
Solving example:

\[(1 - \frac{d}{dz})^{-1} \left( \frac{1}{z} \right) = \left( 1 + \frac{d}{dz} + \frac{d^2}{dz^2} + \ldots \right) \left( z^{-1} \right) \]

\[= z^{-1} + (-1) z^{-2} + (-1)(-2) z^{-3} + \ldots + (-1)^{n-1}(n-1)! z^{-n} + \ldots \]

is a divergent series. Obvious method of trying to invert \(1 - \frac{d}{dz}\) is by

\[(x) \quad \left(1 - \frac{d}{dz}\right)^{-1} f = -e^{z} \int_{z}^{\infty} e^{-t} f(t) \, dt \]

where the initial point for the integration has to be specified. Note that the above has to be related to the asymptotic expansion for the exponential integral

\[\int_{z}^{\infty} \frac{e^{-t}}{t} \, dt = e^{-z} \left( z^{-1} - z^{-2} + 2 z^{-3} - \ldots + (-1)^{n-1}(n-1)! z^{-n} + \ldots \right) \]

Next let us compare both sides of \((x)\) when \(f\) is a polynomial and the initial point is \(z = a\). One has

\[-e^{z} \int_{a}^{z} e^{-t} f(t) \, dt = \left[ \left( 1 + \frac{d}{dz} + \frac{d^2}{dz^2} + \ldots \right) f(z) \right]_{a}^{z} \]

which is not what I want e.g. \(f(z) = z\) gives

\[\left[ z + 1 \right]_{a}^{z} = z - a \]

which is correct only for \(a = -1\), whereas \(f(z) = z^2\) gives

\[\left[ z^2 + 2z + 2 \right]_{a}^{z} \]

which is correct \(\Leftrightarrow a^2 + 2a + 2 = 0\).

However it is clear that what I want is
The formula

\[
(1 - \frac{d}{dz})^{-1} f(z) = e^{z} \int_{z}^{\infty} e^{-t} f(t) dt
\]

To prove this works note

\[
\int_{z}^{\infty} e^{-t} t^n dt = \left[ -e^{-t} t^n \right]_{z}^{\infty} + n \int_{z}^{\infty} e^{-t} t^{n-1} dt
\]

\[
= e^{-z} z^n + n \int_{z}^{\infty} e^{-t} t^{n-1} dt
\]

To

\[
e^{z} \int_{z}^{\infty} e^{-t} t^n dt = z^n + n(z^{n-1} + (n-1)(z^{n-2} - \cdots
\]

\[
= z^n + n z^{n-1} + n(n-1) z^{n-2} + \cdots
\]

\[
= (1 + \frac{d}{dz} + \frac{d^2}{dz^2} + \cdots) z^n
\]

Observe that the constant term is \( n! \) hence this operator on polynomial will not extend to series to give convergence.
March 7, 1977

We consider

\[ \frac{d^2 u}{dx^2} + q(x) u = \lambda^2 u \]

around \( x = 0 \), \( q(x) \) being supposed analytic if we want. We have seen that we can find unique formal solutions of the form

\[ e^{i \lambda x} \left( q_0(x) + q_1(x) \lambda^{-1} + \ldots \right) \]

\[ e^{-i \lambda x} \left( b_0(x) + b_1(x) \lambda^{-1} + \ldots \right) \]

where \( q_1(0) = b_1(0) = 0 \), \( i > 0 \) and \( q_0(x) = b_0(x) = 1 \).

Moreover one has

\[ e^{i \lambda x} \left( q_0(x) + q_1(x) \lambda^{-1} + \ldots \right) = \left(1 - \frac{q_1}{2} \right)^{-1/4} e^{i \int_0^x (\lambda^2 - q) \frac{1}{2} \, dx} \]

\[ + O(\lambda^{-3}) \]

If \( u = a e^{i \lambda x} \), then (1) becomes

\[ - \left( a'' e^{i \lambda x} + 2i \lambda a' e^{i \lambda x} + a(\lambda^2) e^{i \lambda x} \right) + \lambda^2 a e^{i \lambda x} = \lambda^2 a e^{i \lambda x} \]

or

\[ a'' + 2i \lambda a' = \lambda a \]

so (2) is constructed by a process which probably will lead to a divergent series.

Fröman approach (N. and P. Fröman, JWKB approximation, Contributions to the theory, North-Holland, Amsterdam 1965).

One seeks a solution of (1) in the form

\[ u = a(x) e^{i \lambda x} + b(x) e^{-i \lambda x} \]

subject to the variation of constants condition

\[ a' e^{i \lambda x} + b' e^{-i \lambda x} = 0 \]
Then
\[ u = ae^{i\lambda x} + be^{-i\lambda x} \]
\[ u' = a(e^{i\lambda x}) + b(-i\lambda e^{-i\lambda x}) \]
\[ u'' = a'(e^{i\lambda x}) + b'(i\lambda e^{-i\lambda x}) + (-\lambda^2)u \]

so
\[ a'e^{i\lambda x} + b'e^{-i\lambda x} = 0 \]
\[ a'(ie^{i\lambda x}) + b'(-ie^{-i\lambda x}) = gae^{i\lambda x} + geb^{-i\lambda x} \]
\[ a'(2i\lambda e^{i\lambda x}) = gae^{i\lambda x} + geb^{-i\lambda x} \]
\[ b'(2i\lambda e^{-i\lambda x}) = -gae^{i\lambda x} - geb^{-i\lambda x} \]

\[
\begin{pmatrix}
  a' \\
  b'
\end{pmatrix}
= \frac{g}{2i\lambda}
\begin{pmatrix}
  1 & e^{-2i\lambda x} \\
  e^{2i\lambda x} & -1
\end{pmatrix}
\begin{pmatrix}
  a \\
  b
\end{pmatrix}
\]

They put
\[ M = \frac{g(x)}{2i\lambda}
\begin{pmatrix}
  1 & e^{-2i\lambda x} \\
  e^{2i\lambda x} & -1
\end{pmatrix}
\]
and denote

by \( F(x,x_0,\lambda) \) the solution matrix for the above DE starting at \( x_0 \):

\[ F(x,x_0) = \exp \int_{x_0}^{x} M(y) \, F(y,x_0) \, dy \]

Consider finding asymptotic solutions for

\[ A \frac{du}{dx} + Bu = i\lambda u \]

of the form \( u = e^{iSx}v \), where \( S = \sigma_0 + \sigma_1 \lambda + \ldots \).
\[ A \left[ iS_x \nu + \nu_x \right] + B \nu = i \lambda \nu \]

So we want \( \frac{\lambda}{S_x} \) to be an eigenvalue for \( A \).

Supposing the eigenvalues of \( A(x) \) distinct at each \( x \) they should depend smoothly on \( x \), hence we get eigenvalues \( \lambda_1(x), \lambda_2(x) \) for \( A(x) \). This gives the equation

\[ S_x = \frac{\lambda}{\lambda_i(x)} \quad \text{for } S, \quad i=1,2 \]

It follows that \( \nu_0 \) must be a multiple, depending on \( \lambda \), times the eigenvector corresponding to the choice of eigenvalue. Suppose we fix one of the eigenvalues \( \lambda(x) \) and let \( \nu(x) \) be a smooth choice of eigenvector:

\[ A \nu(x) = \lambda(x) \nu(x), \quad \nu(x) \neq 0. \]

\[ S_x = \frac{\lambda}{\lambda(x)} \]

\[ \left[ A i \frac{\lambda}{\lambda(x)} - i \lambda \right] \nu + A \nu_x + B \nu = 0 \]

It is clear \( \nu_0 = f(x) \nu(x) \) some \( f \). The next equation is

\[ i \left( A \frac{1}{\lambda(x)} - 1 \right) \nu_1 + A (\nu_0)_x + B \nu_0 = 0 \]

\[ i \left( A \frac{1}{\lambda(x)} - 1 \right) \nu_1 + f_x \lambda(x) \nu(x) + f(A(\nu_0)_x + B \nu_0) = 0 \]

By proper choice of \( f_x \) one can get an arbitrary multiple of \( \nu(x) \), and by choosing \( \nu_1 \) on the complement of \( \nu \), the first term can be made an arbitrary vector in the complement. Thus it is clear how to grind out the asymptotic series; one determines the \( e \)-part.
of $u_{n-1}$ and the complement to the part of $u_n$ at the
$n^\text{th}$ stage.

So apply this to
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{du}{dx} + \begin{pmatrix} 0 & -\overline{\rho} \\ \rho & 0 \end{pmatrix} u = i\lambda u
\]

Here the eigenvalues of $A$ are $\pm 1$, hence $S_x = \pm 1$
so $e^{iS} = e^{\pm i\lambda x}$.

\[
\begin{pmatrix} \frac{d}{dx} - i\lambda & -\overline{\rho} \\ \rho & -\frac{d}{dx} - i\lambda \end{pmatrix} u = 0
\]

\[
\begin{pmatrix} \frac{d}{dx} & \overline{\rho} \\ \rho & -\frac{d}{dx} - 2i\lambda \end{pmatrix} \nu = 0
\]

Suppose $\nu = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \lambda^{-1} + \ldots$. Then comparing coeffs. of $\lambda^0$
one gets $b_0 = 0$.

\[
(a_0' + a_1' \lambda^{-1} + \ldots) - \overline{\rho} (b_0 + b_1 \lambda^{-1} + \ldots) = 0
\]

\[
p(a_0 + a_1 \lambda^{-1} + \ldots) - (b_0' + b_1' \lambda^{-1} + \ldots) - 2i\lambda (b_0 + b_1 \lambda^{-1} + \ldots) = 0
\]

\[
2i\lambda b_0 = 0 \quad \Rightarrow \quad a_0' = \overline{\rho} b_0
\]

\[
b_0 = 0 \quad \Rightarrow \quad a_0' = 0
\]

So can suppose $a_0 = 1$.

\[
p a_0' - b_0' - 2i b_1 = 0 \quad \Rightarrow \quad b_1 = \frac{p}{2i}
\]

\[
a_1' = \overline{\rho} b_1 = \frac{|p|^2}{2i} \quad \Rightarrow \quad a_1 = \int \frac{|p|^2}{2i} \, dx
\]
\[ 2 b_2 = p a_1 - b_1' = p \int_{2i}^{\infty} \frac{1}{2i} \, dx - \frac{p'}{2i} \]
\[ b_2 = \frac{p}{2i} \int_{0}^{\infty} \frac{1}{2i} \, dx + \frac{p'}{4} \]
\[ a_2' = \bar{p} b_2 = \frac{1}{2i} \int_{0}^{\infty} \frac{1}{2i} \, dx + \frac{p'\bar{p}}{4} \]
\[ a_2 = \frac{1}{2} \left( \int_{0}^{\infty} \frac{1}{2i} \, dx \right)^2 + \int_{0}^{\infty} \frac{p'\bar{p}}{4} \, dx \]

Thus we get
\[ a = e^{-i\lambda^{-1} \int_{0}^{\infty} \frac{1}{2i} \, dx} \left( 1 + \lambda^{-2} \int_{0}^{\infty} \frac{p'\bar{p}}{4} \, dx + \cdots \right) \]
\[ b = e^{-i\lambda^{-1} \int_{0}^{\infty} \frac{1}{2i} \, dx} \left( \lambda^{-1} \frac{p}{2i} + \lambda^{-2} \frac{p'}{4} + \cdots \right) \]

Recall that
\[ \sqrt{\lambda^2 - |p|^2} = \lambda - \lambda^{-1} \frac{|p|^2}{2} + O(\lambda^{-3}) \]
\[ \int_{0}^{\infty} \sqrt{\lambda^2 - |p|^2} \, dx = \lambda x - \lambda^{-1} \int_{0}^{\infty} \frac{|p|^2}{2} \, dx + O(\lambda^3) \]

hence our asymptotic solution can be written
\[ U = e^{\int_{0}^{\infty} \sqrt{\lambda^2 - |p|^2} \, dx} \left( 1 + \lambda^{-2} \int_{0}^{\infty} \frac{p'\bar{p}}{4} \, dx + \cdots \right) \left( \lambda^{-1} \frac{p}{2i} + \lambda^{-2} \frac{p'}{4} + \cdots \right) \]

What is the Frobenius approach here? If \( \frac{dX}{dt} = AX \) has the solution matrix \( S \), then to solve
\[ \frac{dX}{dt} = AX + B \]
put \( X = SY \) with \( Y = e^{\int_{0}^{t} A \, dt} \). Then

\[ \frac{dS}{dt} = AS \]
\[ S(0) = I \]
\[
\frac{d(Sy)}{dt} = A Sy + S \frac{dy}{dt} = A Sy + B
\]
\[
\frac{dy}{dt} = S^{-1}B
\]

or
\[
y = \int_0^t S^{-1}B \, dt + y(0)
\]

Apply this to our system
\[
\frac{du}{dx} = \begin{pmatrix} i & \bar{\rho} \\ \rho & -i \end{pmatrix} u = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} u + \begin{pmatrix} 0 & \bar{\rho} \\ \rho & 0 \end{pmatrix} u
\]

\[
S = \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix}
\]

\[
S^{-1} u = S u(0) + \int_0^x S^{-1} \begin{pmatrix} 0 & \bar{\rho} \\ \rho & 0 \end{pmatrix} u
\]

\[
e^{-i\lambda x} u_1(x) = e^{-i\lambda x} u_1(0) + \int_0^x e^{-2i\lambda y} \bar{\rho}(y) (e^{i\lambda y} u_2(y)) \, dy
\]
\[
e^{i\lambda x} u_2(x) = e^{i\lambda x} u_2(0) + \int_0^x e^{2i\lambda y} \rho(y) (e^{-i\lambda y} u_1(y)) \, dy
\]

Thus if I put
\[
u = Su
\]

one has
\[
v(x) = v(0) + \int_0^x \begin{pmatrix} 0 & e^{-2i\lambda y} \rho(y) \\ e^{2i\lambda y} \bar{\rho}(y) & 0 \end{pmatrix} v(y) \, dy
\]
May 11, 1977

Add to April 1 the following:

\[ \frac{d}{dx} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \]

\[ S = \begin{pmatrix} \varphi & \varphi' \\ \varphi' & \varphi \end{pmatrix} \]

solution matrix starting at \( x = 0 \)

\[ \Delta_b = S(b \lambda)^{-1} p(R) \]

is a circle of radius:

\[ \frac{1}{r(\Delta_b)} = \left| \begin{pmatrix} \varphi(b) & \varphi'(b) \\ \varphi'(b) & \varphi(b) \end{pmatrix} \right| = \lambda \notin \mathbb{R} \]

and center

\[ c(\Delta_b) = \frac{1}{\lambda} \begin{pmatrix} \varphi(b) & \varphi'(b) \\ \varphi'(b) & \varphi(b) \end{pmatrix} \]

But

\[ \frac{d}{dx} \begin{pmatrix} \varphi & \varphi' \end{pmatrix} = \begin{pmatrix} \varphi & \varphi' \\ (\lambda - \lambda) \varphi & (\lambda - \lambda) \varphi' \end{pmatrix} = (\lambda - \lambda) \varphi \varphi' = -2i \text{Im}(\lambda) \varphi \varphi' \]

\[ \begin{pmatrix} \varphi & \varphi' \end{pmatrix}(b) = 1 + \int_0^b (\lambda - \lambda) \varphi \varphi' = 1 - 2i \text{Im}(\lambda) \int_0^b \varphi \varphi' \]

\[ \begin{pmatrix} \varphi & \varphi' \end{pmatrix}(b) = \int_0^b (\lambda - \lambda) \varphi \varphi' = 2i \text{Im}(\lambda) \int_0^b \varphi \varphi' \]

so if \( \text{Im} \lambda > 0 \) one has

\[ \frac{1}{r(\Delta_b)} = 2 \text{Im}(\lambda) \int_0^b |\varphi'|^2 \, dx \]

\[ r(\Delta_b) = \frac{1}{2 \text{Im}(\lambda) \int_0^b |\varphi|^2 \, dx} \]

\[ c(\Delta_b) = \frac{1 - i2 \text{Im}(\lambda) \int_0^b \varphi \varphi' \, dx}{i2 \text{Im}(\lambda) \int_0^b |\varphi|^2 \, dx} \]
But note that if we minimize
\[ \| m\psi + \varphi \|^2 = \int_0^b (m\psi + \varphi)^2 \, dx \]
then we have
\[ (m\psi + \varphi, \varphi) = m\|\psi\|^2 + (\varphi, \varphi) \]
or
\[ m = -\frac{(\varphi, \varphi)}{\|\varphi\|^2} \]

Notice also that
\[ c(\Delta_b) - r(\Delta_b) = -\frac{\int_0^b \varphi \overline{\varphi} \, dx}{\int_0^b |\varphi|^2 \, dx} \]

Therefore we see that
\[ m_b(\lambda) = -\frac{\int_0^b \varphi \overline{\varphi} \, dx}{\int_0^b |\varphi|^2 \, dx} \]
is on \( \Delta_b \).

In fact it seems that it is the point on \( \Delta_b \) closest to the real axis. (Because \( S(x, \lambda) \) shrinks the UHP \( S^{-1}(b, \lambda) \) carries \( \mathbb{P}_1(\mathbb{R}) \) into the lower half plane.)

So we get the formula
\[ m_\infty(\lambda) = \lim_{b \to \infty} -\frac{\int_0^b \varphi \overline{\varphi} \, dx}{\int_0^b |\varphi|^2 \, dx} \]

which might also be valid for \( \lambda \) real.
Simpler formula for $m_\infty$:

$$m_\infty = \lim_{b \to \infty} S(b, \lambda)^{-1}(\lambda) = \lim_{b \to \infty} \frac{r \phi'(b, \lambda) - \phi(b, \lambda)}{r \psi'(b, \lambda) + \psi(b, \lambda)}$$

for any real number $\lambda$ including $\infty$. Thus if $r = 0$ we get

$$m_\infty(\lambda) = \lim_{b \to \infty} -\frac{\phi(b, \lambda)}{\psi(b, \lambda)}$$

---

**Argument:** For each $\lambda$ there should exist a unique up-to-scalar $u(x, \lambda)$ which dies at $x = \pm \infty$. Thus we get a line bundle over $\mathbb{C}$ whose fibre at $\lambda$ is the line of solutions $u$ dying at $\infty$. This line bundle has to be trivial hence we can trivialize it and obtain $u(x, \lambda) = a(\lambda) \psi(x, \lambda) + b(\lambda) \phi(x, \lambda)$ unique up to an invertible function $\tilde{e}^{g(\lambda)}$ of $\lambda$. Now what I want to arrange is for $u(x, \lambda)$ to be of exponential type and rapidly decreasing as $\lambda \to \pm \infty$. 