June 14, 1976

$G = O_{p'}(G)P$. Pick [redacted] a prime l dividing $|O_{p'}(G)|$ and let Q be [redacted] an S_l-subgroup. Then $O_{p'}(G)P = O_{p'}(G)N_G(Q)$, so by conjugating P inside $N_G(Q)$ I can arrange that P normalizes Q. [redacted]

Let $M \subset$ [redacted] G be the stabilizer of the component of $\mathcal{S}_p(G)$ containing P. Then $M \supset P$, so

$$M = (O_{p'}(G) \cap M) \rtimes P$$

Now $M < G$ [redacted], so we can choose [redacted] l so that the l-share of G > l-share of M [redacted] so any S_l-subgp Q will not be contained in any conjugate of M. Thus I can [redacted] choose Q to be normalized by P and such that Q acts without total fixpts on $\pi_0(\mathcal{S}_p(G))$. Then it follows that the image of

$$\pi_0(\mathcal{S}_p(QP)) \longrightarrow \pi_0(\mathcal{S}_p(G))$$

is the Q-orbit of the component containing P, so $\pi_0(\mathcal{S}_p(QP))$ is [redacted] disconnected.

[redacted] Let's now analyze $\mathcal{S}_p(QP)$. Any H in $\mathcal{S}_p(QP)$ can be conjugated into P. One has a map $\mathcal{S}_p(QP) \longrightarrow \mathcal{S}_p(P)$ which is fibred. If $H \subset P$, then the fibre is the set of S_p-subgroups of QH which can be identified with $Q/N_Q(H)$. Now $N_Q(H) = Q \cap N_{QH}(H)$ is normal in $N_{QH}(H)$, and so is H, so H and $N_Q(H)$ have to commute since they are disjoint subgroups. Thus $N_Q H = C_Q(H) = Q^H$. Thus the fibre of $S_p(QP) \to S_p(P)$ over H can be identified with Q/Q^H, and it is clear that $S_p(QP)$ = fibred cat. over $S_p(P)$ assoc. to $H \mapsto Q/Q^H$. Let K = subgp gen. by Q^H as H runs over $S_p(P)$. Its clear that we have a map of functors $Q/Q^H \twoheadrightarrow Q/K$ where the latter is a constant functor. Hence we get at least an epimorphism $\pi_0 S_p(QP) \twoheadrightarrow Q/K$. On the other hand we have a map of functors $Q \twoheadrightarrow Q/Q^H$ where the first is constant, so we get $Q \twoheadrightarrow \pi_0 S_p(QP)$. In fact Q acts transitively on $\pi_0 S_p(QP)$ and its pretty clear that Q^H fixes the component pt $= \pi_0(S_p(P)) \subset \pi_0 S_p(QP)$. Thus its clear that $\pi_0 S_p(QP) = Q/K$.

Assume I can show for any l' group H acting on Q that $(Q/\Phi(Q))^H = Q^H/\Phi(Q)^H$. Then the subspace of $V = Q/\Phi(Q)$ generated by the V^H for $H \in S_p(P)$ is the subspace generated by the $Q^H\Phi(Q)/\Phi(Q)$ which is $K\Phi(Q)/\Phi(Q)$. Since $K < Q$ one has $K\Phi(Q) < G$, hence we get a fixpt. free rep. of P on $Q/K\Phi(Q)$. (First check that if P acts on a vector space V, then $W = \sum V^H$ is invariant. Clear for $g.V^H = V^{gHg^{-1}}$. Also that $1 < H \leq P$ $(V/W)^H = 0$. For $V = W \oplus W'$ and $V^H = W^H \oplus (W')^H$

and $V^H \subset W$, so $(W')^H \simeq (V/W)^H = 0$.)

Lemma: Let an l'-group H act on a [redacted] group Q, [redacted] and let K be an H-invariant l-subgroup. Then $(Q/K)^H = Q^H/K^H$. (Gorenstein p. 187, Th. 3.15)

First suppose K is abelian. Take $\alpha = gK \in (Q/K)^H$. Then $\{g \in Q \mid gK = \alpha\}$ is a K-torsor [redacted] with H-action and is classified by an elt. of $H^1(H,K) = 0$. So next chose an abelian normal subgroup A of K invariant under H. Then $(Q/K)^H = ((Q/A)/(K/A))^H = (Q/A)^H/(K/A)^H$ induction $= (Q^H/A^H)/(K^H/A^H) = Q^H/K^H$. So it works. Have proved

Thm: G p-solvable, [redacted] $\mathcal{S}_p(G)$ disconnected $\Rightarrow$ S_p subgroups are cyclic or generalized quaternion.

Proof: One has $O_p(G) = 1$, and one can assume $P > 1$, [redacted] whence $O_{p'}(G) > 1$ and $O_{p',p}(G) > O_p(G)$. If P is S_p in G, then $P \cap O_{p',p}(G)$ is S_p in $O_{p',p}(G)$. Thus one sees that

$$\pi_0\, \mathcal{S}_p(O_{p',p}(G)) \longrightarrow \pi_0\, \mathcal{S}_p(G)$$

is surjective. Moreover $O_{p',p}(G) = O_{p'}(G)(P \cap O_{p',p}(G)) \subset O_{p'}(G).P$ so we have a factorization

$$\pi_0\, \mathcal{S}_p(O_{p',p}(G)) \twoheadrightarrow \pi_0\, \mathcal{S}_p(O_{p'}(G)P) \longrightarrow \pi_0(\mathcal{S}_p(G))$$

Thus I can replace G by $O_{p'}(G)P$. In fact this part of the argument uses only that $O_{p'}(G)$ acts transitively on $\pi_0 \mathcal{S}_p(G)$.

Assume that $O_{p'}(G)$ acts transitively on $\pi_0 \mathcal{S}_p(G)$. Then

$$\pi_0 \mathcal{S}_p(O_{p'}(G)P) \longrightarrow \pi_0 \mathcal{S}_p(G)$$

is a map of $O_{p'}(G)$-sets which are transitive, hence it is surjective, so $\mathcal{S}_p(O_{p'}(G)P)$ is disconnected. (If M is the stabilizer of the component of P, then $O_{p'}(G)$ acts transitively on G/M implies $G = O_{p'}(G)M$.)

Replace G by $O_{p'}(G)P$, and let M = stabilizer of component containing P. Choose a Sylow l-group Q of G where l divides the index of M so that Q is contained in no conjugate of M. Then all orbits of Q on G/M are non-trivial, and if I pick Q to be normalized by P, then QP will not be contained in M, so the image of

$$\pi_0(\mathcal{S}_p(QP)) \longrightarrow \pi_0(\mathcal{S}_p(G))$$

will be the Q-orbit of the component of P in the latter. Thus $\pi_0 \mathcal{S}_p(QP)$ is disconnected, so I can replace G by QP.

Actually the same argument shows that if P' is any non-trivial subgroup of P, then the image of

$$\pi_0(\mathcal{S}_p(QP')) \longrightarrow \pi_0(\mathcal{S}_p(G))$$

will be the Q-orbit of the component of P. If P is not cyclic or gen. quaternion I can take P' to be an elem abelian p group of rank >1. But then in Gorenstein 5.3.16 one knows that Q is generated by Q^H for $1<H\leq P'$, so $\mathcal{S}_p(QP)$ is connected – a contradiction.

<u>Prop</u>: Let M = stabilizer of the component of P in $\mathcal{S}_p(G)$. Then $M = \langle N_G(H) \mid 1<H\leq P\rangle$.

Proof. Let $L = \langle N_G(H) \mid 1<H\leq P\rangle$. Clearly $L\subset M$. We have to show that $M\subset L$, so let $m\in M$. Then P, mPm^{-1} are in the same component of $\mathcal{S}_p(G)$. If we can show that $mPm^{-1}\in L$, then $mPm^{-1} = lPl^{-1}$ so $m\in l N_G(P)\subset L$. Thus I want to show that if Q_1, Q_2 are S_p subgps with $Q_1 \in L$, $Q_1\cap Q_2 > 1$, then $Q_2\subset L$. ~~Necessary~~ Enough to show $1<H\leq Q$ $H\subset L \Rightarrow Q\subset L$. Proceed by decreasing induction on H. ~~[illegible]~~ Observe that L contains the normalizer of any of its non-trivial p-subgroups. Choose a S_p-group Q' of L containing H, whence $N_{Q'}(H) > H$; if Q'' is an an S_p-grp of G such that $N_{Q''}(H)$ is an S_p-subgp of $N_G(H)$ containing $N_{Q'}(H)$, then induction $(Q''\supset N_{Q''}(H)\subset L,\ N_{Q''}(H)>H)$ shows $Q''\subset L$. By an element of $N_G(H)\subset L$ one can move $N_Q(H)$ into $N_{Q''}(H)$. So we can suppose $Q\supset Q\cap Q''\subset L$, $Q\cap Q''>H$ and use induction to finish.

Simpler version: If $K H \subset L$, and $H < Q$ an S_p group, then $L \supset N_G(H) \supset N_Q(H) > H$, so we can replace H by $N_Q(H)$ and continue until we get $Q \subset L$.

June 16, 1976

Basic problem is to understand the homotopy type of $S_p(G)$ when G is p-nilpotent: $G = O_{p'}(G) \rtimes P$. Is there any possibility $S_p(G)$ spherical?

Consider a group of the form $G = V \rtimes P$ where P is a p-group acting faithfully on an $\mathbb{F}_\ell$-vector space V. I've already seen that $S_p(G)$ disconnected $\Rightarrow$ P cyclic or gen. quaternions. Assume P is elementary abelian of rank r. Is it possible $S_p(G)$ is spherical in dimension $r-1$?

I assume that $r > 1$, whence I know that $S_p(G)$ is connected. If $H \subset P$, then

$$N_G(H) = V^H . P = C_G(H) \qquad P \text{ abelian}$$

$$N_G(H)/H = V^H . P/H$$

If $S_p(N_G(H)/H)$ is disconnected, then I know that P/H is cyclic and that it acts faithfully on V^H.

Since P is abelian one knows that each irreducible representation of P factors through a cyclic quotient, hence we can write uniquely

$$V = \bigoplus_Q V_Q$$ where Q runs over the subgroups of P such that P/Q is cyclic, and where V_Q is a representation of P/Q such that P/Q acts freely away from 0.

Special case: suppose that $Q_1, \ldots, Q_r$ are those subgroups $Q \ni V_Q \neq 0$ and that

$$P \xrightarrow{\sim} P/Q_1 \times \cdots \times P/Q_r \cong (\mathbb{Z}/p\mathbb{Z})^r$$

$$V \xrightarrow{\sim} V_{Q_1} \oplus \cdots \oplus V_{Q_r}$$

Can you see what the homotopy type of $\mathcal{S}_p(G)$ is. Note that $V^H = \bigcap_{H \subset Q_i} V_{Q_i}$. Perhaps it can be shown that we only have to look at the subset of H in P of the form $\bigcap_{i \in \sigma} Q_i$ where $\sigma \subset \{1, \ldots, r\}$.

June 18, 1976

Let J be a poset, let $x \in J$ such that $J_{>x} = \{y > x\}$ is contractible. Then it should be true that $J - \{x\} \subset J$ is a homotopy equivalence. In effect we have removed the vertex x from the simplicial complex BJ. The link of this vertex is the join

$$B(J_{<x}) * B(J_{>x})$$

which is contractible.

Suppose now that $a, b \in J$ and that $J_{>a}, J_{>b}$ are contractible. Remove a first, whence $J-\{a\} \subset J$ is a heq, then remove b. $(J-\{a\})_{>b} = J_{>b}$ if $a \not> b$. Thus if $a \not> b$, $J-\{a,b\} \subset J$ is a heq. On the other hand if $a > b$, then $b \not> a$, so $(J-\{b\})_{>a} = J_{>a}$, so we still win. Thus we have

<u>Prop</u>: Let S be a subset of a poset J such that $J_{>x}$ is contractible for every $x \in S$. Then $J-S \subset J$ is a homotopy equivalence if the dimension (length of a maximal chain) of S is finite

<u>Proof</u>: Use induction on $\dim S$. If $\dim S = 0$, then S is discrete, i.e. in BJ, the subcomplex with vertices S has dim 0, so that we can remove S without changing homotopy types because the links are contractible. If $\dim(S) = d$, let S_0 be the subset of minimal elements of S. Then $\dim(S_0) = 0$, so $J-S_0 \subset J$ is an heq. But $\dim(S-S_0) = d-1$, so $J-S_0-(S-S_0)$ $= J-S$ is heq to J.

<u>Corollary</u>: $\mathcal{S}_p(G)$ is heq to the subposet consisting of p-groups H such that $H = O_p(N_G(H))$

Proof. $\mathcal{S}_p(G)_{>H}$ is heq to $\mathcal{S}_p(N_G(H)/H)$ which is contractible if $H < O_p(N_G(H))$.

Next look at $J = \mathcal{a}_p(G)$. The "link" $J_{>A}$ consists of all elementary abelian p-subgroups $B > A$. Such a B is contained in $C_G(A)$. Thus if $A < \Omega_1 Z C_G A$, this link will be contractible. In particular the link is contractible unless $A \supset \Omega_p Z(G)$.

$G = V \rtimes P$ where P is an elem. abelian p-group and V is a repn. in char l. If $H \subset P$, then
$$N_G(H) = V^H \rtimes P$$
and $O_p(N_G(H)) = \{x \in P \mid x = id \text{ on } V^H\}$. Denote this last group by $(V^H)_0$. In general if $W \subset V$ put $W_0 = \{x \in P \mid x = id \text{ on } W_0\}$. Then $O_p(N_G(H)) = H \iff H = (V^H)_0$. Hence we can identify the poset of $H \subset G$ $\ni$ $O_p(N_G(H)) = H$ with the poset of cosets $v + W$ where W is a subgroup in V of the form V^H for some H, i.e. $W = V^{(W)_0}$.

Because P is abelian we have a canonical decomposition $V = \bigoplus V_Q$ where Q ranges over subgrps $\ni$ P/Q is cyclic and where P/Q acts freely on $V_Q - 0$. I assume to simplify that $V_P = V^P = 0$. Let $Q_1, \ldots, Q_s$ be those codim 1 subgroups such that $V_{Q_i} \neq 0$. To simplify suppose the Q_i are independent: $P \xrightarrow{\sim} P/Q_1 \times \cdots \times P/Q_s$. Then $V = V_1 \oplus \cdots \oplus V_s$. If $H \subset P$, then $V^H = \bigoplus_{i=1}^{s} V_i^H$ and

$$V_i^H = \begin{cases} 0 & H \not\subset Q_i \\ V_i & H \subset Q_i \end{cases}$$

Thus the possible subspaces V^H are $\bigoplus_{i \in \sigma} V_i$ where σ is a subset of $\{1, \dots, s\}$, and the corresponding subgroups are $\bigcap_{i \in \sigma} Q_i$, and $\sigma < \{1, \dots, s\}$.

$s = 2$.

$$\begin{tikzcd} \{1\} \arrow[dr, dash] & & \{2\} \arrow[dl, dash] \\ & \emptyset & \end{tikzcd} \qquad \begin{tikzcd} V/V_1 & & V/V_2 \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Looks like the join. In fact if $P = P_1 \times P_2$, $V = V_1 \oplus V_2$ where V_1 is a repn of P_1 on which P_2 acts trivially, and V_2 is a repn. of P_2 on which P_1 acts trivially, then

$$(V \rtimes P) = (V_1 \rtimes P_1) \times (V_2 \rtimes P_2)$$

so

$$S_p(V \rtimes P) = S_p(V_1 \rtimes P_1) * S_p(V_2 \rtimes P_2).$$

This shows that if the Q_i are independent, then $S_p(G)$ is spherical of dim $=$ rank$(P) - 1$.

I've seen that $\mathcal{S}_p(G)$ is heq to the subcomplex consisting of H such that $H = O_p(N_G(H))$. In the case $G = V \rtimes P$, this means I look at subgroups of the form vHv^{-1} where $\bar{v} \in V/V^H$ and $H = \{x \in P \mid x = id \text{ on } V^H\}$. In particular for such an H I have $V^H > V^P$ if $H < P$. To simplify assume $V^P = 0$.

Assume H such that $V^H > 0$ and $H = \{x \in P \mid x = id \text{ on } V^H\}$. Recall $V = \bigoplus_{i=1}^{s} V_i$ where V_i is a non-zero repn. of the cyclic gp P/Q_i which acts freely on $V_i - 0$. Then

$$V^H = \bigoplus_{i \in \sigma} V_i \qquad \sigma = \{i \mid H \subset Q_i\}$$

$$H = \{x \in P \mid x = id \text{ on } V^H\} = \bigcap_{i \in \sigma} Q_i$$

Since $V^H > 0$ it follows $\sigma \neq \phi$. Can I find a K $K < P$ such that $V^K + V^H = V$? Obviously $V_i \subset V^K$ for $i \notin \sigma$ so that $K \subset Q_i$. Thus it is a question of whether $\bigcap_{H \not\subset Q_i} Q_i$ is > 1.

My idea is to let P act on $\mathcal{S}_p(G)$ or rather this subcomplex, and the hope was that everything but the Sylow groups disjoint from P would be part of the non-free stuff. Note that if $x \in P$, then

$$x(vHv^{-1})x^{-1} = (xvx^{-1})H(xvx^{-1})^{-1}$$

so the [redacted] P-action on the conjugacy class of $H \subset P$ "is" the obvious action of P on V/V^H. If P acts faithfully on V/V^H, then one has non-S_p groups which are free for the P-action.

Observation: [redacted] Assuming $V^P = 0$, we have identified elements of V with the different S_p-subgroups of G. G acts like an affine group, i.e. V acts as translations on V and P acts linearly. Thus the poset we are studying looks like a poset of affine subspaces, as studied by Lusztig.

June 20, 1976

Review ~~notation~~: $G = V \rtimes P$ where P is elem. abelian p group and V is a rep. of P [redacted] over $\mathbb{F}_\ell$. We've identified $\mathcal{S}_p(G)$ with the fibred category over $\mathcal{S}_p(P)$ assoc. to functor $H \mapsto V/V^H$. [redacted] Because P is abelian every non-identity p-subgroup of G is conjugate to a unique H in $\mathcal{S}_p(P)$. The set of subgroups conj. to H can be identified with $G/N_G(H) = G/V^H.P \cong V/V^H$. Assume to simplify that $V^P = 0$. [redacted] Then the S_p-subgroups of G are of the form vPv^{-1}, and this representation is unique. One has $vPv^{-1} \cap v'Pv'^{-1} = vHv^{-1}$ where $H = \{x \in P \mid x\cdot(v-v') = v-v'\}$. Thus it seems clear that the simplicial complex of S_p-groups can be identified with the simplicial complex [redacted] whose vertices are the elements of V and whose simplices are subsets contained in a coset for some subspace V^H, with $KH \leq P$. This is valid without the assumption $V^P = 0$.

Critical case: Assume that $P/Q_1 \times \cdots \times P/Q_r \xleftarrow{\sim} P$ and let Q_{r+1} be another hyperplane in V.

June 21, 1976

Again $G = V \rtimes P$, P elementary abelian p group, V an elementary abelian l group on which P acts faithfully. I assume $V^P = 0$. I claim there is a natural notion of distance between two S_p-subgroups of G. Say that $v_1 P v_1^{-1}$, $v_2 P v_2^{-1}$ are of distance 1 if $v_1 P v_1^{-1} \cap v_2 P v_2^{-1}$ is of codimension 1 in either. This means that $v_1^{-1} v_2$ has stabilizer of codim 1 in P.

$$P \cap v P v^{-1} = \{ x \in P \mid x(v) = v \}.$$

Now define the distance between two S_p-groups to be the minimal length of a sequence starting with one, ending with the other such that consecutive members have distance 1. Thus if we want the distance between P and vPv^{-1} we look at chains

$$P,\ v_1 P v_1^{-1},\ v_2 v_1 P v_1^{-1} v_2^{-1},\ \ldots,\ v_s \cdots v_1 P v_1^{-1} \cdots v_s$$

where $v = v_1 + \cdots + v_s$ (use additive notation) and the stabilizer of v_i is of codim 1. Now we know that

$$V = \bigoplus_Q V^Q$$

where Q ranges over the subgps of codim 1 in P. Thus each v_i must be contained in some V^Q. Thus it's clear that the distance is simply the number of non-zero summands in the decomposition

$$(*) \qquad v = \Sigma v_Q \qquad \text{where } v_Q \in V^Q .$$

Next suppose that we consider a subgroup vHv^{-1}, which we've seen is the same as the coset $v + V^H$. Since $V^H = \bigoplus_{Q \supset H} V^Q$ there is a unique representative for $v \bmod V^H$ whose "support" involves those Q which don't contain H. Terminology: support of v is the set $\{Q \text{ codim } 1 \text{ in } P \mid v_Q \neq 0\}$, $d(0, v) = \text{card}(\text{supp } v)$. It's clear that there is a unique representative for $v + V^H$ of the minimal length. So we define the distance of the coset $v + V^H$ to be the minimal distance of any of its points to 0.

<u>Proposition</u>: Any subgroup vHv^{-1} is contained in a unique S_p-subgroup having the same distance from P. All other S_p-subgroups containing vHv^{-1} are further away from P.

The problem is now to understand the poset of [redacted] H in $Sp(P)$ such that the coset $v + V^H$ has smaller distance from 0 than v itself. Suppose the support of v is $\{Q_1, \ldots, Q_d\}$, say $v = v_1 + \cdots + v_d$, where $v_i \in V^{Q_i} - 0$. Then $v + V^H$ is closer to zero $\iff \exists i \ni v_i \in V^H \iff \exists i \ni H \subset Q_i$. So the poset I have to examine is the poset of all non-zero subspaces H contained in one of the hyperplanes $Q_1, \ldots, Q_d$. This gives rise to a simplicial complex of dimension $r-2$ where $r = \operatorname{rank} P$.

<u>Proposition</u>: Let $Q_1, \ldots, Q_d$ be hyperplanes in a vector space P of dimension r. Then the poset of non-zero subspaces H of P such that $H \subset Q_i$ for some i has the homotopy type of a bouquet of $(r-2)$-spheres.

<u>Proof</u>: Call the poset in question $X(Q_1, \ldots, Q_d)$; it is covered by the sets $X(Q_i)$ which are contractible. Also $X(Q_{i_1}) \cap \cdots \cap X(Q_{i_t})$ is empty or contractible according to whether $Q_{i_1} \cap \cdots \cap Q_{i_t}$ is 0 or not. Thus X is heq to the simplicial complex whose simplices are $\{Q_{i_0}, \ldots, Q_{i_t}\}$ such that $Q_{i_0} \cap \cdots \cap Q_{i_t} > 0$. But any subset of card $r-1$ is a simplex, hence this simplicial

complex has the same $r-2$ skeleton as the full simplex with vertices $Q_1, \dots, Q_d$, so its homotopy groups in dimensions $< r-2$ vanish. Q.E.D.

June 23, 1976

Problem: Is it true that $\mathcal{S}_p(O_{p',p}(G))$ has the homotopy type of $\mathcal{S}_p(G)$?

Put $R = O_{p',p}(G) = V \rtimes Q$, $V = O_{p'}(G)$. Let $K \in \mathcal{S}_p(G)$. We want to show $\mathcal{S}_p(R)^K$ is contractible. $\mathcal{S}_p(R)^K$ consists of non-trivial p-subgroups H of R normalized by K. Because K is a p-group one has $H^K > 1$. Thus $H \mapsto H^K$ deforms $\mathcal{S}_p(R)^K$ into $\mathcal{S}_p(R^K)$. Since I can always choose Q to be normalized by K (take an S_p-gp P of G containing K and put $Q = R \cap P$), it's clear that

$$R^K = V^K \rtimes Q^K.$$

So $\mathcal{S}_p(R^K)$ is apt to be non-contractible unless Q^K acts non-faithfully on V^K. So NO is probable answer

Example: Let $L = \mathbb{F}_\ell(\mu_p)$, and consider the wreath product $(L \rtimes \mu_p)^p \rtimes (\mathbb{Z}/p\mathbb{Z}) = L^p \rtimes (\mu_p^p \rtimes \mathbb{Z}/p\mathbb{Z})$

What are the elements of order p in $\mu_p^p \rtimes \mathbb{Z}/p\mathbb{Z}$?

Consider $\tau\cdot\sigma$ where $\tau \in \mu_p^p$ and σ is a generator of $\mathbb{Z}/p\mathbb{Z}$.

$$(\tau\sigma)(\tau\sigma) = \tau\cdot\tau^\sigma\sigma^2$$

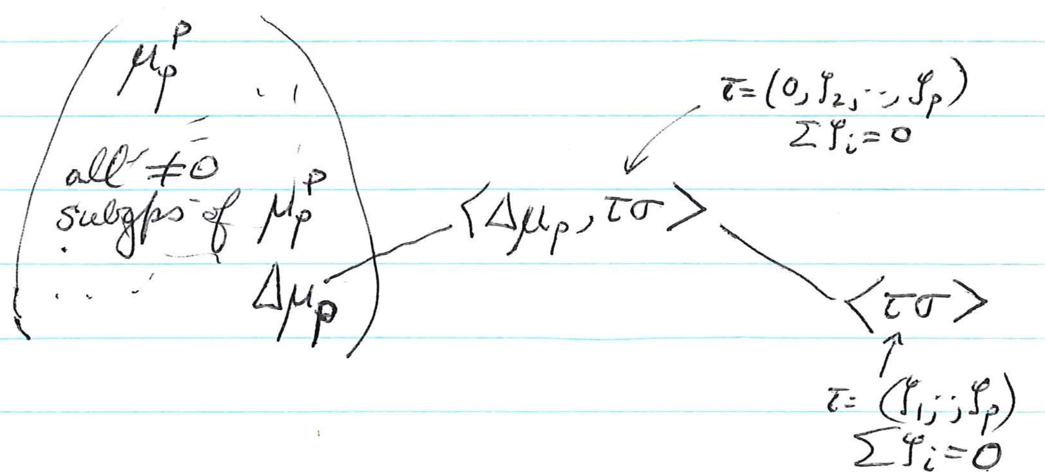
$$(\tau\sigma)^3 = \tau\cdot\tau^\sigma\tau^{\sigma^2}\sigma^3$$

so $$(\tau\sigma)^p = \tau^{1+\sigma+\cdots+\sigma^{p-1}}$$

If
$$\tau = (\zeta_1, \ldots, \zeta_p)$$
$$\tau^\sigma = (\zeta_2, \ldots, \zeta_p, \zeta_1)$$
$$\tau^{\sigma^2} = (\zeta_3, \ldots, \zeta_1, \zeta_2)$$

so $\tau^{1+\sigma+\cdots+\sigma^{p-1}} = 1 \iff \zeta_1+\cdots+\zeta_p = 0$

The centralizer of $\tau\sigma$ is the center $\Delta\mu_p$. Picture of the poset of elem. abelian subgroups:



Suppose P abelian, $V = \bigoplus V_Q$ where Q ranges over subgroups of P such that P/Q is cyclic and where P/Q acts freely on $V_Q - 0$. Assume $V_P = V^P = 0$.

The question is whether I can introduce a natural distance. $P \cap vPv^{-1} = P_v$ stabilizer of v. This is a maximal S_p-intersection $\Longleftrightarrow$ $P \cap wPw^{-1} > P \cap vPv^{-1}$ $\Longrightarrow wPw^{-1} = P$, i.e. $P_w > P_v \Longrightarrow P_w = P$. So if $H = P_v$, then we have a maximal S_p-intersection $\Longleftrightarrow$ $(w \in V^H, P_w > H \Rightarrow w = 0)$ which means P/H acts freely on $V^H - 0$.

Suppose P cyclic whence P has a unique subgroup Q_a of order p^a for $p^a \,\big|\, |P| = p^s$. We have

$$V = \bigoplus_{a=0}^{s-1} V_{Q_a}$$

We can only get to $v \in V_{Q_{s-1}}$ using maximal S_p-intersections.

It is important that each coset $v + V^H$ has a natural center.

$$V^H = \bigoplus_{Q \supset H} V_Q$$

But one knows there is a unique decomposition $V = [H,V] \oplus V^H$ because H is of order prime to l. Thus there is a unique element in the coset whose average $\frac{1}{|H|}\sum_{h\in H} hv$ is zero.

So now it is clear that I want to take a $v \in V$ (i.e. a S_p-grp vPv^{-1}) and look at the poset J_v of $H \in \mathcal{S}_p(P)$ such that the sum $\sum_{h \in H} hv$ is $\neq 0$. Note that

$1 < H' < H$, $H \in J_v \Rightarrow H' \in J_v$. Let P be abelian and suppose

$$v = v_1 + \cdots + v_s$$

$$x \cdot v_i = \chi_i(x) v_i \qquad x \in P$$

where χ_i are distinct characters of P. Then J_v consists of those $H > 1$ contained in some χ_i. Can obviously remove any H from J_v which is not elementary abelian without changing homotopy type. So when P is abelian it is clear J_v has the homotopy type of a bouquet of $(r-2)$-spheres where $r = \mathrm{rank}_p(P)$.

Return now to the example of $V = L^p$ acted on by $P = (\mu_p)^p \rtimes \mathbb{Z}/p\mathbb{Z}$. Here V is an irreducible faithful representation of P, and the center $\Delta\mu_p$ acts as scalars. Let $v = (v_1, \ldots, v_p) \in L^p$. The poset J_v^a consists of elem. abelian p-groups H such that $\sum_{h \in H} hv \neq 0$. Clearly such an H cannot contain $\Delta\mu_p$. Consider the cyclic subgroup generated by $\tau\sigma$ where $\tau = (\zeta_1, \ldots, \zeta_p) \in \mu_p^p$. $\tau\sigma$ has order p if $\prod \zeta_i = 1$.

$$\tau\sigma = (\zeta_1, \ldots, \zeta_p)\sigma$$
$$(\tau\sigma)^2 = (\zeta_1\zeta_2, \zeta_2\zeta_3, \ldots)\sigma^2$$

$$v = (v_1, \ldots, v_p)$$
$$\tau\sigma v = (\zeta_1 v_2, \zeta_2 v_3, \ldots, \zeta_p v_1)$$
$$(\tau\sigma)^2 v = (\zeta_1\zeta_2 v_3, \zeta_2\zeta_3 v_4, \ldots, \zeta_p\zeta_1 v_2)$$

Thus $v + \tau\sigma v + \cdots + (\tau\sigma)^{p-1} v =$

$$= (v_1 + \zeta_1 v_2 + \cdots + \zeta_1 \cdots \zeta_{p-1} v_p,\ v_2 + \zeta_2 v_3 + \cdots + \zeta_2 \cdots \zeta_p v_1,\ v_3 \cdots$$

$$= (w,\ \zeta_1^{-1} w,\ \ldots,\ \zeta_{p-1}^{-1} \cdots \zeta_1^{-1} w)$$

where $w = v_1 + \zeta_1 v_2 + \cdots + \zeta_1 \cdots \zeta_{p-1} v_p$

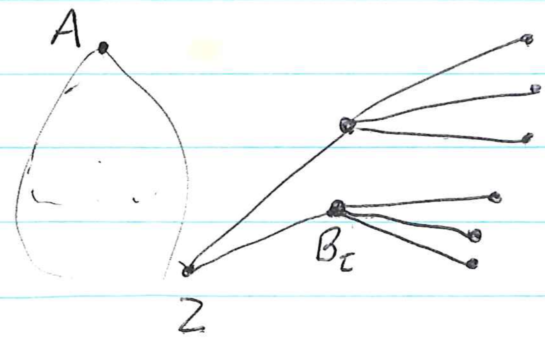
This is non-zero $\iff w \neq 0$. So for example if I take $v_2, \ldots, v_p = 0$, $v_1 \neq 0$ then $w \neq 0$ for any element $\tau\sigma$. So for this case one can see that J_v^a consists of all the subgroups $\langle \tau\sigma \rangle$ and all the subgroups of $1 \times \underbrace{\mu_p \times \cdots \times \mu_p}_{p-1 \text{ times}}$. J_v^a obviously has the homotopy type of a finite set. Perhaps that conclusion holds more generally. Certainly the case for any $(v_1, \ldots, v_p)$ with some $v_i = 0$. If all $v_i \neq 0$, then J_v^a should be the union of a discrete set and a bouquet of $p-2$-spheres.

June 24, 1976

To understand the homotopy type of $\mathcal{S}_p(G)$ when $G = (L \rtimes \mu_p)^p \rtimes (\mathbb{Z}/p\mathbb{Z}) = V \rtimes P$ where $P = \mu_p^p \rtimes (\mathbb{Z}/p\mathbb{Z})$ and V is the faith irreducible repn. on L^p. We already have understood somewhat $\mathcal{S}_p(P)$, or rather $\mathcal{A}_p(P)$. Put $A = \mu_p^p$, $C = \mathbb{Z}/p\mathbb{Z}$ so that $P = A \rtimes C$. An order p subgroup not contained in A has a unique generator $\tau\sigma$, where σ is the elt 1 of C, and $\tau = (\zeta_1, \dots, \zeta_p) \in A$ is such that $\zeta_1 \cdots \zeta_p = 1$. The maximal elementary abelian subgroups of P are A with rank p, and the following of rank 2:

$$B_\tau = \underset{Z}{\underset{\shortparallel}{\Delta\mu_p}} \cdot \langle \tau\sigma \rangle \qquad \tau = (1, \zeta_2, \dots, \zeta_p), \ \prod \zeta_i = 1$$

How many — p^{p-2} different $Z\langle\tau\sigma\rangle$. Diagram



Next point is to understand $\mathcal{A}_p(G)$. It will be the union of $\mathcal{A}_p(V \rtimes A)$ with the different $\mathcal{A}_p(V \rtimes B_\tau)$ amalgamated over $\mathcal{A}_p(V \rtimes Z) \simeq V$. Now I've seen that $\mathcal{A}_p(V \rtimes A)$ has the homot. type of a bouquet of $(p-1)$ spheres, and $\mathcal{A}_p(V \rtimes B_\tau)$ is a bouquet of S^1-s, so it's clear that we get a lot of H_1:

$$0 \longrightarrow \underset{\overset{\|}{0}}{H_1(\mathcal{A}_p(V\rtimes A))} \oplus \bigoplus_i H_1(\mathcal{A}_p(V\rtimes B_i)) \longrightarrow H_1(\mathcal{A}_p(G)) \longrightarrow \mathbb{Z}[V] \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$$

Suppose a simplicial complex [redacted] is a union $X \cup Y$ of two subcomplexes, with $\dim X = r$ and $\dim(X\cap Y) < r$. Then from Mayer-Vietoris

[redacted] $$\longrightarrow \underset{\overset{\|}{0}}{H_r(X\cap Y)} \longrightarrow H_r(X)\oplus H_r(Y) \longrightarrow H_r(X\cup Y)$$

one sees that $H_r(X) \oplus$ [redacted] $H_r(Y)$ embeds in $H_r(X\cup Y)$. Thus in a group of the form $V\rtimes P$ if A is a maximal elementary abelian p-subgroup of P of rank r, then [redacted] take X (resp. Y) to be the [redacted] subcomplex of $\mathcal{A}_p(G)$ consisting of B contained in $V\rtimes A$ (resp. not equal to A). Then $X\cap Y$ has dimension $r-2$, so

$$H_r(\mathcal{S}_p(V\rtimes A)) \hookrightarrow H_r(\mathcal{S}_p(V\rtimes P)).$$

June 25, 1976

G arb. finite group. Recall that $\bigcup_{K \in S_p(P)} S_p(G)^K$ is contractible. This complex consists of chains $1 < H_0 < \cdots < H_s$ normalized by some K in $S_p(P)$. If H is normalized by K in $S_p(P)$, then HK can be put in an S_p-group Q. Then $P \cap Q \supset K$, so if I am going to try to build up $S_p(G)$ by adding Sylow groups and all subgroups, one at a time then perhaps adding just those Q that intersect P gives something contractible? If P is abelian, then if $H_0 < H_1 < \cdots < H_s < Q > K$, then K normalizes $(H_0, \ldots, H_s)$, so in this case $\bigcup S_p(G)^K$ is just those subgroups contained in a Q intersecting P. In general

$$\bigcup_{K \in S_p(P)} S_p(G)^K \subset \bigcup_{Q \cap P > 1} S_p(Q).$$

Question: If V is a faithful repn. of P, then is there some sort of notion of distance of v from 0?

If V is a complex representation, then one could put an invariant unitary metric in V and get a good distance. The property of the distance I want is the following: If $v = v' + v''$ is the decomposition of v relative to $V = [H,V] \oplus V^H$ for some subgroup H, then $d(v') \leq d(v)$ with equality iff $v'' = \frac{1}{|H|}\sum hv$ is 0.

Question: Let V be a representation of a finite group G, say [redacted] over a field of char. 0. Form the fibred category over the poset of non-identity subgroups of G, with the fibre V/V^H over H. What sort of homotopy type does one obtain? Assume $V^G = 0$

Suppose G abelian, and $V = \bigoplus V_\chi$ is the decomposition according to the characters of G. [redacted] Define the length of v to be the number of non-zero components of v relative to this decomposition. Then the [redacted] link on attaching (G, v) to things of lower length is the poset of non-trivial subgroups H of G such that $\exists$ χ in the support of v such that $\chi(H) = 0$. Can obviously restrict [redacted] to [redacted] products of elementary abelian groups

$\exists$ similarity with parabolic (cuspidal [redacted] representations), i.e. a representation which when restricted to the unipotent radical of a parabolic subgp doesn't contain the trivial representation.

~~Thus if W is the G-space of v, I want those subgroups H such that $W^H \neq 0$. Not in general~~

<u>Problem</u>: Let $G = V \rtimes P$ [redacted] with P a p-group and V an elem. ab. l-group. [redacted] Then each coset

$v + V^H$, $H \in \mathcal{S}_p(P)$ has a canonical representative. The problem is to define distance from 0, so that the distance increases for other points of this coset.

If H is any subgroup of P, define
$$d_H(v) = \dim \langle Hv \rangle$$

If $v = v' \oplus v'' \in [H,V] \oplus V^H$, then
$$\langle Hv \rangle = \langle Hv' \rangle \oplus \langle v'' \rangle$$

so that
$$d_H(v') = \begin{cases} d_H(v) & \sum hv = 0 \\ d_H(v) - 1 & \sum hv \neq 0 \end{cases}$$

If H is normalized by K, then $V = [H,V] \oplus V^H$ is K-invariant,

so $\langle Kv' \rangle$ is a quotient of $\langle Kv \rangle$ and
$$d_K(v') \leq d_K(v).$$

June 26, 1976

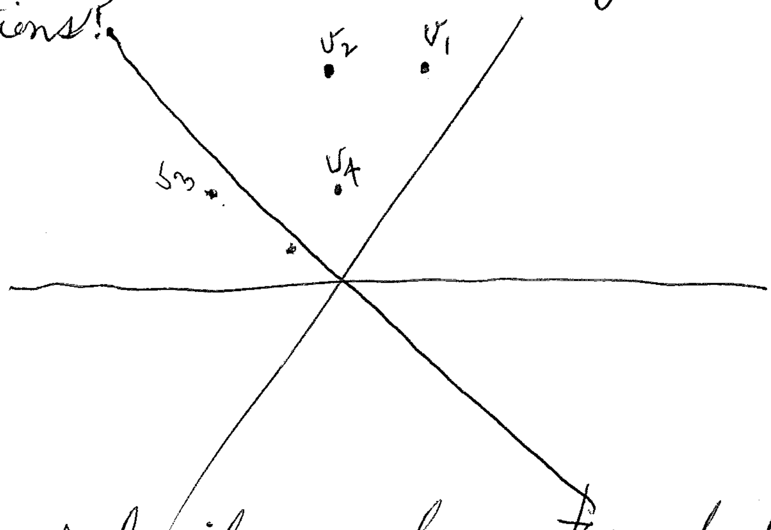
Let V be a char. l repn. of the p-group [redacted] P. I've seen that each coset $v + V^H$, $H \in S_p(P)$ has a unique "center" because of the decomposition $V = [H,V] \oplus V^H$. [redacted] So, given v, one is led to consider those $H \in S_p(P)$ such that v is not the center of $v + V^H$, i.e. such that $\frac{1}{|H|}\sum_{h\in H} hv \neq 0$. [redacted] The real question is whether I can build up $S_p(G)$ by adding one Sylow group vPv^{-1} at a time so that [redacted] vPv^{-1} gets attached via the poset of H in $S_p(P)$ $\ni$ $\sum hv \neq 0$.

[redacted]

Question: Introduce the relation $v_1 \xrightarrow{H} v_2$ to mean that $v_1 = v_2 - \frac{1}{|H|}\sum h v_2$. Write $v_1 \prec v_2$ if $\exists H \in S_p(P)$ such that $v_1 \xrightarrow{H} v_2$. Is the relation $v_1 \prec v_2$ extendable to a linear ordering?

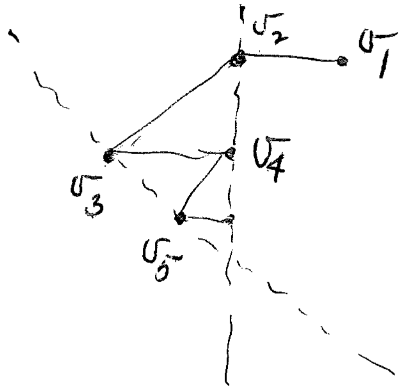
Example: Suppose V is a representation in characteristic 0. If the ordering is not extendable there is a circuit $v_1 \prec v_2 \prec \cdots \prec v_{n-1} \prec v_1$, hence one can suppose V defined over a subfield of $\mathbb{C}$, hence one can suppose V [redacted] is a repn. over $\mathbb{C}$. But then using an invariant unitary structure, one has $v_1 \prec v_2 \implies \|v_1\| \leq \|v_2\|$ with equality $\iff v_1 = v_2$. Hence there are no circuits. Question — can one construct a sequence $v_1 \succ v_2 \succ v_3 \succ \cdots$ tending to zero? (Yes?), see 144)

Generalization: Given a vector space V together with a family $\mathcal{W}$ of subspaces V^H, one forms the poset of cosets $v+V^H$ as in Wagener-Volodin. I can attempt to analyze this poset by adding one v at a time. Here given v I concentrate on those subspaces V^H such that v projects non-trivially onto V^H (say that V is a Hilbert space). What about the case when the group P acting on V is generated by reflections?



pretty clear this sequence goes to zero.

In general, if you have two hyperplanes in the family, then alternately projecting on the orthogonal lines, you get to zero



Difficulty: I have been trying to build up $\mathcal{S}_p(V \rtimes P)$ by adjoining $\mathcal{S}_p(Q)$ for a certain ordering of the S_p-subgroups. Now one thing I want out of my exhaustion of $\mathcal{S}_p(V\rtimes P)$ is to get the contractible set

$$\bigcup_{K \in \mathcal{S}_p(P)} \mathcal{S}_p(G)^K$$

as well as some reason for its contractibility, i.e. if we attach Y along ∂Y, then $\partial Y \subset Y$ is a heq. Now if $H \in \mathcal{S}_p(G)$ and $N_G(H) \cap P = K > 1$ then in the course of getting to H, I would attach probably an S_p-group Q containing KH. Thus I get H by adding $\mathcal{S}_p(Q)$ for some $Q \ni Q \cap P > 1$. But not all of $\mathcal{S}_p(Q)$ is contained in the contractible piece $\bigcup \mathcal{S}_p(G)^K$ when P is non-abelian.

June 28, 1976

Prop: The center $Z(G)$ = the intersection of all maximal abelian subgroups of G.

Proof: If A is abelian, so is $Z(G)A$, hence $Z(G) \subset A$ if A is max. ab.; thus $Z(G) \subset \bigcap A$, A max. ab. If $x \in \bigcap A$ and $y \in G$, then let $\langle y \rangle$ be extended to a maximal abelian subgroup A; then $x, y \in A$ so x, y commute; hence $x \in Z(G)$.

Question: In a p-group P what sort of subgroup is the intersection of the maximal elementary abelian p-subgroups? Can it be bigger than $\Omega_1 Z(P)$?

Let $x \in \bigcap A$, A max. elem. p-ab. If $y \in P$ is of order p, then $\langle y \rangle \subset$ some A, so x, y commute. Thus any such $y \in C_P(x)$, so $\Omega_1 P \subset C_P(x)$. So clearly

$$\bigcap_{\substack{A \text{ max.} \\ \text{in } \mathcal{A}_p(G)}} A = \Omega_1 Z(\Omega_1 P)$$

Let Q be a p'-group acting non-trivially on the p-group P. Assume that Q acts trivially on any Q-invariant subgroup $P' < P$. Then Q acts trivially on $\Phi(P)$. Also $P/\Phi(P)$ must be irreducible and non-trivial, (otherwise the action would be trivial on the whole Lie ring hence trivial on P). Next look at the bracket pairing

$$gr_1(P) \underset{\mathbb{Z}/p}{\otimes} gr_1(P) \twoheadrightarrow gr_2(P).$$

~~Action of Q on $gr_2(P)$ is trivial~~

Cor. 5.3.4: $A \times B \subset \mathrm{Aut}(P)$, A p'-group, B, P are p-groups. If $P^B \subset P^A$, then $P^A = P$.

Put $Q = P \rtimes B$. Then A acts trivially on $P^B \times B = C_Q(B) \cdot B$, so this is equivalent to

<u>Prop:</u> If a p'-group A acts trivially on a p-group Q and acts trivially on a subgroup B of Q such that $C_Q(B) \subset B$, then A acts trivially.

<u>Proof:</u> $B \subset Q^A \Longrightarrow C_Q(Q^A) \subset C_Q(B) \subset B \subset Q^A$, so can suppose $B = Q^A$.

$$N_Q(B)/C_Q(B) \hookrightarrow \mathrm{Aut}(B)$$

so A acts trivially on $N_Q(B)/C_Q(B)$ and $C_Q(B) \Longrightarrow A$ acts trivially on $N_Q(B)$. Thus $N_Q(Q^A) = Q^A$ which in a p-group can happen iff $Q^A = Q$.

<u>Cor.</u> If P^A contains a maximal abelian subgroup of P, then $P^A = P$.

<u>Remark:</u> $C_p(B) \subset B \Longrightarrow C_p(B) = Z(B)$. Let A be a maximal abelian subgroup of B. Then $C_B(A) = A$.

June 29, 1976

A elementary abelian p-group acting faithfully on an l-group V. Assume that the action is not faithful on any proper [redacted] A-invariant subgroup. According to Gor. 5.3.13, V is of class 2 and exponent l, if l is odd. I want to show that V has to be abelian if I can. NO

[redacted]

[redacted] The A-action on $\Phi(V)$ is not faithful; let $B < A$ be the subgp acting trivially. [redacted] A hence B acts faithfully on $V/\Phi(V)$.

~~Case 1: rank $B \geq 2$. We can decompose $V/\Phi(V)$ into pieces associated to the different irreducible reps of~~

[redacted] Since A/B acts faithfully on $\Phi(V)$ if we take any A-invariant $W \subset V/\Phi(V)$ on [redacted] which B acts faithfully, and let $V'/\Phi(V) = W$, then A acts faithfully on V'. Thus B acts [redacted] non-faithfully on any A-invariant proper subspace of $V/\Phi(V)$. So [redacted] if we decompose $V/\Phi(V)$ into irreducible repns. of A thereby obtaining characters of A say $\chi_1, \ldots, \chi_s$ [redacted] the intersection of the kernels of these characters [redacted] with B is trivial, yet for any proper subset of the χ_i it is non-trivial. Thus χ_i as elements of $B^\wedge$ span $B^\wedge$ yet no proper subset does. Thus the χ_i must be a basis for $B^\wedge$.

~~Write $V/\Phi(V) = W_1 \oplus \cdots \oplus W_s$ then has a commutator pairing $W_a \otimes W_b \to \Phi(V)$ ... Because let $C_a = \operatorname{Ker} \chi_a$ on B.~~

[redacted] But the intersection of the $\operatorname{Ker}\chi_i$ on A has to be zero or else A would not act faithfully on $V/\Phi(V)$. Thus $B=A$, so we see A acts trivially on $\Phi(V)$.

Write $V/\Phi(V) = W_1 \oplus \cdots \oplus W_s$, W_i irred. repn of A/Q_i which is cyclic. Now $\Phi(V)$ is a quotient of $\Lambda^2(W_1 \oplus \cdots \oplus W_s)$ on which A acts trivially. Now look at the commutator pairing

$$W_i \underset{\mathbb{F}_\ell}{\otimes} W_j \longrightarrow \Phi(V).$$

Because $Q_i + Q_j = B$, W_i is irreducible over Q_j which acts trivially on W_j. Thus the commutator pairing is 0, so $\Phi(V)$ is [redacted] a A-trivial quotient of $\Lambda^2 W_1 \oplus \cdots \oplus \Lambda^2 W_s$. Thus $V = V_1 \times \cdots \times V_s$ where A/Q_i acts faithfully on V_i, and this is the sort of thing that happens in Cor. 5.3.

A elem. ab. p-group acting faithfully on an ℓ-group V. Suppose $V_1 \triangleleft V$, V_1 A-invariant $\Rightarrow$ A does not act faithfully: $C_A(V_1) > 1$. [redacted]. Put $B = C_A(\Phi(V))$, so that A/B acts faithfully on [redacted] $\Phi(V)$. If $\bar{W}$ is a non-zero A-inv. subspace of $\bar{V} = V/\Phi(V)$ such that $C_B(\bar{W}) = 1$, and W is its inverse image in V, then A acts faithfully on

W [redacted] and [redacted] $\overline{W} \triangleleft V$, so that W [redacted] $= V$, and $\overline{W} = \overline{V}$ by minimality of V. Thus for any A-invariant subspace $\overline{W} < \overline{V}$, $C_B(\overline{W}) > 1$. Write $\overline{V} = \overline{W}_1 \oplus \cdots \oplus \overline{W}_t$ a direct sum of irreducibles over A. I know $\overline{W}_i \cong \mathbb{F}_\ell[\mu_p]$ with A acting via a character $\chi_i : A \to \mu_p$; ~~[illegible]~~ provided $\overline{W}_i$ not trivial. Now B acts faithfully on $\overline{V}$, which means $\bigcap_i \mathrm{Ker}(\chi_i|B) = 0$, yet this becomes false if an i is deleted. Thus the $\chi_i|B$ form a basis for $\mathrm{Hom}(B, \mu_p)$. Also the $\chi_i|A$ must span $\mathrm{Hom}(A, \mu_p)$ since A acts faithfully. Thus $A = B$, so A centralizes [redacted] $\Phi(V)$.

It's also clear that if [redacted] $W/[V,V] = \Omega_1 V/[V,V]$, then A acts faithfully on [redacted] $V/\Phi(V) \Rightarrow A$ acts faith on $V/[V,V] \Rightarrow A$ acts faith on $W/[V,V] \Longrightarrow$ (by min. of V) that $W = V$. Thus [redacted] $V/[V,V]$ is elem. ab. $\Longrightarrow [V,V] = \Phi(V)$.

Next

$$(V, \Phi(V)) = [[A,V], \Phi(V)] \subset \langle [[A, \Phi(V)], V], [A, (V, \Phi(V))] \rangle = 1.$$

$\therefore \quad \Phi(V) \subset Z(V)$.

Note that ~~$[A,V]\Phi(V)$~~ $[A, \overline{V}] = \overline{V}$, so $[A,V]\Phi(V) = V$ so $[A,V] = V$.

Try to prove that if G is a solvable group with $O_p(G)=1$, having a maximal elem. abelian p group A of rank r then $H_{r-1}(\mathcal{A}_p(G)) \neq 0$.

Suppose $\exists$ a subgroup L of G containing A such that $H_{r-1}(\mathcal{A}_p(L)) \neq 0$. ~~Let $|\mathcal{A}_p(L)|$ be the simplicial complex associated to the poset $\mathcal{A}_p(L)$. Let~~ Let $X = \mathcal{A}_p(L)$ and let Y be the subposet of $\mathcal{A}_p(G)$ consisting of elem ab. p-subgroups B which are contained in a max. elem. ab. p-subgroup not in $\mathcal{A}_p(L)$. Thus I divide the max elts. of $\mathcal{A}_p(G)$ into those contained in L and those which aren't and let X, Y be the respective "closures" of these sets. Then $\mathcal{A}_p(G) = X \cup Y$ and $X \cap Y$ has dimension $<$ rank$(A)-1$, so $H_{r-1}(X \cap Y) = 0$. From Mayer-Vietoris

$$\underset{\overset{\parallel}{0}}{H_{r-1}(X \cap Y)} \longrightarrow \underset{\neq}{H_{r-1}(X)} \oplus H_{r-1}(Y) \longrightarrow H_{r-1}(X \cup Y)$$

one sees that $H_{r-1}(\mathcal{A}_p(L)) \hookrightarrow H_{r-1}(\mathcal{A}_p(G))$, hence $H_{r-1}(\mathcal{A}_p(G)) \neq 0$.

Thus if we argue by induction on $|G|$, we can suppose that for any L such that $A \subset L < G$ has one, $O_p(L) > 1$.

Because G is p-solvable and $O_p(G)=1$, one knows that $C_G(H) \subset H$ where $H = O_{p'}(G)$.

<u>Lemma</u>: Let the p-group A act on a p'-group G, and suppose H is a normal subgroup of G normalized by A, such that $C_G(H) \subset H$. If A acts faithfully on G, then A acts faithfully on H.

<u>Proof</u>: Suffices to prove that A ~~acts trivially on~~ centralizes H $\Rightarrow$ A ~~acts trivially on~~ centralizes G. Now $G/C_G(H) \hookrightarrow Aut(H)$, so A centralizes $G/C_G(H)$, hence G/H as well as H. If ℓ is a prime dividing $|G|$, let P denote an S_ℓ-subgroup of G invariant under A. (Existence of P: Form $G^* = G \rtimes A$ and let Q be an S_ℓ-subgroup of G. Then $G^* = GN_{G^*}(Q)$, so $N_{G^*}(Q) \twoheadrightarrow A$, and $A_0 \xrightarrow{\sim} A$ where A_0 is an S_ℓ-subgroup of $N_{G^*}(Q)$. A_0 is conjugate to A, and since $G^* = GA$ the conjugating element x can be assumed to lie in G. So $A = xA_0x^{-1} \subset xN_{G^*}(Q)x^{-1} = N_{G^*}(xQx^{-1})$, and A normalizes xQx^{-1}). Then A centralizes $P/P\cap H \hookrightarrow G/H$ and $P \cap H$. Then A centralizes P by Cor. 5.3.2. Thus $C_G(A)$ contains a S_ℓ-group of G for every ℓ dividing $|G|$, so $C_G(A) = G$. QED.

Return to case G solvable, $O_p(G) = 1$, $H = O_{p'}(G)$. Then $C_G(H) \subset H$, hence A acts faithfully on H. Then $L = H \rtimes A$ has $O_p(L) = 1$, so $G = H \rtimes A$, by the minimality assumption on G. Also A acts non-faithfully on any A-invariant subgroup $H_1 < H$. Let q be a prime dividing $|H|$. One has $C_G(O_{q,q'}(H)) \subset O_{q,q'}(H)$

hence by the lemma A acts faithfully on $O_{qq'}(H)$, so by minimality of H one has $H = O_{qq'}(H)$. Thus H has normal Sylow groups, hence H is nilpotent.

~~We now consider the special case where H is a q-group. A acts faithfully on H, but not faithfully on any A-invariant subgroup $< H$.~~ [redacted]

~~[redacted] I know then that A acts faithfully on [redacted] $H/\Phi(H)$, and hence faithfully on H/H', hence faithfully on $\Omega_1(H/H')$. If $\Omega(H/H') = H_1/H'$, then A acts faithfully on H_1, so by minimality $H = H_1$, i.e. H/H' is elementary abelian and $\Phi(H) = H'$. Also the action of A on $\bar{H} = H/\Phi(H)$ can't contain the trivial representation, for if $[A, \bar{H}] = [A,H]\Phi(H)/\Phi(H) < \bar{H}$, then A would act faithfully on $[A,H]\Phi(H)$ (an elt of A centralizing $[A,H]\Phi(H)$, and $H/[A,H]\Phi(H)$ has to centralize H). Thus $[A,H]\Phi(H) = H$, so $[A,H] = H$.~~

~~Since $\Phi(H) < H$, A is not faithful on $\Phi(H)$, so $\exists\, B \subset A$, $B > 1$, which centralizes~~

Let's next consider the special case where H is a q-group, q prime $\neq p$. A acts faithfully on H but not on any A-invariant subgroup $< H$. I know A acts faithfully on $H/\Phi(H)$, hence faithfully on $H/H' = H_{ab}$, hence faithfully on $\Omega_1(H/H')$. If $H_1/H' = \Omega_1(H/H')$, then A acts faithfully on H_1, so $H_1 = H$ by minimality of H. Thus H/H' is elem. abelian, so $H' = \Phi(H)$.

A does not act faithfully on $\Phi(H)$, let $B = C_A(\Phi(H))$. ~~Then B acts faithfully on $H/\Phi(H)$~~ If $W \subset H/\Phi(H)$ is an A-invariant subspace such that B acts faithfully on W, then A acts faithfully on the inverse image of W in H; Thus $W = H/\Phi(H)$. Write $H/\Phi(H)$ as a sum of irreducible repns. over A

$$H/\Phi(H) = L_1 \oplus \cdots \oplus L_s$$

so we know that that B acts non-faithfully on $L_1 \oplus \cdots \oplus \hat{L_i} \oplus \cdots \oplus L_s$ for any i. Thus each L_i must be a non-trivial repn of B, hence $L_i \cong \mathbb{F}_q(\mu_p)$ with A acting via a character $\chi_i : A \twoheadrightarrow \mu_p$. Since A acts faithfully on $H/\Phi(H)$, it follows that $\chi_1, \ldots, \chi_s$ span $A^* = \mathrm{Hom}(A, \mu_p)$. But by the non-faithfulness of the B-action when some L_i is deleted, it follows that $\chi_1|B, \ldots, \chi_s|B$ is linearly independent. Thus $B = A$, $\chi_1, \ldots, \chi_s$ are a basis for A^*, and A acts trivially on $\Phi(H)$.

Since A acts trivially on $H/[A,H]\Phi(H)$, it must act non-trivially on $[A,H]\Phi(H)$, so $[A,H]\Phi(H) = H$ by minimality of H, so $[A,H] = H$. But

$$H/C_H(\Phi(H)) \hookrightarrow \mathrm{Aut}(\Phi(H))$$

so A centralizes the former, so $H = [A,H] \subset C_H(\Phi(H))$, i.e. $\Phi(H) \subset Z(H)$. Thus $\Phi(H) \to H \to H/\Phi(H) = \bar{H}$ is a central extension, so we get a commutator pairing

$$\bar{H} \otimes_{\mathbb{Z}} \bar{H} \longrightarrow \Phi(H)$$

which is surjective because $\Phi(H) = H'$. It follows that $\Phi(H)$ is an elementary abelian q-group.

Let us consider the homomorphism $L_i \to \Phi(H)$ given by $x \mapsto (x,y)$ where $y \in L_j$, $j \neq i$. This is a homomorphism of representations of $\operatorname{Ker} X_j$, and since X_i maps $\operatorname{Ker} X_j$ onto μ_p, L_i is irreducible so the homom. is zero. Thus $[L_i, L_j] = 0$ for $i \neq j$, so

~~If p is odd, the pth power map gives a homomorphism $\bar{H} \to \Phi(H)$ which has to be zero as $[A, \bar{H}] = \bar{H}$. Thus from the classification for central extensions of elem. abelian q-groups:~~

~~$\operatorname{Ext}(\bar{H}, \Phi(H)) = \operatorname{Hom}(\Lambda^2 H \oplus \bar{H}, \Phi(H))$~~

$$\Phi(H) = \prod_{i=1}^{s} [L_i, L_i] \quad \text{(not a direct product necessarily).}$$

If q is odd, then qth power gives a homomorphism $\bar{H} \to \Phi(H)$ commuting with A, hence this homom is 0 since $[A, \bar{H}] = \bar{H}$. From the classification of central extensions of elem. abelian q-groups, one knows that H is determined up to isom. by $\bar{H}$ and the quotient of $\Lambda^2 \bar{H}$ (over $\mathbb{F}_q$) given by the commutator surjection $\Lambda^2 \bar{H} \twoheadrightarrow \Phi(H)$. So if I start with $\bar{H} = L_1 \oplus \cdots \oplus L_s$ with A acting on L_i via $X_i : A \to \mu_p$ and $A \xrightarrow{\sim} (\mu_p)^s$ via $(X_1, \ldots, X_s)$, then the possible H

up to isomorphism are given by all possible quotients of

$$\left(\Lambda^2 \bar{H}\right)_A = \left(\Lambda^2 L_1\right)_{\mu_p} \oplus \cdots \oplus \left(\Lambda^2 L_s\right)_{\mu_p}.$$

Observe that if $p \mid q-1$, i.e. $\mu_p \subset \mathbb{F}_q^*$, then $\Lambda^2(\mathbb{F}_q) = 0$, so in this case $\Phi(H) = 0$ and H is an elementary abelian q-group.

Suppose r least $\ni$ $q^r - 1 \equiv 0 \pmod{p}$, i.e. $[\mathbb{F}_q(\mu_p) : \mathbb{F}_q] = r$. I want to calculate $\left(\Lambda^2 \mathbb{F}_q(\mu_p)\right)_{\mu_p}$ which ~~make the base extension to~~ is a vector space over $\mathbb{F}_q$. To determine its dimension, we can make the base extension to $E = \mathbb{F}_q(\mu_p)$, whence as an μ_p-module

$$\mathbb{F}_q(\mu_p) \otimes E \simeq E_\chi \oplus E_{\chi^q} \oplus \cdots \oplus E_{\chi^{q^{r-1}}}$$

where $E_{\chi^{q^a}}$ denotes E with μ_p-action

$$\zeta * x = (\zeta)^{q^a} x$$

$\left(\Lambda^2 \mathbb{F}_q(\mu_p)\right) \otimes E$ is the representation with the characters $\chi^{q^a + q^b}$ $0 \leq a < b < r$. So we have ~~we want to know if~~

$$\dim \left(\Lambda^2 \mathbb{F}_q(\mu_p)\right)_{\mu_p} = \operatorname{card} \left\{ (a,b) \;\middle|\; \begin{array}{l} 0 \leq a < b < r \\ q^a + q^b \equiv 0 \quad (p) \end{array} \right\}$$

But $$0 \equiv q^a + q^b = q^a(1 + q^{b-a})$$

so $$\Rightarrow \quad q^{b-a} \equiv -1 \pmod{p}.$$

$$\Longrightarrow \quad q^{2(b-a)} \equiv 1 \pmod{p} \Longrightarrow \quad 2(b-a) \equiv 0 \pmod{r}.$$

But $0 < b-a < r$ so $0 < 2(b-a) < 2r \Longrightarrow 2(b-a) = r$. So r is even and then a can be $0, 1, \dots, r/2 - 1$.

$\therefore$

$$\dim\left(\Lambda^2 \mathbb{F}_q(\mu_p)\right)_{\mu_p} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ r/2 & \text{if } r \text{ is even.} \end{cases}$$

For example, if $q=3$, $p=5$ then $r=4$. $r=2$ isn't possible with q odd for $p \mid q+1 \Longrightarrow p=2$ whence $p \mid q-1$ and $r=1$.

Suppose next that $q=2$. In this case the central extensions of $\bar{H}$ by an elementary abelian 2-group V are classified by homom. $S^2(\bar{H}) \to V$. So we want all quotients of $S^2(\bar{H})_A = (S^2 L_1)_{\mu_p} \times \cdots \times (S^2 L_s)_{\mu_p}$. Let r = least integer >0 such that $2^r \equiv 1 \pmod{p}$. Then

$$\mathbb{F}_2(\mu_p) \otimes E = E_\chi \oplus \cdots \oplus E_{\chi^{2^{r-1}}}$$

so

$$\left(S^2 \mathbb{F}_2(\mu_p)\right) \otimes E \quad \text{is the repn. with characters } \chi^{2^a + 2^b} \quad 0 \le a \le b < r.$$

and

$$\dim\left(S^2 \mathbb{F}_2(\mu_p)\right)_{\mu_p} = \operatorname{card}\left\{(a,b) \;\middle|\; \begin{array}{l} 0 \le a \le b < r \\ 2^a + 2^b \equiv 0 \pmod{p} \end{array}\right\}$$

Because p is odd, $2^a + 2^b \equiv 0 \Longrightarrow a \ne b$. So

$$\dim\left(S^2 \mathbb{F}_2(\mu_p)\right)_{\mu_p} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ r/2 & \text{if } r \text{ is even} \end{cases}$$

For example $p=3$ then $r=2$ so we have an interesting ~~invariant~~ quadratic function on $\mathbb{F}_2(\zeta_3) \simeq \mathbb{F}_4$ invariant for the action of ζ_3. Obviously the norm; Resulting central extension is quaternion group of order 8.

Next ~~suppose~~ look at the general case of an elem. ab. p-group A acting faithfully on a p'-group H such that the action is non-faithful on any invariant subgroup $< H$. We have seen that H is nilpotent, hence the direct product of its Sylow groups:

$$H = \prod_{q \neq p} H_q$$

~~Order the primes $q_1, q_2, \ldots, q_n$ which divide $|H|$. Let A_i be the centralizer of H_{q_i} ... Then $A_i \neq A$ by minimality of A and A_i acts faithfully on H_{q_i}. Moreover if there exists a~~

Let $q_1, \ldots, q_n$ be the primes dividing $|H|$. Let $B_i \subset A$ be the centralizer of H_{q_i}, so that A/B_i acts faithfully on H_{q_i}. If there exists a faithful A/B_i-invariant subgroup $H^*_{q_i} < H_{q_i}$, then $H_{q_1} \times \cdots \times H^*_{q_i} \times \cdots \times H_{q_n}$ is clearly ~~...~~ A-faithful. Thus H_{q_i} is a minimal faithful q_i-group ~~...~~ for A/B_i and one ~~...~~ knows ~~...~~

Let $q_1, \dots, q_n$ be the primes dividing $|H|$, and let $B_i \subset A$ be the centralizer of H_{q_i}, whence A/B_i acts faithfully on H_{q_i}. It's clear that A/B_i is non-faithful on any A/B_i-invariant subgroup $H^*_{q_i} < H_{q_i}$. I am going now to show that $A \xrightarrow{\sim} \prod A/B_i$ by induction on n.

Put $H_1 = H_{q_1}$, $H_2 = H_{q_2} \times \cdots \times H_{q_n}$, $C = B_2 \cap \cdots \cap B_n$ = centralizer of H_2 in A. We've seen that $H_1/\Phi(H_1)$ is a direct sum of irred repn. of A/B_1 over $\mathbb{F}_{q_1}$ associated to characters $\chi_j : A/B_1 \longrightarrow \mu_p$ $j = 1, \dots, s$ such that $\chi_1, \dots, \chi_r$ form a basis for $\mathrm{Hom}(A/B_1, \mu_p)$. Since $C \hookrightarrow A/B_1$ the restrictions of the χ_i to C span $\mathrm{Hom}(C, \mu_p)$. If these restrictions were not independent, then we could find a proper subset of them spanning $\mathrm{Hom}(C, \mu_p)$ and hence find a sub-representation of $H_1/\Phi(H_1)$ on which C is faithful. If H_1^* is the inverse image of this subrep in H_1, then C is faithful on H_1^*, hence A is faithful on $H_1^* \times H_2$, contradiction minimality of H. Thus the $\chi_i|C$ form a basis for $\mathrm{Hom}(C, \mu_p)$, so $C \xrightarrow{\sim} A/B_1$, i.e. $A = B_1 \times C$ and $A \xrightarrow{\sim} A/B_1 \times A/C$. But now A/C acting on H_2 can be handled by induction.

So we see that if H is a solvable group with A acting faithfully and not invariant subgp is faithful, then (H, A) splits as a product of the situations encountered when H is a q-group.

$$H \rtimes A = \prod_{i=1}^{n} (H_i \rtimes A_i)$$

where H_i is a q-group with minimally faithful A_i-action. Thus

$$\mathcal{S}_p(H \rtimes A) = \mathcal{S}_p(H_1 \rtimes A_1) * \cdots * \mathcal{S}_p(H_n \rtimes A_n).$$

Suppose H is a q-group again.

Claim
$$\mathcal{S}_p(H \rtimes A) \xrightarrow{\sim} \mathcal{S}_p(H/\Phi(H) \rtimes A)$$

In effect the subgroup of $H \rtimes A$ projecting onto a subgroup B of A can be identified with H/H^B. But A centralizes $\Phi(H)$, so

$$H/H^B = \bar{H}/\bar{H}^B.$$

But we have that

$$\mathcal{S}_p(\bar{H} \rtimes A) \cong \mathcal{S}_p(L_1 \rtimes \mu_p) * \cdots * \mathcal{S}_p(L_s \rtimes \mu_p)$$

is a non-trivial bouquet of $(s-1)$-spheres.

July 4, 1976:

A elem. ab p-group of rank s, H a q-group on which A acts faithfully. Suppose we choose $H_1 \supset \Phi(H)$ so that H/H_1 is an irreducible A-module, say associated to $\chi: A \twoheadrightarrow \mu_p$. We wish to compare $\mathcal{S}_p(H_1 \rtimes A)$ with $\mathcal{S}_p(H \rtimes A)$. A subgroup in $\mathcal{S}_p(H \rtimes A)$ is of the form hBh^{-1} where the coset hH^B is determined by the subgroup; here $B \in \mathcal{S}_p(A)$. Let $\bar{h}$ = the image of h in H/H_1. I know that $H^B \twoheadrightarrow \bar{H}^B$, hence $\overline{hH^B} = \bar{h}\bar{H}^B$ in $\bar{H}$. Because $\bar{H}$ is irreducible one has

$$\bar{H}^B = 0 \iff B \not\subset \operatorname{Ker} \chi | A$$

$$\bar{H}^B = \bar{H} \iff B \subset \operatorname{Ker} \chi | A$$

Thus

$$0 \in \overline{hH^B} \iff B \subset \operatorname{Ker} \chi | A \text{ or } \bar{h} = 0.$$

If $0 \in \overline{hH^B}$, then we can modify h so that $\bar{h} = 0$, i.e. $h \in H_1$. So the next point is to suppose given $h \notin H_1$ and look at the poset

$$\{B \in \mathcal{S}_p(A) \mid \overline{hH^B} = 0\}$$

$$= \{B \in \mathcal{S}_p(A) \mid \bar{h} \in \bar{H}^B\}$$

$$= \{B \in \mathcal{S}_p(A) \mid \chi(B) = 0\}$$

This means all subgroups of A contained in the hyperplane $\operatorname{Ker} \chi$.

July 6, 1976

Let A be an elementary abelian p-grp. of rank s acting [redacted] on a q-group P. [redacted] Let $B_1, \dots, B_m$ be the hyperplanes in A arranged in some order ($m = p^s-1/p-1$). Claim

$$P = \prod_{i=1}^{m} C_P(B_i)$$

(product as sets, not necessarily direct product).

Case 1: P is an elementary abelian q-group. In this case one has

$$P = P_0 \oplus P_1 \oplus \cdots \oplus P_m$$

where $P_0 = P^A$ and A/B_i acts freely on $P_i - \{0\}$. Thus

$$C_P(B_i) = P^{B_i} = P_0 \oplus P_i$$

so the result is clear.

General case: Look at the A-action on $\Omega = \Omega_1 Z(P)$, [redacted] and put $\bar{P} = P/\Omega$. Then because $P^B \twoheadrightarrow \bar{P}^B$ one has

$$\bar{P} = \prod_{i=1}^{m} \bar{P}^{B_i} \Rightarrow P = \Omega \prod_i P^{B_i}$$

$$= \prod \Omega^{B_i} \prod P^{B_i}$$

$$= \prod P^{B_i}$$

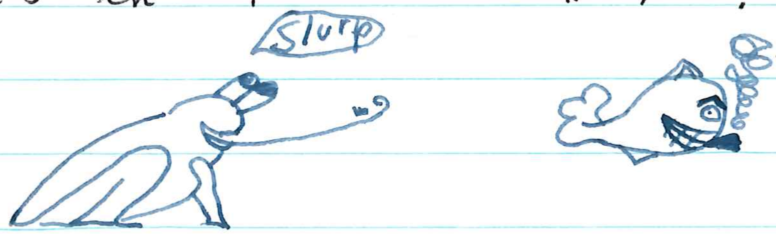
because Ω is central.

Question: Let $B_1, \dots, B_m$ = hyperplanes in A arranged in order. I've seen that the map

$$P^{B_1} \times_{P^A} \cdots \times_{P^A} P^{B_m} \longrightarrow P$$

is surjective. Is this map 1-1?

Suppose $x_1 \cdots x_m = y_1 \cdots y_m$ with $x_i, y_i \in P^{B_i}$. Then $\bar{x}_1 \cdots \bar{x}_m = \bar{y}_1 \cdots \bar{y}_m$. Using induction we know that $\bar{x}_1 = \bar{y}_1 \cdot z$ with $z \in \bar{P}^A$. Since $P^A \twoheadrightarrow \bar{P}^A$, we can lift z to $u \in P^A$, and replace y_1 by $y_1 u$, y_2 by $u^{-1} y_2$. Then $\bar{x}_1 = \bar{y}_1$. Proceeding we can suppose $\bar{x}_i = \bar{y}_i$ for all i. Thus $x_i = y_i \omega_i$ with $\omega_i \in \Omega \cap P^{B_i} = \Omega^{B_i}$. Since ω_i is central $\prod x_i = \prod (y_i \omega_i) = \prod y_i \prod \omega_i$, we have $\prod \omega_i = 1$. But this happens only if $\omega_i \in \Omega^A$ for all i, whence we can modify the y's to be equal to the x's in $P^{B_1} \times_{P^A} \cdots \times_{P^A} P^{B_m}$.



July 8, 1976

$P \rtimes A$. Idea is to associate to a coset vV^B the subgroup $[B,v]$ generated by commutators $bvb^{-1}v^{-1}$. Note that $u \in V^B \Rightarrow$

$$b(vu)b^{-1}(vu)^{-1} = bvb^{-1}v^{-1}$$

so this subgroup depends only on the [redacted] coset.

If P is elementary abelian, then

$$W = [\text{redacted}] \; \mathbb{F}_q[B]v = [B,W] \oplus W^B$$

and $\mathbb{F}_q[B] \twoheadrightarrow W$. Thus W^B is of dim ≤ 1 and $[B,W]$ is generated by $bv - v$, $b \in B$. Thus $[B,W] = [B,v]$.

The idea will be to use the size of $[B,v]$ as a measure of the distance of vV^B from 0. In the case $\Phi(P) = 0$, [redacted] suppose $[B,W] = \oplus L_\chi$ is a decomposition into [redacted] isotypical pieces. Then [redacted] because L_χ is cyclic over $\mathbb{F}_q[B]$, each L_χ must be irreducible. So the size of $[B,v]$ is the number of non-trivial irreducible repns. occurring in $[B,v]$.

If $B \subset A$, then clearly $[B,v] \subset [A,v]$. Put $H = [A,v]$. Then H is A-invariant:

$${}^x(y,z) = (xy,z)(x,z)^{-1}$$

$x, y \in A$, $z = v$

so $\Phi(H)$ is also A-invariant. Question: Is $[A,H]=H$?

Suppose P is generated by ava^{-1} as a ranges over A; v is fixed. Then the same is true of $\bar{P}$ $= P/\Phi(P)$. Thus $\bar{P} = [A,\bar{v}] \oplus \bar{P}^A$ where $\bar{P}^A$ is cyclic generated by $\frac{1}{|A|}\sum a\bar{v} = \bar{v}^\natural$

<u>Case 1</u>: $\bar{v}^\natural = 0$. Then $\bar{P} = [A,\bar{v}]$, so $P/\Phi(P) =$ $[A,v]\Phi(P)$ and so $[A,v] = P$. In this case $H = P$ and $[A,H] = H$.

<u>Case 2</u>: $\bar{v}^\natural \neq 0$. Then we can find a $w \in P^A$ such that $\bar{w} = \bar{v}^\natural$ whence replacing v by vw^{-1} we can assume $\bar{v} = 0$, except that $\langle ava^{-1} \mid a \in A\rangle$ is now smaller. But $[A,v]$ has not changed.

<u>Prop</u>: $H = [A,v] \Longrightarrow [A,H] = H$.

<u>Proof</u>: Changing v to vw with $w \in P^A$ does not change H, hence we can suppose v chosen in vV^A so that $Q = \langle ava^{-1} \mid a \in A\rangle$ has least order. Let $\bar{Q} = Q/\Phi(Q)$. Then $\bar{Q} = \langle a\bar{v}a^{-1}\rangle$ $= [A,\bar{v}] \oplus \bar{Q}^A$ where $\bar{Q}^A$ is cyclic generated by $\bar{v}^\natural = \frac{1}{|A|}\sum_{a\in A} a\bar{v}$. Since $Q^A \twoheadrightarrow \bar{Q}^A$ I can lift $\bar{v}^\natural$ to an elt. u in Q^A. It's clear that $vu^{-1} \in [A,v]\Phi(Q)$ which is A-invariant

and $\sigma u^{-1} \in \sigma P^A$. Thus by minimality $Q = [A,\sigma]\Phi(Q)$, so $Q = [A,\sigma]$. Thus $\sigma \in [A,\sigma]$ so

$$H = [A,\sigma] \subset [A,[A,\sigma]] = [A,H] \subset H$$

QED.

The preceding proof shows:

<u>Prop</u>: $\forall \sigma \in P$, there exists $\tilde{\sigma} \in \sigma P^A$ such that $\tilde{\sigma} \in [A,\tilde{\sigma}] = [A,\sigma]$.

Suppose we look at the map $\sigma P^A \longmapsto [A,\sigma]$. We've seen that $\sigma P^A \cap [A,\sigma] \neq \phi$. Assuming σ chosen so that $\sigma \in [A,\sigma]$, then

$$\sigma P^A \cap [A,\sigma] = \sigma(P^A \cap [A,\sigma]) = \sigma([A,\sigma]^A)$$

Now $[A,\sigma]^A \subset \Phi[A,\sigma]$ so the ambiguity in the choice a representative for the coset σV^A is further down in the group.

Idea: $\ell(\sigma V^B) = |[B,\sigma]|$. Then for $B_1 \subset B$ we have

$$|[B_1,\sigma]| \leq |[B,\sigma]|.$$

Let's analyze when we have $[B,\sigma] = [A,\sigma]$. I assume that $\sigma \in H = [A,\sigma]$. Then $[B,\bar{\sigma}] = [A,\bar{\sigma}] = \bar{H}$

which means that [redacted] $\bar{H}$ doesn't contain the trivial repn. over B. Conversely, if $\bar{H}^B = 0$, then [redacted] each irreducible rep. of A occuring in $\bar{H}$ remains [redacted] irreducible + non-trivial over B. If χ_i are the characters of A involved in $\bar{H}$, then [redacted]

$[B,\sigma] <$ [redacted] $H \iff [B,\bar{\sigma}] < \bar{H}$

$\iff \bar{H}^B \neq 0$ NO.

July 10, 1976:

$S_p(P \rtimes A)$ where A is an elem ab p-group of rank r acting on [redacted] a q-group P [redacted] faithfully. [redacted] An [redacted] element of $S_p(P \rtimes A)$ is of [redacted] the form [redacted] $\sigma B \sigma^{-1}$ for [redacted] a uniquely determined $B \in S_p(A)$ and $\sigma \in P$ which is determined up to the coset σP^B. Thus can identify a subgroup $\sigma B \sigma^{-1}$ with the pair $(B, \sigma P^B)$. To this pair we associate the subgroup

$[B, \sigma$ [redacted] $] =$ subgroup gen. by $[b,\sigma]$, $b \in B$

of P. Note $(b, \sigma u) = (b, \sigma)$ if $u \in P^B$, so this subgroup depends only on $(B, \sigma P^B)$.

I've seen that I can find a representative of the coset vP^B contained in the group $[B,v]$. (Recall one chooses an element v in the coset such that H = B-invariant subgroup gen. by v is minimal, whence $H=[B,v]$). Thus can suppose

$$v \in vP^B \cap [B,v]$$

whence $vP^B \cap [B,v] = v([B,v]^B)$. The other point is that if we put $H=[B,v]$, then $H_{ab}=[B,\bar{v}]$ and so $(H_{ab})^B = 0$. Thus $H^B \subset (H,H)$. In particular:

Proposition: If P is abelian then any coset vP^B has a unique representative contained in $[B,vP^B]$.

July 11, 1976.

Let H be a subgroup of $P \rtimes A$ containing A and vBv^{-1}. Then as $H \twoheadrightarrow A$ we have

$$H = (H \cap P) \rtimes A$$

where $H \cap P$ is an A-invariant subgroup of P. Since B, vBv^{-1} are conjugate in H one can suppose $v \in H \cap P$, whence $[A,v] \subset H \cap P$. I've seen we can suppose $v \in [B,v]$, whence $H \cap P \supset [A,v] = [A,[B,v]]$. Thus $[A,[B,v]] \rtimes A$ is the smallest subgroup of $P \rtimes A$

containing both A and vBv^{-1}. Similarly $[B,v] \rtimes B$ is the smallest subgroup containing B and vBv^{-1}.

The problem I run into seems to be this: [redacted] Given a [redacted] subgroup vBv^{-1}, there [redacted] seem to be many S_p-groups wAw^{-1} containing it which are minimal, i.e. $w \in [A, [redacted] [B,v]]$.

Given vBv^{-1}, i.e. the coset vP^B one can choose v so that $v \in [B,v]$, whence we get a definite [redacted] coset $v \cdot [B,v]^B$. So look at $v \cdot [A,v]^B$ and how many $[A,v]^A$ cosets this involves. Suppose that $P = [A,v]$.

<u>July 13, 1976</u>

A elem. ab. p-grp rank r acting on a q-group P, $q \neq p$. $G = P \rtimes A$. The set of p-subgroups of G over a subgrp $B \subset A$ can be identified with P/P^B by
$$\sigma P^B \longleftrightarrow \sigma B \sigma^{-1}.$$

Write $[B, \sigma P^B]$ for the subgroup of B generated by commutators $b(\sigma u) b^{-1} (\sigma u)^{-1} = b \sigma b^{-1} \sigma^{-1}$; clearly $[B, \sigma P^B] = [B, \sigma]$.

<u>Prop 1:</u> Any coset σP^B in P/P^B contains an element of $[B, \sigma P^B]$, i.e.
$$\sigma P^B \cap [B, \sigma P^B] \neq \phi.$$

<u>Proof:</u> Assume σ chosen in the coset so that the smallest B-invariant subgroup of P containing σ is of least order; this subgroup is gen. by $\{b\sigma b^{-1} \mid b \in B\}$, denote it H. Then $\bar{H} = H/\Phi(H)$ is gen. by $\{b\bar{\sigma} b^{-1}\}$ so it is a quotient of $\mathbb{F}_q[B]$. Thus $\bar{H} = [B, \bar{\sigma}] \oplus \bar{H}^B$ where $\bar{H}^B$ is cyclic of order q or 0 generated by $\frac{1}{|B|}\sum b\bar{\sigma}$, $\bar{\sigma}$ = image of σ in $\bar{H}$. As $H^B \twoheadrightarrow \bar{H}^B$ one can replace σ by σu, $u \in H^B$ so that $\overline{\sigma u} \in [B, \bar{H}]$, whence by minimality of H, one has $\bar{H}^B = 0$, so $\bar{H} = [B, \bar{\sigma}]$, and $H = [B, \sigma]$. Thus $\sigma \in \sigma P^B \cap [B, \sigma P^B]$.

<u>Prop. 2:</u> The smallest subgroup of $G = P \rtimes A$ containing A and $\sigma B \sigma^{-1}$ is $[A, \sigma] \rtimes A$.

Proof: Let $H = \langle A, \sigma B \sigma^{-1} \rangle$; as $H \twoheadrightarrow A$ we have $H = (H \cap P) \rtimes A$. Suppose first that $A = B$. Then $A, \sigma A \sigma^{-1}$ are both S_p-subgroups of H, so $\exists\, x \in H \cap P$ such that $x \in \sigma P^A$. Thus [redacted] $[A, \sigma] = [A, x] \subset H \cap P$, because $H \cap P$ is clearly A-invariant. On the other hand I know from Prop 1 that σ can be chosen in $[A, \sigma]$, whence $\sigma A \sigma^{-1} \subset [A, \sigma] \rtimes A$. $\therefore$ $[A, \sigma] = H \cap P$.

We've used here that A normalizes $[A, \sigma]$. In fact one has

$$(xy, z) = xyzy^{-1}x^{-1}z^{-1}$$
$$= x\, yzy^{-1}z^{-1}x^{-1}\; xzx^{-1}z^{-1}$$
$$= {}^{x}(y, z)(x, z)$$

so that

$${}^{a}(a_1, \sigma) = (aa_1, \sigma)(a, \sigma)^{-1} \quad \text{and}$$
$${}^{a}[A, \sigma] \subset [A, \sigma].$$

Next if $B \neq A$, one has $H \supset B, \sigma B \sigma^{-1}$ so that $H \supset [B, \sigma] \rtimes B$ by applying the above result to $P \rtimes B$. Hence [redacted] $H \cap P \supset [A, [B, \sigma]] \supset [A, \sigma]$ since $\sigma \in [B, \sigma]$ can be assumed. QED.

July 15, 1976

Suppose A is an elementary abelian p-group of rank r acting on a solvable p'-group H. I want to show $\mathcal{S}_p(H \rtimes A)$ has the homotopy type of a bouquet of $(r-1)$-spheres, by using induction with respect to $|H|, r$. We know this is true if $r \leq 1$ or if $|H| = 1$. So assume $H > 1$.

Then $H^{ab} = H/H' > 1$ as H is solvable. We can find a maximal A-invariant subgroup H_0 of H containing H', whence $\bar{H} = H/H_0$ is an irreducible $\mathbb{F}_q[A]$-module for some prime $q \neq p$.

Consider first the case where A acts trivially on $\bar{H}$. I claim in this case that

$$\mathcal{S}_p(H_0 \rtimes A) = \mathcal{S}_p(H \rtimes A).$$

To see this I have to show that for any $B \in \mathcal{S}_p(A)$ one has

$$H_0/H_0^B \simeq H/H^B$$

i.e. that $H_0 H^B = H$ or that $H^B \twoheadrightarrow H/H_0 = (H/H_0)^B$. This will follow from

<u>Lemma</u>: Let $1 \to H_0 \to H \to \bar{H} \to 1$ be an exact sequence of groups on which a p group A acts. Then

$$1 \to H_0^A \to H^A \to \bar{H}^A \to 1$$

is exact, if H_0 is a p'-group.

<u>Proof</u>: Only have to show $H^A \twoheadrightarrow \bar{H}^A$. Let $\alpha \in \bar{H}^A$ and let X be the inverse image of α in H. Then

$G = H \rtimes A$ acts on X and H acts simply-transitively. Hence if K is the isotropy group of a point x of X, K is a complement to H, so K is an S_p-subgroup of $H \rtimes A$, hence K is conjugate to A by an elt g of G: $gKg^{-1} = A$. This implies $gx \in X^A \subset H^A$, so α comes from gx. QED.

Remark: This argument show that if we work in the topos $\mathcal{T}$ of A-sets, then given any A-group H of order prime to A one has $H^1(\mathcal{T}, H) = 0$. Note that this conclusion is equivalent to the conjugacy of complements to H in $H \rtimes A$ (Schur-Zassenhaus).

Next I have to consider the case where A acts non-trivially on $\bar{H}$. In this case I know that there is a hyperplane A_0 in A such that A/A_0 acts freely on $\bar{H} - \{1\}$. Moreover

$$\bar{H}^B = \begin{cases} 0 & B \not\subset A_0 \\ \bar{H} & B \subset A_0. \end{cases}$$

If $\alpha \in \bar{H}$ put

$$X_\alpha = \bigcup_{hH_0 = \alpha} \mathcal{S}_p(hAh^{-1}) \subset \mathcal{S}_p(H \rtimes A)$$

Then $X_{H_0} = \mathcal{S}_p(H_0 \rtimes A)$.

and

$$X_{hH_0} = \bigcup_{x\in H_0} \mathcal{S}_p(hxAx^{-1}h^{-1})$$

$$= h\left[\bigcup_{x\in H_0} \mathcal{S}_p(xAx^{-1})\right]h^{-1} = h\cdot X_{H_0}$$

[redacted] It might be better to say that H permutes the X_α.

Since $H_0 < H$, we know by induction that $\mathcal{S}_p(H_0 \ltimes A)$, hence each X_α, has the homotopy type of a bouquet of $(r-1)$-spheres. [redacted]

Consider next $X_\alpha \cap X_\beta$ where $\alpha \neq \beta$. [redacted] Any subgroup in $X_\alpha \cap X_\beta$ is of the form

$$h_1 B h_1^{-1} = h_2 B h_2^{-1}$$

for some h_1, h_2 such that [redacted] $\alpha = h_1 H_0$, $\beta = h_2 H_0$. Then $h_1^{-1}h_2 \in H^B$ and $h_1^{-1}h_2 H_0 = \alpha^{-1}\beta \neq 1$, so $\bar{H}^B \neq 1$ and so by the above $B \subset A_0$.

Conversely [redacted] let $B \subset A_0$, and let [redacted] h_1 be any element of H [redacted] such that $h_1 H_0 = \alpha$. Because $H^B \twoheadrightarrow \bar{H}^B = \bar{H}$ we can find $x \in H^B$ such that $xH_0 = \alpha^{-1}\beta$, whence

$$h_1 B h_1^{-1} = h_1 x B x h_1^{-1} \qquad h_1 x H_0 = \alpha\cdot xH_0 = \beta.$$

[redacted] belongs to $X_\alpha \cap X_\beta$. Thus we see that

$$X_\alpha \cap X_\beta = \bigcup_{h_1 H_0 = \alpha} S_p(h_1 A_0 h_1^{-1})$$

The point is this is independent of β. So if we fix h_0 in α then

$$X_{h_0 H} \cap X_\beta = h_0 \cdot \bigcup_{x \in H_0} S_p(x A_0 x^{-1})$$

$$= h_0 \cdot S_p(H_0 \rtimes A_0)$$

and this is a bouquet of $(r-2)$-spheres by induction.

So now we can determine the homotopy type of $S_p(H \rtimes A)$. Order the elements of $\bar{H}$: $H_0 = \alpha_1, \alpha_2, \ldots, \alpha_n$, and consider the sequence

$$X_{\alpha_1} \subset X_{\alpha_1} \cup X_{\alpha_2} \subset \cdots \subset X_{\alpha_1} \cup \cdots \cup X_{\alpha_n} = S_p(H \rtimes A)$$

We know that

$$X_{\alpha_i} \cap \left(X_{\alpha_1} \cup \cdots \cup X_{\alpha_{i-1}} \right) = h \cdot S_p(H_0 \rtimes A_0)$$

if $h \in \alpha_i$; by induction $S_p(H_0 \rtimes A_0)$ is a bouquet of $(r-2)$-spheres up to homotopy. Also X_{α_i} is a bouquet of $(r-1)$-spheres. So the rest follows by induction on i using

<u>Lemma</u>: If $X = X_1 \cup X_2$ with $X_1, X_2, X_1 \cap X_2 \sim$ bouquets of $r-1, r-1, r-2$ spheres respectively,

then $X \sim$ bouquet of $(n-1)$-spheres.

Proof: The inclusions $X_1 \cap X_2 \subset X_i$ are null-homotopic. So $X \sim \mathrm{Cyl}\,(X_1 \leftarrow X_1 \cap X_2 \rightarrow X_2) \sim X_1 \vee \Sigma(X_1 \cap X_2) \vee X_2$
[redacted] $\sim$ bouquet of $(n-1)$-spheres.

Let $n(H \rtimes A)$ be the number of these spheres = rank of $\tilde{H}_{n-1}(\mathcal{S}_p(H \rtimes A))$. We should have the formula

$$(*) \qquad n(H \rtimes A) = \begin{cases} |\bar{H}| \cdot n(H_0 \rtimes A) + (|\bar{H}| - 1)\, n(H_0 \rtimes A_0) & \text{if } \bar{H} \text{ non-trivial repn. of } A \\ n(H_0 \rtimes A) & \text{if } \bar{H} \text{ is a trivial repn. of } A. \end{cases}$$

July 18, 1976:

Prop: $n(H \rtimes A) = n(\mathrm{gr}\, H \rtimes A)$ where $\mathrm{gr}\, H$ denotes the associated graded group for a maximal A-invariant chain of subgroups each normal in the succeeding (composition series for H as a A-group)

This follows by induction with respect to $|H|$ and r using $(*)$. Also

$$n(H \rtimes A) = \sum_{0 \leq B \leq A} \mathrm{card}\,(H/H^B) \cdot q^{r(B)(r(B)-1)/2} \, (-1)^{r(A)-r(B)}$$

Check: If $r(A) = 1$, then

$$n(H \rtimes A) = \text{card}(H/H^A) - 1$$

If $r(A) = 2$, then $\mathcal{S}_p(H \rtimes A)$ is a graph with vertices

H/H^B for each line B in A.

H/H^A

and edges H/H^A for each line B in A, so

$$\chi\left(\mathcal{S}_p(H \rtimes A)\right) = \sum_{B \in \mathbb{P}^1(A)} \text{card}(H/H^B) - \text{card}(H/H^A) \cdot q$$

$$n(H \rtimes A) = 1 - \sum_{B \text{ line in } A} \text{card}(H/H^B) + \text{card}(H/H^A) q$$

July 19, 1976

Suppose that M is a subgroup of G such that $|M| \equiv 0 \mod p$ and such that if x is of order p in M, then $C_G(x) \subset M$.

Let $H \in \mathcal{S}_p(M)$, let P be an S_p-subgroup of G containing H. Choose x of order p in H and $y \in \Omega_1 Z(P)$. Then $y \in C_G(x) \subset M$ so $P \subset C_G(y) \subset M$.

Assuming $N_G(P) \subset M$, I want to show. $M \cap xMx^{-1}$ is a p'-group for $x \notin M$. The idea is to use something like Alperin's thm. So assume that one has $N_G(H^*) \subset M$ for all $H^* \subset P$ of order larger than H. It's clear that M contains all non-identity p-subgroups in the same component of $\mathcal{S}_p(G)$ as P, hence M contains all p-elements of $N_G(H)$. These are enough to move around the S_p-groups of $N_G(H)$. Hence given $\alpha \in N_G(H)$ and a S_p-group S of $N_G(H)$, one can modify α by an element of M so that it normalizes S whence $\alpha \in M$ because $S > H$, unless $H = P$ itself in which case we have $N_G(H) \subset M$ by assumption. So we've proved:

<u>Prop</u>: Assume $M \subset G \ni$

$|M| \equiv 0 \mod p$

x order p in $M \Rightarrow C_G(x) \subset M$

S Sylow p-subgp of $M \Rightarrow N_G(S) \subset M$.

Then S is a S_p-group of G and $1 < H \leq S \Rightarrow N_G(H) \subset M$, so M comes from an invariant partition of $\pi_0 \mathcal{S}_p(G)$.

Suppose next $p = 2$ and that $M \neq G$, whence inside $\alpha M \alpha^{-1}$ I can find an involution y not in M. If x is an involution of M, then $|\langle x, y\rangle|$ has to be of order $2 \cdot$ odd number, i.e. xy is of odd order, otherwise we could find an involution $z = (xy)^n$ commuting with x, y. But then x, y are conjugate in G, so G has a single conjugacy class of involutions.

Direct proof of prop: If $M \cap uMu^{-1}$ has order $\equiv 0\ (p)$, then $M \cap uPu^{-1} > 1$ for **some** S_p-subgp P of M, whence $uZ(P)u^{-1} \subset M$, and $uPu^{-1} \subset M$. Then $u \in MN_G(P) = M$ so $uMu^{-1} = M$. Thus there is no need to use Alperin.

July 25, 1976

Maximal subgp thm. (special case) p odd. Suppose $Z(P)$ non-cyclic, that $N_G(H)$ is ~~p-constrained~~ + p-stable for all $H \in \mathcal{S}_p(P)$, and that $\exists\ 1 < B \triangleleft P$ centralizing all P-invariant p'-subgroups. Then there is a largest $N_G(H)$, i.e. if M is the stabilizer of the component of P in $\mathcal{S}_p(G)$, then $O_p(M) \gneq 1$.

Special case: Assume $\exists$ no P-invariant p'-subgps. If $1 < H \triangleleft P$, then $O_{p'}(N_G(H)) = 1$, so Glauberman's thm says that

$$ZJ(P) \triangleleft N_G(H)$$

i.e. $N_G(H) \subset N_G(ZJ(P)) = M$. Thus $N_G(ZJ(P))$ turns out to be the largest local subgroup in this case.

More special case: Suppose P abelian, whence $1 < H \subset P \Rightarrow P \subset N_G(H)$. But $N_G(H)$ p-constrained means that an S_p-subgroup Q of $O_{p',p}N_G(H)$ has its G centralizer contained in $O_{p',p}N_G(H)$. Hence $P \subset O_{p',p}N_G(H)$, so as P normalizes no p' subgroup of G, it

follows that $P = O_p(N_G(H))$, so $N_G(H) \subset N_G(P)$.
Thus $M = N_G(P)$ is a maximum normalizer.