The spectral function of a SL problem start with the problem on a finite interval.

\[ \begin{cases} \dot{u} + \lambda u = 0 \\ u(a) = u(b) = 0 \end{cases} \]

Let \( u_\lambda(x) \) be the solution of

\[ \begin{cases} \dot{u}_\lambda + \lambda u_\lambda = 0 \\ u_\lambda(a) = 0 \\ u_\lambda(b) = 1 \end{cases} \]

so that the eigenvalues are the roots of \( u_\lambda(b) = 0 \). Let \( \lambda_1 < \lambda_2 < \ldots \) be the eigenvalues.

If \( f = \sum a_j u_j \) then

\[ (f, u_j) = a_j \| u_j \|^2 \]

\[ a_j = \frac{(f, u_j)}{\| u_j \|^2} \]

\[ \| f \|^2 = \sum |a_j|^2 \| u_j \|^2 = \sum \frac{|(f, u_j)|^2}{\| u_j \|^2} \]

These formulas can be written

\[ f = \int u_\lambda (f, u_\lambda) d\lambda \]

\[ \| f \|^2 = \int |(f, u_\lambda)|^2 d\lambda \]

where \( d\lambda \) is the measure with mass \( \| u_j \|^2 \) at \( \lambda_j \).

According to the spectral theorem

\[ L = \int A dE_\lambda \]

where \( E_\lambda \) is the orthogonal projection on the subspace spanned by the \( u_j \) with \( \lambda_j < \lambda \). Thus

\[ E_\lambda f = \int E_\lambda (f, u_\lambda) d\lambda \]
so \( E_\mu \) is represented by the kernel
\[
e(x, y, \mu) = \int_{-\infty}^{\mu} u_\lambda(x) \overline{u_\lambda(y)} \, d\mu(\lambda).
\]

Hörmander calls \( e(x, y, \mu) \) the spectral function of the self-adjoint operator defined by (1). He computes an asymptotic expansion of \( e(x, y, \mu) \) as \( \mu \to \infty \). Since
\[
N(\lambda) = \text{number of } \lambda_j \leq \lambda = \text{tr} (E_\lambda) = \int_{-b}^{b} e(x, x, \lambda) \, dx
\]
asymptotically, he obtains from his estimate for \( e(x, y, \lambda) \) an asymptotic estimate for \( N(\lambda) \).

**Note that** \( \int_{-\infty}^{\infty} dE_\lambda = I \), so that one should have
\[
\lim_{\lambda \to \infty} e(x, y, \lambda) = \delta(x-y)
\]

---

**April 13, 1977**

Consider the Schrödinger equation
\[
(1) \quad \frac{d^2 \psi}{dx^2} + (\lambda - g) \psi = 0
\]
and let us try to construct a solution of the form
\[
\psi(x, \lambda) = e^{iS(x, \lambda)} u(x, \lambda).
\]

\[
\frac{d^2 \psi}{dx^2} + (\lambda - g) \psi = \frac{d}{dx} \left( e^{iS} (iS_x u + u_x) \right) + (\lambda - g) \psi
\]
\[
= e^{iS} (-S_x^2 u + 2iS_x u_x + iS_{xx} u + u_{xx} + (\lambda - g) u)
\]

Let us choose \( S \) so that it satisfies the eikonal equation
\[
S_x^2 = \lambda - g
\]
Specifically, put \( S_x = \sqrt{\lambda - g} \) and
\[
S(x, \lambda) = \int_{0}^{x} \sqrt{\lambda - g(t)} \, dt
\]
so that \( S(0, \lambda) = 0 \) and
\[
S(x, \lambda) \sim \sqrt{\lambda} x \quad \lambda \to \infty \quad x \text{ bdd.}
\]
To have \( u \) a solution we need in addition
\[
2i S_x u_x + i S_{xx} u + u_{xx} = 0
\]
\[
2i \sqrt{\lambda-\theta} u_x + \frac{i}{2} \frac{-8(x)}{\sqrt{\lambda-\theta(x)}} u + u_{xx} = 0
\]

or
\[
u_x = \frac{i}{2} (\lambda - \theta)^{-1/2} u_{xx} + \frac{q'(x)}{4} (\lambda - \theta)^{-3/2} u
\]
\[
= \lambda^{-1/2} (1 + \frac{\theta}{2\lambda} + \ldots) \frac{i}{2} u_{xx} + \lambda^{-1/2} [1 + \frac{\theta}{2\lambda} + \frac{\theta^2}{4\lambda^2} + \ldots] \frac{q'}{4} u
\]

From this equation we can construct \( u \) by interation a formal solution
\[
 u(x, \lambda) = u_0(x) + u_1(x) \lambda^{-1/2} + u_2(x) \lambda^{-1} + u_3(x) \lambda^{-3/2} + \ldots
\]
with \( u(0, \lambda) = 1 \). For example from
\[
u_0'(x) + u_1(x) \lambda^{-1/2} + u_2(x) \lambda^{-1} + \cdots = \lambda^{-1/2} (1 + \frac{\theta}{2\lambda} + \frac{3\theta^2}{8\lambda^2} + \cdots) \frac{i}{2} (u_0'' + u_1'' \lambda^{-1/2} + \cdots)
\]
\[
+ \lambda^{-1} (1 + \frac{\theta}{2\lambda} + \frac{\theta^2}{4\lambda^2} + \cdots) \frac{q'}{4} (u_0 + u_1 \lambda^{-1/2} + \cdots)
\]
we get
\[
u_0'(x) = 0 \quad \text{hence} \quad u_0(x) = 1.
\]
\[
u_1'(x) = \frac{i}{2} u_0'' = 0 \quad \text{hence} \quad u_1(x) = 0
\]
\[
u_2'(x) = \frac{q'}{4} u_0 = \frac{q'}{4} \quad \text{hence} \quad u_2(x) = \frac{q(x) - q(0)}{4}
\]

Note on the other hand that if \( q' = 0 \), then \( u(x, \lambda) \)
is to satisfy
\[
2i \sqrt{\lambda-\theta} u_x + u_{xx} = 0
\]
\[
2i \sqrt{\lambda-\theta} x + \ln(u_x) = \text{const}
\]
\[
u_x = u_x(0, \lambda) e^{-2i \sqrt{\lambda-\theta} x}
\]
\[ u(x, \lambda) - u(0, \lambda) = u_x(0, \lambda) \frac{e^{-2i\sqrt{\lambda - \delta}x} - 1}{-2i\sqrt{\lambda - \delta}x} \]

But it's clear that \( u(x, \lambda) \) won't have an asymptotic expansion at \( \lambda \to \infty \) unless \( u(x, \lambda) = 1 \).

So it seems that the Schrödinger equation (1) has a unique solution of the form

\[ \psi(x, \lambda) = e^{iS(x, \lambda)} u(x, \lambda) \]

with \( S(x, \lambda) = \int_0^x \sqrt{\lambda - g(t)} \, dt \) and where \( u \) is a formal series

\[ u(x, \lambda) = u_0(x) + u_1(x) \lambda^{-1/2} + u_2(x) \lambda^{-1} + \cdots \]

where \( u(0, \lambda) = 1 \).

However it should be possible to construct a solution \( \psi^+(x, \lambda) \) of the Schrödinger equation with

\[ \psi^+(0, \lambda) = 1 \]

\[ \psi^+(x, \lambda) \sim e^{i\sqrt{\lambda}x} \quad \text{as} \quad \lambda \to \infty \]

(Now observe that the virtue of the phase function \( S(x, \lambda) \) seems to be that in the asymptotic expansion \( u(x, \lambda) \)

\[ = 1 + O(\lambda^{-1}) \] uniformly for \( x \) fixed. Thus

\[ e^{i\sqrt{\lambda}x} \int_0^x (1 - \lambda^{-1}) dt = e^{i\sqrt{\lambda}x} \int_0^x (-\lambda^{-1/2} + \cdots) dt \]

\[ = e^{i\sqrt{\lambda}x} \int_0^x (1 - \lambda^{-1/2} + \cdots) dt \]

so one could instead find a solution of the form

\[ \psi(x, \lambda) = e^{i\sqrt{\lambda}x} (u_0(x) + u_1(x) \lambda^{-1/2} + \cdots) \quad \psi(0, \lambda) = 1 \]

except that one only has \( u_0(x) = 1 \) and

\[ u_1 \neq 0 \quad \text{in fact} \quad u_1 = -\frac{i}{2} \int_0^x g(t) \, dt \]
April 15, 1977

\[ u_2' \lambda^{-1} + u_3' \lambda^{-3/2} + \ldots = \lambda^{-1/2} \left[ 1 + \frac{\epsilon}{\lambda} + \frac{3 \epsilon^2}{8 \lambda^2} + \ldots \right] \left( \frac{1}{2} \right) (u_2' \lambda^{-1} + u_3' \lambda^{-3/2}) + \lambda^{-1} \left( 1 + \frac{\epsilon}{\lambda} + \frac{3 \epsilon^2}{8 \lambda^2} + \ldots \right) \left( \frac{1}{4} \right) (1 + u_2' \lambda^{-1} + u_3' \lambda^{-3/2}) \]

\[ u_2' = \frac{q'}{q} \quad \text{corol of } \lambda^{-1} \]

\[ u_3' = \frac{1}{2} u_2'' \quad \text{corol of } \lambda^{-3/2} \]

\[ u_4' = \frac{q'}{q} (q + u_2) \quad \text{corol of } \lambda^{-2} \]

so

\[ u_2 = \frac{g(x) - g(0)}{4} \]

\[ u_3 = i \frac{g'(x) - g'(0)}{8} \]

So

\[ \psi(x, \lambda) = e^{i \int_0^x (\lambda - q(x)) \frac{1}{2} dt} \left[ 1 + \frac{1}{4} (g(x) - g(0)) \lambda^{-1} + \frac{i}{8} (g'(x) - g'(0)) \lambda^{-3/2} + \ldots \right] \]

is the asymptotic formula. This shows that we need to assume \( q \) is \( C^\infty \) in order to get the full asymptotic expansion.

**Question.** Let \( f(z) \) be an entire function of \( z \). When is it of the form \( e^{az^2} u_1(\frac{1}{z}) + e^{az^2} u_2(\frac{1}{z}) \) with \( u_1 \) and \( u_2 \) holomorphic at 0, and how unique is this representation?

For example, take a component \( \psi(x, \lambda) \) in the solution matrix for \( \frac{d^2 \psi}{dx^2} + (\lambda - q(x)) \psi = 0 \) with \( q \) analytic near zero. Then does one have a representation

\[ \psi(x, \lambda) = e^{i \lambda^{1/2} x} u_1(x, \lambda) + e^{-i \lambda^{1/2} x} u_2(x, \lambda) \]

where \( u_i \) are analytic in \( x, \lambda^{-1/2} \) near \( \lambda^{1/2} = \infty \)?
Example: Take \( P = \frac{1}{i} \frac{d}{dx} \) on \([0, 2\pi]\) with the periodic boundary conditions \( u(x+2\pi) = u(x) \). Better to replace the line by \( S^1 \). Then
\[
(e^{itP} f)(x) = (e^{\frac{it}{dx}} f)(x) = f(x+t) = \int_S s(x+t-y) f(y) \, dy
\]
so \( s(x+t-y) \) is the kernel representing \( e^{itP} \). On the other hand the spectral function \( e(x,y,\lambda) \) is defined by
\[
P = \int \lambda \, dE_\lambda \quad \quad \quad (E_\lambda f)(x) = \int e(x,y,\lambda) f(y) \, dy
\]
so
\[
e^{itP} = \int e^{it\lambda} \, dE_\lambda
\]
on taking kernels
\[
s(x+t-y) = \int e^{it\lambda} \, e(x,y,\lambda) = \int e^{it\lambda} \, \frac{de(x,y,\lambda)}{d\lambda}
\]
Fourier inversion gives
\[
\frac{de(x,y,\lambda)}{d\lambda} = \frac{1}{2\pi} \int e^{-it\lambda} \, s(x+t-y) \, dt
\]
the image in \( S^1 \) of \( \lambda \) contributes when \( t = y - x \).
\[
= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-i(\lambda - 2\pi n) \lambda}
\]
\[
= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-i2\pi n} \, e^{-i2\pi nx} \, e^{-i2\pi ny}
\]
\[
= \frac{1}{2\pi} \, \delta_{\lambda} (\lambda) \, e^{-i2\pi nx} \, e^{-i2\pi ny}
\]
Thus
\[
e_{\lambda}(x,y) = \sum_{n \in \mathbb{Z}} \frac{e^{inx} e^{-iny}}{2\pi}
\]
Perhaps a better formula, since \( \lambda \) can go to \( -\infty \) is
\[
e_{\lambda}(x,y) - e_{\mu}(x,y) = \sum_{\mu < n \leq \lambda} \frac{e^{inx} e^{-iny}}{2\pi}
\]
April 17, 1977.

Formula for systems:

\[
\frac{d}{dx} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} i\lambda & -p \\ p & -i\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

\[
\frac{1}{i} \begin{pmatrix} \frac{d}{dx} & -p \\ p & -\frac{d}{dx} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

When \( p \) is constant \( u_1, u_2 \) are both killed by

\[
\begin{vmatrix} i\lambda - \frac{d}{dx} & -p \\ p & -i\lambda - \frac{d}{dx} \end{vmatrix} = \frac{d^2}{dx^2} + (\lambda^2 - p^2).
\]

---

I want to understand H"ormander's method for obtaining the eigenvalue distribution.

Example: Consider \( p = \frac{1}{i} \frac{d}{dx} \)

operating on functions \( \phi(x) \) \( 0 \leq x \leq 2\pi \). We require periodic boundary conditions,

\( \phi(x + 2\pi) = e^{i\Theta} \phi(x) \).

Now H"ormander considers the operator \( e^{itP} \). First one notes that \( P \) has the eigenfunctions

\( e^{i\lambda x} \) which have \( \int_{0}^{2\pi} e^{i\lambda x} \cdot e^{i\lambda x} \cdot dx = 2\pi \)

where \( e^{2\pi i\lambda} = e^{i\Theta} \) i.e. \( 2\pi \lambda = \Theta + 2n\pi \)

or \( \lambda = n + \frac{\Theta}{2\pi} \) \( n \in \mathbb{Z} \).
Hence \( e^{itP} \) is represented by the kernel

\[
\sum_n e^{it(n+\frac{\theta}{2\pi})} e^{i(n+\frac{\theta}{2\pi})x} e^{-i(n+\frac{\theta}{2\pi})y} \frac{1}{2\pi}
\]

\[
e^{it(x-y)\frac{\theta}{2\pi}} \sum_n \delta_{2\pi\mathbb{Z}}(t+x-y)
\]

In the above \( 0 \leq x, y \leq 2\pi \) but \( t \in \mathbb{R} \).

Here's how to get the eigenvalue distribution from \( e^{itP} \). Since

\[
P = \sum \lambda dE_{\lambda}, \quad e^{itP} = \sum \text{e}^{it\lambda} dE_{\lambda}
\]

is represented by the kernel \( \int e^{it\lambda} dE_{\lambda}(x,y) \)

where \( E_{\lambda}(x,y) \) is the kernel representing \( E_{\lambda} \). Thus we can get \( \frac{dE_{\lambda}}{d\lambda}(x,y) \) by Fourier inversion:

\[
\frac{dE_{\lambda}(x,y)}{d\lambda} = \frac{1}{2\pi} \int e^{-it\lambda} e^{i(t+x-y)\frac{\theta}{2\pi}} \delta_{2\pi\mathbb{Z}}(t+x-y) dt
\]

\[
= \sum_n \frac{1}{2\pi} e^{-i(-x+y)\frac{\theta}{2\pi}} e^{i\lambda n} e^{i\lambda x - i\lambda y} \delta_{2\pi}(\lambda - \frac{\theta}{2\pi})
\]

\[
e^{i\lambda x - i\lambda y} \delta_{2\pi\mathbb{Z}}(\lambda - \frac{\theta}{2\pi})
\]

For some reason, which is not yet clear, the asymptotics of the eigenvalue distribution depend only on knowing the kernel for small \( t \).

Example: \( \frac{d^2\psi}{dx^2} + (\lambda - q)\psi = 0 \) \( \psi \text{ constant} \)

\( \psi(0) = \psi(\pi) = 0 \).
The eigenfunctions are
\[
\begin{align*}
\phi_n(x) &= \sqrt{\frac{2}{\pi}} \sin(nx) \quad n = 1, 2, \ldots \\
\lambda_n &= b + n^2
\end{align*}
\]

Now consider the wave equation
\[
\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \phi
\]
\[
\phi(0, t) = \phi(L, t) = 0
\]

This has the general solution
\[
\phi(x, t) = \sum_{n=1}^{\infty} \left( a_n e^{i\omega_n t} + b_n e^{-i\omega_n t} \right) \phi_n(x)
\]

where \( \omega_n^2 = \lambda_n \), i.e. \( \omega_n = \sqrt{n^2 + b} \), \( n = 1, 2, \ldots \)

Suppose \( b \) positive so there is no ambiguity in \( \omega_n \).

Introduce \( P^{1/2} \), if
\[
P \text{ is rep. by } \frac{2}{\pi} \sum_{n=1}^{\infty} (n^2 + b) \sin(nx) \sin(ny)
\]

then
\[
P^{1/2} \text{ is rep. by } \frac{2}{\pi} \sum_{n=1}^{\infty} (n^2 + b)^{1/2} \sin(nx) \sin(ny)
\]
and
\[
e^{-itP^{1/2}} \text{ is rep. by } \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-it(n^2 + b)^{1/2}} \sin(nx) \sin(ny)
\]

On the other hand, if \( u(x) = \frac{1}{2\pi} \int e^{ix} \tilde{u}(\xi) d\xi \) has compact support, then
\[
(Pu)(x) = \frac{1}{2\pi} \int e^{ix} \left( \tilde{u}(\xi)^2 + b \right) \tilde{u}(\xi) d\xi
\]
so that the obvious definition of $p^{1/2}$ is

$$(p^{1/2} u)(x) = \frac{1}{2\pi} \int e^{ix\xi} (\xi^2 + g)^{1/2} \tilde{u}(\xi) \, d\xi$$

except that we still have to make $(\xi^2 + g)^{1/2}$ precise.

Follow Hormander and take the positive square root. Then you have

$$e^{-itp^{1/2}} \text{ approximately } \frac{1}{2\pi} \int e^{ix\xi} e^{-it(\xi^2 + g)^{1/2}} e^{-iy\xi} \, d\xi$$

in some sense. The question is how can we use this representation to get at the actual eigenvalue distribution. Note that the above gadget does not depend on the boundary conditions or even on the size of the interval.

Recall

$$e^{-itp^{1/2}} = \sum \delta_{\omega_n} e^{-it\omega_n} \quad \Rightarrow \quad \sum \delta_{\omega_n} \tilde{\psi}_n(x) \tilde{\psi}_n(y)$$

so

$$\text{tr} (e^{-itp^{1/2}}) = \sum \delta_{\omega_n} e^{-it\omega_n} = \int e^{-it\omega} \, dN(\omega)$$

where $N(\omega) = \text{card } \{ \omega_n \mid \omega_n \leq \omega \}$. Thus if I replace the actual kernel of $e^{-itp^{1/2}}$ by the approximate one I find the approximation

$$\text{tr} (e^{-itp^{1/2}}) \approx \frac{1}{2\pi} \int e^{-it(\xi^2 + g)^{1/2}} \, d\xi \int 1 \, dx$$

$$= \frac{1}{2} \int e^{-it(\xi^2 + g)^{1/2}} \, d\xi$$

Suppose $g = 0$. This is $\int e^{-it\xi^2} \, d\xi$ $\boxed{}$ so we get
the approximation \( N(\omega) = \omega \). Suppose \( \theta \) not zero. To evaluate:

\[
\int_0^{\infty} e^{-it(\xi^2 + \theta)} d\xi
\]

\[
= \int_0^{\infty} e^{-it(\xi^2 + \theta)} d\xi = \int_0^{\infty} e^{-it\xi} d\xi
\]

so \( N(\omega) = \int_0^{\infty} e^{-it\xi} d\xi = \omega - \frac{\omega}{2} \) which is consistent

with \( \omega_n = (n^2 + \theta)^{1/2} \approx n + \frac{\theta}{2} \).

So the result to be understood is why is it possible to replace the operator \( e^{-it P^{1/2}} \) which is defined using the boundary conditions by the Fourier integral operator.

So what we maybe should begin with is the case of the resolvent for the operator \( P = \frac{d^2}{dx^2} + \theta \) with same boundary conditions. Here we calculate \( (\lambda - P)^{-1} \) as a pseudo-differential operator.

\[ (P - \lambda) Gf = f \quad \text{for } f \in C_c^\infty(0,\pi). \]

Here \( P = -\frac{d^2}{dx^2} + \theta \).

\[
(P - \lambda) Gf(x) = \frac{1}{2\pi} \int \frac{d}{dx} \left( e^{ix\xi} \left[ (-i\xi) g + g_x \right] \right) f(\xi) d\xi
\]

\[
= \frac{1}{2\pi} \int e^{ix\xi} \\]
\[
(P-\lambda)Gf(x) = \frac{1}{2\pi} \int \left[ e^{ix_1} (\hat{g}(\xi) + g_x) + e^{ix_1} \left[ \hat{g}(\xi) - (1+\lambda)g \right] \right] \tilde{f}(\xi) d\xi
\]

\[
= \frac{1}{2\pi} \int e^{ix_1} \left[ -(i\xi)(i\xi g + g_x) - (i\xi g_x + g_{xx}) + (1+\lambda)g \right] \tilde{f}(\xi) d\xi
\]

We will have \((P-\lambda)Gf = f\) if \(g(x, \xi)\) satisfies

\[(1)\]

\[
\xi^2 g - 2i\xi g_x - g_{xx} + (1+\lambda)g = 1.
\]

I claim one can always find a unique solution of \((1)\) of the form

\[(2)\]

\[
\hat{g}(x, \xi) = \sum_{n \geq 0} a_n(x) \xi^{-n}
\]

In effect we get the recurrence relation

\[
\sum_{n} a_n(x) \xi^{-2n} - 2i a_n'(x) \xi^{-1-n} - a_n''(x) \xi^{-n} + (1+\lambda) a_n(x) \xi^{-n} = 1
\]

\[
a_n(x) - 2i a_{n-1}(x) - a_{n-2}(x) + (1+\lambda) a_n(x) = \left\{
\begin{array}{ll}
1 & n=2 \\
0 & n \neq 2.
\end{array}
\right.
\]

Thus starting with \(a_0(x) = 1\) we can grind out the rest of the coefficients.

The next point is that having constructed the formal solution \((2)\) to \((1)\) we can then find a \(C^\infty\) function \(g(x, \xi)\) which has \(\hat{g}\) as asymptotic expansion as \(\xi \to \infty\). If \(G\) is then defined using \(g(x, \xi)\), we have

\[
(P-\lambda)Gf(x) = \frac{1}{2\pi} \int e^{ix_1} h(x, \xi) \tilde{f}(\xi) d\xi
\]

where \(h\) is a \(C^\infty\) function with 0 asymptotic expansion.
i.e. \( h(x,y) \) is rapidly decreasing as \( \| z \| \to \infty \). It follows that the kernel representing \((P-I)G-I\):

\[
\frac{1}{2\pi i} \int e^{ix\xi} h(x,\xi) e^{-i\xi y} \, d\xi
\]

is a \( C^\infty \) function of \( x,y \).

But suppose now that boundary conditions are given, so that the operator \( R_\lambda = (P-\lambda)^{-1} \) exists for \( \lambda \) not an eigenvalue (e.g. \( \lambda \) not real). By Schwarz kernel thm. \( R_\lambda \) is given by a kernel and from

\[(P-\lambda)G = I + K\]

where \( K \) has a \( C^\infty \) kernel we get

\[(P-\lambda)(G-R_\lambda) = K\]

hence by regularity \( G-R_\lambda \) has a \( C^\infty \)-kernel. Thus modifying the function \( g \) without changing its asymptotic expansion one finds that \( R_\lambda \to G \to 0 \) on \( C_0 \), i.e. the supports of these kernels satisfy the boundary conditions. In particular that \( G \) satisfies the boundary conditions.

But there should be a better reason that once one has exhibited the symbols for \( e^{-iP} \), \( P^s \), \( e^{-iP^s} \), etc.

that the Fourier integral operators associated to these symbols agree with these operators up to \( C^\infty \) kernels,
Here's what I understand of the Hormander eigenvalue distribution theory so far.

Let \( P \) be an elliptic operator, say \( \frac{1}{2} \Delta \) on \( (0, 2\pi) \) to fix the ideas. Let \( \hat{P} \) be a self-adjoint extension, e.g., the one defined by the boundary conditions

\[ \psi(2\pi) = e^{i\theta} \psi(0). \]

By the spectral theorem we can define the operator \( e^{-it\hat{P}} \). Thus, we find the eigenfunctions

\[ \psi_n(x) = e^{i\lambda_n x} \frac{1}{\sqrt{2\pi}} \quad \lambda_n = n + \frac{\theta}{2\pi} \quad n \in \mathbb{Z} \]

whence

\[ e^{-it\hat{P}} \leftrightarrow \sum_n e^{-it(n + \theta/2\pi)} \frac{e^{i(\theta/2\pi)x} e^{-i(n + \theta/2\pi)y}}{2\pi^n} = e^{i(x-y-t)\theta/2\pi} \delta(x-y-t) \]

On the other hand, using the symbol of \( P \), we can write down a Fourier integral operator candidate for \( e^{-it\hat{P}} \):

\[ (Pu)(x) = \frac{1}{2\pi} \int e^{ix\xi} \hat{u}(\xi) d\xi \quad u \in C_0^\infty(0, 2\pi) \]

So the candidate for \( e^{-it\hat{P}} \) has the kernel

\[ \frac{1}{2\pi} \int e^{ix\xi} e^{-it\xi} e^{-iy\xi} d\xi = \delta(x-y-t) \]
What I want to understand. Given a Schrödinger equation
\[ \frac{d^2 \psi}{dx^2} + (\lambda - q) \psi = 0 \]
on \[0 \leq x \leq \infty, \] let \( S(x, \lambda) \) be the solution matrix:
\[ S(x, \lambda) = \begin{pmatrix} \psi & \psi' \\ \psi' & \psi \end{pmatrix} \]
\[ \frac{d}{dx} S(x, \lambda) = \begin{pmatrix} 0 & 1 \\ q - \lambda & 0 \end{pmatrix} S(x, \lambda) \]
\[ S(0, \lambda) = I. \]

Put \( \lambda = \mu^2 \) and write \( S(x, \mu) \) instead of \( S(x, \mu^2) \).

Assume now that \( q(x) \rightarrow +\infty \) as \( x \rightarrow +\infty \), where the spectrum is discrete and one is in the limit point case at \( x = \infty \). Then for each complex number \( \lambda \), there is a unique number \( m(\lambda) \) such that
\[ \psi(x, \lambda) + m(\lambda) \psi(x, \lambda) \]
is square integrable.

Maybe we would do better to introduce the solution \( X(x, \lambda) \) which vanishes at \( \infty \). It should be normalized somehow. Note that \( \tilde{X}(x, \lambda) \)
\( f(1) X(x, \lambda) \) still vanishes at \( x = +\infty \), and that \( f(1) = 0 \)
\[ \Rightarrow \tilde{X}(x, \lambda) = 0. \] Thus perhaps \( X(x, \lambda) \) is uniquely defined
if we require it has some sort of growth at \( \lambda = \infty \).

What we ultimately want is to compute the Fourier transform:
\[ X(x, \lambda) = \frac{1}{2\pi} \int e^{i\lambda \tau} \tilde{X}(x, \tau) d\tau \]
Let's begin again. Suppose \( \psi(x, \lambda) \) is a solution of the Schrödinger equation
\[
-\frac{d^2}{dx^2} + g(x) \psi = \lambda \psi
\]
whose initial values \( \psi(x_0, \lambda), \frac{d\psi}{dx}(x_0, \lambda) \) are independent of \( \lambda \). Does it follow that \( \psi(x, \lambda) \) has an asymptotic expansion
\[
\psi(x, \lambda) = e^{i\frac{\lambda}{\sqrt{2}}(x-x_0)} a_+(x, \lambda) + e^{-i\frac{\lambda}{\sqrt{2}}(x-x_0)} a_-(x, \lambda)
\]
where \( a_+, a_- \) are holom. in \( \lambda^{1/2} \) at \( \infty \)?

Example of the Bessel DE.
\[
\left( -\frac{d^2}{dx^2} + e^{2x} \right) \psi = \mu^2 \psi
\]
with solution
\[
\psi(x, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{e^x}{2}(e^\alpha + e^{-\alpha})} e^{i\mu \alpha} d\alpha
\]
\[
= \frac{1}{2\pi} \int_0^\infty e^{-\frac{e^x}{2}(t+t^{-1})} t^{i\mu} \frac{dt}{t}
\]
\[
= \frac{1}{2\pi} K(e^x)
\]

Suppose we take the Fourier transform of \( \psi \)
\[
u(x, \mu) = \mathcal{F}(x, \mu) = \int_{-\infty}^{\infty} e^{-i\mu \alpha} \psi(x, \mu) d\mu
\]
\[
= e^{-\frac{e^x}{2}(e^\alpha + e^{-\alpha})}
\]
In general this will satisfy the DE.
\[
-\frac{d^2 u}{dx^2} + e^{2x} u = \int \mu^2 e^{-i\mu \alpha} \psi(x, \mu) d\mu = -\frac{d^2 u}{d\alpha^2}
\]
Thus $u(x, x)$ satisfies the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - e^{2x} u,$$

so now our problem appears to be to locate a potential $g(x)$ such that the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - g(x) u$$

has a solution $u(x, t)$ rapidly decreasing as $x \to \pm \infty$ and $t \to \pm \infty$, with $u(0, t)$ prescribed.