April 20, 1977.

Philosophy: Up to now you have been thinking in terms of solutions of

$$ -\frac{d^2 y}{dx^2} + \lambda^2 y = 0 $$

with $\lambda$ constant and $x$ varying globally. But what you want to do is to think globally in $\lambda$ and locally in $x$ so that you can take the Fourier transform with respect to $\lambda$ and get the wave equation.

So suppose we work around $x = 0$. Let $c(x, \lambda)$ and $s(x, \lambda)$ denote the solutions of (1) with

$$
\begin{pmatrix}
    c & s \\
    c' & s'
\end{pmatrix}
(0, \lambda) =
\begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
$$

It should be true that $c$ and $s$ have asymptotic expansions in $\lambda$

$$ c(x, \lambda) = e^{i\lambda x} \left( a_0(x) + a_1(x) \lambda^{-1} + \ldots \right) + e^{-i\lambda x} \left( a_0(x) + a_1(x) \lambda^{-1} + \ldots \right) $$

which can be found formally. Note that better asymptotic formulas to the first (2nd) order can be found if one uses

$$ e^{i\int_0^x \sqrt{1 - q}} \approx e^{i\lambda x - \frac{\lambda^2 x}{2} \lambda^{-2}} $$

I think that (2) implies the Fourier transform of $c(x, \lambda)$ with respect to $\lambda$ has support in $[-|x|, |x|]$ with singularities at the ends.
Suppose $c(x, \lambda)$ as above. Then put
\[ \tilde{c}(x, y) = \int e^{i \lambda y} c(x, \lambda) \, d\lambda. \]

One has
\[ \tilde{c}(0, y) = \int e^{-i \lambda y} \, d\lambda = \frac{i}{2\pi} \delta(y), \]
\[ \frac{\partial \tilde{c}}{\partial x}(0, y) = 0 \]
and $\tilde{c}$ satisfies the wave equation
\[ \frac{\partial^2 \tilde{c}}{\partial y^2} = \frac{\partial^2 \tilde{c}}{\partial x^2} - q(x) \tilde{c}. \]

Similarly $\tilde{s}(x, \lambda)$ satisfies the wave equation with the initial condition
\[ \tilde{s}(0, y) = 1 \]
\[ \frac{\partial \tilde{s}}{\partial x}(0, y) = \int e^{-i \lambda y} \frac{\partial \tilde{s}}{\partial x}(x, \lambda) \, d\lambda = \int e^{-i \lambda y} \, d\lambda = \frac{i}{2\pi} \delta(y). \]

Put
\[ \psi(x, \lambda) = \int \frac{\sin \frac{\lambda(x-y)}{\lambda}}{\lambda} f(y) \, dy. \]
Then
\[ \psi(0, \lambda) = 0 \]
\[ \psi_x(x, \lambda) = \int_0^x \cos \frac{\lambda(x-y)}{\lambda} f(y) \, dy \]
\[ \psi_{xx}(0, \lambda) = f(x) + \int_{-\lambda}^{\lambda} \sin (x-y) f(y) \, dy. \]
Thus \( \frac{\partial^2 \psi}{\partial x^2} + \lambda^2 \psi = f(x) \) and \( \psi(0) = \psi_x(0) = 0 \).

It follows that \( \psi(x, \lambda) \) satisfies the integral equation
\[
\psi(x, \lambda) = \cos(\lambda x) + \int_0^x \frac{\sin \frac{\lambda(x-y)}{\lambda}}{\lambda} g(y) \psi(y, \lambda) \, dy
\]
because \( \frac{d^2 \psi}{dx^2} + \lambda^2 \psi = g \).

A basic fact about wave equations is unique solvability of the Cauchy problem across non-characteristic hypersurfaces. In particular, singularities propagate along characteristic hypersurfaces. Let's consider the operator
\[
L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} + g
\]
and a fundamental solution for it:
\[
L \psi = \delta_{(0,0)}
\]
Then the singularities of \( \psi \) can lie on the characteristics issuing from the origin.
It would seem that there exists a Green's function $g$ which is zero for $x < 0$. Then $g$ would be supported in $\{(x, y) | x \geq 0, \ |y| \leq x\}$.

Example: if $g = 0$, then the function

$$g(x, y) = \begin{cases} 1 & 0 \leq x > y \\ 0 & \text{otherwise} \end{cases}$$

which is 1 in the x-forward cone and 0 outside satisfies $\nabla^2 g = 0$ away from 0 because locally $g$ is a function of $x-y$ or of $x+y$ away from 0. In fact

$$g(x, y) = H(x-y) H(x+y)$$

up to a constant.

$$L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} = -\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \frac{\partial}{\partial u} + \frac{\partial}{\partial y} \frac{\partial}{\partial u} \quad \begin{cases} x = u + v \\ y = -u + v \end{cases}$$

$$L = -\frac{\partial^2}{\partial u \partial v}$$

$$H(x-y) H(x+y) = H(2u) H(2v)$$

$$L \{H(x-y) H(x+y)\} = -\frac{\partial H(2u)}{\partial u} \cdot \frac{\partial H(2v)}{\partial v} = -\frac{1}{4} \delta(u) \delta(v)$$

$$= -\frac{1}{2} \delta(x) \delta(y)$$

Change $\nabla^2$ to

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 8$$

$$\oint M dx + N dy = \int \int \left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) dx \; dy$$
If \( g = H(x+y)H(x-y) \) and \( f \in C_c^\infty(\mathbb{R}^2) \)

\[
\iint g \, f = \iint (\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}) \, dx \, dy = \]

\[
= \iint \frac{\partial f}{\partial u \partial v} \, 2 \, du \, dv = 2 \int -\frac{\partial f}{\partial u}(u_0, v_0) \quad u_0 \neq 0 \quad v_0 \neq 0
\]

\[
= 2f(0, 0)
\]

Thus

\[
\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \left(\frac{H(x+y)H(x-y)}{2}\right) = \delta(x) \delta(y)
\]

Use the variables \( \xi, \eta \) for \( u, v \):

\[
x = \xi + \eta
\]
\[
y = -\xi + \eta
\]

\[
\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \xi \partial \eta}
\]

The fundamental solution \( g \) of

\[
\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} - 8g = \delta
\]

can be used to solve the Cauchy problem. Thus given a solution \( u \) one can express \( u(0,0) \) as an integral of the values \( u(x_0, y) \)

\[
u_{x}(x_0, y) \quad \text{for} \quad |y| \leq x.
\]
Suppose good boundary conditions for
\[ \frac{d^2 y}{dx^2} + (\lambda^2 - \theta) y = 0 \]
are given on \( 0 \leq x \leq b \), so that we have eigenvalues \( \pm \lambda_n \) and eigenfunctions \( \psi_n(x) \) for \( n \geq 1 \). Suppose \( \theta \) is not an eigenvalue and \( \int_0^b |\psi_n|^2 dx = 1 \).

Form the kernel
\[ K_t(x, x') = \frac{1}{2} \sum_{n \geq 1} e^{i \lambda_n t} \psi_n(x) \overline{\psi_n(x')} + \frac{1}{2} \sum_{n \geq 1} e^{-i \lambda_n t} \psi_n(x) \overline{\psi_n(x')} \]
\[ = \sum_{n \geq 1} \cos \lambda_n t \ \psi_n(x) \overline{\psi_n(x')} \]

Then if \( (K_t \ast f)(x) = \int_0^b K_t(x, x') f(x') dx' \),

one has
\[ \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \theta \right) K_t \ast f = 0 \]
\[ K_0 \ast f = f \]
\[ \left( \frac{\partial K_t \ast f}{\partial t} \right)_{t=0} = 0 \]

Thus \( (K_t \ast f)(x) \) is a solution of the Cauchy problem for the wave equation with the initial values \( f, 0 \) along the line \( t=0 \).
So we know from the theory of these wave equations that $\mathcal{K}_t(x,x')$ should have its support in $|x-x'| \leq t$. Assume so.

Next let's shift to systems, $\Psi = (\psi_1, \psi_2)$

$$
\frac{1}{i} \begin{pmatrix}
\frac{d}{dx} & \bar{P} \\
P & -\frac{d}{dx}
\end{pmatrix} \Psi = \lambda \Psi
$$

$P$ is a first order self-adjoint operator which is elliptic as its symbol is $\tilde{\sigma}(\xi, \eta)$. (In fact it has the property that the full symbol 

$$
\begin{vmatrix}
\xi & -\frac{\bar{P}}{x_2} \\
\frac{P}{x_2} & -\xi
\end{vmatrix} = -\xi^2 + \frac{P\bar{P}}{x_2} = -\left(\xi^2 + \frac{P\bar{P}}{x_2}\right)
$$

doesn't vanish for $\xi$ real.)

Next consider $e^{-itP}$ which will yield solutions of the Cauchy problem

$$
\left(\frac{1}{i} \frac{d}{dt} + P\right) (e^{-itP} f) = 0
$$

$$
(e^{-itP} f)_{t=0} = f
$$

As above if $P \psi_n = \lambda_n \psi_n$ are the eigenfunctions and eigenvalues, $P$ is represented by the kernel

$$
\sum_n \lambda_n \psi_n(x) \psi_n(x')^* 
$$

2x2 matrix

so $e^{-itP}$ is represented by the kernel.
\[ k_t(x, x') = \sum_{n} e^{-i\theta_n} \psi_n(x) \psi_n(x')^* \]

The wave equation under consideration is

\[
\left( \frac{\partial}{\partial t} + i \mathbf{p} \right) u = 0
\]

or

\[
\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \right] u = 0
\]

Weyl equation (neutrinos)

\[
\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial z} \right] u = 0
\]

where the \( \sigma_i \) are the Pauli spin matrices (inf. rotation around \( x, y, z \) axes).

The example I want to handle is \( p = e^x \). If I put \( r = e^x \), then

\[
\frac{d}{dx} = \frac{d}{dr} \frac{dr}{dx} = \frac{d}{dr} e^x = r \frac{dr}{dx}
\]

hence we get the equation

\[
\frac{\partial}{\partial t} u + r \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] u = 0
\]

which we want to view as analogous to

\[
\frac{\partial^2 u}{\partial t^2} = \left( \frac{n}{\partial r} \right)^2 u - \frac{r^2}{n^2} u
\]
Which has the solution
\[ u = e^{-x^2/2 \cos^2 t} \]

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Consider again
\[ Lu = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + g u \quad \text{on} \quad 0 \leq x < \infty. \]

Put
\[ H = \frac{1}{2} \int_{0}^{\infty} (u_t^2 + u_x^2 + g u^2) \, dx \]

assuming this converges. Then \( H = H(t) \) satisfies
\[
\frac{dH}{dt} = \int_{0}^{\infty} (u_t u_{tt} + u_x u_{xt} + g u u_t) \, dx
\]

\[ = \left[ u_x u_t \right]_{0}^{\infty} + \int_{0}^{\infty} u_t (u_{tt} - u_{xx} + g u) \, dx \]

\[ = -u_x(0,t) u_t(0,t) \]

so the energy remains constant if \( u_x(0,t) = 0 \) which means \( x = 0 \) is a reflecting endpoint, or if \( u(0,t) = 0 \) which means that the endpoint 0 is held fixed. Another case to consider is
\[ u_x(0,t) = c \, u(0,t) \]

for then
\[ -u_x(0,t) u_t(0,t) = -c \, u(0,t) u_t(0,t) = \]

\[ = -\frac{c}{2} \frac{d}{dt} [u(0,t)^2] \]

Here
\[ H(t) = -\frac{c}{2} \, u(0,t)^2 + \text{constant}. \]
Example: \( u = e^{-r \cos \lambda t} \) which satisfies
\[
\frac{\partial^2 u}{\partial t^2} = \left( r \frac{\partial}{\partial r} \right)^2 u - r^2 u
\]

Now \( \psi(r, \lambda) = \int e^{i \lambda t} e^{-r \cos \lambda t} \, dt = K_{i \lambda}(r) \)
is never identically zero in \( r \) for any \( \lambda \), and \( K_{i \lambda}(r) \to 0 \) rapidly uniformly in \( \lambda \) as \( r \to +\infty \).

In general given \( \frac{\partial^2}{\partial x^2} + (1 - q) \psi = 0 \) on \( 0 \leq x \leq \infty \)

where \( q(x) \to +\infty \) as \( x \to +\infty \), it should be true that the spectrum is discrete for any of the boundary conditions at 0. Moreover there should exist for any complex number \( \lambda \) a solution \( \psi(x, \lambda) \) unique up to a scalar which is square integrable. I conjecture that it should always be possible to normalize \( \psi(x, \lambda) \) as a function of \( \lambda \) in the following way:

1) \( \psi(x, \lambda) \) should be holomorphic in \( \lambda \) and of exponential type rapidly decreasing along the real axis. This means that its Fourier transform \( u(x, \lambda) \) should be rapidly decreasing.
2) \( \psi(x, \lambda) \) not identically zero in \( x \) for each \( \lambda \).

The thing to prove first is that the eigenvalues are discrete. A possible method is to prove, using WKB, the existence of \( \psi(x, \lambda) \) of the form
\[
\psi(x, \lambda) = (q - \lambda)^{-1/4} e^{-\int \sqrt{(q - \lambda)} \, dx}
\]
Once you have this, you have \( \psi(x, \lambda) \) defined and it only
remains to establish the properties for fixed \( z \).

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\[
\frac{1}{2} \begin{pmatrix}
\frac{d}{dx} & -P \\
-\frac{d}{dx} & P
\end{pmatrix} \psi = \lambda \psi
\]

Take \( p = e^x \) and change independent variable to \( x = \log r \), \( \frac{d}{dx} = r \frac{d}{dr} \), \( e^x = r \)

\[
\begin{pmatrix}
\frac{d}{dr} & -1 \\
1 & -\frac{d}{dr}
\end{pmatrix} \psi = \frac{id}{r} \psi = \frac{k}{r} \psi \quad \text{if} \quad k = i\omega
\]

\[
\frac{d\psi_1}{dr} - \frac{k}{r} \psi_1 = \psi_2
\]

\[
\frac{d\psi_2}{dr} + \frac{k}{r} \psi_2 = \psi_1
\]

\[
\left( \frac{d}{dr} + \frac{k}{r} \right) \left( \frac{d}{dr} - \frac{k}{r} \right) \psi_1 = \psi_1
\]

\[
\left( \frac{d^2}{dr^2} + \frac{k}{r^2} - \frac{k^2}{r^2} \right) \psi_1 = \psi_1
\]

\[
\frac{d^2\psi_1}{dr^2} - \frac{k(k-1)}{r^2} \psi_1 = \psi_1
\]

\[
\frac{d^2\psi_2}{dr^2} - \frac{k(k+1)}{r^2} \psi_2 = \psi_2
\]

I'm interested in working on the interval \([a, \infty)\) for some \( a > 0 \). Hence asymptotically as \( r \to \infty \) one should have

\[
\begin{align*}
\psi_1 & \sim c e^{-r} \\
\psi_2 & \sim -c e^{-r}
\end{align*}
\]

\( c \) constant.
Since I expect the solutions $\psi$ to be something like Bessel functions, let's try power series expansions

$$\psi = r^\mu \sum_{n=0}^{\infty} \left( \frac{a_n}{b_n} \right) r^n$$

$$\frac{d\psi}{dx} = \mu r^{\mu-1} \sum (a_n) r^n + r^\mu \sum (a_n) nr^{n-1}$$

$$\frac{k}{r} \left( \frac{\psi_2}{\psi_1} \right) = k r^{\mu-1} \sum (-a_n) r^n$$

$$\left( \frac{\psi_2}{\psi_1} \right) = r^\mu \sum (-\frac{b_n}{a_n}) r^n$$

$$0 = r^{\mu-1} \left[ (\mu-k)(a_n) + k (-a_n) \right]$$

$$+ \sum_{n=1}^{\infty} \left\{ (\mu+n)(a_n) + k (-a_n) - \left( \frac{b_{n-1}}{a_{n-1}} \right) r^{n-1} \right\}$$

The indicial equation is:

$$(\mu-k) a_0 = 0$$

$$(\mu+k) b_0 = 0$$

Other equations:

$$(\mu+n-k) a_n = b_{n-1}$$

$$(\mu+n+k) b_n = a_{n-1}$$

so the roots are $\mu = \pm k$. Assume the difference of the indicial roots is not integral i.e. $2k \not\in \mathbb{Z}$. Then from each root we get a solution and the two solutions are linearly independent.

The root $\mu = k$, $a_0 = 1$, $b_0 = 0$

$$a_n = \frac{b_{n-1}}{\mu+n-k} = \frac{a_{n-2}}{(\mu+n-k)(\mu+n+k-1)} = \frac{a_{n-2}}{n(n+2k-1)}$$
\[ b_n = \frac{a_{n-1}}{(\mu+n+k)} = \frac{b_{n-2}}{(\mu+n+k)(\mu+n-k-1)} = \frac{b_{n-2}}{(n+2k)(n-1)} \]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \frac{r}{2k+1} & \frac{r^2}{2(2k+1)} & 0 \\
0 & \frac{r^3}{2(2k+1)(2k+3)} & \frac{r^4}{2(2k+1)(2k+3)(2k+5)} & 0 \\
0 & & & \frac{r^5}{2(2k+1)(2k+3)(2k+5)} \\
\end{array}
\]

It seems that \( \frac{d}{dr} (r^{-k} \psi_1) = r^{-k} \psi_2 \)

\( \frac{d}{dr} (r^k \psi_2) = r^k \psi_1 \)

as they should be.

\[ \psi_1 = r^{-k} \sum_{m=0}^{\infty} \frac{r^{-2m}}{2 \cdot 4 \cdot \cdots 2m} \frac{1}{(2k+1)(2k+3) \cdots (2k+2m-1)} \]

\[ = r^{-k} \sum_{m=0}^{\infty} \frac{(\frac{r}{2})^{2m}}{m!} \frac{1}{(k+\frac{1}{2})(k+\frac{1}{2}+1) \cdots (k+\frac{1}{2}+m-1)} \cdot \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2}+m)} \]

\[ = r^{-k} \sum_{m=0}^{\infty} \frac{(\frac{r}{2})^{2m}}{m!} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2}+m)} \]
\[
\psi_1 = \left( \frac{2}{\lambda} \right)^{k-1/2} \Gamma(\frac{k+1}{2}) \frac{1}{2^{k-1/2}} \sum_{m=0}^{\infty} \left( \frac{r}{2} \right)^{2m} \frac{1}{m!} \frac{1}{\Gamma(m+1+\frac{k-1}{2})}
\]

But

\[
J_\lambda(iz) = \frac{(iz)^\lambda}{\Gamma(\lambda)} \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m}}{m!} \frac{1}{\Gamma(m+1+\lambda)}
\]

so it appears that

\[
\psi_1 = \left( \frac{2}{\lambda} \right)^{k-1/2} \Gamma(\frac{k+1}{2}) \frac{1}{2^{k-1/2}} J_{k-1/2}(ir)
\]

Similarly

\[
\psi_2 = r^k \frac{r}{2^{k+1/2}} \sum_{m=0}^{\infty} \frac{r^{2m}}{2^{2m} m!} \frac{1}{(k+\frac{3}{2}) \cdots (k+\frac{1}{2}+m)} \frac{\Gamma(k+1/2)}{\Gamma(k+1/2)}
\]

\[
= \frac{1}{2} \left( \frac{2}{\lambda} \right)^{k+1/2} \Gamma(\frac{k+1}{2}) \left( \frac{i\lambda}{2} \right)^{k+1/2} \frac{1}{2^{k+1/2}} \sum_{m=0}^{\infty} \frac{(\frac{r}{2})^{2m+1}}{m!} \frac{1}{\Gamma(m+1+\frac{k+1}{2})}
\]

\[
= \frac{1}{2} \left( \frac{2}{\lambda} \right)^{k+1/2} \Gamma(\frac{k+1}{2}) \frac{1}{i} \frac{1}{2^{k+1/2}} J_{k+1/2}(ir)
\]

so

\[
\psi_1 = c \frac{1}{\lambda} \frac{1}{2^{k-1/2}} J_{k-1/2}(ir)
\]

\[
\psi_2 = \frac{1}{\lambda} c \frac{1}{2^{k+1/2}} J_{k+1/2}(ir)
\]

where \( c \) is a constant (\( k \) fixed). Can check this using

\[
\frac{d}{dz} J_\lambda(z) = \frac{\lambda}{2} J_{\lambda-1}(z) - J_{\lambda+1}(z)
\]
\[
\frac{d}{dr} \left( r^{\frac{1}{2}} J_{k-\frac{1}{2}} (ir) \right) = r^{\frac{1}{2}} \left[ \frac{k-\frac{1}{2}}{ir} J_{k-\frac{1}{2}} (ir) - \frac{1}{2} r^{-\frac{1}{2}} J_{k-\frac{1}{2}} (ir) \right] i + \frac{1}{2} r^{-\frac{1}{2}} J_{k-\frac{1}{2}} (ir)
\]

\[
= \frac{k}{r} \left( r^{\frac{1}{2}} J_{k-\frac{1}{2}} (ir) \right) + \frac{i}{r} \left( r^{\frac{1}{2}} J_{k-\frac{1}{2}} (ir) \right)
\]

so it works. Other solution should be

\[
\psi_1 = r^{\frac{1}{2}} J_{k-\frac{1}{2}} (ir)
\]

\[
\psi_2 = \frac{i}{2} r^{\frac{1}{2}} J_{k-\frac{1}{2}} (ir)
\]

Check

\[
\frac{d\psi_2}{dr} = i r^{\frac{1}{2}} \left[ \frac{-k-\frac{1}{2}}{ir} J_{k-\frac{1}{2}} (ir) - \frac{1}{2} r^{-\frac{1}{2}} J_{k-\frac{1}{2}} (ir) \right] i + \frac{i}{2} r^{-\frac{1}{2}} J_{k-\frac{1}{2}} (ir)
\]

\[
= -\frac{k}{r} \left( r^{\frac{1}{2}} J_{k-\frac{1}{2}} (ir) \right) + r^{\frac{1}{2}} J_{k+\frac{1}{2}} (ir)
\]

\[
K_s (r) = \int_{-\infty}^{\infty} e^{-r \cosh t} e^{st} \, dt
\]

\[
\frac{d}{dr} K_s (r) = \int e^{-r \cosh t} (-\cosh t) e^{st} \, dt
\]

\[
= -\frac{1}{2} \int e^{-r \cosh t} (e^t + e^{-t}) e^{st} \, dt
\]

\[
\frac{dK_s (r)}{dr} = -\frac{1}{2} \left( K_{s+1} (r) + K_{s-1} (r) \right)
\]

\[
s K_s (r) = \int_{-\infty}^{\infty} e^{-r \cosh t} s e^{st} \, dt = -\int (e^{-r \cosh t})' e^{st} \, dt
\]
\[ sK_s(r) = \int e^{-r \cos \theta} r \sin \theta \, dt = \frac{1}{2} \int e^{-r \cos \theta} (e^t - e^{-t}) e^s \, dt \]

\[
\frac{s}{r} K_s(r) = \frac{1}{2} (K_{s+1}(r) - K_{s-1}(r))
\]

\[
\frac{dK_s}{dr} + \frac{s}{r} K_s = -K_{s-1}(r) \quad \frac{dK_s}{dr} - \frac{s}{r} K_s = -K_{s+1}
\]

\[
\frac{dK_{s-\frac{1}{2}}}{dr} - \frac{(s-\frac{1}{2})}{r} K_{s-\frac{1}{2}} = -K_{s+\frac{1}{2}}
\]

\[
\frac{dK_{s+\frac{1}{2}}}{dr} + \frac{(s+\frac{1}{2})}{r} K_{s+\frac{1}{2}} = -K_{s-\frac{1}{2}}
\]

Hence

\[
\frac{dK_{s-\frac{1}{2}}}{dr} - \frac{(s-\frac{1}{2})}{r} K_{s-\frac{1}{2}} = -K_{s+\frac{1}{2}}
\]

Now if we put \( \psi = r^{\frac{1}{2}} \phi \) in the equations

\[
\frac{d\psi_1}{dr} - \frac{s}{r} \psi_1 = \psi_2
\]

\[
\frac{d\psi_2}{dr} + \frac{s}{r} \psi_2 = \psi_1
\]

we get

\[
\frac{d\phi_1}{dr} + \frac{1}{2} r^{-\frac{1}{2}} \phi_1 - \frac{s}{r} r^{\frac{1}{2}} \phi_1 = r^{\frac{1}{2}} \phi_2
\]

\[
\frac{d\phi_1}{dr} - \frac{(s-\frac{1}{2})}{r} \phi_1 = \phi_2 \quad \text{etc.}
\]

hence we see that the equations \( \bigcirc \) have the solution
\[ y = \begin{pmatrix} r^{1/2} K_{s-1/2}(r) \\ -r^{1/2} K_{s+1/2}(r) \end{pmatrix} \]

This should be the unique solution of \( \otimes \) which vanishes as \( r \to +\infty \). Consequently I should know that for any real \( \Theta \) the equation

\[ K_{i\lambda - \frac{1}{2}}(r) = e^{i\Theta} K_{i\lambda + \frac{1}{2}}(r) \]

has only real solutions in \( \lambda \) for \( r \) real > 0.

Since

\[ y = \int_{-\infty}^{\infty} r^{1/2} e^{-r \cos \Theta} \begin{pmatrix} e^{-\frac{1}{2}t} \\ -e^{\frac{1}{2}t} \end{pmatrix} e^{st} dt \]

one has

\[ u(r, t) = r^{1/2} e^{-r \cos \Theta} \begin{pmatrix} e^{-\frac{1}{2}t} \\ -e^{\frac{1}{2}t} \end{pmatrix} \]

is "the "privileged” solution of the wave equation

\[ \frac{1}{i} \frac{\partial u}{\partial t} + \frac{1}{r} \left( \frac{2}{r} - \frac{\partial^2}{\partial r^2} \right) u = 0 \]

Furthermore:

Does "privileged” have a sense?

Continued fraction expansion for \( K_{s-1}/K_s \)
Consider again

\[(1) \quad \frac{d}{dx}\left( \frac{\psi}{x} \right) = \lambda \psi \quad \text{p real} \]

on \(0 < x < \infty\) and let the solution matrix be

\[S(x, \lambda) = \left( \begin{array}{c} \psi_1(x, \lambda) \\ \psi_2(x, \lambda) \end{array} \right) \]

The columns of \(S(x, \lambda)\) are vector functions \(\psi_1(x, \lambda), \psi_2(x, \lambda)\) satisfying the DE with initial values \((i)\) and \((i)\) at \(x=0\). Hence that \(S(x, \lambda)\) I prefer the notation:

\[S(x, \lambda) = \left( \psi_1(x, \lambda), \psi_2(x, \lambda) \right) \]

For example:

\[S(x, \lambda) = \left( \begin{array}{cc} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{array} \right) \]

Now let's suppose that \(p \to +\infty\) as \(x \to +\infty\). Then there should be a unique solution \(S\) which dies at \(\infty\), unique up to a scalar multiple, which we can write uniquely as

\[m(\lambda) \psi_1(x, \lambda) + \psi_2(x, \lambda) \]

Here \(m(\lambda)\) is a meromorphic function of \(\lambda\) whose poles occur at those \(\lambda\) such that \(\psi'(x, \lambda)\) dies at \(\infty\). So if we factor

\[m(\lambda) = \frac{m_1(\lambda)}{m_2(\lambda)} \]

with \(m_2(\lambda)\) entire and relatively prime, then we get a solution.
\[ \psi(x, \lambda) = m_1(\lambda) \psi^1(x, \lambda) + m_2(\lambda) \psi^2(x, \lambda) \]

entire in \( \lambda \), not identically zero in \( x \) for any \( \lambda \), which dies at \( x = \pm \infty \). Clearly \( \psi(x, \lambda) \) is unique up to multiplying by an invertible entire function of \( \lambda \). If we can produce a \( \psi(x, \lambda) \) which is of exponential type, then the only possible variation of it would be by a \( g \) of the form \( e^{a \lambda + b} \), \( a, b \) constants.

Put \( u(x, t) = (e^{-itP} f)(x) \), where \( P = \frac{i}{t} (\frac{d}{dx} - \frac{p}{p} \frac{d}{dx}) \).

Then \( u \) satisfies

\[
\frac{1}{i} \frac{\partial u}{\partial t} = -Pu \quad u
\]

(2)

\[
\left( \frac{\partial u}{\partial t} + \left( \frac{\partial}{\partial x} - \frac{p}{p} \frac{\partial}{\partial x} \right) \right) u = 0
\]

and \( u(x, 0) = f(x) \). Thus the operator \( e^{-itP} \) solves the Cauchy problem on \( t = 0 \) for the wave equation (2). Notice also that if \( \psi(x, \lambda) \) satisfies (1)

i.e.

(1) \[ P \psi(x, \lambda) = \lambda \psi(x, \lambda) \]

then assuming we can Fourier transform in \( \lambda \) we get that

(3) \[ u(x, t) = \int e^{-it\lambda} \psi(x, \lambda) d\lambda \]

satisfies

\[
\frac{1}{i} \frac{\partial u}{\partial t} = -\int e^{-it\lambda} \lambda \psi(x, \lambda) d\lambda = -Pu
\]

Thus the Fourier transform (3) sets up a correspondence between solutions of (1) and (2).
The problem now is to understand solutions of the wave equation (2). Think globally in t, x and (more or less) locally in x. Philosophy:

The totality of all $u(x, t)$ solving the wave equation vanishing at $x = \infty$ and rapidly decreasing in $t$ can be identified with the totality of functions $a(\lambda) \Phi(\lambda, x)$ where $\Phi(x, \lambda)$ is the good solution near $x = \infty$ described at the top of page 19.

April 26, 1977 I want to consider the problem of relating solution matrices to fundamental solutions for the wave equation. Let's start with the example

$$P = \frac{1}{i} \frac{d}{dx}$$

Here we want to relate the solution of

$$\begin{cases}
\frac{d^2 \psi}{dx^2} = \lambda \psi \\
\psi(0) = 1
\end{cases}$$

which is $\psi(x, \lambda) = e^{i\lambda x}$ to a fundamental solution $E$ for $P$ which is a solution of

(1) $PE = S$.

Better to write $(P - \lambda) E_1 = S$. Suppose $\lambda = 0$. Then the solutions of $P \psi = 0$ are the constants and a particular fundamental solution for (1) is

$$E = i \theta$$

where $\theta$ is the Heaviside fnc. $$\begin{cases}
0 & x < 0 \\
1 & x = 0 \\
\theta & x > 0
\end{cases}$$
Thus the possible fundamental solutions for $P$ are

$$i \Theta + \text{const}.$$ 

Hence there are unique forward and backward fundamental solutions. The same holds for

$$\lambda (P-I) \mathcal{E}_\lambda = \delta.$$ 

The solutions are

$$E_\lambda = e^{i \Theta} e^{i \lambda x} + (\text{const}) e^{-i \lambda x} = e^{i \lambda x} (i \Theta + \text{const})$$

so again there are unique forward and backward solutions. Notation

forward

$$E_\lambda^+ = e^{i \lambda x} i \Theta(x)$$

backward

$$E_\lambda^- = -e^{i \lambda x} i \Theta(-x) = e^{i \lambda x} i (\Theta(x) - 1)$$

One has

$$E_\lambda^+ - E_\lambda^- = e^{i \lambda x} i = i \psi(x)$$

(this solution of $P \psi = \lambda \psi$ with initial value $i$).

Now take Fourier transform:

$$\tilde{E}_\lambda^+ = \int e^{-i \lambda t} e^{i \lambda x} (i \Theta(x) + c) \, dx = 2\pi \delta(x-t) (i \Theta(x) + c)$$

$$= 2\pi i \delta(x-t) [\Theta(x) + c]$$

$$\tilde{E}_\lambda^+(x,t) = 2\pi i \delta(x-t) \Theta(x)$$

should be solutions of

$$\frac{1}{i \partial_t} \tilde{E} + P \tilde{E} = 2\pi \delta(x) \delta(t)$$
\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \delta(x-t) \Theta(x) = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \delta(x-t) \left[ \Theta(x) + \delta(x-t) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \Theta(x) \right] \\
\quad = \delta(x-t) \delta(x) \\
\quad = \delta(t) \delta(x)
\]

**Prop.** For the wave equation \( \left( \frac{1}{i} \frac{\partial}{\partial t} + \frac{1}{i} P \right) \psi(x) = 0 \), one has the fundamental solutions:

- **Forward:** \( F^+(x, t) = i \delta(x-t) \Theta(x) = \frac{1}{2\pi} \int e^{-i\lambda t} (e^{i\lambda x} \Theta(x)) \, dx \)
- **Backward:** \( F^-(x, t) = i \delta(x-t) [\Theta(x) - 1] \)

\[
F^+ - F^- = i \delta(x-t) = \frac{1}{2\pi} \int e^{-i\lambda t} (e^{i\lambda x} - 1) \, dx
\]

solution of \( \frac{1}{i} \frac{dx}{dx} = \lambda K \)

with \( \psi(0, x) = 1 \).

Next consider the system

\[
P \psi = \frac{1}{i} \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & -\frac{d}{dx} \end{pmatrix} \psi = \lambda \psi
\]

The solution matrix for initial values at \( x = 0 \) is

\[
\psi = \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix}
\]

**Forward fundamental solution:**

\[
E^+(x, \lambda) = \begin{pmatrix} e^{i\lambda x} & i \Theta(x) \\ -e^{-i\lambda x} & i \Theta(x) \end{pmatrix}
\]

**Backward fundamental solution:**

\[
E^-(x, \lambda) = \begin{pmatrix} e^{i\lambda x} (\Theta(x) - 1) & 0 \\ 0 & e^{-i\lambda x} (\Theta(x) - 1) \end{pmatrix}
\]
Fundamental solns.

\[ E^+(x,\lambda) = \begin{pmatrix} e^{i\lambda x} i\Theta(x) & 0 \\ 0 & -e^{i\lambda x} i\Theta(x) \end{pmatrix} \]

\[ E^-(x,\lambda) = \begin{pmatrix} e^{i\lambda x} i(\Theta(x)-1) & 0 \\ 0 & -e^{-i\lambda x} i(\Theta(x)-1) \end{pmatrix} \]

Again \( E^+ - E^- = i \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix} \)

so for the wave equation

\[ \frac{i}{i} \frac{\partial u}{\partial t} + Pu = i \begin{pmatrix} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \end{pmatrix} u = 0 \]

one has the fundamental solutions

\[ F^+(x,t) = \begin{pmatrix} i\delta(x-t)\Theta(x) & 0 \\ 0 & -i\delta(x+t)\Theta(x) \end{pmatrix} \]

\[ F^-(x,t) \text{ same with } \Theta(x) \mapsto \Theta(x)-1 \]

and \( F^+ - F^- = \int \frac{1}{2i\pi} e^{-i\lambda t} \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix} dx \)

matrix solution of \( Pu = f \) with initial value \( u \) at \( x = 0 \)
Note that $\delta(x-t) \theta(x) = \delta(x-t) \theta(x+t)$, so we can write

$$F^+(x,t) = \begin{pmatrix} i \delta(x-t) \theta(x+t) & 0 \\ 0 & -i \delta(x+t) \theta(x-t) \end{pmatrix}$$

Return to Hörmander's analysis in the case of

$$P \psi = \frac{i}{\hbar} \begin{pmatrix} \frac{\partial}{\partial x} & -P \\ P & -\frac{\partial}{\partial x} \end{pmatrix} \psi = \lambda \psi$$

and suppose we work on $0 \leq x \leq b$ finite with given self-adjoint boundary conditions. Then we get eigenfunctions $\psi_n(x)$ and can form

$$e^{-itP} = \sum_n e^{-it\lambda_n} \psi_n(x) \psi_n(x)^* \quad \|\psi_n\| = 1.$$

This satisfies the Cauchy problem

$$\frac{i}{\hbar} \frac{\partial K}{\partial t} + PK = 0$$

$$K_0(x,y) = \delta(x-y)$$

as well as the boundary conditions at $x=0$, $x=b$. But the point is that the value at $(x,t)$ is determined by $\delta(x',y)$ for $|x'-x| \leq |t|$. 

\[\text{Diagram:}\]

\[(x,t)\]

\[x', \ldots\]
Hence $K_t(x,y)$ is independent of the boundary conditions for $t$ small, i.e. $|t| \leq x$ and $|t| \leq b-x$. Now one chooses a $\hat{p}(t)$ such that $\hat{p}(t)$ is supported in $|t| \leq x$. Then $\hat{p}(t)K_t(x,y)$ or at least its singularities in $x$ is known.

To be specific suppose $P = \frac{1}{i\pi} \hat{\phi}$ whence the wave equation is $\frac{\partial K}{\partial t} + \frac{\partial K}{\partial x} = 0$ so $K$ is a function of $x-t$. Then

$$K_t(x,y) = \delta(x-t-y)$$

near $t=0$. So $\hat{\delta}(t)\delta(x-t-y)$ has an inverse transform

$$\frac{1}{2\pi} \int e^{i\lambda t} \hat{\delta}(t) \delta(x-t-y) \, dy = \frac{1}{2\pi} \int e^{i\lambda t} K_t(x,y) \hat{p}(t) \, dt$$

$$\int \mu(\lambda) e^{-i\mu(x-y)} \, d\mu = \sum p(\lambda-\lambda_n) \psi_n(x) \psi_n(y)^*$$

so if we take $x=y$ we get

$$\int \mu(\mu) \, d\mu = \sum p(\lambda-\lambda_n) \psi_n(x) \psi_n(x)^*$$

showing the right side is independent of $\lambda$. If one thinks of the RHS as giving an average of $\psi_n(x) \psi_n(x)^*$ for $\lambda_n$ in some big neighborhood of $\lambda$, the above is clearly consistent with the measure

$$d\nu(x,x) = \sum \psi_n(x) \psi_n(x)^* \delta(\lambda-\lambda_n)$$

being asymptotically equivalent to Lebesgue measure $dx$. 