June 19, 1976

$G = O_p(G)P$. Pick a prime dividing $(O_p(G))$ and let $Q$ be an $S_p$-subgroup. Then $O_p(G)P = O_p(G)N_G(Q)$, so by conjugating $P$ inside $N_G(Q)$ I can arrange that $P$ normalizes $Q$.

Let $M \subset G$ be the stabilizer of the component of $Sp(G)$ containing $P$. Then $M \supset P$, so

$$M = \langle O_p(G) \cap M \rangle \times P$$

$M \subset G$.

Now $(O_p(G) \cap M) \times P \subset Sp(G)$, so we can choose $l$ so that the $l$-share of $G$ is $l$-share of $M$ contained in any conjugate of $M$. Thus I can choose $Q$ to be normalized by $P$ and such that $Q$ acts without fixed points on $\pi_0(Sp(G))$. Then it follows that the image of

$$\pi_0(\mathbb{P}(QP)) \rightarrow \pi_0(Sp(G))$$

is the $Q$-orbit of the component containing $P$, so $\pi_0(\mathbb{P}(QP))$ is connected.

---

Let's now analyze $Sp(QP)$. Any $H$ in $Sp(QP)$ can be conjugated into $P$. One has a map $Sp(QP) \rightarrow Sp(P)$ which is fibred. If $H \subset P$,
then the fibre is the $g$ $S_p$-subgroups of $QH$ which can be identified with $Q/N_Q(H)$. Now $N_Q(H) = Q \cap N_{QH}(H)$ is normal in $N_{QH}(H)$, and so is $H$, so $H$ and $N_Q(H)$ have to commute since they are disjoint subgroups. Thus $N_{QH} = C_Q(H) = Q^H$. Thus the fibre of $S_p(QP) \to S_p(P)$ over $H$ can be identified with $Q/Q^H$, and it is clear that $S_p(QP) =$ fibred cat. over $S_p(P)$ assoc. to $H \to Q/Q^H$.

Let $K = \text{Sub}_g$ gen. by $Q^H$ as $H$-forms over $S_p(P)$. It's clear that we have a map of functors $Q/Q^H \to Q/K$ where the latter is a constant functor. Hence we get at least an epimorphism $\pi_0 S_p(QP) \to Q/K$. On the other hand we have a map of functors $Q \to Q/Q^H$ where the first is constant, so we get $Q \to \pi_0 S_p(QP)$. In fact $Q$ acts transitively on $\pi_0 S_p(QP)$ and it's pretty clear that $Q^H$ fixed the component $\text{pt} = \pi_0 S_p(P) \subset \pi_0 S_p(QP)$. Thus it's clear that $\pi_0 S_p(QP) = Q/K$.

Assume I can show $\text{for any } g', H$ acting on $Q$ that $(Q/H(Q)^{H'}) = Q^H/H(Q)^{H'}$. Then the subspace of $V=Q/\Phi(Q)$ generated by the $V^H$ for $H \in S_p(P)$ is the subspace generated by the $Q^H \Phi(Q)/\Phi(Q)$ which is $K(Q)/\Phi(Q)$. Since $K \subset Q$ one has $K(Q) < G$, hence we get a finite, free rep. of $P$ on $Q/K\Phi(Q)$. (First check that if $P$ acts on a vector space $V$, then $W = \sum V^H$ is invariant. Clear for $g \cdot V^H = V^g H g^{-1}$. Also that $K \in S_p(V/W)^H = 0$. For $V = W \oplus W'$ and $W^H = W H \oplus (W')^H$)
and \( V^H \leq W \), so \( (W')^H = (V/W)^H = 0 \).

**Lemma:** Let an \( H \)-group \( H \) act on a \( G \)-group \( Q \), and let \( K \) be an \( H \)-invariant subgroup. Then \((Q/K)^H = Q^H/K^H\). (Borelstein p. 187, Th. 3.15)

First suppose \( K \) is abelian. Take \( \alpha = gK \in (Q/K)^H \). Then \( \{ g \in Q \mid gK = \alpha \} \) is a \( K \)-torsor with \( H \)-action and is classified by an elt. of \( H^1(H, K) = 0 \). So next choose an abelian normal subgroup \( A \) of \( K \) invariant under \( H \). Then \((Q/K)^H = (Q/A)/(K/A)^H = (Q/A)^H/(K/A)^H \) inducibly \( = (Q^H/A^H)/(K^H/A^H) = Q^H/K^H \), so it works. Have proved

**Thm:** \( G \) \( p \)-solvable, \( \implies \text{Sp}(G) \) disconnected \( \implies \) Sp subgroups are cyclic or generalized quaternion.

**Proof:** One has \( O_p(G) = 1 \), and one can assume \( P > 1 \), whence \( O_{p'}(G) > 1 \) and \( O_{p'}(G) > O_p(G) \).

If \( P \) is Sp in \( G \), then \( P \cap O_{p'}(G) \) is Sp in \( O_{p'}(G) \). Thus one sees that

\[
\pi_0 \text{Sp}(O_{p'}(G)) \rightarrow \pi_0 \text{Sp}(G)
\]
is surjective. Moreover \( O_{p'}(G) = O_p(G)(P \cap O_{p'}(G)) < O_p(G) \), so we have a factorization

\[
\pi_0 \text{Sp}(O_{p'}(G)) \rightarrow \pi_0 \text{Sp}(O_p(G)P) \rightarrow \pi_0 \text{Sp}(G)
\]
Thus I can replace $G$ by $O_p(G)P$. In fact this part of the argument uses only that $O_p(G)$ acts transitively on $\pi_0 \delta_p(G)$.

Assume that $O_p(G)$ acts transitively on $\pi_0 \delta_p(G)$. Then

$$\pi_0 \delta_p(O_p(G)P) \rightarrow \pi_0 \delta_p(G)$$

is a map of $O_p(G)$-sets which are transitive, hence it is surjective, so $\delta_p(O_p(G)P)$ is disconnected. (If $M$ is the stabilizer of the component of $P$, then $O_p(G)$ acts transitively on $G/M$ implies $G = O_p(G)M$.)

Replace $G$ by $O_p(G)P$, and let $M = \text{stabilizer of component containing } P$. Choose a Sylow $p$-group of $G$ where $l$ divides the index of $M$ so that $Q$ is contained in no conjugate of $M$. Then all orbits of $Q$ on $G/M$ are non-trivial, and if I pick $Q$ to be normalized by $P$, then $QP$ will not be contained in $M$, so the image of

$$\pi_0 \delta_p(QP) \rightarrow \pi_0 \delta_p(G)$$

will be the $Q$-orbit of the component of $P$ in the latter. Thus $\pi_0 \delta_p(QP)$ is disconnected, so we can replace $G$ by $QP$.

Actually the same argument shows that if $P'$ is any non-trivial subgroup of $P$, then the image of

$$\pi_0 \delta_p(QP') \rightarrow \pi_0 \delta_p(G)$$
will be the $Q$-orbit of the component of $P$. If $P$ is not cyclic or gen. quaternion I can take $P'$ to be an elem.
abelian $p$ group of rank $> 1$. But then in Gorenstein 5.3.16
one knows that $Q$ is generated by $Q^H$ for $K H \leq P'$, so
$\delta_p(QP)$ is connected - a contradiction.

\textbf{Prop.} Let $M = \text{stabilizer of the component of } P \text{ in } \text{Sp}(G)$. Then $M = \langle N_G(H_i) | 1 \leq H \leq P \rangle$.

\textbf{Proof.} Let $L = \langle N_G(H) | 1 \leq H \leq P \rangle$. Clearly $L \subset M$. We have to show that $M \leq L$, so let $m \in M$. Then $P_j m P_{m^{-1}}$
are in the same component of $\text{Sp}(G)$. If we can show
that $m P_{m^{-1}} \in L$, then $m P_{m^{-1}} = l P_l$ for $m \in L$.
Thus I want to show that if $Q_1, Q_2$ are $Sp$ subgps with
$Q_1 \subset L$ and $Q_1 \cap Q_2 > 1$, then $Q_2 \subset L$. 

Observe that $L$ contains the
normalizer of any $P$-subg. Choose a $Sp$-group
$Q'$ of $L$ containing $H$, whence $N_{Q'}(H) > H$; if $Q''$ is an
$Sp$-subg. of $G$ such that $N_{Q''}(H)$ is an $Sp$-subg.
of $N_G(H)$ containing $N_{Q'}(H)$, then induction
($Q'' \subset N_{Q''}(H) < L$, $N_{Q''}(H) > H$) shows $G'' \subset L$. By an
element of $N_G(H) \subset L$ one can move $N_G(H)$ into $N_{Q''}(H)$.
So we can suppose $Q \subset Q \cap Q'' \subset L$, $Q \cap Q'' > H$ and use
induction to finish.
Simpler version: If $K < H < L$, and $H < Q$ an $S_p$ group, then $L > N_G(H) > N_Q(H) > H$, so we can replace $H$ by $N_Q(H)$ and continue until we get $Q = L$.

June 16, 1976

Basic problem is to understand the homotopy type of $S_p(G)$ when $G$ is $p$-nilpotent: $G = Q_p(G) \times P$. Is there any possibility $S_p(G)$ spherical?

Consider a group of the form $G = V \times P$ where $P$ is a $p$-group acting faithfully on an $E$-vector space $V$. I've already seen that $S_p(G)$ disconnected $\Rightarrow$ $P$ cyclic or gen. quaternion. Assume $P$ is elementary abelian of rank $r$. Is it possible $S_p(G)$ is spherical in dimension $r - 1$?

I assume that $r > 1$, whence I know that $S_p(G)$ is connected. If $H \triangleleft P$, then

$$N_G(H) = V^H \cdot P = C_G(H) \quad \text{P-abelian}$$

$$N_Q(H) / H = V^H \cdot P / H$$

If $S_p(N_G(H) / H)$ is disconnected, then I know that $P / H$ is cyclic and that it acts faithfully on $V^H$.

Since $P$ is abelian one knows that each irreducible representation of $P$ factors through a cyclic quotient, hence we can write uniquely...
\[ V = \bigoplus_{Q} V_Q \quad \text{where } Q \text{ runs over the subgroups of } P \text{ such that } P/Q \text{ is cyclic and where } V_Q \text{ is a representation of } P/Q \text{ such that } P/Q \text{ acts freely away from } 0. \]

**Special case:** Suppose that \( Q_1, \ldots, Q_r \) are these subgroups and \( V_Q \neq 0 \) and that
\[
P \to P/Q_1 \times \cdots \times P/Q_r \cong (\mathbb{Z}/p\mathbb{Z})^r
\]
\[
V \to V_{Q_1} \oplus \cdots \oplus V_{Q_r}
\]

Can you see what the homotopy type of \( B_\mu(\mathbb{R}) \) is? Note that \( V^H = \bigcap_i V_{Q_i} \). Perhaps it can be shown that we only have \( H = \bigcap_i Q_i \) to look at the subset of \( H \) in \( P \) of the form \( \bigcap_i Q_i \) where \( \sigma \subseteq \{1, \ldots, n\} \).

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**June 18, 1976**

Let \( T \) be a poset, let \( x \in T \) such that \( T = \{ y \geq x \} \) is contractible. Then it should be true that \( T - \{ x \} \subset T \) is a homotopy equivalence. In effect we have removed the vertex \( x \) from the simplicial complex \( BT \). The link of this vertex is the join
\[
B(T_{<x}) \ast B(T_{>x})
\]
which is contractible.
Suppose now that \( a, b \in T \) and that \( T \uparrow a, T \uparrow b \) are contractible. Remove a first, whence \( T - \{a\} \subset T \) is a local, then remove \( b \). \( (T - \{a\}) \uparrow b = T \uparrow b \) if \( a \not> b \). Thus if \( a \not> b \), \( T - \{a, b\} \subset T \) is a local. On the other hand if \( a > b \), then \( b \not> a \), so \( (T - \{b\}) \uparrow a = T \uparrow a \), so we still win. Thus we have

**Prop.** Let \( S \) be a subset of a poset \( T \) such that \( T \uparrow x \) is contractible for every \( x \in S \). Then \( T - S \subset T \) is a homotopy equivalence if the dimension (length of a maximal chain) of \( S \) is finite.

**Proof.** Use induction on \( \dim S \). If \( \dim S = 0 \), then \( S \) is discrete, i.e., in \( BT \), with vertices \( S \) has \( \dim 0 \), so that we can remove \( S \) without changing homotopy types because the links are contractible. If \( \dim(S) = d \), let \( S_0 \) be the subset of minimal elements of \( S \). Then \( \dim(S_0) = 0 \), so \( T - S_0 \subset T \) is a local. But \( \dim(S - S_0) = d - 1 \), so \( T - S_0 - (S - S_0) = T - S \) is local to \( T \).

**Corollary:** \( S_p(G) \) is local to the subposet consisting of \( p \)-groups \( H \) such that \( H = \mathbb{Q}_p(N_e(G)) \).

**Proof.** \( S_p(G) \uparrow H \) is local to \( S_p(N_e(G)/H) \) which is
Contractible if $H \leq O_p(N_G(H))$.

Next look at $J = A_p(G)$. The "link" $J_A$ consists of all elementary abelian $p$-subgroups $B > A$. Such a $B$ is contained in $C_G(A)$. Thus if $A \leq \Omega_1ZC_G(A)$, this link will be contractible. In particular the link is contractible unless $A \leq Z(G)$.

$G = V \times P$ where $P$ is an elem. abelian $p$-group and $V$ is a repn. in char $p$. If $H \lhd P$, then $N_G(H) = V^H \times P$

and $O_p(N_G(H)) = \{x \in P | x = \text{id} \text{ on } V^H\}$. Denote this last group by $(V^H)_0$. In general if $W \leq V$ put $W_0 = \{x \in P | x = \text{id} \text{ on } W\}$. Then $O_p(N_G(H)) = H \iff H = (V^H)_0$.

Hence we can identify the poset of $H \lhd G$

$\geq O_p(N_G(H)) = H$ with the poset of cosets $V\setminus W$ where $W$ is a subgroup in $V$ of the form $V^H$ for some $H$, i.e.,

$W = V^{(W)_0}$.

Because $P$ is abelian we have a canonical decomposition $V = \bigoplus V_Q$, where $Q$ ranges over subgps of $P/Q$ is cyclic and where $P/Q$ acts freely on $V_Q$ - 0. I assume to simplify that $V_P = V^P = 0$. Let $Q_1, \ldots, Q_d$ be those
codim 1 subgroups such that $V_{Q_i} \neq 0$. To simplify suppose the $Q_i$ are independent: $P \sim P/Q_1 \times \cdots \times P/Q_s$. Then $V = V_1 \oplus \cdots \oplus V_s$. If $H \subset P$, then $V^H = \bigoplus_{i \in \sigma} V_i^H$ and $V_i^H = \begin{cases} 0 & H \cap Q_i \\
V_i & H \subset Q_i \end{cases}$

Thus the possible subspaces $V^H$ are $\bigoplus_{i \in \sigma} V_i$ where $\sigma$ is a subset of $\{1, \ldots, s\}$, and the corresponding subgroups are $\bigcap_{i \in \sigma} Q_i$, and $\sigma < \{1, \ldots, s\}$.

$s = 2$.

\[
\begin{array}{c}
\{1\} \quad \{2\} \\
\downarrow \quad \downarrow \\
\phi \\

V/V_1 \quad V/V_2 \quad V
\end{array}
\]

Looks like the join. In fact if $P = P_1 \times P_2$ $V = V_1 \oplus V_2$ where $V_1$ is a rep of $P_1$ on which $P_2$ acts trivially, and $V_2$ is a rep of $P_2$ on which $P_1$ acts trivially, then

$$(V \times P) = (V_1 \otimes P_1) \times (V_2 \times P_2)$$

so

$$s_p(V \times P) = s_p(V_1 \times P_1) \ast s_p(V_2 \times P_2)$$

This shows that if the $Q_i$ are independent, then $s_p(G)$ is spherical of $\dim = \text{rank}(P) - 1$. 
I've seen that \( S_p(G) \) is beg to the subcomplex consisting of \( H \) such that \( H = Q_p(N_G(H)) \). In the case \( G = V \times P \), this means I look at subgroups of the form \( vHv^{-1} \) where \( v \in V/V^H \) and \( H = \{ x \in P | x = \text{id on } V^H \} \).

In particular for such \( H \), I have \( V^H > V^{P} \) if \( H < P \).

To simplify assume \( V^{P} = 0 \).

Assume \( H \) such that \( V^H > 0 \) and \( H = \{ x \in P | x = \text{id on } V^H \} \).

Recall \( V = \bigoplus_i V_i \) where \( V_i \) is a non-zero repn. of the cyclic gp \( P/Q_i \) which acts freely on \( V_i - 0 \). Then

\[
V^H = \bigoplus_{i \in \sigma} V_i, \quad \sigma = \{ i | H < Q_i \}
\]

\[
H = \{ x \in P | x = \text{id on } V^H \} = \bigcap_{i \in \sigma} Q_i
\]

Since \( V^H > 0 \) it follows \( \sigma \neq \emptyset \).

Can I find a \( K < P \) such that \( V^K + V^H = V \)? Obviously \( V_i < V^K \) for \( i \notin \sigma \) so that \( K < Q_i \). Thus it is a question of whether \( \bigcap_{H \in Q_i} Q_i \) is \( > 1 \).

My idea is to let \( P \) act on \( S_p(G) \) or rather this subcomplex, and the hope was that everything but the Sylow groups disjoint from \( P \) would be part of the non-free stuff. Note that if \( x \in P \), then

\[
x(vHv^{-1})^{-1} = (xvx^{-1})H(xvx^{-1})^{-1}
\]
so the P-action on the conjugacy class of $H \subset P$ is the obvious action of $P$ on $V/VH$. If $P$ acts faithfully on $V/VH$, then one has non-$Sp$ groups which are free for the $P$-action.

Observation: Assuming $V^P = 0$, we have identified elements of $V$ with the different $Sp$-subgroups of $G$. $G$ acts like an affine group, i.e. $V$ acts as translations in $V$ and $P$ acts linearly. Then the poset we are studying looks like a poset of affine subspaces, as studied by Lusztig.

June 20, 1976

Review notation: $G = V \times P$ where $P$ is an abelian $p$-group and $V$ is a rep. of $P$ over $F_q$. We've identified $Sp(G)$ with the fibered category over $Sp(P)$ assoc. to functor $H \mapsto V/VH$. Because $P$ is abelian, every non-identity $p$-subgroup of $G$ is conjugate to a unique $H$ in $Sp(P)$. The $p$-subgroups conj. to $H$ can be identified with $Sp(G) = G/VH. P \cong V/VH$. Assume to simplify that $V^P = 0$. Then the $Sp$-subgroups of $G$ are of the form $V^{P \cdot v^{-1}}$, and this representation is unique. One has $vPv^{-1} = vHv^{-1}$ where $H = \{ x \in P \mid x \cdot (v - v') = v - v' \}$. Thus it seems clear that the simplicial complex of $Sp$-groups can be identified with the simplicial complex whose vertices are
The elements of $V$ and whose simplifies are subsets contained in a coset for some subspace $V^H$, with $1 < H \leq P$. This is valid without the assumption $V^P = 0$.

Critical case: Assume that $P/Q_1 \times \cdots \times P/Q_r \cong P$ and let $Q_{r+1}$ be another hyperplane in $V$.

June 21, 1976

Again $G = V \times P$, $P$ elementary abelian $p$ group, $V$ an abelian $d$ group on which $P$ acts faithfully. I assume $V^P = 0$. I claim there is a natural notion of distance between two $S_P$-subgroups of $G$. Say that $v_1 P v_1^{-1}$, $v_2 P v_2^{-1}$ are of distance $1$ if $v_1 P v_1^{-1} \cap v_2 P v_2^{-1}$ is of codimension $1$ in either. This means that $\mathfrak{g}_{\mathfrak{h}}$ has stabilizer of codim $1$ in $P$.

$P \cap \sigma P \sigma^{-1} = \{x \in P \mid x(\sigma) = \sigma z\}$.

Now define the distance between two $S_P$-groups to be the minimal length of a sequence starting with one, ending with the other such that consecutive members have distance $1$. Thus, if we want the distance between $P$ and $\sigma P \sigma^{-1}$ we look at chains $P, v_1 P v_1^{-1}, \ldots, v_k P v_k^{-1}, \ldots$.
where \( v = \sigma_1 + \cdots + \sigma_k \) (use additive notation) and the stabilizer of \( \sigma_i \) is of codim 1. Now we know that
\[
V = \bigoplus Q
\]
where \( Q \) ranges over the subgps of codim 1 in \( P \). Thus each \( \sigma_i \) must be contained in some \( V_Q \). Thus it's clear that the distance is simply the number of non-zero summands in the decomposition
\[
(v) \quad v = \sum v_Q \quad \text{where} \quad v_Q \in V_Q
\]
Next suppose that we consider a subgroup \( vHv^{-1} \), which we've seen is the same as the coset \( vV^H \). Since \( V^H = \bigoplus Q \), there is a unique representative for \( v \mod V^H \) whose "support" involves those \( Q \) which don't contain \( H \). Terminology: support of \( v \) is the set \( \{ Q \text{ codim 1 in } P \mid v_Q \neq 0 \} \), \( \text{card} (\text{supp } v) \). It's clear that there is a unique representative for \( v + V^H \) of the minimal length, so we define the distance of the coset \( v + V^H \) to be the minimal distance of any of its points to 0.

**Proposition:** Any subgroup \( vHv^{-1} \) is contained in a unique \( S_p \)-subgroup having the same distance from \( P \). All other \( S_p \)-subgroups containing \( vHv^{-1} \) are further away from \( P \).
The problem is now to understand the poset of $H$ in $Sp(P)$ such that the set $v + VH$ has smaller distance from $O$ than $v$ itself. Suppose the support of $v$ is \{Q_1, \ldots, Q_d\}, say $v = v_1 + \ldots + v_d$, where $v_i \in V_{Q_i} \cap O$. Then $v + VH$ is closer to zero $\iff \exists i \neq j \in Q_i$. So the Poet I have to examine is the poset of all non-zero subspaces $H$ contained in one of the hyperplanes $Q_1, \ldots, Q_d$. This gives rise to a simplicial complex of dimension $r - 2$ where $r = \text{rank } P$.

**Proposition:** Let $Q_1, \ldots, Q_d$ be hyperplanes in a vector space $P$ of dimension $r$. Then the poset of non-zero subspaces $H$ of $P$ such that $H \subseteq Q_i$ for some $i$ has the homotopy type of a bouquet of $(r - 2)$-spheres.

**Proof:** All the poset in question is covered by the sets $X(Q_i)$ which are contractible. Also $X(Q_1) \cap \ldots \cap X(Q_d)$ is empty or contractible according to whether $Q_1 \cap \ldots \cap Q_d$ is 0 or not. Thus, $X$ is hom to the simplicial complex whose simplifies are $\{Q_0, \ldots, Q_i\}$ such that $Q_0 \cap \ldots \cap Q_i > 0$. But any subset of card $r - 1$ is a simplex, hence this simplicial
complex has the same $k-2$ skeleton as the full simplex with vertices $Q_1, \ldots, Q_d$, so its homotopy groups in dimensions $< k-2$ vanish.  A.E.D.

June 23, 1976

Problem: Is it true that $Sp(O_p)Sp(6)$ has the homotopy type of $Sp(6)$?

Put $R = O_pSp(6) = V \times Q$, $V = O_pSp(6)$. Let $K \leq Sp(6)$.

We want to show $Sp(R)^K$ is contractible. $Sp(R)^K$ consists of non-trivial $p$-subgroups of $R$ normalized by $K$. Because $K$ is a $p$-group, one has $H^K > 1$. Thus $H^K \rightarrow H^K$ deforms $Sp(R)^K$ into $Sp(R^K)$. Since $K$ can always choose $Q$ to be normalized by $K$ (take an $Sp$-$gP$ of $G$ containing $K$ and put $Q = RnP$), it's clear that

$$R^K = V^K \times Q^K.$$ 

So $Sp(R^K)$ is apt to be non-contractible unless $Q^K$ acts non-faithfully on $V^K$. So NO is probable answer.

**Example:** Let $L = H_p$, and consider the wreath product 

$$(L \times \mu_p)^p \times (Z/pZ) = L^p \times (\mu_p^p \times Z/pZ)$$

What are the elements of order $p$ in $\mu_p^p \times Z/pZ$?
Consider $\tau \sigma$ where $\tau \in \mu_p$ and $\sigma$ to a generator of $\mathbb{Z}/p\mathbb{Z}$.

$$(\tau \sigma)(\tau \sigma) = \tau \cdot \tau^\sigma \cdot \tau^2$$

$$(\tau \sigma)^3 = \tau \cdot \tau^\sigma \cdot \tau^2 \cdot \tau^3$$

So

$$(\tau \sigma)^p = \tau^1 + \cdots + \tau^p$$

If $\tau = (\tau_1 \tau_2 \cdots \tau_p)$

$\tau^\sigma = (\tau_2 \tau_3 \cdots \tau_p \tau_1)$

$\tau^\sigma^2 = (\tau_3 \tau_4 \cdots \tau_1 \tau_2)$

So

$\tau^{1 + \sigma + \cdots + \sigma^{p-1}} = 1 \iff \tau_1 + \cdots + \tau_p = 0$

The centralizer of $\tau \sigma$ is the center $\Delta \mu_p$. Picture of the poset of elem. abelian subgroups:

Suppose $P$ abelian, $V = \bigoplus V_Q$ where $Q$ ranges over subgroups of $P$ such that $P/Q$ is cyclic and where $P/Q$ acts freely on $V_Q \neq 0$. Assume $V_p = V = 0$. 
The question is whether I can introduce a natural distance. \( P_w P_{w^{-1}} = P_w \) stabilizer of \( w \). This is a maximal \( S_p \)-intersection \( \iff \ P_{w^{-1}} P_w > P_w \vee P_{w^{-1}} \iff w P_{w^{-1}} = P \), i.e. \( P_w > P_w \Rightarrow P_w = P \). So if \( H = P_w \), then we have a maximal \( S_p \)-intersection \( \iff (w \in V^H, P_w \supset H \Rightarrow w = 0) \) which means \( P/H \) acts freely on \( V^H \). Suppose \( P \) cyclic whence \( P \) has a unique subgroup of order \( p^n \) for \( p^n \mid |P| = p^k \). We have
\[
V = \bigoplus_{q \in H} V_{q^n}
\]
We can only get to \( w \in V_{q^n} \) using maximal \( S_p \)-intersections.

It is important that each coset \( v + V^H \) has a natural center.
\[
V^H = \bigoplus_{q \supset H} V_q
\]
But one knows there is a unique decomposition \( V = [H, V] \oplus V^H \) because \( H \) is of order prime to \( p \). Thus there is a unique element in the coset whose average \( \sum_{h \in H} v h v^{-1} \) is zero.

So now it is clear that I want to take a \( v \in V \) (i.e. a \( S_p \)-group \( v P_{w^{-1}} \)) and look at the poset \( \tilde{J}_w \) of \( H \cap \tilde{H} \) such that the sum \( \sum_{h \in H} v h v^{-1} \) is \( \neq 0 \). Note that
$1 < H' < H$, $H \in J_0 \Rightarrow H' \in J_0$. Let $P$ be abelian and suppose $v = v_1 + \ldots + v_4$.

$x \cdot v_i = x(x)v_i \quad x \in P$

where $x_i$ are distinct characters of $P$. Then $J_0$ consists of those $H > 1$ contained in some $x_i$. Can obviously remove any $H$ from $J_0$ which is not elementary abelian without changing homotopy type. So when $P$ is abelian it is clear $J_0$ has the homotopy type of a bouquet of $(r-2)$-spheres where $r = \text{rank}_P(P)$.

Return now to the example of $V = L^p$ acted on by $P = \mu_p \times \mathbb{Z}/p\mathbb{Z}$. Here $V$ is an irreducible faithful representation of $P$, and the center $A_{mp}$ acts as scalars. Let $v = (v_1, \ldots, v_p) \in L^p$. The poset $J_0^a$ consists of elem. abelian groups $H$ such that $\sum_{h \in H} v = 0$. Clearly such an $H$ cannot contain $A_{mp}$. Consider the cyclic subgroups generated by $v \sigma$

where $\tau = (v_1, \ldots, v_p) \in \mu_p$. $\tau$ has order $p$ if $\prod \tau_i = 1$.

$v = (v_1, \ldots, v_p)$

$\tau v \sigma = (v_1 v_2, v_3, \ldots, v_p, v_1)$

$(\tau v)^2 = (v_1, v_2, v_3, \ldots, v_p, v_1)$
Thus \[ y + \sum_{i=1}^{p} y_i = (v_1 + \sum_{i=1}^{p} y_i, v_2 + \sum_{i=1}^{p} y_i, \ldots, v_p + \sum_{i=1}^{p} y_i) \]

\[ = (w, \sum_{i=1}^{p-1} y_i, \ldots, \sum_{i=1}^{p} y_i) \]

where \[ w = v_1 + y_1 v_2 + \ldots + y_{p-1} v_p \]

This is non-zero \( \Leftrightarrow w \neq 0 \). So for example if I take \( v_1, \ldots, v_p = 0 \), \( y_i \neq 0 \) then \( w \neq 0 \) for any element of \( \mathbb{T} \). So for this case one can see that \( T^a \) consists of all the subgroups \( \langle T^a \rangle \) and all the subgroups of \( 1 \times \mu^p \times \ldots \times \mu^p \). \( T^a \) obviously has the homotopy type of a \( p-1 \) times finite set. Perhaps that conclusion holds more generally. Certainly the case for any \( (v_1, \ldots, v_p) \) with some \( v_i = 0 \). If all \( v_i \neq 0 \), then \( T^a \) should be the union of a discrete set and a bouquet of \( p-2 \)-spheres.
To understand the homotopy type of \( \text{Ap}(G) \) when \( G = (L \times \mu_p)^p \times (Z/pZ) = V \times P \) where \( P = \mu_p^p \times (Z/pZ) \) and \( V \) is the faith irreducible repn. on \( L \). We already have understood somewhat \( \text{Sp}(P) \), or rather \( \text{Ap}(P) \). Put \( A = \mu_p^p \), \( C = Z/pZ \) so that \( P = A \times C \).

An order \( p \) subgroup not contained in \( A \) has a unique generator \( t \sigma \), where \( \sigma \) is the elt 1 of \( C \), and \( t = (t_1, \ldots, t_p) \in A \) is such that \( t_1 \cdots t_p = 1 \).

The maximal elementary abelian subgroups of \( P \) are \( A \) with rank \( p \), and the following of rank 2:

\[
\begin{align*}
B_t &= \Delta \mu_p \cdot \langle t \sigma \rangle \\
Z &= \langle 1, \tau_1, \ldots, \tau_p \rangle,
\end{align*}
\]

\( \prod \tau_i = 1 \)

How many different \( Z \langle t \sigma \rangle \), Diagram

Next point is to understand \( \text{Ap}(G) \). It will be the union of \( \text{Ap}(V \times A) \) with the different \( \text{Ap}(V \times B_t) \) amalgamated over \( \text{Ap}(V \times Z) \). Now I've seen that \( \text{Ap}(V \times A) \) has the homot. type of a bouquet of \((p-1)\) spheres, and \( \text{Ap}(V \times B_t) \) is a bouquet of \( S^{1-3} \).
so it's clear that we get a lot of $H_1$:

$$0 \rightarrow H_1(A_p(V \times A)) \oplus \bar{H}_1(A_p(V \times B_L)) \rightarrow H_1(A_p(G)) \rightarrow \mathbb{Z}^2 \rightarrow 0$$

Suppose a simplicial complex is a union $X \cup Y$ of two sub-complexes, with $\dim X = n$ and $\dim (X \cap Y) < n$. Then from Mayer-Vietoris

$$0 \rightarrow H_n(X \cap Y) \rightarrow H_n(X) \oplus H_n(Y) \rightarrow H_n(X \cup Y)$$

one sees that $H_n(X) \oplus H_n(Y)$ embeds in $H_n(X \cup Y)$. Thus in a group of the form $V \times P$ if $A$ is a maximal elementary abelian $p$-subgroup of $P$ of rank $r$, then take $X$ to be the subcomplex of $A_p(G)$ consisting of $B$ contained in $V \times A$ (resp. not equal to $A$). Then $X \cap Y$ has dimension $r-2$, so

$$H_n(A_p(V \times A)) \hookrightarrow H_n(A_p(V \times P))$$
G is a finite group. Recall that $U_{Sp}(G)^K$ is contractible. This complex consists of chains $1 < H_0 < \cdots < H_s$ normalized by some $K$ in $Sp(P)$. If $H$ is normalized by $K$ in $Sp(P)$, then $HK$ can be put in an $Sp$-group $Q$. Then $P \cap Q = K$, so if I am going to try to build up $Sp(G)$ by adding Sylow groups and all subgroups, one at a time then perhaps adding just those $Q$ that intersect $P$ gives something contractible? If $P$ is abelian, then if $H_0 < H_1 < \cdots < H_s < Q > K$, then $K$ normalizes $(H_0, H_s)$, so in this case $U_{Sp}(G)^K$ is just those subgroups contained in a $Q$ intersecting $P$. In general

$$U_{Sp}(G)^K \subseteq U_{Sp}(Q)$$

$$K \in Sp(P)$$

$$Q \cap P > 1$$

Question: If $V$ is a faithful rep. of $P$, then is there some sort of notion of distance of $v$ from 0?

If $V$ is an invariant complex representation, then one could put an unitary metric in $V$ and get a good distance. The property of the distance I want is the following: If $v = v^1 + v^2$ is the decomposition of $v$ relative to $V = [H, V] \oplus V^H$ for some subgroup $H$, then $d(v^1) \leq d(v^2)$ with equality iff $v^2 = \frac{1}{|H|} \sum w$ is 0.
Question: Let $V$ be a representation of a group $G$, say over a field of char. $0$. Form the fibred category over the poset of non-identity subgroups of $G$, with the fibre $V/H$ over $H$. What sort of homotopy type does one obtain? Assume $V^G = 0$.

Suppose $G$ abelian, and $V = \bigoplus V_\lambda$ is the decomposition according to the characters of $G$. Define the length of $V$ to be the number of non-zero components of $V$ relative to this decomposition. Then the link on attaching $(G, V)$ to things of lower length is the poset of non-trivial subgroups $H$ of $G$ such that $\exists \lambda$ in the support of $V$ such that $\chi(H) = 0$. Can obviously restrict to products of elementary abelian groups.

If similarity with parabolic (cuspidal representations), i.e. a representation which when restricted to the unipotent radical of a parabolic subgroup doesn't contain the trivial representation.

If $V \in \mathfrak{g}$ is the $G$-space of $V$, I suggest these notations and the $G$-invariants. A short answer:

Problem: Let $G = V \times P$ with $P$ a $p$-group and $V$ an elem. ab. $p$-group. Then each coset...
If $H$ is any subgroup of $P$, define
\[ d_H(v) = \dim \langle Hv \rangle \]

If $v = v' \oplus v'' \in [H, V] \oplus V^H$, then
\[ \langle Hv \rangle = \langle Hv' \rangle \oplus \langle Hv'' \rangle \]

so that
\[
\begin{align*}
    d_H(v') &= d_H(v) \\
    d_H(v'') &= 1
\end{align*}
\]

so that $\sum ho = 0$ and $\sum ho \neq 0$.

If $H$ is normalized by $K$, then $V = [H, V] \oplus V^H$ is $K$-invariant.

\[ \langle Kv' \rangle \]

is a quotient of $\langle Kv \rangle$ and
\[ d_K(v') \leq d_K(v) . \]

\[ \text{canonical representative} \]

\[ \text{problem is to define distance from 0, so that the distance increases for other points of this coset.} \]
Let $V$ be a char. repn. of the p-group $P$. I've seen that each coset $v + V^H$, $H \in \mathfrak{sp}(P)$ has a unique "center" because of the decomposition $V = [H, V] \oplus V^H$. So, given $v$, one is led to consider those $H \in \mathfrak{sp}(P)$ such that $v$ is not the center of $v + V^H$, i.e., such that \( \frac{1}{|H|} \sum_{h \in H} v = 0 \). The real question is whether I can build up $\mathfrak{sp}(6)$ by adding one Sylow group $\varphi(P_0)^{-1}$ at a time so that $\varphi(P_0)^{-1}$ gets attached via the poset of $H$ in $\mathfrak{sp}(B)$ to $\Sigma H = 0$.

**Question:** Introduce the relation $v_1 \xrightarrow{H} v_2$ to mean that $v_1 = v_2 - \frac{1}{|H|} \sum_{h \in H} v_2$. Write $v_1 < v_2$ if $\exists H \in \mathfrak{sp}(P)$ such that $v_1 \xrightarrow{H} v_2$. Is the relation $v_1 < v_2$ extendable to a linear ordering?

**Example:** Suppose $V$ is a representation in characteristic 0. If the ordering is not extendable there is a circuit $v_1 < v_2 < \cdots < v_{n-1} < v_1$, hence one can suppose $V$ defined over a subfield of $C$, hence one can suppose $V$ is a repn. over $C$. But then using an invariant unitary structure one has $v_1 < v_2 \implies \|v_1\| < \|v_2\|$ with equality $\iff v_1 = v_2$. Hence there are no circuits. **Question:** can one construct a sequence $v_1 \xrightarrow{H} v_2 \xrightarrow{H} v_3 \cdots$ tending to zero? (Yes, see 14)
Generalization: Given a vector space $V$ together with a family of subspaces $V^H$, one forms the poset of cosets $v + V^H$ as in Wagner-Volodin. I can attempt to analyze this poset by adding one $v$ at a time. Here given $v$ I concentrate on those subspaces $V^H$ such that $v$ projects non-trivially onto $V^H$ (say that $V$ is a Hilbert space). What about the case when the group $P$ acting on $V$ is generated by reflections?

Pretty clear this sequence goes to zero.

In general, if you have two hyperplanes in the family, then alternately projecting on the orthogonal lines, you get to zero.
Difficulty: I have been trying to build up $Sp(V \times P)$ by adjoining $Sp(Q)$ for a certain ordering of the $Sp$-subgroups. Now one thing I want out of my exhaustion of $Sp(V \times P)$ is to get the contractible set $\bigcup_{K \in Sp(P)} Sp(K)$, as well as some reason for its contractibility, i.e., if we attach $Y$ along $\partial Y$, then $\partial Y \subset Y$ is a hom. Now if $H \in Sp(G)$ and $N_G(H) \cap P = K$, then in the course of getting to $H$, I would attach probably an $Sp$-group $Q$ containing $K$. Thus I get $H$ by adding $Sp(Q)$ for some $Q \supset Q \cap P > 1$. But not all of $Sp(Q)$ is contained in the contractible piece $\bigcup_{K \in Sp(P)} Sp(K)$ when $P$ is non-abelian.

June 28, 1976

Prop: The center $Z(G)$ is the intersection of all maximal abelian subgroups of $G$.

Proof: If $A$ is abelian, so is $Z(G)A$, hence $Z(G)A$ if $A$ is maxim. ab. Thus $Z(G) < \bigcap A$, A maxim. ab. If $x \in NA$ and $y \in G$, then let $<y>$ be extended to a maximal abelian subgroup $A'$; then $xy \in A'$ so $xy$ commutes; hence $x \in Z(G)$.
Question: In a $p$-group $P$ what sort of subgroup is the intersection of the maximal elementary abelian $p$-subgroups? Can it be bigger than $\Omega_1 Z(P)$?

Let $x \in \bigcap A$, $A$ max. elem. p-sub. If $y \in P$ is of order $p$, then $\langle y \rangle \leq$ some $A$, so $x, y$ commute. Thus any such $y \in C_p(x)$, so $\Omega_1 P \leq C_p(x)$. So clearly

$$\bigcap A = \Omega_1 Z(\Omega_1 P)$$

Let $Q$ be a $p'$-group acting on the $p$-group $P$. Assume that $Q$ acts trivially on any $Q$-invariant subgroup $P' \leq P$. Then $Q$ acts trivially in $\Phi(P)$. Also $R/\Phi(P)$ must be irreducible and non-trivial (otherwise the action would be trivial on the whole Lie ring, hence trivial on $P$). Next look at the bracket pairing

$$[g_1, g_2]_P : g_1^*(P) \otimes g_2^*(P) \rightarrow g_{21}(P).$$

Action of $Q$ on $g_{21}(P)$ is trivial.
Cor. If $p^A$ contains a maximal abelian subgroup of $P$, then $P^A = P$.

**Remark:** $C_p(B) < B$ implies $C_p(B) = Z(B)$. Let $A$ be a maximal abelian subgroup of $B$. Then $C_B(A) = A$. 

**Proof:** If $A$ acts trivially on $Q$ and acts trivially on a subgroup $B$ of $Q$ such that $C_Q(B) < B$, then $A$ acts trivially.

$B < Q^A \implies C_Q(Q^A) < C_Q(B) < B < Q^A$,

so we can suppose $B = Q^A$, $N_Q(B)/C_Q(B) \twoheadrightarrow \text{Aut}(B)$.

$N_Q(Q^A) = Q^A$ which is a $p$-group can happen iff $Q^A = Q$. 

Put $Q = P \times B$. Then $A$ acts trivially on $p^B \times B = C_Q(B) \cdot B$, so this is equivalent to...
June 29, 1976

A elementary abelian $p$-group acting on an $l$-group $V$. Assume that the action is not faithful on any proper $A$-invariant subgroup. According to Cor. 5.3.13, $V$ is of class 2 and exponent $l$, if $l$ is odd. I want to show that $V$ has to be abelian if I can.

The $A$-action on $\Phi(V)$ is not faithful; let $B < A$ be the subgroup acting trivially. Then $B < A$ hence $B$ acts faithfully on $V/\Phi(V)$.

Since $A/B$ acts faithfully on $\Phi(V)$ if we take any $A$-invariant $W \subset V/\Phi(V)$ on which $B$ acts faithfully, and let $V/\Phi(V) = W$, then $A$ acts faithfully on $V$. Thus $B$ acts non-faithfully on any $A$-invariant subspace of $V/\Phi(V)$. So if we decompose $V/\Phi(V)$ into irreducible reps of $A$ thereby obtaining characters of $A$ say $\chi_1, \ldots, \chi_s$, the intersection of the kernels of these characters with $B$ is trivial, yet for any subset of the $\chi_i$ it is non-trivial. Thus $\chi_i$ as elements of $B^\wedge$ span $B^\wedge$ yet no proper subset does. Thus the $\chi_i$ must be a basis for $B^\wedge$. 

Write $V/\Phi(V) = W^\perp$. Then the $\text{char} \{ \text{ker } \chi \}$ of $\chi$ for $\chi \in \{ \chi_1, \ldots, \chi_s \}$ is $W^\perp$. 

Due to $A$ has $5$-commutator rank $2$, $W^\perp \subset \text{char } 5$. If $B^\wedge$ is the $5$-commutator rank $2$-subrep
But the intersection of the \( \ker \chi_i \) on \( A \) has to be zero or else \( A \) would not act faithfully on \( V/\overline{\phi}(V) \). Thus \( B = A \), so we see \( A \) acts trivially on \( \overline{\phi}(V) \).

Write \( V/\overline{\phi}(V) = W_1 \oplus \cdots \oplus W_s \). \( W_i \) are \( A \)-irreducible repns of \( A/\mathbb{Q}_i \), which is cyclic. Now \( \overline{\phi}(V) \) is a quotient of \( \Lambda^2 (W_1 \oplus \cdots \oplus W_s) \) on which \( A \) acts trivially. Now look at the commutator pairing

\[
W_i \otimes W_j \rightarrow \overline{\phi}(V).
\]

Because \( Q_i + Q_j = B \), \( W_i \) is irreducible over \( \mathbb{Q}_j \), which acts trivially on \( W_j \). Thus the commutator pairing is 0, so \( \overline{\phi}(V) \) is a trivial quotient of \( \Lambda^2 W_1 \oplus \cdots \oplus \Lambda^2 W_s \). Thus \( V = V_1 \times \cdots \times V_s \) where \( A/\mathbb{Q}_i \) acts faithfully on \( V_i \), and this is the sort of thing that happens in Cor. 5.3.

A comm. ab. \( p \)-group acting faithfully on an \( l \)-group \( V \). Suppose \( V_i < V \), \( V \) \( A \)-invariant \( \implies \) \( A \) does not act faithfully: \( C_A(V_i) > 1 \).

Put \( B = C_A(\overline{\phi}(V)) \), so that \( A/B \) acts faithfully on \( \overline{\phi}(V) \). If \( W \) is an \( A \)-two-subspace of \( \overline{V} = V/\overline{\phi}(V) \) such that \( C_B(W) = 1 \), \( \overline{W} \) is its inverse image in \( V \), then \( A \) acts faithfully on
\( W \) and \( \overline{W} \leq V \), so that \( W \oplus = V \), and \( \overline{W} = \overline{V} \) by minimality of \( V \). Thus for any \( A \)-invariant subspace \( \overline{W} \leq \overline{V} \), \( C_B(\overline{W}) > 1 \). Write \( \overline{V} = \overline{W}_1 \oplus \cdots \oplus \overline{W}_k \) a direct sum of irreducibles over \( A \). I know \( \overline{W}_i \leq \mathbb{F}[\mu_p] \) with \( A \) acting via a character \( \chi_i : A \rightarrow \mathbb{F}^* \), provided \( \overline{W}_i \) not trivial. Now \( B \) acts faithfully on \( \overline{V} \), which means \( \bigcap \ker(\chi_i | B) = 0 \), yet this becomes false if on \( i \) is deleted. Thus the \( \chi_i | B \) form a basis for \( \text{Hom}(B, \mathbb{F}_p) \). Also the \( \chi_i | A \) must span \( \text{Hom}(A, \mathbb{F}_p) \) since \( A \) acts faithfully. Thus \( A = B \), so \( A \) centralizes \( \Phi(\overline{V}) \).

It's also clear that if \( W/\langle \overline{V}, V \rangle = \overline{W}/\langle \overline{V}, V \rangle \), then \( A \) acts faithfully on \( V/\Phi(\overline{V}) \Rightarrow A \) acts faith on \( W/\langle \overline{V}, V \rangle \) \( \Rightarrow A \) acts faith on \( W/\langle \overline{V}, V \rangle \) \( \Rightarrow \) (by min. of \( V \)) that \( W = V \). Thus \( V/\langle \overline{V}, V \rangle \) is elem. ab. \( \Rightarrow \langle V, V \rangle = \Phi(\overline{V}) \).

Next \( V, \Phi(\overline{V}) = \langle [A, V], \Phi(\overline{V}) \rangle < \langle [A, \Phi(\overline{V})], V \rangle, [A, [V, \Phi(\overline{V})] \rangle = 1 \).

\( \Rightarrow \Phi(\overline{V}) < Z(\overline{V}) \).

Note that \( \langle [A, V], \Phi(\overline{V}) \rangle = \overline{V} \) so \( [A, V] \Phi(\overline{V}) = V \)
so \( [A, V] = V \).
Try to prove that if \( G \) is a solvable group, having a maximal elem. abelian \( p \)-group \( A \) of rank \( r \) then \( H_{r-1}(\mathbb{A}_p(G)) \neq 0 \).

Suppose \( A \) a subgruop \( L \) of \( G \) containing \( A \) such that \( H_{r-1}(\mathbb{A}_p(L)) \neq 0 \). Let \( X = \mathbb{A}_p(L) \) and let \( Y \) be the subposet of \( \mathbb{A}_p(G) \) consisting of elem ab. \( p \)-subgroups \( B \) which are contained in a max. elem.ab. \( p \)-subgroup not in \( \mathbb{A}_p(L) \). Thus I divide the max elts. of \( \mathbb{A}_p(G) \) into those contained in \( L \) and those which aren't and let \( X, Y \) be the respective "closures" of these sets. Then \( \mathbb{A}_p(G) = XY \) and \( XY \) has dimension \( < \text{rank}(A) - 1 \), so \( H_{r-1}(XY) = 0 \).

From Mayer-Vietoris

\[
H_{r-1}(XY) \rightarrow H_{r-1}(X) \oplus H_{r-1}(Y) \rightarrow H_{r-1}(XY)
\]

one sees that \( H_{r-1}(\mathbb{A}_p(L)) \rightarrow H_{r-1}(\mathbb{A}_p(G)) \), hence \( H_{r-1}(\mathbb{A}_p(G)) \neq 0 \).

Thus if we argue by induction on \( |G| \), we can suppose that for any \( L \) such that \( A < L < G \) has \( \mathbb{A}_p(L) > 1 \).

Because \( G \) is \( p \)-solvable and \( \mathbb{A}_p(G) = 1 \), one knows that \( C_G(H) < H \) where \( H = \mathbb{A}_p(G) \).
Lemma: Let the $p$-group $A$ act on a $p$-group $G$, and suppose $H$ is a normal subgroup of $G$ normalized by $A$, such that $C_G(H) < H$. If $A$ acts faithfully on $G$, then $A$ acts faithfully on $H$.

Proof: Sufficient to prove that $A$ centralizes $H \Rightarrow A$ centralizes $G$. Now, $G/C_G(H) \to \text{Aut}(H)$, so $A$ centralizes $G/C_G(H)$, hence $G/H$ as well as $H$. If $l$ is a prime dividing $|G|$, let $P$ denote an $S_l$-subgroup of $G$ invariant under $A$. (Existence of $P$.

From $G^x = G \rtimes A$ and let $P$ be an $S_l$-subgroup of $G$. Then $A^x G_{N_{l}(Q)}$, so $N_{l}(Q) \to A$, and $A_0 \to A$ where $A_0$ is an $S_l$-subgroup of $N_{l}(Q)$, $A_0$ is conjugate to $A$, and since $G^x = GA$, the conjugating element $x$ can be assumed to lie in $G$. So $A = x A_0 x^{-1} \subset x N_{l}(Q)x^{-1} = N_{l}(Qx^{-1})$, and $A$ normalizes $x Q x^{-1}$). Then $A$ centralizes $P/P \cap H \leq G/H$ and $P \cap H$. Then $A$ centralizes $P$ by Cor. 5.3.2. Thus $C_G(A)$ contains a $S_l$-group of $G$ for every $l$ dividing $|G|$, so $C_G(A) = G$. Q.E.O.

Return to case $G$ solvable, $O_p(G) = 1$, $H = O_p(G)$. Then $C_G(H) < H$, hence $A$ acts faithfully on $H$. Then $L = H \rtimes A$ has $O_p(L) = 1$, so $G = H \rtimes A$, by the minimality assumption on $G$. Also, $A$ acts non-faithfully in any $A$-invariant subgroup $H < H$. Let $q$ be a prime dividing $|H|$. One has $C_G(O_{p}(q)) < O_{p^{2}}(q)$. 

hence by the lemma A acts faithfully on \( \Omega \Phi(H) \), so by minimality of \( H \) one has \( H = \Omega \Phi(H) \). Thus \( H \) has normal Sylow groups, hence \( H \) is nilpotent.

We now consider the special case where \( H \) is a \( p \) group. A acts faithfully on \( H \), but not faithfully on any \( A \)-invariant subgroup \( H' \). I know then that \( A \) acts faithfully on \( H/\Phi(H) \) and hence faithfully on \( H/H' \). Hence faithfully on \( \Omega_1(H/H') \). If \( \Omega_1(H/H') = H/H' \), then \( A \) acts faithfully on \( H \), so by minimality \( H = H_1 \), i.e. \( H/H' \) is elementary abelian. Also the action of \( A \) on \( H/\Phi(H) \) doesn't contain the trivial representation, for if \( [A, H] = [A, H]\Phi(H)/\Phi(H) \) \( < \Phi(H) \), then \( A \) would act faithfully on \( [A, H]\Phi(H) \) (an elt of \( A \) centralizes \( [A, H]\Phi(H) \), and \( H/[A, H]\Phi(H) \) has to centralize \( H \)). Thus \( [A, H]\Phi(H) = H \), so \( [A, H] = H \). Since \( \Phi(H) < H \), \( A \) is not faithful on \( \Phi(H) \), so \( B = A, B > 1 \), which centralizes.

Let's next consider the special case where \( H \) is a \( p \) group, \( p \) prime \( \neq \pi \). A acts faithfully on \( H \), but not on any \( A \)-invariant subgroup \( H' \). I know \( A \) acts faithfully on \( H/\Phi(H) \), hence faithfully on \( H/H' \). Hence faithfully on \( \Omega_1(H/H') \). If \( H/H' = \Omega_1(H/H') \), then \( A \) acts faithfully on \( H_1 \), so \( H_1 = H \) by minimality of \( H \). Thus \( H/H' \) is elem. abelian, so \( H' = \Phi(H) \).
A does not act faithfully on $\hat{\Pi}(H)$, let $B = C_A(\hat{\Pi}(H))$. If $W = H/\hat{\Pi}(H)$ is an $A$-invariant subspace such that $B$ acts faithfully on $W$, then $A$ acts faithfully on the inverse image of $W$ in $H$. Thus $W = H/\hat{\Pi}(H)$.

Write $H/\hat{\Pi}(H)$ as a sum of irreducible reps, over $A$:

$$H/\hat{\Pi}(H) = L_1 \oplus \cdots \oplus L_s$$

so we know that $B$ acts non-faithfully on $L_1 \oplus \cdots \oplus L_s$ for any $i$. Thus each $L_i$ must be a non-trivial rep of $B$, hence $L_i \cong \mathbb{F}_p(\mu_p)$ with $A$ acting via a character $X_i : A \rightarrow \mathbb{F}_p^*$. Since $A$ acts faithfully on $H/\hat{\Pi}(H)$, it follows that $X_1, \ldots, X_s$ span $A^* = \text{Hom}(A, \mathbb{F}_p^*)$. But by the non-faithfulness of the $B$-action, when some $L_i$ is deleted, it follows that $X_1|B, \ldots, X_s|B$ is linearly independent. Thus $B = A$, $X_1, \ldots, X_s$ are a basis for $A^*$, and $A$ acts trivially on $\hat{\Pi}(H)$.


$$H/C_H(\hat{\Pi}(H)) \hookrightarrow \text{Aut}(\hat{\Pi}(H))$$

so $A$ centralizes the former, so $H = [A, H] \subseteq C_H(\hat{\Pi}(H))$, i.e. $\hat{\Pi}(H) \subseteq Z(H)$. Thus $\hat{\Pi}(H) \hookrightarrow H \twoheadrightarrow H/\hat{\Pi}(H) = H$ is a central extension, so we get a commutator pairing.
\[ \overline{H} \otimes \overline{H} \rightarrow \overline{\Phi(H)} \]

which is surjective because \( \overline{\Phi}(H) = H \). It follows that \( \overline{\Phi}(H) \) is a elementary abelian \( q \)-group.

Let us consider the homomorphism \( L_i \rightarrow \Phi(H) \) given by \( x \mapsto (x, y) \) where \( y \in L_j, j \neq i \). This is a homomorphism of representations of \( \text{Ker} \, X_j \), and since \( X_i \) maps \( \text{Ker} \, X_j \) onto \( \mu_p \), \( L_i \) is irreducible so the homomorphism is zero. Thus \( [L_i, L_j] = 0 \) for \( i \neq j \), so

\[ [\overline{L}_i, \overline{L}_j] = 0 \]

If \( p \) is odd, the path-fover makes \( p \)-a homomorphism \( H \rightarrow \overline{\Phi(H)} \) which has to be zero as \( [A, H] = 0 \). Thus from the classification of central extensions of elem. abelian \( q \)-groups:

\[ \text{Ext}(H, \overline{\Phi(H)}) = \text{Hom}(\overline{A^2H} \oplus \overline{L}_i, \overline{\Phi(H)}) \]

\[ \overline{\Phi(H)} = \prod_{i=1}^{\frac{q^2}{p}} [L_i, L_i] \quad (\text{not a direct product necessarily}) \]

If \( q \) is odd, then \( q \)-th power gives a homomorphism \( H \rightarrow \overline{\Phi(H)} \) commuting with \( A \), hence this homomorphism is zero since \( [A, H] = 0 \). From the classification of central extensions of elem. abelian \( q \)-groups, one knows that \( H \) is determined up to isomorphism by \( \overline{H} \) and the quotient of \( A^2 \overline{H} \) (over \( \overline{F} \)) given by the commutator surjection \( A^2 \overline{H} \rightarrow \overline{\Phi(H)} \). So if I start with \( \overline{H} = L_1 \oplus \cdots \oplus L_s \) with \( A \) acting on \( L_i \) via \( X_i : A \rightarrow \mu_p \) and \( A \rightarrow (\mu_p)^2 \) via \( (X_1, \ldots, X_s) \), then the possible \( H \)
up to isomorphism are given by all quotients of
\[
(A^2 H) = (A^2 L_1)_{\mu_p} \oplus \cdots \oplus (A^2 L_\ell)_{\mu_p}.
\]

Observe that if \( p \mid q-1 \), i.e. \( \mu_p \subset F_q^* \), then
\( \Lambda^2(F_q) = 0 \), so in this case \( H = 0 \) and
\( H \) is an elementary abelian \( q \)-group.

Suppose \( r \) least \( \equiv q^r-1 \equiv 0 \) \( \text{mod } p \), i.e.
\( [F_q(\mu_p) : F_q] = r \). I want to calculate \( (\Lambda^2 F_q(\mu_p))_{\mu_p} \), which
is a vector space
over \( F_q \). To determine its dimension, we can make
the base extension to \( E = F_q(\mu_p) \), whence as an \( \mu_p \)-module
\( F_q(\mu_p) \otimes E \approx E_x \oplus E_{y^a} \oplus \cdots \oplus E_{y^{q-1}} \)
where \( E_{y^a} \) denotes \( E \) with \( \mu_p \)-action
\( f \cdot x = \circ (f) y^a \)
\( \Lambda^2 F_q(\mu_p) \otimes E \) is the representation with the character
\( x^{a+b} \quad 0 \leq a \leq b < r \). So we have
\[
\dim (\Lambda^2 F_q(\mu_p))_{\mu_p} = \text{card } \{ (a, b) \mid 0 \leq a \leq b < r, \quad y^a + y^b \equiv 0 \quad \text{mod } p \}\]

But \( 0 \equiv y^a + y^b = y^a(1 + y^{b-a}) \)
\( \circ \Rightarrow y^{b-a} \equiv -1 \) \( \text{mod } p \).
\[ g^{2(b-a)} \equiv 1 \pmod{p} \Rightarrow 2(b-a) \equiv 0 \pmod{r}. \]

But \(0 < b-a < r\) so \(0 < 2(b-a) < 2r \Rightarrow 2(b-a) = r\).

So \(r\) is even and then \(a\) can be \(0, 1, \ldots, \frac{r}{2} - 1\).

\[
\dim \left( \Lambda^2 F_q(\mu_p) \right)_{\mu_p} = \begin{cases} 
0 & \text{if } r \text{ is odd} \\
\frac{r}{2} & \text{if } r \text{ is even}
\end{cases}
\]

For example, if \(g = 3\), \(p = 5\) then \(r = 4\). \(r = 2\) isn't possible with \(g\) odd for \(p \mid g+1 \Rightarrow p = 2\) whereas \(p \mid g-1\) and \(r = 1\).

Suppose next that \(g = 2\). In this case the extensions of \(F\) by an elementary abelian 2-group \(V\) are classified by homomorphisms \(S^1(F) \rightarrow V\). So we want to find \(S^1(F)_A = (S^1(F_1))_{\mu_p} \times \cdots \times (S^1(F_s))_{\mu_p}\). Let \(r = \) least integer \(> 0\) such that \(2^r \equiv 1 \pmod{p}\).

Then

\[
F_2(\mu_p)^E = E \oplus \cdots \oplus E, \chi^{2^{a-1}}
\]

so

\[
(S^2 F_2(\mu_p)) \otimes E \text{ is the repn. with characters } \chi^{2^{a+2^{b}}}, 0 \leq a \leq b < r,
\]

and

\[
\dim (S^2 F_2(\mu_p))_{\mu_p} = \text{card } \left\{(a, b) \mid 0 \leq a \leq b < r, 2^{a+2^b} \equiv 0 \pmod{p} \right\}
\]

Because \(p\) is odd, \(2^a + 2^b \equiv 0 \Rightarrow a \neq b\) so

\[
\dim (S^2 F_2(\mu_p))_{\mu_p} = \begin{cases} 
0 & \text{if } r \text{ is odd} \\
\frac{r}{2} & \text{if } r \text{ is even}
\end{cases}
\]
For example $p = 3$ then $r = 2$ so we have an interesting quadratic function on $H_2(q_3) \approx \mathbb{F}_4$ invariant for the action of $S_3$. Obviously the normal $\text{Res}_H^G$ central extension is quaternion group of order 8.

Next look at the general case of an elem. ab. pg. $A$ acting faithfully on a $p'$-group $H$ such that the action is non-faithful on any invariant subgroup $< H$. We have seen that $H$ is nilpotent, hence the product of its Sylow groups:

$$H = \prod_{i \neq p} H_i$$

Define the primes $\mathcal{S} = \{\mathfrak{p}, \mathfrak{q}, \ldots\}$ for which there exists a set $\mathcal{S}_i \subset \mathcal{S}$ such that $H_i \not| \mathfrak{p}$. Let $B_i$ be the centralizer of $H_i$ in $H$, so that $A/B_i$ acts faithfully on $H_i$. If there exists a faithful $A/B_i$-invariant subgroup $H_i < H$, then $H \times H_i \times H_i \times \cdots \times H_i$ is clearly $A$-faithful. Thus $H_i$ is a minimal faithful $g_i$-group for $A/B_i$ and one knows that.
Let $g_1, \ldots, g_n$ be the primes dividing $|H_1|$ and let $B_i < A$ be the centralizers of $g_i$, whence $A/B_i$ acts faithfully on $H_{g_i}$. It's clear that $A/B_i$ is non-faithful on any $A/B_i$-invariant subgroup $H_i^* < H_{g_i}$. I am going now to show that $A \to T(A/B_i)$ by induction on $n$.

Put $H_1 = g_1, \quad H_2 = H_2 \times \cdots \times H_2, \quad C = B_2 \cap \cdots \cap B_n = \text{centralizer of } H_2 \text{ in } A$. We've seen that $H_1/\Phi(H_1)$ is a direct sum of irreducible reps. of $A/B_1$, over $\mathbb{F}_{g_1}$, associated to characters $\chi_j: A/B_1 \to \mu_{g_1}, \quad j=1, \ldots, s$, such that $\chi_1, \ldots, \chi_s$ form a basis for $\text{Hom}(A/B_1, \mu_{g_1})$. Since $C \to A/B_1$, the restrictions of the $\chi_i$ to $C$ span $\text{Hom}(C, \mu_{g_1})$. If these were not independent, then we could find a proper subset of them spanning $\text{Hom}(C, \mu_{g_1})$ and hence find a sub-representation of $H_1/\Phi(H_1)$ on which $C$ is faithful. If $H_1^*$ is the inverse image of this subrep in $H_1$, then $C$ is faithful on $H_1^*$, hence $A$ is faithful on $H_1^* \times H_2$, contradiction minimality of $H$. Thus the $\chi_i|C$ form a basis for $\text{Hom}(C, \mu_{g_1})$, so $C \to A/B_1$, i.e. $A = B_1 \times C$ and $A \to A/B_1 \times A/C$. But now $A/C$ acting on $H_2$ can be handled by induction.

So we see that if $H$ is a solvable group with $A$ acting faithfully and not invariant subgroup is faithful, then $(H, A)$ splits as a product of the situations
encountered when $H$ is a $q$-group.

$$H \times A = \prod_{i=1}^{n} (H_i \times A_i)$$

where $H_i$ is a $q$-group with minimally faithful $A_i$-action. Thus

$$\mathcal{S}_p(H \times A) = \mathcal{S}_p(H_1 \times A_1) \ast \cdots \ast \mathcal{S}_p(H_n \times A_n).$$

Suppose $H$ is a $q$-group again.

Claim

$$\mathcal{S}_p(H \times A) \cong \mathcal{S}_p(H/\Xi(H) \times A)$$

In effect the subgroup of $H \times A$ projecting onto a subgroup $B$ of $A$ can be identified with $H/H_B$. But $A$ centralizes $\Xi(H)$, so

$$H/H_B = \overline{H}/\overline{H}_B.$$

But we have that

$$\mathcal{S}_p(\overline{H} \times A) \cong \mathcal{S}_p(L_1 \times M_1) \ast \cdots \ast \mathcal{S}_p(L_n \times M_n)$$

is a non-trivial bouquet of $(s-1)$-spheres.
July 4, 1976:

A elem. ob p group of rank r, H, a g-group on which A acts faithfully. Suppose we choose \( \phi \rightarrow \overline{\phi(A)} \) so that \( H/H_1 \) is an irreducible \( A \)-module, say associated to \( \chi: A \rightarrow \mathbb{C} \). We wish to compare \( \phi(H_1) \) with \( \phi(H) \).

A subgroup in \( \phi(H \times A) \) is of the form \( \phi(BH) \) where the coset \( hH^B \) is determined by the subgroup, here \( B \in \phi(A) \). Let \( \overline{H} = \phi(h)^B \) the image of \( h \) in \( H/H_1 \). I know that \( H^B \rightarrow \overline{H}^B \), hence \( \overline{hH^B} = \overline{H} \cdot \overline{H}^B \) in \( \overline{H} \).

Because \( \overline{H} \) is irreducible one has

\[ \overline{H}^B = 0 \iff B \notin \ker \chi \mid A \]

\[ \overline{H}^B = \overline{H} \iff B \subset \ker \chi \mid A \]

Thus

\[ 0 \subset \overline{hH^B} \iff B \subset \ker \chi \mid A \]

or \( \overline{h} = 0 \).

If \( 0 \in \overline{hH^B} \), then we can modify \( h \) so that \( \overline{h} = 0 \), i.e. \( h \in H_1 \). So the next point is to suppose given \( h \in H_1 \) and look at the set

\[ \{ B \in \phi(A) \mid \overline{hH^B} = 0 \} \]

\[ = \{ B \in \phi(A) \mid \overline{h} \in \overline{H}^B \} \]

\[ = \{ B \in \phi(A) \mid \chi(B) = 0 \} \]

This means all subgroups of \( A \) contained in the hyperplane \( \ker \chi \).
July 6, 1976

Let $A$ be an elementary abelian $q$-group acting on a $q$-group $P$. Let $B_1, \ldots, B_m$ be the hyperplanes in $A$ arranged in some order ($m = q^{q-1}/p - 1$). Claim

$$P = \prod_{i=1}^{m} C_P(B_i)$$

(product as sets, not necessarily direct product).

**Case 1:** $P$ is an elementary abelian $q$-group. In this case one has

$$P = P_0 \oplus P_1 \oplus \cdots \oplus P_m$$

where $P_0 = P^A$ and $A/B_i$ acts freely on $P_i$. Thus

$$C_P(B_i) = P^{B_i} = P_0 \oplus P_i$$

so the result is clear.

**General case:** Look at the $A$-action on $\Omega_1Z(P)$ and put

$$\bar{P} = P/\Omega.$$ Then because $P^{B_i} \to \bar{P}^{B_i}$ one has

$$\bar{P} = \prod_{i=1}^{m} \bar{P}^{B_i} \Rightarrow P = \Omega \prod \bar{P}^{B_i}$$

$$= \prod \Omega^{B_i} \prod P^{B_i}$$

because $\Omega$ is central.
Question: Let $B_1, \ldots, B_m = \text{hyperplanes in } A$ arranged in order. I've seen that the map
\[ p^{B_1} \times p^A \times \cdots \times p^A p^{B_m} \rightarrow p \]
is surjective. Is this map 1-1?

Suppose $x_1, \ldots, x_m = y_1, \ldots, y_m$ with $x_i, y_i \in p^{B_i}$. Then $\overline{x}_1, \ldots, \overline{x}_m = \overline{y}_1, \ldots, \overline{y}_m$. Using induction we know that $\overline{x}_i = \overline{y}_i \cdot z$ with $z \in p^A$. Since $p^A \rightarrow p^A$, we can lift $z$ to $u \in p^A$, and replace $y_1$ by $y_1 u$, $y_2$ by $u^{-1} y_2$. Then $\overline{x}_1 = \overline{y}_1$. Proceeding we can suppose $\overline{x}_i = \overline{y}_i$ for all $i$. Thus $x_i = y_i \cdot \omega_i$ with $\omega_i \in 2 \cdot p^{B_i}$.

Since $\omega_i$ is central, $\Pi x_i = \Pi (y_i \cdot \omega_i) = \Pi y_i \cdot \Pi \omega_i$, we have $\Pi \omega_i = 1$. But this happens only if $\omega_i \in 2 \cdot p^A$ for all $i$, whence we can modify the $y$'s to be equal to the $x$'s in $p^{B_1} \times p^A \times \cdots \times p^A p^{B_m}$.
PvA. Idea is to associate to a coset $vV^B$ the subgroup $[B,v]$ generated by commutators $bu b^{-v}u^{-1}$. Note that $u \in V^B \implies b(uv)b^{-1}(uv)^{-1} = b(vb^{-v}u^{-1})$ so this subgroup depends only on the coset.

If $P$ is elementary abelian, then

$$W = \begin{bmatrix} \mathbb{F}_q [B] \end{bmatrix} \cong [B,W] \oplus W^B$$

and $\mathbb{F}_q [B] \rightarrow W$. Thus $W^B$ is of dimension 1 and $[B,W]$ generated by $bu-v$, $b \in B$. Thus $[B,W] = [B,v]$.

The idea will be to use the size of $[B,v]$ as a measure of the distance of $vV^B$ from $O$. In the case $\Delta(P) = 0$, suppose $[B,W] = \bigoplus L_x$ is a decomposition into isotypical pieces. Then because $L_x$ is cyclic over $\mathbb{F}_q[B]$, each $L_x$ must be irreducible. So the size of $[B,v]$ is the number of non-trivial irreducible summands occurring in $[B,v]$.


$$x(\gamma,\varepsilon) = (xy,\varepsilon)(x,\varepsilon)^{-1}$$

$x,y \in A$, $\varepsilon = \sigma$.
so \( \Phi(H) \) is also \( A \)-invariant. Question: \( \Phi([A,H]) = H? \)

Suppose \( P \) is generated by \( av^{-1} \) as a range over \( A; v \) is fixed. Then the same is true of \( P \).

Thus \( P = [A,v] \oplus P^A \) where \( P^A \) is cyclic generated by \( \frac{1}{|A|} \sum a_v = \bar{v} \).

Case 1: \( \bar{v} \neq 0 \). Then \( \bar{P} = [A,v] \) and \( \Phi(P) = [A,v] \Phi(P) \) and so \( [A,v] = P \). In this case \( H = P \) and \( [A,H] = H \).

Case 2: \( \bar{v} = 0 \). Then we can find a \( w \in P^A \) such that \( \bar{w} = \bar{v} \). Hence replacing \( v \) by \( vw^{-1} \) we can assume \( \bar{v} = 0 \), except that \( \langle a \bar{v}^{-1}, a \in A \rangle \) is now smaller. But \( [A,v] \) has not changed.


Proof: Changing \( v \) to \( vw \) with \( w \in P^A \) does not change \( H \), hence we can suppose \( v \) chosen in \( v \cdot P^A \) so that \( Q = \langle av^{-1} \mid a \in A \rangle \) has least order.

Let \( \bar{Q} = Q/\Phi(Q) \). Then \( \bar{Q} = \langle a \bar{v}^{-1} \rangle \) where \( \bar{Q}^A \) is cyclic generated by \( \bar{v}^A = \frac{1}{|A|} \sum a_v \bar{v} \). Since \( Q^A \to \bar{Q}^A \) I can lift \( \bar{v}^A \) to an elt. \( u \) in \( Q^A \). It's clear that \( v^{-1} \bar{u} \in [A,v] \Phi(Q) \) which is \( A \)-invariant.
and \( \sigma^{-1} \in \sigma P^A \). Thus by minimality \( Q = [A, \sigma] \Phi(Q) \) so \( Q = [A, \sigma] \). Thus \( \sigma \in [A, \sigma] \) so

\[
H = [A, \sigma] \subseteq [A, [A, \sigma]] \subseteq [A, H] \subseteq H
\]

The preceding proof shows:

**Prop:** \( \forall \sigma \in P \), there exists \( \tilde{\sigma} \in \sigma P^A \) such that \( \tilde{\sigma} \in [A, \tilde{\sigma}] = [A, \sigma] \).

Suppose we look at the map \( \sigma P^A \rightarrow [A, \sigma] \). We've seen that \( \sigma P^A \cap [A, \sigma] \neq \emptyset \).

Assuming \( \sigma \) chosen so that \( \sigma \in [A, \sigma] \), then

\[
\sigma P^A \cap [A, \sigma] = \sigma (P^A \cap [A, \sigma]) = \sigma ([A, \sigma]^A)
\]

Now \( [A, \sigma]^A \subset \Phi [A, \sigma] \) so the ambiguity in the choice \( \sigma \) is representative for the coset \( \sigma V^A \) is further down in the group.

**Idea:** \( e(\sigma V^B) = |[B, \sigma]|. \) Then for \( B \subset B \) we have

\[
|[B, \sigma]| \leq |[B, \sigma]|.
\]

Let's analyze when we have \( [B, \sigma] = [A, \sigma] \). I assume that \( \sigma \in H = [A, \sigma] \). Then \( [B, \bar{\sigma}] = [A, \bar{\sigma}] = \overline{H} \).
which means that \( \overline{H} \) doesn't contain the trivial repn. over \( B \). Conversely, if \( \overline{H}^B = 0 \), then each irreducible repn. of \( A \) occurring in \( \overline{H} \) remains irreducible + non-trivial over \( B \). If \( \chi_i \) are the characters of \( A \) involved in \( \overline{H} \), then

\[
[B, \sigma] < \iota \overline{H} \iff [B, \sigma] < \overline{H}
\]

\[
\iff \overline{H}^B \neq 0
\]

July 10, 1976

\[S_p(P \rtimes A) \text{ where } A \text{ is an elem ab } p\text{-group of rank } r \text{ acting on } \rtimes \text{ a } q\text{-group } P \text{ faithfully.} \]

An element of \( S_p(P \rtimes A) \) is of the form \( \iota \cdot \sigma \cdot b \cdot \sigma^{-1} \) for a uniquely determined \( B \in S_p(A) \) and \( \sigma \in P \) which is determined up to the coset \( \sigma P^B \). Thus can identify a subgroup \( \sigma B \sigma^{-1} \) with the pair \((B, \sigma P^B)\). To this pair we associate the subgroup

\[\left[ B, \sigma \right] = \text{subgroup gen. by } [b, \sigma], \ b \in B\]

of \( P \). Note \( (b, \nu u) = (b, \nu) \) if \( u \in P^B \), so this subgroup depends only on \((B, \sigma P^B)\).

I've seen that I can find a representative of the
coset $vP^B$ contained in the group $[B,v]$. (Recall one chooses an element $v$ in the coset such that $H = B$-invariant subgroup gen. by $v$ is minimal, whence $H = (B,v)$. Thus can suppose $v = vP^B \cdot [B,v]$.

where $vP^B \cdot [B,v] = v \cdot [B,v]^B$). The other point is that if we put $H = [B,v]$, then $H_{av} = [B,v]$ and so $(H_{av})^B = 0$. Thus $H^B \leq (H,v,H)$. In particular:

Proposition: If $P$ is abelian, then any coset $vP^B$ has a unique representative contained in $[B,vP^B]$.

July 11, 1976.

Let $H$ be a subgroup of $P \times A$ containing $A$ and $vBv^{-1}$. Then as $H \to A$ we have $H = (H \cap P) \times A$.

where $H \cap P$ is an $A$-invariant subgroup of $P$. Since $B,vBv^{-1}$ are conjugate in $H$ one can suppose $v \in H \cap P$.

whence $[A,v] \leq H \cap P$. I've seen we can suppose $v \in [B,v]$, whence $H \cap P \geq [A,v] = [A,[B,v]]$. Thus $[A,[B,v]] \leq A$ is the smallest subgroup of $P \times A$.
containing both $A$ and $vBv^{-1}$. Similarly $[B,v] \times B$ is the smallest subgroup containing $B$ and $vBv^{-1}$.

The problem I run into seems to be this:

Given a subgroup $vBv^{-1}$, there seem to be many $S_p$-groups containing it which are minimal, i.e. $w \in [A, [B, v]]$.

A elem. ab. $p$-grp. rank $r$ acting on a $q$-group $P$, $q 
eq p$. $G = P \times A$. The set of $p$-subgroups of $G$ over a subgroup $B \triangleleft A$ can be identified with $P/P^B$ by $vP^B \leftrightarrow vBo^{-1}$.

Write $[B, vP^B]$ for the subgroup of $B$ generated by commutators $b^v = b^v b^{-1} v^{-1}$; clearly $[B, vP^B] = [B, v]$. 

Prop.1: Any coset $vP^B$ in $P/P^B$ contains an element of $[B, vP^B]$, i.e. $vP^B \cap [B, vP^B] \neq \emptyset$.

Proof: Assume $v$ chosen in the coset so that the smallest $B$-invariant subgroup containing $v$ is of least order; this subgroup is generated by $\{ b^v b^{-1} \}; b \in B$, denote it $H$. Then $H = H/[H,H]$ is generated by $\{ b^v b^{-1} \}$ so it is a quotient of $F[B]$. Thus $H = [B, H] \oplus H^B$ where $H^B$ is cyclic of order $q$ or $0$, generated by $\sum_{v} b^{-1}$, $v =$ image of $v$ in $H$. As $H^B \rightarrow H^B$, one can replace $v$ by $uv$, $u \in H^B$ so that $uv \in [B, v]$, hence by minimality of $H$, one has $H^B = 0$, so $H = [B, v]$, and $H = [B, vP^B]$. Thus $v \in vP^B \cap [B, vP^B]$.

Prop.2: The smallest subgroup of $G = P \times A$ containing $A$ and $vBo^{-1}$ is $[A, v] \times A$. 
Proof: Let $H = \langle A, vB_0^{-1} \rangle$; as $H \rightarrow A$ we have $H = (H \cap R) \times A$. Suppose first that $A = B$. Then $A, vAv^{-1}$ are both $S^2$-subgroups of $H$, so $\exists x \in H \cap P$ such that $x \in vPA$. Thus $[A, v] = [A, x] \subset H \cap P$, because $H \cap P$ is clearly $A$-invariant. On the other hand, I know from Prop 1 that $v$ can be chosen in $[A, v]$, whence $vAv^{-1} \subset [A, v] \times A$. \[ [A, v] \subset H \cap P. \]

We've used here that $A$ normalizes $[A, v]$. In fact one has

\[
(xy, z) = x y z y^{-1} x^{-1} z^{-1} = x y z y^{-1} z^{-1} x z x^{-1} z^{-1} = x (y, z) (x, z) \]

so that $a(a_1, v) = (aa_1, v)(a, v)^{-1}$ and $[A, v] \subset [A, v]$.

Next if $B \neq A$, one has $H \supset B_v B_0^{-1} \supset B_v$ so that $H \supset [B_v] \times B$ by applying the result to $P \times B$. Hence $H \cap P \supset [A, [B_v]] \supset [A, v]$ since $v \in [B_v]$ can be assumed. QED
Suppose $A$ is an elementary abelian $p$-group of rank $r$ acting on a solvable $p$-group $H$. I want to show $S_p(H \times A)$ has the homotopy type of a bouquet of $(r-1)$-spheres, by using induction with respect to $|H|$, $r$. We know this is true if $r=1$ or if $|H|=1$. So assume $H > 1$.

Then $H^{ab} = H/H' > 1$ as $H$ is solvable. We can find a maximal subgroup $H'$ of $H$ containing $H'$, whence $H = H/H'$ is an irreducible $F[Z]^A$-module for some prime $q 
eq p$.

Consider first the case where $A$ acts trivially on $H$. I claim in this case that

$$S_p(H_0 \times A) = S_p(H \times A).$$

To see this I have to show that for any $B \leq S_p(A)$ one has

$$H_0/H_0^B \to H/H^B$$

i.e. that $H_0 \cdot H^B = H$ or that $H^B \to H/H_0 = (H/H_0)^B$.

This will follow from

**Lemma:** Let $1 \to H_0 \to H \to H/\sim \to 1$ be an exact sequence of groups on which a $p$-group $A$ acts. Then $1 \to H_0^A \to H^A \to H^A /\sim A \to 1$

is exact, if $H_0$ is a $p'$-group.

**Proof:** Only have to show $H^A \to H^A$. Let $x \in H^A$ and let $X$ be the inverse image of $x$ in $H$. Then
\( G = H \times A \) acts on \( X \) and \( H \) acts simply transitively.

Hence if \( K \) is the isotropy group of a point \( x \), \( K \) is a complement to \( H \), so \( K \) is an \( S_p \)-subgroup of \( H \times A \), hence \( K \) is conjugate to \( A \) by an elt \( g \) of \( G \): \( gKg^{-1} = A \). This implies \( gx \in X^A \subset A(H) \), so \( x \) comes from \( gx \). QED.

Remark: This argument show that if we work in the topos of \( A \)-sets, then given any \( A \)-group \( H \) of order prime to \( A \) one has \( H^1(G, T_A) = 0 \). Note that this conclusion is equivalent to the conjugacy of complements to \( H \) in \( H \times A \) (Schur-Zassenhaus).

Next I have to consider the case where \( A \) acts non-trivially on \( H \). In this case I know that there is a hyperplane \( A_0 \) in \( A \) such that \( A/A_0 \) acts freely on \( H - \{1\} \). Moreover

\[
\overline{H}B = \begin{cases} 0 & B \not\subset A_0 \\ \overline{H} & B \subset A_0. \end{cases}
\]

If \( x \in \overline{H} \) put

\[
X_x = \bigcup_{b \in H} \mathcal{S}_p(bA_0 b^{-1}) \subset \mathcal{S}_p(H \times A)
\]

\( \forall H_0 = x \)

Then

\[
X_{H_0} = \mathcal{S}_p(H_0 \times A).
\]
\[ X_{h,H_0} = \bigcup_{x \in H_0} S_p(hx Ax^{-1}h^{-1}) \]
\[ = h \left[ \bigcup_{x \in H_0} S_p(xAx^{-1}) \right] h^{-1} = h \cdot X_{H_0} \]

It might be better to say that \( H \) permutes the \( X_\alpha \).

Since \( H_0 \subset H \), we know by induction that \( S_p(H_0 \times A) \), hence each \( X_\alpha \), has the homotopy type of a bouquet of \((r-1)\)-spheres.

Consider next \( X_\alpha \cap X_\beta \) where \( \alpha \neq \beta \).

Any subgroup in \( X_\alpha \cap X_\beta \) is of the form
\[ h_1Bh_1^{-1} = h_2Bh_2^{-1} \]
for some \( h_1, h_2 \) such that \( \alpha = h_1H_0 \), \( \beta = h_2H_0 \). Then \( h_1^{-1}h_2 \in H^B \) and \( h_1^{-1}h_2 \cdot H_0 = \alpha^{-1} \beta \neq 1 \), so \( H^B \neq 1 \) and so by the above \( B \subset A_0 \).

Conversely, let \( B \subset A_0 \), and let \( h_1 \) be any element of \( H \) such that \( h_1H_0 = \alpha \). Because \( H^B \rightarrow H^B = H \) we can find \( x \in H^B \) such that \( xH_0 = \alpha^{-1} \beta \), whence
\[ h_1Bh_1^{-1} = h_1xBxh_1^{-1} \]
\[ h_1xH_0 = \alpha \cdot xH_0 = \beta. \]

Belongs to \( X_\alpha \cap X_\beta \). Thus we see that
\[ X_\alpha \cap X_\beta = \bigcup_{h,H_0=\alpha} s_p(h,A_0,h) \]

The point is this is independent of \( \beta \). So if we fix \( h_0 \) isn't \( \alpha \) then

\[ X_{h_0 H} \cap X_\beta = h_0 \cdot \bigcup_{x \in H_0} s_p(xA_0x^{-1}) \]

\[ = h_0 \cdot s_p(H_0 \times A_0) \]

and this is a bouquet of \((r-2)\)-spheres by induction.

So now we can determine the homotopy type of \( s_p(H \times A) \). Order the elements of \( H : H_0 = x_1, x_2, \ldots, x_n \), and consider the sequence

\[ X_0 \subset X_{x_1} \subset X_{x_1 \cdot x_2} \subset \ldots \subset X_{x_1 \cdot \ldots \cdot x_n} = s_p(H \times A) \]

We know that

\[ X_{x_i} \cap (X_{x_1 \cdot \ldots \cdot x_i} \cdot \ldots \cdot \cdot x_{i-1}) = h_0 \cdot s_p(H_0 \times A_0) \]

if \( h_0 x_1 \ldots x_i \), by induction \( s_p(H_0 \times A_0) \) is a bouquet of \((r-2)\)-spheres up to homotopy. Also \( X_{x_i} \) is a bouquet of \((r-1)\)-spheres. So the rest follows by induction on \( i \) using

**Lemma.** If \( X = X_1 \cup X_2 \) with \( X_1, X_2, X_1 \cap X_2 \) bouquets of \( r_2-1, r_1-1, r_2-2 \) spheres respectively,
then $X$ is bouquet of $(n-1)$-spheres.

**Proof.** The inclusions $X_1 \cap X_2 \subset X_i$ are null-homotopic. So $X \sim Cyl (X_1 \leftarrow X_1 \cap X_2 \rightarrow X_2) \sim X_1 \vee \Sigma (X_1 \cap X_2) \vee X_2 \sim$ bouquet of $(n-1)$-spheres.

Let $n(H \times A)$ be the number of these spheres = rank of $\tilde{H}_{n-1}(fp(H \times A))$. We should have the formula

$$n(H \times A) = \begin{cases} \mid H \mid \cdot n(H_0 \times A) + (|H|-1) n(H_0 \times A_0) & \text{if } \tilde{H} \text{ non trivial repn. of } A \\ n(H_0 \times A) & \text{if } \tilde{H} \text{ is a trivial repn. of } A \end{cases} \tag{*}$$

**July 18, 1976:**

**Prop:** $n(H \times A) = n(gr H \times A)$ where $gr H$ denotes the associated graded group for a maximal A-invariant chain of subgroups each normal in the succeeding (composition series for $H$ as a $A$-group).

This follows by induction with respect to $|H|$ and using $(*)$. Also,

$$n(H \times A) = \sum_{0 \leq B \leq A} \text{card } (H/H_B) \cdot q^{r(B)(r(B)-1)/2} (-1)^{r(A)-r(B)}$$
Check: If \( r(A) = 1 \), then
\[
\eta(H \times A) = \text{card}(H/H^A) - 1
\]
If \( r(A) = 2 \), then \( sp(H \times A) \) is a graph with vertices
\[
H/H^B \quad \text{for each line } B \text{ in } A,
\]
\[
H/H^A
\]
and edges \( H/H^A \) for each line \( B \) in \( A \), so
\[
\chi(sp(H \times A)) = \sum_{B \in P(A)} \text{card}(H/H^B) - \text{card}(H/H^A) \cdot q
\]
\[
\eta(H \times A) = 1 - \sum_{B \text{ line in } A} \text{card}(H/H^B) + \text{card}(H/H^A) \cdot q
\]

July 19, 1976
Suppose that \( M \) is a subgroup of \( G \) such that \( |M| \equiv 0 \mod p \) and such that if \( x \) is of order \( p \) in \( M \), then \( C_G(x) \leq M \).

Let \( H \in sp(M) \), let \( P \) be a \( sp \)-subgroup of \( G \) containing \( H \). Choose \( x \) of order \( p \) in \( H \) and \( y \in Z(P) \). Then \( y \in C_G(x) \leq M \) so \( P \leq C_G(y) \leq M \).

Assuming \( N_G(P) \leq M \), I want to show \( M \neq Mx \leq M^{x^{-1}} \) is a \( p' \)-group for \( x \notin M \). The idea is to use something like Alperin's thm. So assume that
one has $N_G(H) \subseteq M$ for all $H \in \mathcal{P}$ of order larger than $H$. It's clear that $M$ contains all $p$-subgroups in the same component of $\mathcal{P}$, hence $M$ contains all $p$-elements of $N_G(H)$. These are enough to move around the $S_p$-groups of $N_G(H)$. Hence given $x \in N_G(H)$ and a $S_p$ group $S$ of $N_G(H)$ we can modify $x$ by an element of $M$ so that it normalizes $S$ where $x \in M$ because $S > H$, unless $H = P$ itself in which case we have $N_G(H) \subseteq M$ by assumption. So we've proved:

**Proof:** Assume $M \leq G$ \Rightarrow

$|M| \equiv 0 \mod p$

$x$ order $p$ in $M \Rightarrow C_G(x) \subseteq M$

$S$ Sylow $p$-subg of $M \Rightarrow N_G(S) \subseteq M$.

Then $S$ is a $S_p$-group of $G$ and \textit{whence $1 \leq H \leq S \Rightarrow N_G(H) \subseteq M,$}

so $M$ comes from an invariant partition of $T_0 \mathcal{P}(G)$.

Suppose next $p = 2$ and that $M \neq G$, whence inside $G^2$ I can find an involution $y$ not in $M$. If $x$ is an involution of $M$, then $|\langle x, y \rangle|$ has to be of order $2 \cdot$ odd number, i.e. $xy$ is of odd order, otherwise we could find an involution $z = \langle xy \rangle^m$ commuting with $x^y$. But then $x, y$ are conjugate in $G$, so $G$ has a single conjugacy class of involutions.
Direct proof of prop: If $M \cup M_{\mu}^{-1}$ has order $\equiv 0 \ (p)$ then $M \cup Pu^{-1}$ for some $S_p$-subgrp. of $M$, whence $u \in Z(P)u^{-1} \subset M$, and $uPu^{-1} \subset M$. Then $u \in \mathcal{N}_G(P) = M$ so $uM_{\mu}^{-1} = M$. Thus no need to use Alperin.

July 25, 1976

Maximal subgrp thin. (special case) Suppose $Z(P)$ non-cyclic, so that $N(H)$ is $p$-constrained & $p$-stable for all $H \in S_p(P)$, and that $\exists \ L \in B < P$ centralizing all $P$-invariant $p'$-subgroups. Then there is a largest $N_G(H)$, i.e. if $M$ is the stabilizer of the component of $P$ in $S_p(G)$, then $O_p(M) > 1$.

Special case: Assume $\exists \ no \ P$-invariant $p'$-subgps. If $1 < H < P$, then $O_p'(N_G(H)) = 1$, so Glauberman's thin says that

$$ZJ(P) = N_G(H)$$

i.e. $N_G(H) = N_G(ZJ(P)) = M$. Thus $N_G(ZJ(P))$ turns out to be the largest local subgroup in this case.

More special case: Suppose $P$ abelian, whence $1 < H < P \Rightarrow P \subset N_G(H)$. But $N_G(H)$ $p$-constrained means that an $S_p$-subgp. $Q$ of $O_p \times N_G(H)$ has its $G$ centralizer contained in $O_p \times p, N_G(H)$. Hence $P < O_p \times p, N_G(H)$ so as $P$ normalizes no $p'$-subgrp of $G$, it
follows that \( P = G_p (N_G (H)) \), so \( N_G (H) \subseteq N_G (P) \). Thus \( M = N_G (P) \) is a maximum normalizer.