May 3, 1976

So far we have this method: Given $\alpha \in H^*(P)$, in order that α comes from $H^*(G)$ it suffices that $\alpha|H$ be $N_G(H)$-invariant for certain subgroups H in P. I can assume $N_p(H)$ is an S_p-subgroup of $N_G(H)$. This yields Frobenius' thm. for $N_G(H)$ is generated by the S_p-subgroup $N_p(H)$ and by the set of p'-elements.

Note: $\alpha|N_p(H)$ comes from $N_G(H)$ $\Rightarrow$ $\alpha|H$ comes from $N_G(H)$ $\Rightarrow$ $\alpha|H$ is $N_G(H)$-invariant.

So I ought to consider an $H \subset P$ such that $N_p(H)$ is an S_p-subgp of $N_G(H)$ and such that $\alpha|N_p(H)$ does not come from $N_G(H)$, and such that $|N_p(H)|$ is maximal.

May 4, 1976

Let M be a subgroup of G containing $N_G(P)$, and suppose $\alpha \in H^*(M)$ is given. To extend α to $H^*(G)$ we have equalization conditions on α for each double coset MxM. If $M \cap xMx^{-1}$ contains a S_p-gp Q, then Q, $x^{-1}Qx$ are two S_p-subgps of M, so $\exists\, m \in M$ with $xm \in N_G(Q)$. If $m_1 Q m_1^{-1} = P$, then

$$m_1(xm)m_1^{-1} \in m_1 N_G(Q) m_1^{-1} = N_G(P) \subset M$$

so $x \in M$ and so the condition resulting from MxM is vacuous.

Recall also that $M \supset N_G(P) \Rightarrow N_G(M) = M$.

<u>Example:</u> Let K char P. Then $N_G(P) \subset N_G(K) = M$.

Relative to α, we can now describe a double coset MxM as being good or bad. Take a bad MxM with $|M \cap xMx^{-1}|_p$ maximal. Let H be <u>any</u> S_p-subgroup of $M \cap xMx^{-1}$.

Digression: Let Q_1 be a p-subgp. $M_1 = N_G(Q_1)$. Q_1 is a normal p-subgroup of M_1 so if $Q_2 = O_p(M_1)$, then $Q_1 \triangleleft Q_2$. M_1 normalizes Q_2 so $M_1 \subset M_2 = N_G(Q_2)$. Put $Q_3 = O_p(M_2)$. Now $M_1 \subset M_2 \Longrightarrow O_p(M_1) \supset O_p(M_2) \cap M_1$. $Q_3 \triangleleft M_2 \Longrightarrow M_2 \subset N_G(Q_3) = M_3$. $Q_2 \triangleleft M_2 \Longrightarrow Q_2 \triangleleft Q_3$. So

$$Q_3 \cap M_1 \subset Q_2 \subset Q_3 \qquad\qquad Q_2 \subset Q_3 \cap M_1$$

$$\therefore \quad Q_2 = Q_3 \cap M_1$$

$$\begin{array}{ccc} Q_1 \triangleleft & Q_2 \triangleleft & Q_3 \\ & & \\ M_1 \subset & M_2 & \end{array} \qquad\qquad \begin{array}{ccc} Q_1 \triangleleft Q_2 & \triangleleft & Q_3 \\ | & & | \\ M_1 & \subset & M_2 \end{array}$$

So these chains stop eventually.

Special case: Suppose Q_1 is normal in some

Sylow group P.

Then $P \subset M_1 \subset M_2 \subset \cdots$ so $Q_i = O_p(M_{i-1})$ = int. of S_p-subgroups contained in M_{i-1} is also in P. Thus $Q_2 = Q_3 = \cdots$ and also $M_2 = M_3 = \cdots$

The way to state this is as follows. Let Q be a p-group normal in some S_p-gp. P. Then $Q' = O_p(N_G(Q))$ = intersection of the S_p-subgroups in which Q is normal is closed under this operation:

$$Q' = O_p(N_G(Q'))$$

<u>Proof:</u> $Q' \subset O_p(N_G(Q'))$ clear. But $P \subset N_G(Q) \subset N_G(Q')$ so $O_p(N_G Q')$ is a subgp of P, hence of $N_G(Q)$, and it is normal in $N_G Q$. Thus

$$O_p(N_G Q') \subset O_p(N_G Q) = Q'. \qquad \text{QED.}$$

<u>Question:</u>

May 5, 1976

Prop: Let A be a maximal abelian + normal subgroup of a p-group P. Then A is maximal abelian in P, i.e. $C_P(A) = A$.

Proof: Assume $C_P(A) > A$. Then $(C_P(A)/A)^P > 1$. Choose $xA \in (C_P(A)/A)^P$ not the identity. Then $\langle x \rangle A \triangleleft P$ and $\langle x \rangle$ is abelian, so A is not maximal normal + abelian.

Suppose A is maximal normal + elem. abelian in P, and p is odd. I want to show $\Omega_1 C_P(A) = A$ ($\Omega_1 =$ subgp generated by elements of order p).

First I show that A is maximal normal + elem. ab. in $C_P(A)$. Let B/A be any normal elem. ab. subgroup of $C_P(A)$. Then $A \longrightarrow B \longrightarrow B/A$ is a central extension of elem. ab. p-groups so we have a canonical homomorphism $B/A \longrightarrow A$ $\quad bA \longmapsto b^p$. (Here we use $p \neq 2$). Take B/A to be ${}_pZ(C_P(A)/A)$, and let $B'/A = \operatorname{Ker}\{B/A \longrightarrow A\}$. Note that B' is normal in P, so if $B' > A$, then $(B'/A)^P > 1$, so if $xA \in (B'/A)^P$, then $\langle x \rangle A$ will be normal + elem. abelian in P strictly containing A, which is impossible. $\therefore B' = A$. But if A' is normal + elem ab in $C_P(A)$ and $A' > A$, then $A'/A \cap {}_pZ(C_P(A)/A) > 1$, and $A' \cap B > A$; also $A' \cap B < B'$. Thus we see that A is maximal normal + elem. abelian in $C_P(A)$.

So now we can suppose that $A = {}_pZ(P)$ is maximal normal + elem. abelian in P, and we have to show $\Omega_1(P) = A$.

Review:

<u>Lemma</u>: Let A be a normal elem. ab subgroup of a p group P, p odd. Because p is odd we have a homom.

$$_pZ(C_P(A)/A) \longrightarrow A \qquad xA \mapsto x^p$$

Then A is maximal normal + elem. ab. in $P \iff$ this homomorphism is injective.

<u>Proof</u>: Put $B/A = {}_pZ(C_P(A)/A)$, and let $B'/A =$ the Kernel of the above homomorphism from B/A to A. Every element of B' is of order p. If $B' > A$, then $(B/A)^p > 1$, so $\exists\ xA \in (B'/A)^p$ $xA \neq 1$, and $\langle x \rangle A$ is elem. abelian and normal in P. $\therefore$ A not max normal + elem. ab. On the other hand if A not max. normal + elem. ab., $\exists\ A_1 > A$, A_1 elem. ab, $A_1 \triangleleft P$. Then $A_1/A \cap B/A > 1$, also A_1 elem. ab. $\Rightarrow A_1 \cap B \subset B'$ so $B'/A > 1$. QED.

~~Now what I am trying to prove is false i.e. $\exists A$ normal max elem ab in P but~~

<u>Corollary</u>: A normal + elem. ab. in P. Then A is max. normal + elem ab. in $P \iff A$ max. normal + elem. ab. in $C_P(A)$.

$(\Leftarrow)$ is trivial

So now let's seek a minimal counterexample to
A max. normal + elem. ab. $\Longrightarrow$ A max. elem. abelian.

A max. normal + elem. ab. in $P \Longrightarrow A$ max. normal + elem. ab. in $C_P(A)$

A not max elem. abelian in $P \Longrightarrow A$ not max. elem. ab. in $C_P(A)$

So in $P = C_P(A)$, so $A = {}_pZ(P)$.

Put $B/A = {}_pZ(P/A)$. Assume B/A is max. normal elem. abel in P/A. Then by induction it is maximal elementary abelian, so $B/A = \Omega_1(P/A)$. Thus if x is of order p in P, then $xA \in B/A$ so $x \in B \Rightarrow x \in A$ as we know $B/A \hookrightarrow A$ by $(xA) \mapsto x^p$.

~~Thus B/A is not max. normal elem. ab. in P/A, so ${}_pZ(P/B) \longrightarrow B/A$ $xB \mapsto x^pA$ is not injective.~~

Let B_1/A be a maximal normal + elem. ab. subgroup of P/A. Then we have a canonical homomorphism $B_1/A \longrightarrow A$ whose kernel B_1'/A is P-normal and has all elements of order p. So again if $B_1'/A > 1$ one can choose a non-identity element of $(B_1'/A)^P$ and construct a normal elem. abelian subgroup $> A$. Thus $B_1/A \hookrightarrow A$, so P acts trivially on B_1/A, hence $B_1/A = {}_pZ(P/A)$ is a maximal normal elem. abelian subgp of P/A and we win.

<u>Theorem</u>: If p is odd, then any maximal normal elementary abelian subgroup A of P is maximal elementary abelian.

<u>Proof</u>: Let B/A be a maximal normal elementary abelian subgp of $C_p(A)/A$. Then

$$A \longrightarrow B \longrightarrow B/A$$

is a central extension of elem. ab. p groups, hence there is a canonical p-th power homomorphism

$$B/A \longrightarrow A \qquad\qquad xA \longmapsto x^p$$

because p is odd. Let B_1/A be the kernel, so that every element of B_1 has order p. (In other words because $p \neq 2$, the set of elements of order $/p$ is a subgp. of B). Clearly $B_1 \triangleleft P$, so if $B_1/A > 1$, then $\exists\ xA \in (B_1/A)^P$ with $xA \neq 1$. Then $\langle x \rangle A$ is normal in P, and elementary abelian, which contradicts the maximality assumption of A.

Consequently $B/A \hookrightarrow A$, and so $C_p(A)$ acts trivially on B/A. Hence $B/A \subset Z(C_p(A)/A)$ so by the maximality of B one has

$$B/A = {}_pZ(C_p(A)/A)$$

is a maximal normal elementary abelian subgroup of $C_p(A)/A$. Now apply induction to conclude that every element of order p in $C_p(A)/A$ is contained in

B/A. ~~[illegible]~~ Hence every element of order p in $C_p(A)$ is contained in B, hence in ${}_pB = A$. QED.

~~Maioranas~~ Maiorana's paper:

G finite group, V faithful real representation, F = flag ~~[illegible]~~ manifold of V. The isotropy groups of G on F are elementary-abelian 2 groups. Let r = p-rank of G. We have

$$PS\{H^*(G)\} \cdot PS\{H^*(F)\} = PS\{H^*_G(F)\}$$

$$PS\{H^*(F)\} = \frac{PS\{H^*(BO_1^n)\}}{PS\{H^*(BO_n)\}} = \frac{(1-t)(1-t^2)\cdots(1-t^{n-1})}{(1-t)^n}$$

I want

$$c(2,G) = \lim_{t\to 1} (1-t)^r PS\{H^*(G)\} .$$

$$c(2,G)\cdot n! = c(2, H^*_G(F)).$$

Now ~~[illegible]~~ by decomposing F into the parts involving the elementary ~~[illegible]~~ abelian 2 groups of rank r and the rest, one sees the rest is of dim $< r$, so $c(2, H^*_G(F))$ is the same as for

$$\coprod_A GF^A$$

where A ranges over the ~~[illegible]~~ $A \in \mathcal{Q}_2(G)$ of rank r. As

$$GF^A = G \times^{N(A)} F^A$$

because A is a maximal isotropy group of F, we have

$$H^*_G(GF^A) = H^*_{N(A)}(F^A).$$

Now fix A, and let $X(A)$ be the set characters of A appearing in V:

$$V = \bigoplus_{\lambda \in X(A)} V_\lambda \qquad V_\lambda \neq 0.$$

F^A is the set of flags fixed by A, i.e. $V = L_1 \oplus \cdots \oplus L_n$ where each L_i is contained in some V_λ. Thus

$$F^A = \coprod_{\phi : \{1,\dots,n\} \twoheadrightarrow X(A)} F^A_\phi$$

where ϕ runs over the set of map $\{1,\dots,n\} \to X(A)$ $\ni$ $\phi^{-1}(\lambda)$ has card $= \dim V_\lambda$, and where F^A_ϕ consists of flags $\{L_i\}$ with $V_\lambda = \bigoplus_{\phi(i)=\lambda} L_i$. Thus

$$F^A_\phi \cong \prod_{\lambda \in X(A)} F(V_\lambda)$$

is a component of $F(A)$. $N(A)$ acts on $\pi_0(F^A) = \{\phi\}$. $C(A)$ normalized each V_λ, hence $C(A)$ normalizes each F^A_ϕ, so we get an action of $N(A)/C(A) = W(A)$ on $\{\phi\}$. It is the action induced by $W(A)$ acting on $X(A)$.

Let ϕ be fixed by $g \in N(A)$. Thus if $\{L_i\}$ is such that $L_i \subset V_{\phi(i)}$, then $\{gL_i\}$ is such that $gL_i \subset V_{\phi(i)}$. It follows that $g\phi(i) = \phi(i)$ for each i hence g acts trivially on $X(A)$. But since V is a faithful repn. of G, [redacted] this means the λ in $X(A)$ generates $\mathrm{Hom}(A, \mathbb{C}^*)$, so g must centralize A. Thus we see that $W(A)$ acts freely on $\pi_0 F^A$.

$$H^*_{N(A)}(F^A) \Longleftarrow H^*(W(A), H^*_{C(A)}(F^A)) = E_2^{pq}$$

$$H^*_{C(A)}(F^A) = \bigoplus_{\phi} H^*_{C(A)}(F^A_\phi)$$

so $H^+(W(A), \cdots) = 0$, and we get

$$H^*_{N(A)}(F^A) \cong \bigoplus_{\phi \in S} H^*_{C(A)}(F^A_\phi)$$

where S is a set of representatives for the $W(A)$ orbits on $\pi_0 F^A$.

But since I am only interested in numbers we get

$$c\left(H^*_{N(A)}(F^A)\right) = \frac{|\pi_0 F^A|}{|W(A)|}\, c\left(H^*_{C(A)}\right) \dim H^*(F^A_\phi)$$

$$= \frac{1}{|W(A)|}\, c\left(H^*_{C(A)}\right) \cdot \dim H^*(F^A)$$

because [redacted] Smith theory says

$$H_A^{*[e_A^{-1}]} \otimes H^*(F^A) \cong H_A^*(F)[e_A^{-1}]$$

so it should be true that $\dim H^*(F) = \dim H^*(F^A)$.
Check directly

$$\dim H^*(F^A) = \operatorname{card}\{\phi\} \cdot \dim H^*\{\prod_\lambda F(V_\lambda)\}$$
$$= \frac{n!}{\prod_\lambda (\dim V_\lambda)!} \cdot \prod_\lambda (\dim V_\lambda)! = n!$$

Thus I get the formula

$$\boxed{c(H_G^*) = \sum_A \frac{1}{|W(A)|} c(H^*_{C(A)})}$$

where A runs over representatives for the conjugacy classes of [redacted] A of maximal rank r.

Check M's calculations:

Heisenberg group of order p^3 $\left\{ e = x^p = y^p = z^p,\ (x,y) = z \atop z \text{ central} \right\}$ p odd

has $p+1$ [redacted] elementary groups of order p^2 which are maximal abelian so
$$c = (p+1)\frac{1}{p}.$$

May 6, 1976

Assume J is a normal subgroup of the p-group P. Let A be a maximal P-normal + elementary abelian subgroup of J. Then $A \lhd J$. If A is not maximal normal elem. ab. in J, then I know that

$$ {}_pZ(C_J(A)/A) \longrightarrow A \qquad xA \mapsto x^p $$

is not injective; ~~let~~ B'/A be the kernel. B' is normal in P, so $B'/A > 1 \implies (B'/A)^P > 1$. If $A \neq xA \in (B'/A)^P$, then $\langle x \rangle A$ is a subgroup of J which is P-normal and elementary abelian. This contradicts maximality of A. So

Prop: Let $J \lhd P$, P a p-group, p odd, and let A be a subgroup of J which is normal in P and elementary abelian, and which is maximal among subgps of J with these properties. Then A is a maximal normal elem. ab. subgrp of J, hence A is a maximal elementary abelian subgroup of J.

May 7, 1976: Σ_n symmetric group. The p-rank is $[\frac{n}{p}]$ and there is a unique maximal rank A namely $(\mathbb{Z}/p)^m \subset (\Sigma_p)^m \subset \Sigma_{pm} \subset \Sigma_n$ $\quad m = [\frac{n}{p}]$
Then

$$ N(A) = \Sigma_m \ltimes \left((\mathbb{Z}/p)^* \ltimes \mathbb{Z}_p\right)^m \times \Sigma_{n-pm} $$

$$ C(A) = (\mathbb{Z}/p)^m \times \Sigma_{n-pm} $$

so $$W(A) = N(A)/C(A) = \Sigma_m \ltimes \left(\mathbb{Z}/p^*\right)^m$$

Now [redacted] if $n = a_0 + a_1 p + \cdots + a_r p^r$,
$0 \le a_i < p$, then $m = a_1 + \cdots + a_r p^{r-1}$

$$\left[\frac{n}{p^i}\right] = a_i + a_{i+1} p + \cdots + a_r p^{r-i}$$

$$\left[\frac{m}{p^{i-1}}\right] = a_i + \cdots + a_r p^{r-i}$$

so $$\operatorname{ord}_p(n!) = \underbrace{\left[\frac{n}{p}\right]}_{=m} + \cdots + \left[\frac{n}{p^r}\right] = m + \operatorname{ord}_p(m!)$$

Thus [redacted]

$$\frac{|\Sigma_n|}{|N(A)|} = \frac{n!}{m!\,(p-1)^m p^m a_0!} \not\equiv 0 \pmod p$$

so A is normal in some Sylow group of Σ_n.

$$H^*(C(A)) = H^*(A) \quad ! \ldots$$

$$c\left(H^*(\Sigma_n)\right) = \frac{1}{m!\,(p-1)^m}$$

$$c\left(H^*(S_p(\Sigma_n))\right) = \frac{1}{p^{\operatorname{ord}_p(m!)}}$$

Symmetric group Σ_{2p}. p odd [redacted]

[redacted] A non-trivial p-subgroup of Σ_{2p} divides the set $\{1, \ldots, 2p\}$ into 2 [redacted] invariant subsets of order p. There is at least one non-trivial orbit, the other is its complement. [redacted] Let Σ_{2p} act on partitions of $\{1, 2, \ldots, 2p\}$ into 2 subsets of order p, i.e. on $\Sigma_{2p}/\Sigma_2 \wr \Sigma_p$. Each p-subgp has exactly one fix pt. Thus $\mathcal{S}_p(\Sigma_{2p})$ breaks up according to these partitions. So next we have to consider $\mathcal{S}_p(\Sigma_2 \wr \Sigma_p) = \mathcal{S}_p(\Sigma_p^2)$.

What is $\mathcal{S}_p(G_1 \times G_2)$? [redacted]

Is [redacted]

$$\mathcal{S}_p(G_1 \times G_2) \sim \mathcal{S}_p(G_1) * \mathcal{S}_p(G_2) \quad ?$$

[redacted] $\mathcal{S}_p(G)$ is homotopy equivalent to the simplicial complex $X(G)$ whose vertices are the subgroups C of order p, and whose simplices are subsets $\{C_0, \ldots, C_n\}$ which mutually commute. In effect $\mathcal{S}_p(G) \sim \mathcal{A}_p(G)$, and we can cover $X(G)$ by the subcomplexes $X(A)$ for each $A \in \mathcal{A}_p(G)$. Note

$$X(A) \cap X(B) = X(A \cap B)$$

and $X(A)$ is the full simplex whose vertices are the "lines" in A. Now $X(G_1 \times G_2) \to X(G_1) \times X(G_2)$ but there are not equal. Too bad.

Recall that $X * Y = \mathrm{Cyl}\{X \leftarrow X \times Y \rightarrow Y\}$. Now [redacted] we can divide up $\mathcal{S}_p(G_1 \times G_2)$ according to subgroups having non-trivial projections in

G_1 resp. G_2. Thus let U_i be the open subset of $P < G_1 \times G_2$ such that $pr_i(P) > 1$.

We can identify $\mathcal{S}_p(G_1)$ with the subset of U_1 consisting of P with $pr_2(P) = 1$ i.e. $P < G_1$. We can deform U_1 into $\mathcal{S}_p(G_1)$ by sending P to $\mathrm{Ker}\{pr_1 : P \to G_1\} = P \cap G_2$?

The point is that $\mathcal{S}_p(G_1 \times G_2)$ can be deformed into the subset consisting of P of the form $P_1 \times P_2$ where P_i is a p-subgroup of G_i and not both P_1, P_2 are trivial. Then it's clear that this subset has the homotopy type of the join . . .

<u>Prop</u>: $\mathcal{S}_p(G_1 \times G_2) \sim \mathcal{S}_p(G_1) * \mathcal{S}_p(G_2)$.

So $\mathcal{S}_p(\Sigma_{2p})$, p odd, is not "spherical." In effect

$$\mathcal{S}_p(\Sigma_p^{\,2}) \simeq \mathcal{S}_p(\Sigma_p) * \mathcal{S}_p(\Sigma_p)$$

will be a non-trivial bouquet of 1-spheres, when $p \geq 5$.

Check by counting χ.

$$\chi\left(S_p(\Sigma_p)\right) = \frac{p!}{p(p-1)} = (p-2)!$$

$$\chi\left(S_p(\Sigma_p^{\wr 2})\right) = \chi(S_p(\Sigma_p)) + \chi(S_p(\Sigma_p)) - \chi(S_p(\Sigma_2))^2$$

$$= 2(p-2)! - \left((p-2)!\right)^2$$

$$\chi(S_p(\Sigma_{2p})) = \left|\Sigma_{2p}/\Sigma_2 \wr \Sigma_p\right| \cdot \chi(S_p(\Sigma_p^{2}))$$

$$= (2p)!/2(p!)^2 \cdot \left(2(p-2)! - (p-2)!^2\right)$$

This should be $\equiv 1 \mod (p^2)$.

$$= \frac{(p+1)\cdots(2p-1)\,[2-(p-2)!]}{p-1}$$

Check $p=3$: $\quad \frac{4\cdot 5}{2}[2-1] = 10 \equiv 1 \quad (9)$

$p=5$: $\quad \frac{6\cdot 7\cdot 8\cdot 9}{4}[2-6] = -6\cdot 7\cdot 8\cdot 9$

$$\equiv 2\cdot 13 \equiv 1 \quad (25)$$

Conjecture: $S_p(G) \sim pt \Rightarrow O_p(G) > 1$.

<u>Prop</u>: The conjecture is true if $l_p(G) \leq 2$.

<u>Proof</u>: If $l_p(G) = 1$, $\mathcal{A}_p(G)$ finite set, so if $\mathcal{S}_p(G) \sim pt$, then $\mathcal{A}_p(G)$ consists of a single point.

If $l_p(G) = 2$, then $\mathcal{A}_p(G)$ is a graph, which is a tree. By Serre any <u>finite group</u> acting on a tree has a fixpoint.

Next go over proof of the fixpoint property for G on a tree X. Define a subset of X to be <u>convex</u> if it contains the geodesic joining any two of its points. Take an orbit Gx and put it in a finite convex subset. e.g. the ball of given radius is convex. Better: take the union of the geodesics from a point y to the points of Gx. Now take intersections to get a finite G-invariant convex set. Let K be a minimal G-invariant convex set which is non-empty. Then we could remove extreme points from K, except if K is a point.

Philosophy: In order to prove the conjecture I want to use a similar sort of argument with a space like $\mathcal{S}_p(G)$.

Start with a faithful representation of G over $\mathbb{F}_p$, V, then take

$$X = \bigcup_{H \in \mathcal{S}_p(G)} T(V)^H$$

to be the required simplicial complex. $\forall x \in X$, I want $\{H \in \mathcal{S}_p(G) \mid H \subset G_x\} = \mathcal{S}_p(G_x)$ to be contractible. Doesn't work.

<u>Question</u>: Consider the poset whose elements are pairs $H \triangleleft K \subset G$ where $H \in \mathcal{S}_p(G)$. Define the ordering:

$$(H_1, K_1) \leq (H_2, K_2) \iff H_2 \subset H_1 \subset K_1 \subset K_2.$$

Does this have the homotopy type of $\mathcal{S}_p(G)$?
Recall $\tilde{\mathcal{S}}_p(G)$ = poset of subgroups $K \ni O_p(K) > 1$.
Put the order $K \leq K'$ to mean $K \subset K'$ and $O_p(K') \cap K > 1$, i.e. $\exists$ non-trivial p-subgp. $H \subset K$ normalized by K'. Have map

$$\mathcal{S}_p(G) \longrightarrow \tilde{\mathcal{S}}_p(G) \qquad ???$$

May 8, 1976:

Let a p-group P act on $V \cong \mathbb{F}_q^n$, $q = p^d$. I want to go over the proof that the poset of flags σ in V "stabilized" by P in the sense of group theory is contractible. Put $J = \{\sigma \mid P\sigma = \sigma,\ P \text{ acts trivially on } gr(\sigma)\}$. This means that $P \subset$ the unip radical B_σ^u, where B_σ = the parabolic fixing σ. Let V' be the subspace $\ni$ $H_0(P, V) = V/V'$. For each W, $V' \subset W < V$, let J_W be the subset of $\sigma = \{0 < W_1 < \cdots < W_s < V\}$ in J such that $W_s \subset W$. Then $J_{W_1} \cap J_{W_2} = J_{W_1 \cap W_2}$ and each J_W is contractible by induction: $J_W = J(P, W)$.

Notice that it is desirable here to allow all chains $0 = W_0 < W_1 < \cdots < W_s = V$ stabilized by P, so as to get a good induction. Thus if P acts trivially I have to get, not $\mathrm{Tits}(V)$, but a contractible gadget.

So again let G be a subgroup of $GL_n(\mathbb{F}_q)$. To each $H \in \mathcal{S}_p(G)$, let $T_H \subset \mathrm{Tits}(V)$ be the subposet consisting of flags $\sigma \ni H \subset B_\sigma^u$. As $H > 1$, this subset is contractible. Also $H_1 \subset H_2 \Rightarrow T_{H_2} \supset T_{H_1}$. Finally let $\sigma \in \bigcup_{H \in \mathcal{S}_p(G)} T_H$. As $\sigma \in T_H \iff H \subset B_\sigma^u$, there is a largest $H \ni \sigma \in T_H$ namely $G \cap B_\sigma^u$. Thus

Prop. $G \subset GL_n(\mathbb{F}_q)$. Then $\mathcal{S}_p(G) \sim$ the open subset of $\mathrm{Tits}(V)$ consisting of σ such that $G \cap B_\sigma^u > 1$.

Conjecture: $S_p(G)$ contractible $\iff O_p(G) > 1$.

Checks: 1) $O_p(G_1) > 1 \implies O_p(G_1 \times G_2) > 1$

$$S_p(G_1) \sim pt \implies S_p(G_1 \times G_2) = S_p(G_1) * S_p(G_2) \sim pt \times S_p(G_2) \sim pt.$$

2) Is it true that

$$N \triangleleft G, \quad S_p(N) \sim pt \implies S_p(G) \sim pt \quad ?$$

$$S_p(G) \sim pt \implies S_p(G/O_{p'}(G)) \sim pt \quad ?$$

This is a special case of:

$$S_p(G) \sim pt \implies S_p(N) \sim pt \text{ or } S_p(G/N) \sim pt ?$$

Let us fix a normal subgroup N of G. To each p-group P we associated PN/N, $P \cap N$. This gives a map

$$S_p(G) \longrightarrow S_p(G/N) * S_p(N).$$

An important case is where $N = O_{p'}(G)$, whence we get an epimorphism

$$S_p(G) \longrightarrow S_p(G/N).$$

It seems possible that this is connected with the question of homotopy fixpts. In effect X contractible $\implies$ space of homotopy fixpts is contractible. You

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want the space of fixpts to be contractible.

Go back to Alperin's thm. It seems that $H^*(G) = H^*(N_G(P))$ provided this is true for all normalizers of non-identity p-subgroups.

Call a p-subgroup H of G <u>good</u> if whenever $H \subset P, Q$ with P, Q Sylow groups, then α_P and α_Q have the same restriction to $P \cap Q$. Let H be a maximal bad group. Then we've seen $H = P \cap Q$ is a tame intersection, and also $HC_G(H)/H$ is a p'-group. The problem is that $N_G(H)$ acts non-trivially on $\alpha_P | H \in H^*(H)$. In fact we know the set of possible restrictions $\alpha_Q | H$ forms an orbit under $N_G(H)$ acting on $H^*(H)$. Let $R = N_P(H)$, whence $H < R \subseteq P$, and R is an S_p-subgp of $N_G(H)$.

$$\begin{tikzcd}
 & & N_G(R) & \\
R = N_P(H) \arrow[d] \arrow[r, dash] & N_N(R) \arrow[ur, dash] \arrow[r, dash] & N = N_G(H) & \\
H = P \cap Q & & &
\end{tikzcd}$$

Because $R > H$ it follows that R is good. It should follow that $\exists !$ class $\alpha_{N_G(R)} \in H^*(N_G(R))$ compatible with the classes we have on the S_p-subgroups containing R. Thus the class $\alpha_P | R$ extends to $N_N(R)$, hence as we are assuming $H^*(N) = H^*(N_N(R))$ for every normalizer $N = N_G(H)$, it follows that α_R comes from $H^*(N)$. Hence $\alpha_p|H$ is N-invariant and H is good.

It seems that $H^*(G) = H^*(N_G(ZP))$ provided this is true for all normalizers of non-identity p-subgrps. Again suppose H is maximal bad.

$$\begin{tikzcd}
 & N_G(ZR) & \\
R = N_p(H) \arrow[r, dash] \arrow[d, dash] & N_N(ZR) \arrow[u, dash] \arrow[r, dash] & N = N_G(H) \\
H = P \cap Q & &
\end{tikzcd}$$

Different induction used here. The new point is that although R is good, this doesn't get α up to $N_G(ZR)$, so we can't handle P containing $Z(R)$ but not R. The key here is this. Suppose $H \triangleleft P$ so that $R = P$. Then by hypothesis α_R comes from $N_G(ZR)$, hence by hypothesis: $H^*(N) = H(N_N(ZR))$, α comes from $H^*(N)$.

<u>Lemma</u>: Assume $\alpha \in H^*(P)$ comes from $H^*(N_G(ZR))$ and that for $H^*(N) = H^*(N_N(ZS_p(N)))$ for any normalizer of a non-identity p-group. Then α comes from $H^*(N_G(H))$ for any $K H \triangleleft P$.

<u>Proof</u>: P is a S_p-subgp of $N = N_G(H)$, so to extend α

to $H^*(N)$, we need show it extends to $H^*(N_N(ZP))$. As $N_N(ZP) \subset N_G(ZP)$, this is clear.

So now argue as follows: To show $\alpha \in H^*(P)$ comes from $H^*(G)$ it suffices to show for each $H \subset P$ such that $N_P(H)$ is an S_p subgp of $N_G(H)$ that $\alpha | N_P(H)$ comes from $H^*(N_G(H))$. Use decreasing induction on $|N_P(H)|$. If $N_P(H) = P$, then OKAY.

$$\begin{tikzcd}
 & & N_G(ZR) & \\
R = N_P(H) \arrow[r, dash] \arrow[d, dash] & N_N(ZR) \arrow[ur, dash] \arrow[rr, dash] & & N = N_G(H) \\
H & & &
\end{tikzcd}$$

Check that the condition $\alpha_P | N_P(H)$ comes from $N_G(H)$ is independent of the choice of P such that P contains a S_p-subgroup of $N_G(H)$. If $N_Q(H)$ is also an S_p-subgp. then $\exists\, x \in N_G(H) \ni$ $x N_P(H) x^{-1} = N_Q(H)$. As $x \alpha_P x^{-1} = \alpha_Q$, can suppose $N_P(H) = N_Q(H)$.

Start again. Assume we know $H^*(N) = H^*(N_N(ZR))$ if $N = N_G(H)$, $H \in \mathcal{S}_p(G)$, and R an S_p-subgp of N. To show that the same is true for G. Start with $\alpha \in H^*(N_G(ZP))$. Then we get a family of α_P for each S_p-subgroup compatible under conjugation. Next let $KH \triangleleft P$ some S_p group P. Claim that if $P, Q \subset N_G(H)$, then α_P, α_Q agree on $P \cap Q$. In effect from:

$$\begin{tikzcd}[arrows=dash]
 & & N_G(ZP) \\
P \arrow[r] \arrow[d] & N_N(ZP) \arrow[ur] \arrow[r] & N = N_G(H) \\
H & &
\end{tikzcd}$$

we see α_P comes from $N_N(ZP)$ where P is an S_p-subgp of N, hence by hypothesis on N, α_P comes from $H^*(N_G(H))$. ~~Thus $\alpha_P|H$ is invariant under $N_G(H)$, so $\alpha_P|H = \alpha_Q|H$ for all P, Q in $N_G(H)$~~. But if α_P extends to $N_G(H)$ so does α_Q, so α_P and α_Q agree on $P \cap Q$.

Next suppose $H \in \mathcal{S}_p(G)$ is such that an S_p-subgroup of $N_G(H)$ is normal in some S_p-group P. Thus

$$\begin{array}{lcl}
P & & \\
\triangledown & & \\
R = N_P(H) & \xrightarrow[\;p'\text{-index}\;]{} & N_G(H) \\
\triangledown & & \\
H & &
\end{array}$$

I have defined α_R unambiguously, and I want to show it extends to $N_G(H) = N$.

$$\begin{tikzcd}[arrows=dash]
P \subset N_G(R) \subset & N_G(ZR) & \\
R \arrow[u] \arrow[d] \arrow[r] & N_N(ZR) \arrow[u] \arrow[r] & N = N_G(H) \\
H & &
\end{tikzcd}$$

Because ZR char in $R \triangleleft P$, $P \triangleright ZR$, so α_P extends to $N_G(ZR)$ by the above, hence α_R extends to $N_N(ZR)$, hence to $N_G(H)$ as I wanted.

Generalization: Let $H \in \mathcal{S}_p(G)$ be such that a S_p-group R of $N_G(H)$ is such that $N_G(ZR)$ contains an S_p-subgroup P. Then we can suppose $R \subset P$ and $R = N_P(H)$. We get

$$\begin{tikzcd} P \arrow[rr, dash] \arrow[d, dash] & & N_G(ZR) \\ R \arrow[r, dash] \arrow[d, dash] & N_N(ZR) \arrow[r, dash] \arrow[ur, dash] & N = N_G(H) \\ H & & \end{tikzcd}$$

I have seen that if P, Q are two S_p-subgroups of $N_G(ZR)$, then α_P, α_Q agree on $P \cap Q$. (Check: Put $K = ZR$, then

$$\begin{tikzcd} & N_G(ZP) & \\ P \arrow[r, dash] & N_N(ZP) \arrow[r, dash] \arrow[u, dash] & N_G(K) = N \end{tikzcd}$$

so α_P extends to N, similarly α_Q extends ~~to~~ N. As P, Q are conjugate by an element of N and α_P, α_Q agree it follows α_P, α_Q come from the same element of $H^*(N)$, so α_P, α_Q agree on $P \cap Q$). Thus it follows I get a well-defined element $\alpha_R \in H^*(R)$.

Let's suppose $S_p(G)$ is not connected and let M be the stabilizer of the component containing P. It's clear that $H^*(M) \xleftarrow{\sim} H^*(G)$ because all conditions arising from double cosets PxP with $P \cap xPx^{-1} > 1$ occur in M. Thus $P \cap xPx^{-1} > 1 \Rightarrow x \in M$. M is self-normalizing and meets its conjugates along p'-groups.

Let $S(G)$ be the poset of non-identity solvable subgroups of G. ~~Assume $S(G)$ not connected (hence G not solvable) and let M be the stabilizer of a component. Assume G minimal simple. Then M is solvable as it is a proper subgroup. Moreover it is maximal solvable~~ Assume G minimal simple. Consider the action of G on $\pi_0 S(G)$, and suppose it is non-trivial. Then $\exists\ \alpha \in \pi_0 S(G)$ whose stabilizer M is a proper subgroup of G. Note that if $H \in \alpha$, then $H \subset M$. Since M is solvable M belongs to some component of $\pi_0 S(G)$ which has to be α. Thus α consists of all subgroups >1 in M, and M is a maximal subgroup of G, equal to its normalizer. The components of G in the orbit of α correspond to the different conjugates of M. Thus $M \cap xMx^{-1} \neq 1 \Rightarrow M = xMx^{-1}$. Now apply the Frobenius thm. to get a contradiction

Perhaps $\pi_0 S(G)$ can be identified with a partition of the set of primes dividing G. Since G acts trivially on $\pi_0(G)$ it follows that any two S_p-subgroups lie in the same component. So it's clear we get the following:

<u>Prop</u>: Let $\mathcal{S}(G)$ be the poset of proper subgroups of the simple group G. Then

1) G acts trivially on $\pi_0 \mathcal{S}(G)$.

2) $\pi_0 \mathcal{S}(G)$ can be identified with the quotient of the set of primes dividing $|G|$ by collapsing the subset of primes dividing $|H|$ for each $H \in \mathcal{S}(G)$.

Let $J(G)$ be a poset of non-identity subgroups of G containing at least the products of the elementary abelian p-groups. Let α be a component of $J(G)$ and M its stabilizer. Take an H in α and replace it by a cyclic group of order p, then by the elements of order p in the center of an S_p-subgroup P. Then $N_G(P) \subset M$ so M is its own normalizer. If $H \in \alpha$, then $H \subset N_G(H) \subset M$. I can't seem to show that $m \in M \Rightarrow \langle m \rangle \in \alpha$. Unless I were to know $M \in J(G)$. Then I know that M is the largest member of α

Let G be a solvable group with $|\text{Primes}(|G|)| \geq 3$. If $\alpha \in \pi_0(\mathcal{S}(G))$, then α contains an S_q-group H for some q, and H is contained in an $S_{p,q}$-group, so α contains an S_p-group. Thus G acts transitively on $\pi_0 \mathcal{S}(G)$. If M is the stabilizer of α, then $M < G$ assuming $\mathcal{S}(G)$ not connected. But M contains an S_p-subgroup for each p, so we have a contradiction. $\therefore$

<u>Prop</u>: If G is solvable and $|G|$ is divisible by ≥ 3 distinct primes, then $\mathcal{S}(G)$ is <u>connected</u>.

Let K be a fixed p-subgroup of G. If $H \in \mathcal{S}_p(G)$ and $H > K$, then $N_H(K) > K$, consequently

$$H \longmapsto H \cap N_G(K)$$

deforms $\{H \in \mathcal{S}_p(G) \mid H > K\}$ into $\mathcal{S}_p(N_G(K))$. Let $B \in \mathcal{A}_p(G)$, then $A > B \Rightarrow A < C_G(B)$.

$$\mathcal{A}_p(G)_{>B} = \mathcal{A}_p(C_G(B))_{>B}$$

So when might this be contractible?

Example: $GL_n(R)$ where R contains $\frac{1}{p}, \mu_p$. Let A be an elementary abelian p-subgroup of G. Better suppose $G = GL_n(\mathbb{F}_q)$ where we work with elementary abelian l groups and l divides $q-1$. I can decompose $V = \mathbb{F}_q^n$ according to the characters of A, say

$$V = \bigoplus_{\chi \in Hom(A, \mu_l)} V_\chi$$

Then $C_G(A) = \prod_{\chi \in Hom(A,\mu_l)} GL(V_\chi)$ is the same as for the $\tilde{A} = \prod_\chi \mu_l =$ subgroup of the center of $C_G(A)$ killed by l.

Lemma: If $B \in \mathcal{A}_p(G)$ is not the $\tilde{B}$ subgroup of elements of order dividing p in $C_G(B)$, then $\mathcal{A}_p(G) > B$ is contractible.

Proof: For if $\tilde{B} = {}_pZ(C_G(B))$, then $A > B \Rightarrow A \leq A\tilde{B} \geq \tilde{B}$ gives a canonical contraction of $\mathcal{A}_p(G) > B$.

Question: The poset of $B \in \mathcal{A}_p(G) \ni B = {}_pZ(C_G(B))$, does it have the same homotopy type as $\mathcal{A}_p(G)$?

If $B \subset B'$, then $C_G(B) \supset C_G(B')$, so what?

If $G = P$, then $B = {}_pZ(C_P(B)) \supset {}_pZ(P)$ so the poset has a least element.

In the case of GL_n we have identified the poset $\mathcal{J}$ of subgroups in question with the poset of splittings $V = \bigoplus_{W \in \sigma} W$. Change to SL_n where $n \not\equiv 0 \mod l$. Then we want non-trivial splittings. If P is an l-group, then $\mathcal{J}^P$ is non-empty because the degree of an irreducible repn. is a power of l. If P is a maximal elementary l-group, then $\mathcal{J}^P$ can be identified with the poset of non-trivial partitions of $\{1, \ldots, n\}$. ~~This is not contractible~~ ~~$(n > 2)$ so the poset~~ ~~$\mathcal{J}$~~

Lemma: Let p be a prime, $\mathbb{F}_q$ a finite field such that $p \mid q-1$. Let $\mathcal{J}$ be the poset of non-trivial partitions of V (i.e. a family σ of proper subspaces W of V such that $V = \bigoplus_{W \in \sigma} W$). If P is a non-identity p-subgroup of $GL(V)$, then $\mathcal{J}^P$ is contractible or empty depending on whether V is irreducible or not under P.

Proof: An element of $\mathcal{J}^P$ is a partition of

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Let X be a G-space, let A be a cyclic subgroup of order p contained in the center of G which acts trivially on X.

$$1 \to A \to G \to G/A \to 1$$

$$PG \times^G X$$

More generally, let A be a normal subgroup of G, let X be a G/A-space. The problem is to relate $H^*_{G/A}(X)$ with $H^*_G(X)$.

$$PG \times^G X = (PG/A) \times^{G/A} (X/A)$$

Now $PG/A \sim BA$ and presumably the action of G/A on PG/A corresponds to the conjugation action of G/A on BA up to homotopy.

$$PG \times^G (X \times P(G/A)) = (PG/A) \times^{G/A} (P(G/A) \times X)$$

This leads to a spectral sequence

$$E_2^{st} = H^s(P(G/A) \times^{G/A} X,\ H^t(BA)) \Longrightarrow H^*(PG \times^G X)$$

If G/A acts trivially on A, then this becomes

$$E_2^{st} = H^s_{G/A}(X) \otimes H^t_A \Longrightarrow H^*_G(X)$$

Suppose now that A is cyclic, $p \mid |A|$ and let $u \in H^2_A$ be the canonical generator. Then for some k I know u^k comes from $c \in H^{2k}_G$. (Take an irred repn. of G on which A acts non-trivially). I recall that multiplication by u^k on E_2 sets up an isomorphism $E^{st}_r \xrightarrow{\sim} E^{s,t+2k}_r$ for all s,t, $t \geq 0$. Check this: True for $r=2$. Assume true for r.

$$\begin{tikzcd}
E_r^{s-r,t+r-1} \arrow[r, "d_r"] \arrow[d, "\wr"'] & E_r^{st} \arrow[r, "d_r"] \arrow[d, "\wr"'] & E_r^{s+r,t-r+1} \arrow[d] \\
E_r^{s-r,t+r-1+2k} \arrow[r, "d_r"'] & E_r^{s,t+2k} \arrow[r, "d_r"] & E_r^{s+r,t+2k-r+1}
\end{tikzcd}$$

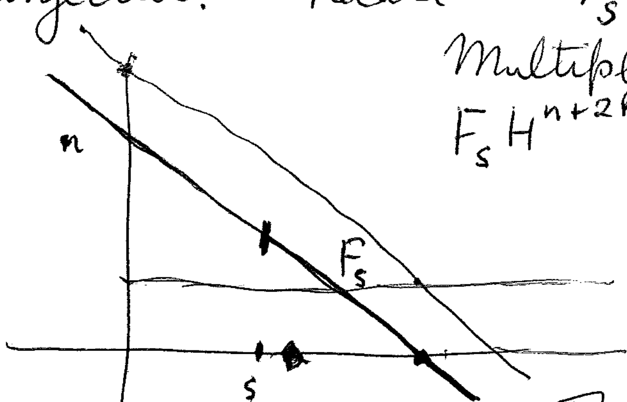
an isom for $t-r+1 \geq 0$

injective if <0 because then $E_r^{s+r,t-r+1} = 0$

$\cancel{t}+r-1 > 0$ $\qquad t \geq 0$

So one sees that $\operatorname{Im} d_r$, and $\operatorname{Ker} d_r$ are the same in E^{st}_r and $E^{s,t+2k}_r$, so one has $E^{st}_{r+1} \xrightarrow{\sim} E^{s,t+2k}_{r+1}$ for $t \geq 0$. It follows that $E^{st}_\infty \xrightarrow{\sim} E^{s,t+2k}_\infty$.

Now look at the abutment. To prove $H^n \xrightarrow{c} H^{n+2k}$ is injective. Recall $F_s H^n / F_{s+1} H^n = E^{s,n-s}_\infty$.



Multiplacation by c carries $F_s H^n$ into $F_s H^{n+2k}$. So it seems that

$$c : H^n \xrightarrow{\sim} H^{n+2k} / F_{n+1} H^{n+2k}$$

In H^n one has $F_{n-2k+1} H^n$. Thus it appears that if we put

$F^{s}H^{n} = F_{n-s}H^{n}$, then

$$F^{2k-1}H^{*}[c] \xrightarrow{\sim} H^{*} \qquad \deg(c) = 2k.$$

This is strange to have a canonical subset of H^* act as generators.

Suppose that the elements of order 1 or p of $Z(G)$ form a maximal elementary abelian p-subgp A of G. If $l = \operatorname{rank}(A)$, then choose irred. reps. $V_1, \dots, V_l$ whose restrictions to A give a basis for $\mathrm{Hom}(A, \mu_p)$, and put $c_i = e(V_i)$. Then I know that H^*_G is Cohen-Macaulay with $c_1, \dots, c_l$ as regular sequence, and

$$H^*_G/(c_1, \dots, c_l) \cong H^*_G(SV_1 \times \cdots \times SV_l)$$

Observe that A acts freely on $SV_1 \times \cdots \times SV_l$, so no element of order p of G has a fixpt on $SV_1 \times \cdots \times SV_l$, so the isotropy groups are p'-groups. $\therefore$

$$H^*_G(SV_1 \times \cdots \times SV_l) = H^*((SV_1 \times \cdots \times SV_l)/G)$$

Suppose P is a p-group such that $\Omega_1 P = {}_pZ(P)$. Put $A = {}_pZ(P)$. Assume $\mathrm{rank}(A) = \dim H^1(P)$. Claim $H^*(P) \cong \Lambda H^1 \otimes SA^\vee$ if p is odd.? One knows from group theory that P/A also has $\Omega_1 \subset Z$, so we can use induction. First case: $A \not\subset \Phi(P)$. Let $A = A \cap \Phi(P) \oplus B$. Then we have $B \hookrightarrow P/\Phi(P)$ and so we can find a homomorphism $P/\Phi(P) \twoheadrightarrow B$ so that $B \hookrightarrow P \longrightarrow B$ is the identity. Since B is in ${}_pZ(P)$, it follows $P = B \times \underbrace{\mathrm{Ker}\{P \to B\}}_{P'}$. $A = \Omega_1 P = B \times \Omega_1 P'$ so $\Omega_1 P' \subset ZP'$. Now use induction. 2nd case: $A \subset \Phi(P)$, whence $H^1(P/A) = H^1(P)$. Now use the spectral sequence

$$E_2^{st} = H^s(P/A) \otimes H^t(A) \Longrightarrow H^*(P)$$

Now show (1) $$H^2(P/A) \xleftarrow[(\mathrm{cup},\, d_2)]{\sim} \Lambda^2 H^1(P/A) \oplus H^1(A)$$

Have
$$\begin{tikzcd} 0 \arrow[r] & \Lambda^2 H^1(P/A) \arrow[r] & H^2(P/A) \arrow[r] & (B/A)^\vee \arrow[r] & 0 \\ & & H^1(A) \arrow[u] \arrow[ur, "\sim"'] & & \end{tikzcd}$$
$B/A = \Omega_1(P/A)$

by induction and also

so (1) is OKAY. Also show

(2) $$\mathrm{Im}\{H^2(P) \to H^2(A)\} \supset A^\vee$$

$$\begin{array}{|lll} \Lambda^2 H^1(A) \oplus A^\vee & & \\ H^1(A) & H^1(P/A) \otimes H^1(A) & \\ \hline & H^2(P/A) & H^3(P/A) \end{array}$$

Calculation leaves following after $H^1(A) \hookrightarrow H^2(P/A)$ etc. are used with d_2.

$$\begin{array}{cccc}
\hat{A} & & & \\
0 & 0 & 0 & 0 \\
\mathbb{Z}/p & H^1(P/A) & \Lambda^2 H^1(P/A) & \Lambda^3 H^1(P/A).
\end{array}$$

so there is a possible map $d_3 \hat{A} \to \Lambda^3 H^1(P/A)$. I see no reason why this has to be zero. So the proof has problems.

May 15, 1976

Return to the fusion problem again. Consider the following property for a finite group

$$(*) \qquad H^*(G) = H^*(N_G(ZJP)) \qquad \text{if } P \text{ is an } S_p\text{-subgp.}$$

Gorenstein claims that if this property holds for $N_G(H)$ $\forall H \in \mathcal{S}_p(G)$, then it holds for G.

Let $\alpha \in H^*(P)$ come from $N_G(ZJP)$. Since ZJP char P, $N_G(P) \subset N_G(ZJP)$, so we have seen how to define α on any S_p-grp Q. We have to show that α_P, α_Q have the same restrictions on H when $H \subset P, Q$. ~~Argue by descending induction on H. We've seen that we have only to show that $\alpha_P|H$ is $N_G(H)$ invariant, and that we can assume $N_P(H)$ is an S_p subgroup of~~ We've seen that it is enough to show that if $N_P(H)$ is an S_p-subgroup of $N_G(H)$, then $\alpha_P | N_P(H)$ comes from $N_G(H)$; moreover we can assume α_P, α_Q have the same rest. to $P \cap Q$ if $|P \cap Q| > |H|$. In this case we are free to vary P to another S_p-subgp $Q \ni N_Q(H) = N_P(H)$. So put $N_1 = N_G(H)$, $H_1 = N_P(H)$,

$$N_2 = N_G(ZJH_1) \supset H_1$$

choose H_2 an S_p subgp of N_2 containing H_1.

I can suppose $P \supset H_2$. Repeat this process:

$$
\begin{array}{ccccccccc}
H = H_0 < & H_1 & < & H_2 & < \cdots < & H_r = P & & & \\
 & \cap & & \cap & & \cap & & & \leftarrow S_p \text{ inclusions} \\
 & N_1 & & N_2 & & N_r & & N_{r+1} & \\
 & \| & & \| & & \| & & \| & \\
 & N_G(H_0) & & N_G(ZJH_1) & & N_G(ZJH_r) & & N_G(ZJP) &
\end{array}
$$

To show $\alpha_p | H_1$ comes from $N_1 = N_G(H_0)$ it suffices to show it comes from $N_{N_1}(ZJH_1)$ (H_1 is an S_p-subgp of N_1). Since $N_{N_1}(ZJH_1) \subset N_G(ZJH_1) = N_2$, it suffices to prove $\alpha_p | H_2$ comes from N_2. So inductively one is reduced to knowing that α_p comes from $N_G(ZJP)$ which is the hypothesis. QED.

May 16, 1976

Go over the fusion question. Let G be a finite group, P an S_p-subgroup. We want to describe the image of $H^*(G) \hookrightarrow H^*(P)$. We know it consists of classes α equalized by the two homoms $P \cap xPx^{-1} \rightrightarrows P$ for any double coset PxP. (This follows from the double coset formula).

We analyze these conditions in the order of decreasing $|P \cap xPx^{-1}|$, i.e. in the order of increasing $|PxP/P|$. If $|PxP/P| = 1$, then $x \in N_G(P)$ and conversely, so the first set of conditions is that α be invariant under $N_G(P)$. This being satisfied, one can define $\alpha_P \in H^*(P)$ for all S_p-groups so as to be compatible with conjugation.

~~Then I can define a p-subgp H to be good if α_P and α_Q have the same restriction to $P \cap Q$ for all P, Q S_p gps containing H. Even better~~

Suppose now that α_P, α_Q have the same restriction to $P \cap Q$ when $|P \cap Q| > |H|$. Let us then consider the problems presented by the α_P for $P \supset H$.

(i) $\alpha_P | H = \alpha_Q | H$ $\quad \forall$ S_p-grp P, Q containing H.

(ii) $\alpha_P | H$ is $N_G(H)$-invariant for some S_p-gp P cont. H

(iii) ~~If $N_p(H)$ is Sylow in $N_G(H)$, then~~ $\alpha_P | N_P(H)$ comes from $H^*(N_G(H))$ for some S_p-gp P containing H.

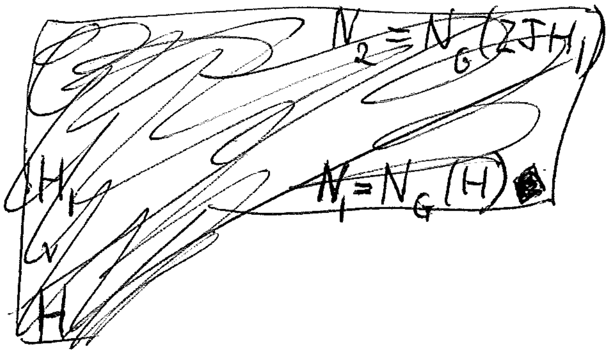
We know for each $H' \in \mathcal{S}_p(G)$ of order $> |H|$ that we get a unique restriction $\alpha_{H'} \in H^*(H')$ from the P containing H'. This in particular applies to the S_p-subgroups of $N_G(H)$. Thus the only conditions remaining to get a cohomology class on $N_G(H)$ is the conditions coming from intersections $H = P' \cap Q'$. where P', Q' are S_p-subgps of $N_G(H)$. Since $N_G(H)$ acts transitively on the S_p-subgps of $N_G(H)$, the possible $\alpha_p | H$ where P runs over the S_p-groups containing H is an $N_G(H)$-orbit. $\therefore$

<u>Lemma</u>: Assume $\alpha_P | P \cap Q = \alpha_Q | P \cap Q$ if $|P \cap Q| > |H|$. Then $\{\alpha_P | H \mid P \text{ is an } S_p\text{-gp of } G \text{ cont. } H\}$ is an $N_G(H)$-orbit in $H^*(H)$.

It is clear that then the conditions (i), (ii), (iii) are equivalent. Note $N_p(H) > H$.

The restrictions $\alpha_p | H$ depends only on the component of P in $\{H' \in \mathcal{S}_p(G) \mid H' > H\}$ which is heq to $\mathcal{S}_p(N_G(H)/H)$.

So



$$\begin{array}{llll} H_2 & & & N_2 = N_G(ZJH_1) \\ \vee & & \diagup\!\!\!\!\cup & \\ H_1 & \subset N_{N_1}(ZJH_1) & \subset & N_1 = N_G(H) \\ \vee & & & \\ H & & & \end{array}$$

We argue by induction that to extend α_{H_1} to N_1 it suffices by hypothesis to extend it to $N_{N_1}(ZJH_1)$, hence to N_2, etc. This process stops

$$\begin{array}{l} P \subset N_{N_n}(ZJP) \subset N_G(ZJH_{n-1}) = N_n \quad N_G(ZJP) \\ H_{n-1} \end{array}$$

To carry out this argument I need to know that if H' is an S_p-subgp. of $N_G(H'')$, then there is a charac. subgroup $H_c' \subset H'$ such that the coh. of $N = N_G(H'')$ is the same as $N_N(H_c')$.

$$\begin{array}{llll} P & & & N = N_G(H^*_{n-1}) \\ \vee & & \cup & \\ H_{n-1} & \subset N_{N_{n-1}}(H^*_{n-1}) & \subset & N_2 = N_G(H_1^*) \\ & & \cup & \\ H_1 & \subset N_{N_1}(H_1^*) & \subset & N_1 = N_G(H) \\ & & & \\ H & & & \end{array}$$

<u>Assertion:</u> For any finite G, [redacted] $\exists$ a char. subgp P^* of a S_p-subgp P such that $H^*(G) = H^*(N_G(P^*))$.

Suppose we try to prove this by induction on $|G|$. [redacted] Assume $\alpha \in H^*(P)$ is given invariant under $N_G(P)$. Let H be a non-identity p-subgp $\ni$ $\alpha_P | P \cap Q = \alpha_Q | P \cap Q$ for $|P \cap Q| > |H|$.

I think the critical case is where [redacted] $P \subset N_G(H)$. [redacted] If $N_G(H) = G$, then $H \triangleleft G$. So we are faced with the problem of whether $O_p(G) > 1$ $\Rightarrow$ $\exists$ a non-trivial characteristic subgroup P^* of P which is normal in G, or at least which has the property that $H^*(G) = H^*(N_G(P^*))$.

<u>Glauber[redacted]man's thm.</u> Under suitable conditions (p-constraint + p-stability),

$$O_p(G) > 1 \implies G = O_{p'}(G)\, N_G(ZJP)$$

If P abelian, this says $G = O_{p'}(G) N_G(P)$.

May 20, 1976:

$F(G) = \prod_p O_p(G)$ is the largest nilpotent normal subgroup of G; it is called the Fitting subgroup.

~~Let G be a p-solvable group such that $O_{p'}(G)=1$. Then $F(G)=O_p(G)$. Put $H=O_p(G)$ and consider $C_G(H)$; it is a characteristic subgroup of G. Hence $O_p C_G(H) = H$, $O_{p'}(C_G(H))=1$. But G p-solvable $\Rightarrow$ $C_G(H)$ p-solvable. Apply Schur-Zassenhaus to get~~

$$\cancel{O_{pp'}(C_G(H)) = H \rtimes K}$$
$$\cancel{= H \times K}$$

~~whence necessarily $K = O_{p'}(C_G(H))$. Thus $O_{pp'}(C_G(H)) = H$,~~

Assume $O_p(G)$ is abelian. Put $H = O_p(G)$, so $H \subset C_G(H)$. $C_G(H)$ is characteristic in G

$$O_p\big(C_G(H)\big) = H$$
$$O_{p'}\big(C_G(H)\big) = O_{p'}(G)$$
$$O_{p,p'}\big(C_G(H)\big) = H \times O_{p'}(G)$$

So you see that

$$O_{p'}(G) = 1,\ C_G(H) \text{ } p\text{-solvable} \Longrightarrow C_G(H) = H.$$
$$\Longrightarrow \quad G/H \hookrightarrow \mathrm{Aut}(H).$$

Suppose G is a p-solvable group with abelian Sylow groups and $O_{p'}(G) = 1$. Put $H = O_p(G)$. Then we see that $O_{p,p'}(C_G(H)) = H$, so as $C_G(H)$ is p-solvable, this means that $C_G(H) = H$, whence $H = P$. Thus G is a semi-direct product of a p'-group acting faithfully on an abelian p-group.

More generally suppose G is p-solvable with abelian Sylow groups. Then $G/O_{p'}(G)$ also has abelian Sylow groups, so from the above we see that

$$G/O_{p'}(G) = O_{p',p}(G)/O_{p'}(G) \rtimes K$$

where K is a <u>p'-group acting faithfully.</u>

Next suppose G is p-solvable with $O_{p'}(G) = 1$. Put $H = O_p(G)$. Let C be the subgroup of G which acts trivially on $H/\Phi(H)$:

$$C = \mathrm{Ker}\{G \longrightarrow \mathrm{Aut}(H/\Phi(H))\}$$

Recall that any p'-auto. of H is trivial if it is trivial on $H/\Phi(H)$. C is char. in G, so $O_p(C)=H$ and $O_{p'}(C)=1$. Look at $O_{p,p'}(C) = H \rtimes K$ where K is a p'-group. K acts trivially on $H/\Phi(H) \Rightarrow K$ centralizes $H \Rightarrow K$ char in $O_{p,p'}(C) \Rightarrow K \subset O_{p'}(C) = 1$. Thus $O_{p,p'}(C) = H$, so as C is p-solvable, one has $C=H$. Thus

<u>Prop</u>: Assume $O_{p'}(G)=1$, and that $C = \mathrm{Ker}\{G \to \mathrm{Aut}(H/\Phi H)\}$ is p-solvable where $H = O_p(G)$, (this holds if G is p-solvable) Then $C=H$, so G/H acts faithfully on $H/\Phi(H)$.

<u>Summary</u>: $G/O_{p',p}(G)$ faithfully acts on $H_1(O_{p,p'}(G)/O_{p'}(G))$.

<u>Prop</u>. G p-solv., $O_{p'}(G)=1 \Rightarrow C_G(O_p(G)) \subset O_p(G)$

<u>Cor</u>. G p-solvable $\Rightarrow C_G(P \cap O_{p',p}(G)) \subset O_{p',p}(G)$.

<u>Cor</u>: " $\Rightarrow Z(P) \subset O_{p',p}(G)$

Now try to generalize the last corollary to get A normal abelian in $P \Rightarrow A \subset O_{p',p}(G)$. Can suppose $O_{p'}(G)=1$, put $H = O_p(G)$. Then G/H acts faithfully on $V = H/\Phi(H)$. A normal abelian $\Rightarrow [P, A, A] = 1 \Rightarrow [H, A, A] = 1 \Rightarrow [V, A, A] = 1$ $\Rightarrow$ each $x \in A$ has minimal poly $\mid (X-1)^2$. Now special analysis of this case, using the fact that $O_p(G/H) = 1$ shows $SL_2(\mathbb{F}_p)$ is a subquotient of G/H. If $p \geq 5$, this contradicts p-solvability, and if $p = 3$ one assumes $SL_2(\mathbb{F}_3) = \mathbb{Z}_3 \ltimes Q_8$ is not involved in G. Conclude $A \subset H$.

G has abelian Sylow subgroups. Look at the map $Z(G) \to G/G'$. To show injective it suffices to show $Z(G)_{(p)} \longrightarrow (G^{ab})_{(p)}$ inj. for any p. Because P is abelian I know

$$(G^{ab})_{(p)} = H_0(N_G(P), P)$$

and because $N_G(P):P$ is prime to p I know

$$H^0(N_G(P), P) \xrightarrow{\sim} H_0(N_G(P), P)$$

But $Z(G)_{(p)}$ injects into the former:

$$Z(G)_{(p)} = Z(G) \cap P.$$

Can you classify simple groups with abelian Sylow groups? example: A_5, $60 = 3 \cdot 4 \cdot 5$, so all Sylow groups are abelian.

Look at fusion again. Let H be a p-subgroup of G such that $\mathcal{S}_p(N_G(H)/H)$ is disconnected. Then $O_p(N_G(H)) = H$. I wanted to know that if $P > H$, then $\alpha_P|H$ is [redacted] invariant under $N_G(H)$. [redacted] I know [redacted] $HC_G(H)$ acts trivially on $H^*(H)$. [redacted] Assume $HC_G(H) : H$ is divisible by p. Let P, Q be Sylow subgps of $G \ni N_P(H), N_Q(H)$ are [redacted] S_p-subgroups of $N_G(H)$. I want to show $\alpha_P|H = \alpha_Q|H$. Now [redacted] because $HC_G(H) \triangleleft N_G(H)$, $P \cap HC_G(H)$, $Q \cap HC_G(H)$ are S_p-subgroups of $HC_G(H)$, hence $\exists$ x in $HC_G(H)$ with

$$P \cap HC_G(H) = x(Q \cap C_G(H))x^{-1}$$

Now because $P \cap HC_G(H) > H$, this means that P, xQx^{-1} [redacted] lie in the same component of $\{T \in \mathcal{S}_p(G) \mid T > H\}$, hence $\alpha_P, x\alpha_Q x^{-1}$ have the same restriction to H. But [redacted] because x acts trivially on [redacted] $H^*(H)$, this means that that α_P, α_Q have the same restriction to H.

[redacted] So if H is critical for the fusion question, then $HC_G(H)/H$ is a p'-group

$$1 \longrightarrow Z(H) \longrightarrow C_G(H) \longrightarrow \overbrace{HC_G(H)/H}^{p'\text{-group}} \longrightarrow 1$$

so it's clear that

$$(*) \qquad HC_G(H) = H \times O_{p'}(C_G(H))$$

Suppose one is interested in 1-dimensional cohomology. Then instead of $HC_G(H)$ let us consider the group

$$C = \operatorname{Ker}\{N_G(H) \longrightarrow \operatorname{Aut}(\underset{\substack{\| \\ H_1(H,\mathbb{Z}/p)}}{H/\Phi(H)})\}$$

This is normal in $N_G(H)$, and if $C:H$ is divisible by p, then again there is no fusion problem. So if H is critical (for H' fusion), then C/H has to be a p'-group, so

$$C = H \rtimes K$$

with K a p'-group. But then K centralizes $H/\Phi(H)$, so K centralizes H, so $C = H \times K$, so K char C $\Rightarrow$ K normal in $N_G(H)$ $\Rightarrow$ $K = O_{p'}(C_G(H)) \Rightarrow$

$$C = H \times O_{p'}(C_G(H)) = H \cdot C_G(H)$$

(More generally, suppose H is maximal critical for the fusion problem relative to an $\alpha_p \in H^i(P)$. Then let $V \subset H^i(H)$ be the subspace invariant under $N_G(H)$ generated by $\alpha_p|H$, and let $C = \operatorname{Ker}\{N_G(H) \longrightarrow \operatorname{Aut}(V)\}$. Then we see that C/H has to be a p'-group, so $C \subset O_{p,p'}(N_G(H))$.)

If H is critical for H'-fusion, then we have proved that

$$N_G(H)/H \times O_{p'}(C_G(H)) \hookrightarrow \operatorname{Aut}(H/\Phi(H)) \qquad H = O_p(N_G(H))$$

Also note that $H \times O_{p'}(N_G(H)) \subset O_{p,p'}(N_G(H))$. It seems that in this case

$$HC_G(H) = H \times O_{p'}(C_G(H)).$$

Check this:

Proposition: Suppose given $\alpha \in H^i(P)$. Let H be a critical p-subgroup for the fusion problem relative to α, let $V \subset H^i(H)$ be the $N_G(H)$-invariant subspace generated by the $\alpha_P|H$ as P ranges over the S_p-groups containing H. Let

$$C = \operatorname{Ker}\{N_G(H) \longrightarrow \operatorname{Aut}(V)\}.$$

Then C/H is a normal p'-subgp of $N_G(H)/H$, so $C \subset O_{p,p'}(N_G(H))$. Also $HC_G(H) \subset C$ so that $HC_G(H)/H$ is also a p'-group. If H is critical for H'-fusion, then

$$\begin{cases} H = O_p(N_G(H)) \\ N_G(H)/HC_G(H) \hookrightarrow \operatorname{Aut}(H/\Phi(H)) \\ HC_G(H) = H \times O_{p'}(C_G(H)) \end{cases}$$

Review: If H critical for H'-fusion, and if $C = \operatorname{Ker}\{N_G(H) \longrightarrow \operatorname{Aut}(H/\Phi H)\}$, then I've seen C/H is a p'-grp. Thus $C = H \rtimes K$, with K a p'-gp. As K centralizes $H/\Phi H$, it centralizes H, so $C = H \times K$ and K char $C \Rightarrow$

$K \triangleleft N_G(H) \Rightarrow K \subset O_{p'}(N_G(H))$. But $O_{p'}(N_G(H))$ centralizes H, so $O_{p'}(N_G(H)) \subset C_G(H) \subset C \Rightarrow K = O_{p'}(N_G(H))$. Thus

$$HC_G(H) = H \times O_{p'}(N_G(H)) = C$$

as claimed. Note that in general if $HC_G(H)/H$ is a p'-gp. then $C_G(H) = Z(H) \times O_{p'}(N_G(H))$ and $HC_G(H) = H \times O_{p'}(N_G(H))$.

Conclusion: (1) H critical for any fusion problem

$$\Rightarrow \quad H = O_p(N_G(H))$$
$$HC_G(H) = H \times O_{p'}(C_G(H))$$

(2) H critical for H'-fusion

$$\Rightarrow \quad H = O_p(N_G(H))$$
$$HC_G(H) = H \times O_{p'}(C_G(H))$$
$$N_G(H)/HC_G(H) \hookrightarrow \mathrm{Aut}(H/\Phi(H))$$

See p. 117

May 21, 1976

Try to find a minimal counterexample to $H^*(G) = H^*(N_G(ZJP))$. Let G be a counterexample of least order.

Let $\bar{G} = G/O_{p'}(G)$ whence $H^*(\bar{G}) \xrightarrow{\sim} H^*(G)$. Let $\bar{P} = PO_{p'}(G)/O_{p'}(G)$, so $P \xrightarrow{\sim} \bar{P}$, and $ZJP \xrightarrow{\sim} ZJ\bar{P}$. Let N be the inverse image of $N_{\bar{G}}(ZJ\bar{P})$; then $N = N_G(ZJP \cdot O_{p'}(G))$. Since ZJP is a S_p-gp of the normal subgroup $ZJP \cdot O_{p'}(G)$ of N one has

$$\begin{aligned} N &= N_N(ZJP) \cdot ZJP \; O_{p'}(G) \\ &= N_N(ZJP) \; O_{p'}(G) = N_G(ZJP) O_{p'}(G) \end{aligned}$$

Thus

$$\begin{aligned} H^*(N_{\bar{G}}(ZJ\bar{P})) &= H^*(N_G(ZJP)/N_G(ZJP) \cap O_{p'}(G)) \\ &\cong H^*(N_G(ZJP)) \end{aligned}$$

$$H^*(\bar{G}) = H^*(G)$$

so by the minimality of G we conclude $O_{p'}(G) = 1$.

~~Let H be a critical subgroup for the fusion problem in G. Then we get~~

~~Let $\alpha \in H^*(N_G(ZJP))$ be an element not coming from G. Then we know $\exists$ $H \in S_p(G)$ such that $N_p(H)$ is an S_p-subgp of $N_G(H)$, and such that $\alpha | N_p(H)$ doesn't extend to $N_G(H)$. Pick such an H with $|N_p(H)|$ maximal. Then:~~

~~$N_G(ZJR)$~~

By assumption there exists elements $\alpha \in H^*(N_G\, ZJP)$ not coming from G. Since $N_G(P) \subset N_G(ZJP)$, I can define α_Q for each S_p-subgp Q of G so as to be compatible with conjugation. I know that $\exists\ H \in \mathcal{S}_p(G)$ such that $\forall Q \ni N_Q(H)$ is S_p in $N_G(H)$ one has that $\alpha_Q | N_Q(H)$ does not extend to $N_G(H)$. Choose such an H with $|N_G(H)|_p$ maximal. Then form:

$$\begin{tikzcd}
N_Q(ZJR) \arrow[rr, dash] & & N_G(ZJR) & \\
R = N_Q(H) \arrow[u, dash] \arrow[r, dash] & N_N(ZJR) \arrow[ur, dash] \arrow[r, dash] & & N = N_G(H) \arrow[ll, dash] \\
H \arrow[u, dash] & & &
\end{tikzcd}$$

Assume $N = N_G(H) < G$. Then by induction

$$(*) \qquad H^*(N) = H^*(N_N(ZJR))$$

so $R = Q$ is impossible. Thus $R < Q$, so ZJR is a p-subgroup whose normalizer $N_G(ZJR)$ has larger p-share. I can assume Q chosen so that $N_Q(ZJR)$ is S_p in $N_G(ZJR)$. Then I know $\alpha_Q | N_Q(ZJR)$ extends to $N_G(ZJR)$, so $\alpha_Q | R$ extends to $N_N(ZJR)$, hence to $N_G(H)$ by $(*)$, which is a contradiction. Conclude that $N_G(H) = G$.

Thus I've shown that $O_p(G) > 1$. Therefore the minimal counterexample to the theorem has $O_p(G) > 1$. Also $O_p(G) < P$ (otherwise $N_G(ZJP) = G$).

Put $H = O_p(G)$ and $\bar{G} = G/H$, $\bar{P} = P/H$. By induction

$$H^*(\bar{G}) = H^*(N_{\bar{G}}(ZJ\bar{P})).$$

Let $N_1/H = N_{\bar{G}}(ZJ\bar{P})$, $P_1/H = ZJ\bar{P}$. Then $N_1 = N_G(P_1)$. Since $H < P$, $\bar{P} > 1$, so $P_1 > H$ and $N_1 < G$. Also P_1/H char $\bar{P} \Rightarrow P_1$ char in $P \Rightarrow P < N_1$. Thus by induction $H^*(N_1) = H^*(N_{N_1}(ZJP))$.

$$H^*(G) \Longleftarrow H^*(\bar{G}, H^*(H))$$

$$H^*(N_1) \Longleftarrow H^*(N_1/H, H^*(H))$$

Thus my induction isn't strong enough. So let's work with H^1:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\bar{G}) \arrow[r] \arrow[d, "\wr"] & H^1(G) \arrow[r] \arrow[d] & H^1(H)^{\bar{G}} \arrow[r] \arrow[d] & H^2(\bar{G}) \arrow[r] \arrow[d, "\wr"] & H^2(G) \arrow[d] \\
0 \arrow[r] & H^1(\bar{N}) \arrow[r] & H^1(N_1) \arrow[r] & H^1(H)^{\bar{N}} \arrow[r] & H^2(\bar{N}) \arrow[r] & H^2(N_1)
\end{tikzcd}$$

?

Let examine the fusion problem abstractly. Let F be a contravariant functor on finite groups such as cohomology modulo p. Assume that it satisfies

$$F(G) = \varprojlim_{H} F(H)$$

where H runs over the category of finite p-subgroups with arrows $g : H \to H'$ to mean $gHg^{-1} \subset H'$. Note that this forces $F(G) \hookrightarrow F(P)$, so that $F(G) = F(P)$ if G has a normal p-complement. ~~Assume that~~

May 22, 1976:

Suppose X is a G-space $\ni$ X^H is p-acyclic for each H in $\mathcal{S}_p(G)$. Then I know that $\hat{H}^*_G \xrightarrow{\sim} \hat{H}^*_G(X)$. In effect, it suffices to restrict to P, but then $\bigcup_{1<H\leq P} X^H$ is p-acyclic, and P acts freely on the complement. In order to conclude X is p-hog to $\mathcal{S}_p(G)$ I need to know that $\mathcal{S}_p(G_x)$ is p-contractible for each $x \in X$.

So even when $O_p(G) > 1$, it might be possible to find a G-space X which reduces Tate cohomology

NO: Suppose $G = C \rtimes H$ with H a p-group > 1 and C cyclic. Then X^H p-acyclic $\Longrightarrow$ $\chi(X^G) = 1$ by Lefschetz so $X^G \neq \phi$.

Suppose G acts faithfully on the $\mathbb{F}_q$-vector space V. I believe I showed that $\mathcal{S}_p(G)$ is heq to some subset of $\mathrm{Tits}(V)$ invariant under G. To each H in $\mathcal{S}_p(G)$ I let $X_H \subset \mathrm{Tits}(V)$ be the open subset consisting of flags σ strictly stabilized by H, i.e. σ is H-invariant and H acts trivially on $gr(\sigma)$. One has $X_H \subset X_{H'}$ if $H' \subset H$, and X_H is contractible as $H > 1$. So I get

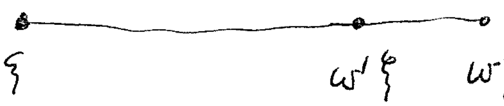
$$\begin{tikzcd} {} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod\limits_{H_0 \subset H_1} X^{H_1} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, "\text{heq}"'] & \coprod\limits_{H_0} X^{H_0} \arrow[r] \arrow[d] & \mathrm{Tits}(V) \\ & & \text{nerve of } \mathcal{S}_p(H) & \end{tikzcd}$$

Finally, if $\sigma \in \bigcup X^H$, then G_σ $= G \cap B_\sigma$ contains a non-trivial p-subgroup H with $H \subset G \cap B_\sigma^u$. In fact if σ is fixed one has $\sigma \in X^H \Leftrightarrow H \subset G \cap B_\sigma^u$. Since $G \cap B_\sigma^u$ is a p-group, it follows that $\{H \mid \sigma \in X^H\}$ is either empty or contractible. Thus $\bigcup X^H \sim \mathcal{S}_p(H)$.

In $GL_n(\mathbb{F}_q)$, let P, Q be two ~~Borel~~ S_p-subgroups. Suppose P, Q normalized by a torus T and that $B = T \ltimes P$ is the fundamental chambre. Than $P \cap Q$ is the unipotent group with the roots α such that $\alpha > 0$ and $w(\alpha) > 0$ where $Q = P^w$. These are the roots one does not cross in going from the fundamental chambre to the chambre belonging to Q. Now suppose

~~we write $w = w's\xi$ where $l(w') = l(w) - 1$. Because $s\xi$ is closer to $w\xi$ we have $P^s \cap P^w = (P \cap P^{w'})^s$~~

we factor: $w = w's$ with $l(w') = l(w) - 1$.



Then any root hyperplane not separating $\xi, w\xi$ doesn't separate $w, w'\xi$, and there is one hyperplane separating $\xi, w\xi$ which doesn't separate $\xi, w'\xi$. Thus

$$P \cap P^{w'} > P \cap P^{w}$$

One the other hand $P \cap P^w \leq P^{w'} \cap P^w$ with equality iff no roots separate ξ and $w'\xi$ i.e. $w' = e$. Thus $P, P^{w'}, P^w$ are in the same component of $\mathcal{S}_p(G) > P \cap P^w$ except when $l(w) = 1$

May 24, 1976 (Jeanie leaves for Spain)

Let G act on X such that $\operatorname{card}(X^g) \leq 1$ for $g \neq e$. Case 1: G acts transitively on X say $X = G/H$. Then H is a Frobenius subgroup of G. By Frobenius–Thompson etc. H has a normal complement N which is nilpotent. So for each p dividing $|G|$ but not dividing $|H|$, i.e. $p \mid (G:H)$ as $|H|, (G:H)$ are rel. prime, G has a unique S_p-subgroup.

General case: Let Y be a G-orbit of X. Then $\operatorname{card}(Y^g) \leq \operatorname{card}(X^g) \leq 1$ for $g \neq e$ so I know that for each prime p dividing $|Y|$, G has a unique S_p-subgroup. Thus I see that the primes dividing $|G|$ fall into 2 classes, namely those whose S_p groups act freely on Y and those with semi-free action. Assuming $Y \neq pt$, the primes in the second class have non-unique S_p-subgroups, yet unique ones in the first class. This ought to mean that the normal complement of any of the subgroups G_x is the same group N which is the product of the normal Sylow groups. It follows from Schur-Zassenhaus, (N solvable) that any of the groups G_x are conjugate, so X is a disjoint union of copies of the basic Frobenius G-set Y.

So it would seem that if I tried to prove Frob. + Thompson by induction on $|G|$, then for any subgroup $K < G$, I know that all the groups $K \cap gHg^{-1}$ are conjugate, and that K has a unique S_p-group for each p dividing $[G:H]$. Unfortunately, it is necessary that $K \cap gHg^{-1} > 1$ before I can make these conclusions about the S_p-subgroups.

Let θ be a fixpoint-free automorphism of prime order l of G.

<u>May 31, 1976</u>:

If K is a non-trivial p-subgroup of $GL_n(\mathbb{F}_q) = \mathrm{Aut}(V)$, then I looked at the subset F_K of x in $X = \mathrm{Tits}(V)$ such that $K \subset B_x^u$. This maybe is the Interior of X^K? Clearly F_K is open and contained in X^K. On the other hand, let $x \in \mathrm{Int}\, X^K$. This means that each chambre containing x is contained in X^K, i.e. each Borel B contained in B_x contains K. So $B \subset B_x \Rightarrow K < B^u$. But the intersection of the B^u as B runs over the Borels in

B_x has to be B_x^u, so $K \subset B_x^u$, i.e. $x \in F_K$.

June 13, 1976: $H^*(G) \xrightarrow{\sim} H^*(N_G(ZJP))$?

I have seen that P abelian $\Rightarrow H^*(G) = H^*(N_G P)$ for any G. If G is p-solvable and $O_{p'}(G) = 1$, then I know that $G/H \hookrightarrow \mathrm{Aut}(\Phi(H))$ where $H = O_p(G)$. (Precisely: Let $K = \mathrm{Ker}\{G \to \mathrm{Aut}(\Phi(H))\}$, whence $O_p(K) = H$ and $O_{p'}(K) = 1$. One shows $O_{p,p'}(K) = H$ as follows: $O_{p,p'}(K) = H \rtimes R$ by Shur-Zass, R acts trivially on $\Phi(H) \Rightarrow R \subset C_G(H) \Rightarrow O_{p,p'}(K) = H \times R \Rightarrow R = O_{p'}(O_{p,p'}(K)) \Rightarrow R \subset O_{p'}(G) = 1$. Thus K p-solvable $\Rightarrow K = H$.) It follows that G p-solv $+\ O_{p'}(G) = 1 \Rightarrow C_G(O_p(G)) \subset O_p(G)$, hence if P is abelian that $O_p(G) = P$. Thus $P \triangleleft G$.

Prop. P abelian, G p-solvable $\Rightarrow G = O_{p'}(G)\, N_G(P)$.

(Proof: $\bar{G} = G/O_{p'}(G)$ has a unique S_p-group $\bar{P} = O_{p'}(G)P/O_{p'}$, so if $g \in G$, then gPg^{-1} and P are both S_p-subgps of $O_{p'}(G)P$, hence $\exists\ h \in O_{p'}(G)$, $y \in P \ni y^{-1}h^{-1}g \in N_G(P) \Rightarrow g \in O_{p'}(G)\, N_G(P)$.)

Summary:

$$P \text{ abelian} \Rightarrow \begin{cases} H^*(G) \cong H^*(N_G(P)) & \text{in general} \\ G \text{ } p\text{-solv} \Rightarrow G = O_{p'}(G)\, N_G(P) \end{cases}$$

Glauberman: G strongly p-solv (p odd) $\Rightarrow G = O_{p'}(G)\, N_G(ZJP)$

Question: Is $H^*(G) = H^*(N_G(ZJP))$ in general?

[redacted] Let H be a p-subgroup of G.

$$\begin{tikzcd} N_G(H)/C_G(H) \arrow[r, hook] \arrow[d] & \mathrm{Aut}(H) \arrow[d, "\rho"] \\ N_G(H)/HC_G(H) \arrow[r, "\rho_1"] & \mathrm{Aut}(H/\Phi(H)) \end{tikzcd}$$

We know $\mathrm{Ker}(\rho)$ is a p-group, hence if $\mathrm{Ker}\,\rho_1 = K/HC_G(H)$, then $K/HC_G(H)$ is a p-group. I also know that if H is critical for H^*-fusion, then $HC_G(H)/H = C_G(H)/Z(H)$ is a p'-group, so

$$C_G(H) = Z(H) \times O_{p'}(C_G(H))$$
$$HC_G(H) = H \times O_{p'}(C_G(H)).$$

An important fact might be that the possible restrictions $\alpha_P|H$ depend on the orbit of $[P] \in \pi_0\, \mathcal{S}_p(N_G(H)/H)$ modulo the action of $C_G(H)$.

[redacted] We assume that H is such that $\alpha_P|^{P\cap Q} = \alpha_Q|^{P\cap Q}$ if $P \cap Q > H$, P, Q are S_p-groups. Then $\alpha_P|H$ depends only on the component of $N_P(H)/H$ in $\mathcal{S}_p(N_G(H)/H)$. If $g \in N_G(H)$, then $g(\alpha_P|H) = \alpha_{gPg^{-1}}|H$, so $\alpha_P|H = \alpha_Q|H$ when P, Q are conjugate via an element of [redacted] $C_G(H)$.

~~Observe that if $G \to G/N$ is surjective and N does not contain a S_p subgroup of G, then one has a surjection~~

~~$\pi_0\, \mathcal{S}_p(G) \to \pi_0\, \mathcal{S}_p(G/N)$.~~

Suppose that R is a normal subgroup of $N_G(H)$ containing H which acts trivially on the subspace of $H^*(H)$ generated by the restrictions $\alpha_P|H$ as P ranges over the S_p-subgroups containing H. Assume R/H is a p'-group, so that we have a surjective map $\mathcal{S}_p(N_G(H)/H) \to \mathcal{S}_p(N_G(H)/R)$ sending K/H to KR/R. Assume $\alpha_P|P\cap Q = \alpha_Q|P\cap Q$ where P, Q are S_p-groups $\ni$ $P \cap Q > H$, so that for each p-subgroup $K > H$ we have a well-defined class α_K compatible with conjugation and restriction. Let T/R be a non-trivial p-subgroup of $N_G(H)/R$. Then $T/H = K/H \ltimes R/H$ where K/H is unique up to conjugacy in T/H, hence unique up to conjugacy by an element of R. It follows that $\alpha_K|H$ should depend only on T/H. More precisely given T/R one picks a Sylow grp. K in T. Then $T = K\cdot R = RK$ and K is unique up to conjugacy by an element of R. Thus $\alpha_K|H$ should not depend on the choice of K, but only up $T/R \in \mathcal{S}_p(N_G(H)/R)$. Next if $T/R \subset T'/R$, then if K is S_p in T we can choose K' S_p in T' so that $K \subset K'$. Then $\alpha_K|H = \alpha_{K'}|H$ showing that $T/H \mapsto \alpha_K|H$ is a well-defined function on $\pi_0\,\mathcal{S}_p(N_G(H)/R)$

If R/H is not a p' group, then we know that $\pi_0(\mathcal{S}_p(R/H)) \longrightarrow \pi_0(\mathcal{S}_p(N_G(H)/H))$ is surjective and hence the $\alpha_P|H$ have to coincide.

Therefore given an H^*-fusion problem

and a bad subgrp H for this problem, I can take $R = HC_G(H)$ and I can conclude i) $HC_G(H)/H$ is a p'-group. ii) $\mathcal{S}_p(N_G(H)/HC_G(H))$ is disconnected, hence $N_G(H)/HC_G(H)$ has no non-id normal [redacted] p-subgroup. Thus $N_G(H)/HC_G(H) \hookrightarrow \mathrm{Aut}(H/\Phi(H))$

[redacted] Addition to p. 106. If H is a bad [redacted] subgroup for an H^*-fusion problem, then

$$H = O_p(N_G(H))$$

$$HC_G(H) = H \times O_{p'}(C_G(H))$$

$$N_G(H)/HC_G(H) \hookrightarrow \mathrm{Aut}(H/\Phi(H))$$

$\mathcal{S}_p(N_G(H)/HC_G(H))$ disconnected, so $O_p(N_G(H)/HC_G(H)) = 1$

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Suppose G p-solvable, $O_p(G) = 1$, and $\mathcal{S}_p(G)$ is disconnected. Forget the case $O_{p'}(G) = G$, so that $O_{p',p}(G) > O_{p'}(G)$. Let [redacted] P be an S_p-subgroup of G. Then $O_{p',p}(G) \cap P$ is an S_p-subgroup of $O_{p',p}(G)$. So it's clear that the inclusion $\mathcal{S}_p(O_{p',p}(G)) \subset \mathcal{S}_p(G)$ induces a surjection on π_0.

$$\begin{tikzcd} \pi_0(\mathcal{S}_p(O_{p',p}(G))) \arrow[r] \arrow[rr, bend left, two heads] & \pi_0(\mathcal{S}_p(O_{p'}(G)P)) \arrow[r] & \pi_0(\mathcal{S}_p(G)) \end{tikzcd}$$

So I see that $\mathcal{S}_p(O_{p'}(G)P)$ is disconnected. [redacted] I want to show P is cyclic or gen. quaternion, so I can assume