April 7, 1976

G finite group, p prime number

Say $G \in \mathcal{G}_p^1$ if G has a normal p-subgroup P such that G/P is cyclic. $\mathcal{G}_p^1$ closed under subgroups.

If $G \in \mathcal{G}_p^1$ and X is an $\mathbb{F}_p$-acyclic G-space, then X^P is $\mathbb{F}_p$-acyclic (Smith), and so

$$\chi(X^G) = \chi\left((X^P)^{G/P}\right) = 1 \qquad \text{(Lefschetz)}.$$

Stratification of a G-space X: Let J_G be the set of isomorphism classes of transitive G-sets; same as conjugacy classes of subgroups. Partially order J_G by saying $X \leq Y$ iff $\exists$ G-map $Y \to X$; equivalently $G/H \leq G/K$ iff K conjugate to a subgroup of H. Reason for this ordering is that a general G-space X will be built up starting from X^G and ending with free orbits. ~~[illegible]~~

[redacted]

A [redacted] "family" of subgroups $\mathcal{F}$ in G is the same as an open subset of J_G (corresponds to an open subset of X). A "cofamily" (subgroups closed under enlarging) corresponds to [redacted] a family of supports, i.e. a closed set of X).

Suppose Y is a G-space, and let's consider the problem of embedding Y in an $\mathbb{F}_p$-acyclic acyclic G-space without changing the fixpoint set. (Thus $Y \subset CY$ is out).

If Y is $\mathbb{F}_p$-acyclic, nothing to do. If Y not $\mathbb{F}_p$-acyclic, we might try to attach orbits of type G to Y, to get an $\mathbb{F}_p$-acyclic X. If so, then G acts freely on $X-Y$, so $X^H = Y^H$ for $1 < H \leq G$. It follows that Y^H has to be $\mathbb{F}_p$-acyclic for H a p-group, and that $\chi(Y^H) = 1$ if $H \in \mathcal{G}_p'$.

We consider this special case: for all $1 < H \leq G$ if H is a p-group, then Y^H is $\mathbb{F}_p$-acyclic, and if $H \in \mathcal{G}_p'$ then $\chi(Y^H) = 1$. By attaching free G-orbits to Y we can obtain a G-space Y_1 which is an $(n-1)$-connected n-complex. Claim $\tilde{H}_n(Y_1)$ ($\mathbb{F}_p$-coeffs.) is a projective $\mathbb{F}_p[G]$-module. Pf: Let G_p be a Sylow p-subgroup of G. If $1 < H \leq G_p$, then $Y_1^H = Y^H$ is $\mathbb{F}_p$-acyclic by hypothesis, so

$$\bigcup_{1 < H \leq G_p} Y_1^H \quad \text{is } \mathbb{F}_p\text{-acyclic}$$

so $\tilde{H}_n(Y_1) = \tilde{H}_n(Y_1 / \bigcup Y_1^H)$. If $Y_2 = Y_1 / \bigcup Y_1^H$, then one has Y_2 is a $(n-1)$-connected n-complex, so $\exists$ exact sequence

$$0 \to \tilde{H}_n(Y_2) \longrightarrow C_n(Y_2, *) \longrightarrow \cdots\cdots \to C_0(Y_2, *) \longrightarrow 0,$$

and G_p acts freely on $Y_2 - \{*\}$ $\Rightarrow$ $C_i(Y_2, *)$ are $\mathbb{F}_p[G_p]$-free $\Rightarrow$ $\tilde{H}_n(Y_2)$ is $\mathbb{F}_p[G_p]$-projective $\Rightarrow$ $\tilde{H}_n(Y_1)$ is $\mathbb{F}_p[G]$-proj.

Claim $\tilde{H}_n(Y_1)$ is a free $\mathbb{F}_p[G]$-module. We will use the known fact that $K_0(\mathbb{F}_p[G])$ embeds in $R_{\mathbb{F}_p}(G)$ which in turn embeds in complex central functions on G via the Brauer character. Let $g \in G$ be of order prime to p. Since $\tilde{H}_n(Y_1)$ lifts to $\tilde{H}_n(Y_1, \mathbb{Z})$ which is free over $\mathbb{Z}$, the Brauer character of $\tilde{H}_n(Y_1)$ evaluated on g is the trace of g on $\tilde{H}_n(Y_1, \mathbb{Z})$ which by Lefschetz and fact $Y_1 \sim \vee S^n$ is $\pm\{\chi(Y^g) - 1\}$:

$$\chi(Y^g) = 1 - (-1)^n \, tr_g \text{ on } \tilde{H}_n(Y, \mathbb{Z})$$

By hypothesis $\chi(Y^g) = 1$ if $g \neq e$. Thus $\tilde{H}_n(Y_1)$ and $\mathbb{F}_p[G]$ have proportional Brauer characters.

~~It seems necessary now to assume in addition that~~ $\cancel{\chi(Y) \equiv 1 \pmod{|G|}}$~~; this is necessary that~~ $\cancel{\exists\, X}$ ~~$\mathbb{F}_p$-acyclic with~~ $\cancel{X - Y}$ ~~G-free. If I assume this then $|G|$ divides the rank of~~ $\cancel{\tilde{H}_n(Y_1)}$~~, so I then know that~~ $\cancel{\tilde{H}_n(Y_1)}$ ~~is~~ $\cancel{\mathbb{F}_p[G]}$~~-free.~~

To finish I need to know the dimension of $\tilde{H}_n(Y_1)$ is divisible by $|G|$. We know it is divisible by $|G_p|$, since it is free over $\mathbb{F}_p[G_p]$. If q is a prime $\neq p$, then the character of $\tilde{H}_n(Y_1, \mathbb{Z})$ as a rep of G_q vanishes at all elements $\neq e$, hence $\tilde{H}_n(Y_1, \mathbb{Z}) \otimes \mathbb{Q}$ is proportional to $\mathbb{Q}[G_q]$, which means $\tilde{H}_n(Y_1,\mathbb{Z})\otimes\mathbb{Q}$ is free over $\mathbb{Q}[G_q]$. (The point is the trivial repn. occurs only once in the regular repn.) So $|G_q|$ divides $\dim \tilde{H}_n(Y_1)$, for all q, so we win. Therefore the fact involved is:

<u>Assertion</u>: If M is a projective $\mathbb{F}_p[G]$-module whose Brauer character vanishes at all p'-elements not the identity, then M is free.

<u>Proof</u>: M free over $\mathbb{F}_p[G_p] \Longrightarrow |G_p|$ divides $\dim M$. Over G_q, M has the same character as a multiple of $\mathbb{F}_p[G_q]$, hence M is free over $\mathbb{F}_p[G_q]$ because the trivial repn. occurs only once in $\mathbb{F}_p[G_q]$. Thus $|G_q|$ divides $\dim M$. $\therefore$ $|G|$ divides $\dim M$ so M has the same character as an integral multiple of $\mathbb{F}_p[G]$, so M is free over $\mathbb{F}_p[G]$. <u>QED.</u>

Summarizing we have proved:

<u>Proposition</u>: Y a G-space such that for all $1<H\leq G$ one has

a) $H \in \mathcal{P}_p$ (p-groups) $\Longrightarrow Y^H$ $\mathbb{F}_p$-acyclic

b) $H\in \mathcal{P}'_p \Longrightarrow \chi(Y^H)=1$.

Then $\exists\ Y\subset X$ with X $\mathbb{F}_p$-acyclic and $X-Y$ G-free.

Next suppose we have a G space Y which we want to embed in an acyclic G-space, without changing fixpoint set. There is no problem if a) b) hold for all $1 < H \le G$. So let us assume this is not true and let H be maximal such that either a) or b) fails. We want then to attach G/H orbits to Y, so as to remedy the situation.

Suppose H is a p-group. Let $N = N_G(H)$. Consider Y^H as a N/H-space. Then for all $1 < H'/H \le N/H$ we have

$$(Y^H)^{H'/H} = Y^{H'}$$

so a) and b) hold for Y^H as an N/H-space. Then by the prop. I get an $\mathbb{F}_p$-acyclic N/H-space Z containing Y^H such that $Z - Y^H$ is N/H-free. Put

$$Y_1 = (G \times^N Z) \cup_{G \times^N Y^H} Y$$

Then $Y_1 - Y = G \times^N (Z - Y^H)$ consists of G/H-orbits, and $Y_1^H = Z$ is $\mathbb{F}_p$-acyclic.

Next suppose $H \in \mathcal{G}_p^1$ but that H is not a p-group. Here I want to embed

Y^H in $Z \ni Z - Y^H$ is N/H-free such that $\chi(Z) = 1$. Clearly necessary & sufficient that $\chi(Y^H) \equiv 1 \bmod |N/H|$.

I get stuck at this point, so it is necessary to introduce some extra condition. The point is that $\chi(X^H) = 1 \Longrightarrow \chi(\bigcup_{H<K\le N} X^K) \equiv 1 \bmod (N/H)$, which is a condition involving subgroups larger than H. Oliver's method to get around this point is to suppose given an element $\varphi = [V]$ in $A(G)$ satisfying the Euler conditions:

$$\chi(V^H) = 1 \qquad H \in \mathcal{G}_p' \qquad \forall\ 1 \le H \le G$$

Next one wants to construct an $\mathbb{F}_p$ acyclic X with $[X] = [V]$, so one wants the conditions for each $1 \le H \le G$

α) $H \in \mathcal{G}_p \Longrightarrow X^H$ $\mathbb{F}_p$ acyclic

β) $\chi(X^H) = \chi(V^H)$.

Suppose Y is a G-space, and let H be a maximal subgroup not satisfying both α) and β). If $H \in \mathcal{G}_p$, then we can apply the proposition to the (N/H)-space Y^H to get a Y_1 satisfying α) and β) for H and for those subgroups preceding H. If $H \notin \mathcal{G}_p$, then we want to attach orbits of type N/H to

Y^H to get a Z with $\chi(Z) = \chi(V^H)$. This is possible iff $\chi(Y^H) \equiv \chi(V^H) \mod |N/H|$. But

$$\chi(V^H) \equiv \chi\Big(\bigcup_{H<K\le N} V^K\Big) = \chi\Big(\bigcup_{H<K\le N} Y^K\Big) \equiv \chi(Y^H)$$

where we use the induction. So it marches. ∴

<u>Theorem</u>: Let $[V] \in A(G)$ satisfy $\chi(V^H) = 1$ for all $1 \le H \le G$, $H \in \mathcal{G}_p^1$. Let Y be a G-space and $\mathcal{F}$ a family of subgroups such that the conditions

α) $H \in \mathcal{G}_p \Longrightarrow Y^H$ $\mathbb{F}_p$-acyclic.

β) $\chi(Y^H) = \chi(V^H)$

hold for all $H \notin \mathcal{F}$. Then $\exists$ embedding $Y \subset X$ with

i) X $\mathbb{F}_p$-acyclic

ii) $[X] = [V]$ in $A(G)$

iii) isotropy groups of $X - Y$ are in $\mathcal{F}$.

Application: Take $\mathcal{F}$ to be the family of all H, $1 \le H < G$.

Let $J_p \subset A(G)$ be the ideal of $[V] \ni \chi(V^H) = 1$ for $H \in \mathcal{G}_p^1$.

Then we have $J_p \xrightarrow{\chi_G} \mathbb{Z}$, $[V] \longmapsto \chi(V^G)$. Let Y be a complex $\ni$ $\chi(Y) \in \chi_G(J_p)$, better, such that $\exists [V] = -1$ in J_p $\ni$ $\chi(Y) = \chi(V^G)$. If G is not a p-group, then conditions α), β) hold for all $H \notin \mathcal{F}$, so we get an $\mathbb{F}_p$-acyclic space X, with $[X] = [V]$ in $A(G)$, such that $X^G = Y$. In general $\chi_G(J_p) = m_p(G)\mathbb{Z}$, where $m_p(G)$ in principle can be determined by doing some algebra in $A(G)$.

April 9, 1976 More Oliver.

Let X be a G-complex such that $\forall\ 1 < H \leq G$, X^H is contractible [redacted] or [redacted] empty, and X^H is contractible for each non-zero p-subgroup for each p dividing $|G|$. Claim one can add free orbits to X to make it contractible.

We can make X [redacted] an $(n-1)$-connected n complex by attaching free G-orbits. We need to know that $\tilde{H}_n(X)$ is stably $\mathbb{Z}[G]$-free, and we know it is projective because X^H is contractible for each non-zero p-subgroup H. A theorem of Swan tells us that $\tilde{H}_n(X) \otimes \mathbb{F}_p$ is $\mathbb{F}_p[G]$-free (same theorem used by Brown: $\tilde{H}_n(X) \otimes \mathbb{Q}$ is $\mathbb{Q}[G]$-free + Brauer char. theory). Hence we can attach free G-orbits to X to get an [redacted] $\mathbb{F}_p$-acyclic X_p [redacted] containing X such that $X_p^H = X^H$ for $1 < H \leq G$.

X_p $\mathbb{F}_p$-acyclic $\Rightarrow$ X_p acyclic except at a finite set of primes. Recall that the reduced homology of the join $A * B$ is [redacted] $\tilde{H}_*(A) \otimes \tilde{H}_*(B)$ shifted [redacted] up one degree:

$$0 \longrightarrow \tilde{H}_q(A*B) \xrightarrow{\partial} \tilde{H}_{q-1}(A \times B) \twoheadrightarrow \tilde{H}_{q-1}(A) \oplus \tilde{H}_{q-1}(B) \longrightarrow 0.$$

Thus for some choice of primes $Y = X_{p_1} * X_{p_2} * \cdots * X_{p_k}$ will be contractible. Then

$$Y^H = X_{p_1}^H * \cdots * X_{p_k}^H = X^H * \cdots * X^H$$

[illegible] is contractible ~~or~~ empty when X^H is for all $1<H\leq G$. Now ~~[illegible]~~ let $f: X \to Y$ be the inclusion $X \hookrightarrow X_{p_1} \hookrightarrow X_{p_1} * \cdots * X_{p_k} = Y$, and let $\tilde{Y} = \mathrm{Cone}(f: X \to Y)$. Then

i) $\tilde{Y}^H \sim pt \quad \forall\ 1<H\leq G$

ii) $\tilde{H}_*(\tilde{Y},\mathbb{Z}) = \tilde{H}_{*-1}(X,\mathbb{Z})$.

From i) we know $\bigcup_{1<H\leq G} \tilde{Y}^H \sim pt$, so $H_{n+1}(\tilde{Y},\mathbb{Z}) = \tilde{H}_n(X,\mathbb{Z})$ is $\mathbb{Z}[G]$-stably free. Thus there are no obstructions to attaching G-orbits to X to make it contractible.

Suppose now $\mathcal{F}$ is a separating family of subgroups and we want to construct a G-space X such that X^H is contractible or empty according to whether H is in $\mathcal{F}$ or not. Start with a maximal H in $\mathcal{F}$ and with $Y = G/H$. $Y^H = (G/H)^H = NH/H$ is a point because $NH = H$. (Recall that $H \triangleleft K$ and K/H solvable $\Rightarrow$ K, H both in or both outside of $\mathcal{F}$. Thus H maximal in $\mathcal{F} \Rightarrow NH = H$). ~~Suppose constructed a G-space Y with all isot gps in $\mathcal{F}$ and $\exists$ X^H [illegible] contractible or empty, let H be a maximal subgroup in $\mathcal{F}$ $\ni$ X^H not~~

Suppose given a G-space Y will all isotropy groups in $\mathcal{F}$, let H be a maximal subgroup in $\mathcal{F}$ such that Y^H is not contractible. Then for $H<K\leq NH$ we have $(Y^H)^{K/H}=Y^K$ is contractible if $K\in\mathcal{F}$, ϕ if $K\notin\mathcal{F}$, so [redacted] by the preceding stuff, we can attach free NH/H orbits to Y^H to make it contractible. [redacted] Then we have enlarged Y to X by adding G/H-orbits so that X^H is contractible without changing other orbit types. It follows that X has isot. gps in $\mathcal{F}$, that $X^K = Y^K$ unless $(G/H)^K \neq \phi$ i.e. $K \dashrightarrow H$. (Maybe the good way is to [redacted] consider the family of $H \ni$ X [redacted] ^{K} is not contractible [redacted] for some $K\geq H$, $K\in\mathcal{F}$).

Seems OKAY.

Should $\exists$ similarity between Oliver theory + Hatcher theory.

Suppose X is a G-space with solvable isotropy groups $\ni$ X^H is contractible or empty according to whether H is solvable or not. Remove from X the free orbits to obtain a G-space $Y = \bigcup_{1<H\leq G} X^H$.

Let J = poset of solvable non-trivial subgroups of G. Then $\forall H \in J$ we have a subset Y^H of Y which is contractible.

$$\begin{array}{ccc} \left\{ q \mapsto \coprod_{H_0<\cdots<H_q} Y^{H_q} \right\} & \longrightarrow & Y \\ \downarrow & & \\ \left\{ q \mapsto \coprod_{H_0<\cdots<H_q} pt \right\} & & \end{array}$$

So it seems then that Y is of the homotopy type of the poset J. However Y need not be G-homotopy equivalent to J, for there might be a non-zero solvable subgroup whose normalizer is not solvable.

Let X be a G-space such that $1 < H \leq G \Rightarrow X^H \sim pt$ or ϕ. Attaching free G-orbits to X, I can assume X is an $(n-1)$-connected n-complex. $\tilde{H}_{2n+1}(X * X) = \tilde{H}_n X \otimes \tilde{H}_n X$. If P is a projective $\mathbb{Z}[G]$-module is $P^{\otimes m}$ stably-free for some m?

$H \subset G$, X an H-space. Then we have Serre's induction process:

$$\tilde{X} = \text{sections}\,\{G \times^H X \longrightarrow G/H\}$$

Change notation:

$$X = \text{sections}\,\{G \times^{G'} X' \longrightarrow G/G'\}.$$

Let $H \subsetneq G$. What is X^H? It is a product over $H\backslash G/G'$ of some sort. $HgG'/G' \simeq H/H \cap gG'g^{-1}$.

$$X^H = \prod_{HgG'} (X')^{g^{-1}Hg \cap G'}$$

So note that this is contractible provided $(X')^{g^{-1}Hg \cap G'}$ is contractible $\forall g$. Suppose X' such that $(X')^{H'}$ contractible for all $1 \leq H' < G'$ and empty for $H' = G'$. Better, suppose $X'^{H'}$ contractible or empty for all $H' \leq G'$. Then the same is true for X.

Suppose $(X')^{H'} \sim pt$ for $1 \leq H' < G'$, yet $(X')^{G'} = \phi$. Then

$$X^H \sim pt \quad \text{if } G' \not\subset g^{-1}Hg \text{ any } g$$
$$= \phi \quad \text{if } G' \subset g^{-1}Hg \text{ for some } g.$$

Here might be another approach to Oliver's theorem once the minimal simple groups were understood. The problem is to construct G-spaces such that $\forall H$, $1 \leq H \leq G$, X^H is contractible or empty. Call these special. For each such X we get a separating family of subgroups. Separating families are the same as closed subsets in the poset of conjugacy classes of perfect subgroups. Call this poset I. Assume inductively that I can find for any G' perfect $< G$ a special G'-space X' without fixpoints such that any $1 \leq H' < G'$ has $(X')^{H'}$ contractible. Then inducing X' up to G multiplicatively gives a special G-space with $X^H = \phi$ iff G' is conjugate to a subgroup of H. So this means that for each $x \in I$ I get a special G-space associated to the complement of $\{y \geq x\}$, except for $x = [G]$.

X_1, X_2 are special $\Longrightarrow$ $X_1 \times X_2$ and $X_1 * X_2$ are special for

$$(X_1 * X_2)^H = X_1^H * X_2^H = \begin{cases} \phi & \text{if } X_1^H, X_2^H = \phi \\ \sim pt & \text{if } X_1^H \text{ or } X_2^H \sim pt. \end{cases}$$

These give us the usual operations of union + intersection for the "supports" in J, etc.

So suppose $\mathcal{U}$ is a family of open sets in J closed under $\cup$, $\cap$ and containing $\{y \geq x\}$ for all $x \neq$ largest element of J. Let $x_1, \ldots, x_r$ be the maximal elements of J not the largest. Then if $r \geq 2$ $\{y \geq x_1\} \cap \{y \geq x_2\}$ would be the largest element of J. So there is a problem if G contains a perfect subgroup $G' < G$ such that every other perfect subgroup is conjugate to a subgroup of G'. For example if G has a minimal simple quotient group G/N.

Let J_p be the poset of non-zero p-subgroups of G. Then for any non-zero p-subgroup H we have J_p^H is contractible (Brown). If g is a p'-element, I need $\chi(J_p^g) = 1$ in order to complete J_p to an $\mathbb{F}_p$ acyclic space.

Let G be a perfect group. By Oliver $\exists$ a special G-space with $X^H \sim pt$ $1 \leq H < G$ and $X^G = \phi$. Then consider the non-free part of X:

$$Y = \bigcup_{1 < H \leq G} X^H$$

This has the homotopy type of the poset J of proper subgroups of G, but not the G-homotopy type since $J^G \neq \phi$ if G not simple. If G is simple, then if

$K \in J$ is normalized by H, then $K \cdot H \subset NK \in J$ so J^H is contractible for all $1 < H < G$.

April 11, 1976:

Frobenius thm: $H \subset G$ finite $\ni$ $H \cap H^x = 1$ for $x \notin H$ $\Longrightarrow$ $N = \{e\} \cup G - \bigcup_{x \in G} H^x$ is a subgroup of G.

Such an H called a Frobenius subgroup. Let $X = G/H$. Then G acts transitively on X and card $X^g \leq 1$ for $g \neq e$. Conversely such an X is of the form G/H where H is a Frobenius group, or $H = 1$.

If K is a subgroup of G, then each orbit of K on X is of the form $K/K \cap xHx^{-1}$ where $K \cap xHx^{-1}$ is a Frobenius subgp of K, or $=1$. If K is nilpotent then the only self-normalizing subgroup of K is K itself. Thus

$$K \text{ nilpotent} \Longrightarrow K \cap xHx^{-1} = 1 \text{ or } K.$$

In particular the set N contains all subgroups of order prime to $|H|$. (Counting shows

$$|N| = 1 + |G| - (G:H)(|H| - 1) - 1 = (G:H)$$

so we know $|N|$, $|H|$ are rel. prime divisors of $|G|$.)

Let $\mathcal{N}$ be the poset of subgroups contained in the set N. Consider the poset of cosets of $\mathcal{N}$, which one might denote $G/\mathcal{N}$. Since $\mathcal{N}$ is closed under intersections we have

$$\begin{array}{ccccc} G/\mathcal{N} & \longrightarrow & B\mathcal{N} & \longrightarrow & BG \\ & & \| & & \\ & & \bigcup_{K\in\mathcal{N}} BK & & \end{array}$$

Assuming Frobenius' theorem, $\mathcal{N}$ has a greatest element, so $G/\mathcal{N} \sim G/N$. Maybe you can directly show that H acts simply-transitively on $\pi_0(G/\mathcal{N})$. In any case you have succeeded in geometrically constructing the right representation of G, assuming Frob. thm.

Calculate the character of the representation of G on $H_*(G/\mathcal{N}, \mathbb{Q})$. If $h \in H$, and

$$h\,(gK_0 \subset \cdots \subset gK_m) = (gK_0 \subset \cdots \subset gK_m)$$

then $hgK_0 = gK_0 \Longrightarrow g^{-1}hg \in K_0 \subset N \Longrightarrow h = e$. Thus the character vanishes on $G-N$, because there are no fixpts. If on the other hand, $K \in \mathcal{N}$, then $(G/\mathcal{N})^K$ is the poset consisting of cosets gK_0 such that $KgK_0 = gK_0$.

?

Feit's calculation: If χ is an irred. char. of H of degree m non-trivial, then because H is a Frobenius group $(\chi - m 1_H)^G$ has the same norm as $\chi - m 1_H$, namely $1 + m^2$. $(\chi - m 1_H)^G = \chi^G - m(1_H)^G$ and χ^G doesn't contain 1_G (as $\chi \neq 1_H$), and $(1_H)^G$ contains 1_G once. Thus

$$(\chi - m 1_H)^G = \sum a_i \chi_i - m 1_G$$

χ_i irred repns of $G \neq 1_G$, $a_i \in \mathbb{Z}$, $a_i \geq 0$. So

$$\|(\chi - m 1_H)^G\| = \sum a_i^2 + m^2 = 1 + m^2$$

$\Rightarrow$ exactly one $a_i = 1$. $\therefore$ $(\chi - m 1_H)^G = \chi_i - m 1_G$ and so each non-trivial irred. repn of H comes from G.

This shows that

$$(\mathbb{Z}[H] - |H| \cdot \mathbb{Z})^G = \mathbb{Z}[G] - |H| \cdot \mathbb{Z}[G/H]$$

is isomorphic in $R(G)$ to $\mathbb{Z}[G/N] - |H| \cdot \mathbb{Z}$, i.e.

$$\mathbb{Z}[G] = |H| \cdot \mathbb{Z}[G/H] \oplus \mathbb{Z}[G/N]$$

which one can test also by characters.

H acts freely on $G/\mathcal{N}$ $(hgK = gK \Rightarrow ghg^{-1} \in K \Rightarrow h = e)$, so consider $H\backslash G/\mathcal{N}$. I can describe this as the poset $X/\mathcal{N}$ formed out of the orbits of the subgroups of $\mathcal{N}$ on $X = G/H$. It would be nice to show $X/\mathcal{N}$ is contractible. Why connected. I have to show that any two points are connected by a chain:

$$x,\ n_1 x,\ n_2 n_1 x, \ldots, n_k \cdots n_2 n_1 x$$

which $n_i \in N$. So one considers the components of X defined in this way. Because $\mathcal{N}$ is closed under conjugation, the components are permuted under G. Let us fix $x_0 = eH$ and let S be the subgp of G normalizing the component containing x_0. Then S contains H and all subgroups in $\mathcal{N}$, so S must be all of G (it contains a Sylow subgp for each prime dividing $|G|$). One can assume that G is generated by N.

April 15, 1976

G finite group, H subgroup of G.
H is called a Frob subgroup if $H \cap H^x = 1$ for $x \notin H$.

Alternative interp. Put $X = G/H$. Then $H^x = xHx^{-1}$ is the stabilizer of xH, so H is a Frobenius group $\Leftrightarrow$ card [redacted] $(X^g) \leq 1$ for all $g \in G$. This [redacted] condition persists to subgroups K of G. Thus $K \cap H^x$ is a Frobenius subgp. in K for any x in G.

Note that if H is Frob. in G, and $H \neq 1$, then H is its own normalizer, for $\exists\, h \neq e$ $h \in H$ so $x \notin H \Rightarrow xhx^{-1} \in H^x$ so $xhx^{-1} \notin H$ otherwise $xhx^{-1} = e$, which is impossible. Thus K nilpotent in G $\Rightarrow K \cap H^x = K$ or 1, since [redacted] $H' < K \Rightarrow H'$ not its own normalizer.

In particular any [redacted] p-subgroup P [redacted] is contained in some H^x or else intersects each H^x in 1 which means it acts freely on X. Let N be the subset of G consisting of the identity and elements without fixpoints on X. We know

[redacted]

$$|G| = |N| + (G:H)(|H|-1)$$

or $$|N| = (G:H)$$

and we have seen that $|N|$, $|H|$ are relatively prime factors of $|G|$. (This is because any Sylow p-subgroup of G where p divides $|H|$ must intersect H^x non-trivially for some x, hence must be contained in this H^x).

Frobenius' theorem says N is a subgroup, and Thompson's theorem says N is nilpotent. I want to really understand these theorems.

~~Let χ be a character of degree 1 of~~

If p divides $|H|$, then $(G:H) \not\equiv 0 \ (p)$ so

$$\text{res}: \ H^*(G, \mathbb{F}_p) \longrightarrow H^*(H; \mathbb{F}_p)$$

is injective by transfer. But more is true because

$$H^*(G, \mathbb{F}_p) \longrightarrow H^*(H; \mathbb{F}_p) \rightrightarrows H^*_G((G/H)^2, \mathbb{F}_p)$$

is exact and the G action on $X \times X$ is free off the diagonal. Thus one sees that

$$H^*(G, \mathbb{F}_p) \xrightarrow{\sim} H^*(H, \mathbb{F}_p)$$

Specifically this works as follows. Given $\alpha \in H^+(H, \mathbb{F}_p)$ induce α up to G. Then by Mackey formula

$$\text{Res}_{H \to G} \text{Ind}_{H \to G}(\alpha) = \alpha + \sum_{\substack{HxH \\ x \notin H}} \text{Ind}_{1 \to H} \text{Res}_{1 \to H} \alpha$$

$$= \alpha$$

Let $u: H \longrightarrow A$ be a homomorphism with A abelian. Then we can induce to G ~~to~~ get a homomorphism $G \longrightarrow A$. Suppose u is a char. $\chi: H \longrightarrow \mathbb{C}^*$. Then $\mathrm{Ind}_{H \to G} \chi$ is a $(G:H)$-dimensional repn. of G. Take its determinant and you get $\chi' : G \longrightarrow \mathbb{C}^*$, which restricts to χ on H. Why does χ' vanish on N? Because any subgroup K of N acts freely on X. Hence $\mathrm{Res}_{K \to G} \mathrm{Ind}_{H \to G} \chi$ is $\mathrm{card}(K \backslash G / H)$ copies of the reg. rep of K.

?

What goes wrong is that the determinant of the regular repn. can be a non-trivial character of a group. Thus it appears that det of the induced repn. is not the induction we seek.

(When is det of regular repn. non-trivial? Fix g. Then g is a cyclic permutation on $\langle g \rangle$ so we get

$$\det(g) = \left(\det g \text{ on } \langle g \rangle\right)^{[G:\langle g \rangle]}$$

$$= \begin{pmatrix} +1 & \text{order } g \text{ odd} \\ -1 & \text{order } g \text{ even} \end{pmatrix}^{[G:\langle g \rangle]}$$

Thus the regular repn. has a non-trivial determinant iff the Sylow 2 subgroup is cyclic of even order.)

So in any case one sees that for $u : H \to A$ abelian, the induction of u: $\tilde{u} : G \to A$ restricts to u and is trivial on every subgroup of N.

Thus we see easily that if H is solvable, then N has to be a group. Use induction on the length of the derived series for H.

Frobenius method of proof. Start with an irred. char $\chi \neq 1_H$ of H of degree m. Then

$$(\chi - m 1_H)^G = \chi^G - m\,\mathbb{C}[G/H]$$

has the same norm as $\chi - m 1_H$ because H is Frob. in G.

$$\|\chi - m 1_H\|^2 = 1 + m^2$$

since χ^G doesn't contain 1_G, and $\mathbb{C}[G/H]$ contains 1_G once we have

$$\chi^G - m\,\mathbb{C}[G/H] = \sum_{\chi_i \neq 1} a_i \chi_i - m 1_G \qquad a_i \in \mathbb{Z}$$

$$\|\quad '' \quad\|^2 = \sum a_i^2 + m^2.$$

$\therefore \sum a_i^2 = 1$ so $\chi^G - m\,\mathbb{C}[G/H] = \chi_i - m 1_G$ where $m = \deg(\chi_i)$. χ_i stands for a hom. $G \to GL_m\mathbb{C}$. It remains to see that this homomorphism kills N. But $\chi_i - m 1_G = 0$ on N, hence $\chi_i = m$ on N.

Now one uses the fact that the value of χ_i is a sum of m roots of unity. Using complex absolute values this can happen only if all roots are $= 1$, whence N has to be killed by χ_i.

Review representations & characters for a finite group G. The group ring $\mathbb{C}[G]$ can be identified with functions on G

$$f \longleftrightarrow \sum f(g)\,g \qquad \left(\text{maybe } f \mapsto \frac{1}{|G|}\sum f(g)\,g \text{ might be better}\right)$$

Then
$$\sum f_1(g)g \sum f_2(g)g = \sum f_1(x)f_2(y)xy$$
$$= \sum_g \Big(\sum_{xy=g} f_1(x)f_2(y)\Big) g$$

Hence product in $\mathbb{C}[G]$ corresponds to convolution of functions

$$(f_1 * f_2)(g) = \sum_{xy=g} f_1(x)f_2(y)$$

$$g\sum f(x)\,x = \sum f(x)\,gx = \sum f(g^{-1}x)\,x$$

Thus the left action of G on $\mathbb{C}[G]$ is $g, f \mapsto f(g^{-1}\cdot)$
and the right mult action " " " is $(g,f) \mapsto f(\cdot\, g)$.

As a $G\times G$-module, $\mathbb{C}[G]$ is a direct sum of $V_i\otimes V_i^*$ where V_i runs over the different irreducible representations of G.

Each irreducible representation V_i of G determines a central idempotent e_i in $\mathbb{C}[G]$, which corresponds to a function on G, which ought to be the character of the representation.

Suppose V is an irreducible representation of G.

$$V\otimes V^* \xrightarrow{\alpha} \mathbb{C}^G \qquad (v,\lambda)\longmapsto (g\mapsto (g^{-1}v,\lambda))$$

$$\begin{tikzcd}
(v,\lambda) \arrow[r, mapsto] \arrow[d, mapsto] & (g\mapsto (g^{-1}v,\lambda)) \arrow[d, mapsto] \\
(g_1 v, g_2\lambda) \arrow[rd, mapsto] & (g\mapsto ((g_1^{-1}gg_2)^{-1}v,\lambda)) \\
 & (g_2^{-1}g^{-1}g_1v,\lambda) \arrow[u, phantom, "\shortparallel"] \\
 & (g^{-1}g_1v,\, g_2\lambda) \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

This shows α is a $G\times G$ map where $G\times G$ acts on $f\in\mathbb{C}^G$ by $(g_1,g_2)f = (g\mapsto f(g_1^{-1}gg_2))$. Now we have

$$\mathbb{C}^G \xrightarrow{\beta} \mathbb{C}[G] \qquad f\longmapsto \int_g f(g)g$$

$$\begin{tikzcd}
(g\mapsto f(g)) \arrow[r, mapsto] \arrow[d, mapsto] & \int_g f(g)g \arrow[d, mapsto] \\
(g\mapsto f(g_1^{-1}gg_2)) \arrow[r, mapsto] & \int_g f(g_1^{-1}gg_2)g = \int_g f(g)\,g_1gg_2^{-1}
\end{tikzcd}$$

so β is also a $G\times G$-map. And we have

$$\begin{aligned} \mathbb{C}[G] &\xrightarrow{\ \gamma\ } V\otimes V^* = \mathrm{End}(V) \\ g &\longmapsto (v \mapsto gv) \end{aligned}$$

Then $\gamma\beta\alpha$ is a $G\times G$ map from $V\otimes V^*$ to itself, so by Schur's lemma (as $V\otimes V^*$ is irreducible), $\gamma\beta\alpha$ must be a multiple of 1. Thus we have

$$\int_g (g^{-1}v,\lambda)\, gx = c\, v\, (x,\lambda) \qquad \forall x\in V$$

for any $v\in V$, $\lambda\in V^*$, where c is a scalar to be determined. Rewrite

$$\int_g \overline{(g\lambda, v)}\,(gx, v) = c\,(v,v)(x,\lambda)$$

Now let v run over an orthonormal basis v_i and add up

$$\int_g \sum_i \overline{(g\lambda, v_i)}\,(gx, v_i) = c \sum_i (v_i, v_i)\,(x,\lambda)$$

$$\shortparallel$$

$$\int_g (gx, g\lambda) = c\cdot d\cdot (x,\lambda)$$

$$\shortparallel$$

$$(x,\lambda). \qquad\qquad \therefore\ c = \frac{1}{d}$$

where $d = \dim(V)$.

Therefore one sees that the identity $\overset{\sum v_i \otimes v_i^*}{\in V \otimes V^*}$ which goes to the function $g \mapsto$ [redacted] $\sum (g^{-1} v_i, v_i^*)$ = trace g^{-1} on V, [redacted] which goes to the element

$$\frac{1}{|G|} \sum_{g \in \,\text{[redacted]}\, G} \chi(g^{-1}) g \quad \text{in } \mathbb{C}[G]$$

is $\frac{1}{d}$ times the central [redacted] idempotent associated to V.

Good method from Lang's book

$$e_i = \sum_{\tau \in G} a_\tau \tau$$

where $a_\tau = \dfrac{d_i}{|G|} \chi_i(\tau^{-1})$

$$\boxed{e_i = d_i \int_g \chi_i(g^{-1}) g}$$

[redacted]

To understand the Frobenius thm., I have to see why $\chi^G - m\mathbb{C}[G/H]$ is [redacted] $\chi_i - m 1_G$

April 17, 1976

I want to consider the action of a finite group G on a Euclidean space E, to consider various G-spaces inside of E, and the geometry of distances. The basic tools will be an integral lattice inside of E, and the metric, so we have the usual machinery from algebraic number theory, a mixture of rigid geometry and integers. This is what character theory also has, so the point is to see if you can get anything new.

Example: $\mathbb{C}[G]$ contains the integral lattice $\mathbb{Z}[G]$.

Suppose E is a Euclidean space on which G acts linearly. Then as a G-space E decomposes

$$E = E_1 \times \cdots \times E_k$$

into irreducible representations

Positive definite function on G is the same thing as a unitary representation of G together with a cyclic vector. Specifically let V be a unitary representation of G, and let v be a non-zero element of V. Then we get a map

$$\mathbb{C}[G] \longrightarrow V \qquad g \mapsto gv$$

which is onto if v is a cyclic vector for V. The inner product on V lifts ~~to~~ give a (possibly degenerate) inner product on $\mathbb{C}[G]$.

$$(*) \qquad \left\| \int f(g_1) g_1 \right\|^2 = \left(\int_{g_1} f(g_1) g_1 v , \int_{g_2} f(g_2) g_2 v \right)$$

$$= \iint_{g_1, g_2} f(g_1) \overline{f(g_2)} \, (g_2^{-1} g_1 v, v) .$$

The function $\lambda(g) = (gv, v)$ is an example of a positive-definite function on G. Positive-definite means simply that the [redacted] sesqui-linear form $(*)$ is ≥ 0, i.e. that $\forall\, g_1, \ldots, g_k$ in G ~~distinct~~ the matrix $\lambda(g_i^{-1} g_j)$ is positive semi-definite.

Example: [redacted] Take an [redacted] irreducible repn. of $G \times G$ of the form $W \otimes W^*$ where W is an irreducible repn. of G.

April 19, 1976

Suppose $\lambda(g) = (gv, v)$ is a positive definite function on G. Then
$$\lambda(g) = \lambda(e) \qquad \text{i.e.} \quad (gv - v, v) = 0$$
implies $\quad gv = gv - v + v \quad$ is an orth. decomp.
so $\quad \|v\|^2 = \|gv\|^2 = \|gv - v\|^2 + \|v\|^2$

i.e. $gv = v$. Thus $\{g \in G \mid \lambda(g) = \lambda(e)\}$ is a subgroup of G; it is the subgroup leaving v fixed.

For example taking a representation W of a group G_0 and letting $V = W \otimes W^*$, $G = G_0 \times G_0$ and $v = id = \sum e_i \otimes e_i^*$, then
$$\lambda(g_1, g_2) = ((g_1 g_2^{-1} \otimes 1) v, v)$$
$$= \sum_{i,j} (g_1 g_2^{-1} e_i \otimes e_i^*, e_j \otimes e_j^*)$$
$$= \sum_i (g_1 g_2^{-1} e_i, e_i^*) = \chi_V(g_1 g_2^{-1})$$
Then
$$\{(g_1, g_2) \in G_0 \times G_0 \mid \chi(g_1 g_2^{-1}) = \chi(e) = \dim W\}$$
is a subgroup of $G_0 \times G_0$ containing ΔG_0; such subgps correspond to normal subgroups of G_0

<u>Prop</u>: $\lambda(g)$ positive definite on $G \Rightarrow \{g \mid \lambda(g) = \lambda(e)\}$ is a subgroup of G.

Proof: (direct). By definition, for any $g_1, \ldots, g_n$ the matrix $\lambda(g_i^{-1} g_j)$ is ≥ 0. Hence if g_1, g_2 are given the matrix

$$\begin{matrix} \lambda(e) & \lambda(g_1) & \lambda(g_2) \\ \lambda(g_1^{-1}) & \lambda(e) & \lambda(g_1^{-1} g_2) \\ \lambda(g_2^{-1}) & \lambda(g_2^{-1} g_1) & \lambda(e) \end{matrix}$$

is ≥ 0. Say $\lambda(e) = \lambda(g_1) = \lambda(g_2) = 1$. Then $\lambda(g_1^{-1}) = \overline{\lambda(g_1)} = 1$, all $\lambda(g_2^{-1}) = 1$. So we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \alpha \\ 1 & \bar{\alpha} & 1 \end{pmatrix} \geq 0 \qquad \alpha = \lambda(g_1^{-1} g_2)$$

which implies first that $1 - |\alpha|^2 \geq 0$ i.e. $|\alpha| \leq 1$. Also the determinant is ≥ 0, so

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha - 1 \\ 0 & \bar{\alpha} - 1 & 0 \end{vmatrix} = -(\alpha - 1)(\bar{\alpha} - 1) \geq 0$$

$$\overset{\shortparallel}{-|\alpha - 1|^2}$$

This is possible only if $\alpha = 1$.

Unfortunately, pos. def. functions versus representations is a tautology.

Classify positive-def functions on $G \times G$ right invariant under ΔG. Such a function λ is of the form $\lambda((g_1, g_2)) = ((g_1, g_2)v, v)$ where of $v \in$ some repn. of $G \times G$. Then we've seen that $(g,g)v = v \quad \forall g \in G \Longleftrightarrow \lambda$ right invariant under ΔG. Better:

$$\forall g \in G \quad \lambda(g,g) = \lambda(e,e) \Longrightarrow (g,g)v = v \quad \text{all } g$$

$$\Longrightarrow \lambda \text{ is biinvariant}$$

We should generalize the prop on page 11 to

<u>Prop</u>: If λ is pos. def on G, then $H = \{g \mid \lambda(g) = \lambda(e)\}$ is a subgp of G and λ is H bi-invariant

<u>Proof</u>: $\lambda(g) = (gv, v)$. We've seen $\lambda(g) = \lambda(e) \Longleftrightarrow gv = v$, so H = stabilizer of G is a subgp. But

$$\lambda(hg) = (hgv, v) = (gv, h^{-1}v) = (gv, v) = \lambda(g).$$

QED.

<u>Prop</u>: Pos. def. functions on $G \times G$ invariant under ΔG can be identified with those central functions on G which are positive (≥ 0) linear combinations of irred characters.

Proof: We know that any such pos. def. function is of the form $\lambda(g_1, g_2) = ((g_1, g_2)v, v)$, where v is fixed under ΔG, hence if v is supposed to be cyclic, the repn V is a quotient of $\mathbb{C}[G\times G]/\mathbb{C}[\Delta G] \cong \mathbb{C}[G]$ with G^2 acting by left + right mult. But we know $\mathbb{C}[G]$ is multiplicity 1 so v must be a multiple of the identity in each irreducible component of $\mathbb{C}[G]$ occurring in V. Rest is clear.

The problem is now to start from a Frobenius subgroup and produce a positive-definite function on G which will give a proper normal subgroup.

Let's see how positive definite translates into for functions on G.

Suppose then $\lambda(g_1, g_2) = \phi(g_1 g_2^{-1})$ is positive definite on $G\times G$. This means that if I select

$$(g_1, \bar{g}_1), \ldots, (g_n, \bar{g}_n) \in G\times G$$

the matrix

$$\lambda\left((g_i, \bar{g}_i)(g_j, \bar{g}_j)^{-1}\right) \text{ is } \geq 0.$$

$$\parallel$$

$$\lambda(g_i g_j^{-1}, \bar{g}_i \bar{g}_j^{-1}) = \phi(g_i g_j^{-1} \bar{g}_j \bar{g}_i^{-1}) = \phi(\bar{g}_i^{-1} g_i g_j^{-1} \bar{g}_j)$$

$$= \phi\left((\bar{g}_i^{-1} g_i)(\bar{g}_j^{-1} g_j)^{-1}\right)$$

where I have used that λ is biinvariant under $\Delta G \Rightarrow \phi$ central

Therefore λ is pos. definite if $\forall\ g_1, \dots, g_n \in G$ the matrix $\phi(g_i g_j^{-1})$ is ≥ 0

This means just that ϕ is positive definite & central as a function on G.

~~Suppose H is a Frobenius group~~ [redacted] ~~such that~~ [redacted] ~~$H\backslash G/H$ has 2 elements. This means H acts transitively on the elements of G/H different from H. I consider the problem of constructing a positive definite~~ [redacted] ~~H-biinvariant on G. The space of H-biinvariant functions on G is 2-dimensional. If I have the representation $\mathbb{C}L$~~

If H is a Frobenius group, one has

$$\mathrm{Res}_{H\to G}\,\mathrm{Ind}_{H\to G}\,\alpha = \alpha + \sum_{HxH \neq H} \mathrm{Ind}_{e\to H}\,\mathrm{Res}_{e\to H}\,\alpha$$

[redacted] where this formula takes place in in any abelian [redacted] monoid valued functor F in which one has induction. Does this imply $F(G) \to F(H)$ is onto? [redacted] The answer is yes if F is group-valued for then we can split α into

$$\alpha - \pi^*\varepsilon(\alpha) + \pi^*\varepsilon(\alpha) \qquad \pi^*\varepsilon(\alpha) = \mathrm{Res}_{H\to e}\,\mathrm{Res}_{e\to H}\,\alpha$$

and now it is clear that each piece $\alpha - \pi^*\varepsilon(\alpha)$, $\pi^*(\varepsilon\alpha)$ comes from G.

<u>April 20, 1976</u>. H Frobenius subgroup of G.

Here is a possible way to construct representations of G starting from H. Consider a prime p dividing $|H|$, and consider the poset of non-trivial p-subgroups of G; denote this $S_p(G)$. Obviously

$$S_p(G) = \coprod_{xH \in G/H} S_p(xHx^{-1})$$

and the same would hold for any family of subgroups, maybe?

~~Let R be a subgroup of G. Assume $R \cap N = 1$. We know $R \cap H$ is Frobenius in R. Assume $R \cap H \neq 1$ and let N' be the normal subgroups of R complementary to $R \cap H^r$ for $r \in R$. Then N'~~

Let R be a subgroup of G such that $R \cap N = 1$. Then consider $X = G/H$ as an R-spaces. Assume all proper subgroups of R have a fixpoint on X.

Let Y be the orbit under R of a point x of X such that some non-trivial element of R fixes x. (i.e $x = gH$ [redacted] where $R \cap gHg^{-1} > 1$). By induction the elements of R not having fixpts on X together with 1 form a **normal** subgroup $M < R$ of R. [redacted] If $M \neq 1$, then ~~M has~~ a fixpt [redacted] on X by induction, and then R has to preserve this fixpoint set, since $M \triangleleft R$. If $M = 1$, then since we know M acts transitively on Y, it follows that Y must be the single point x. So we have proved.

<u>Prop</u>. If R is a subgroup of G such that $R \cap N = 1$, then R is contained in a conjugate of H.

<u>Prop</u>:

Suppose R is a group acting on a set X such that $\mathrm{card}(X^r) = 1$ for all $1 \neq r \in R$. Claim that R acts semi-freely on X, i.e. $\exists$ one fixpoint and [redacted] the action is free on the complement of the fixpt.

<u>April 21, 1976</u>

G, H Frobenius.

Recall

$$\mathrm{Res}_{H\to G}\left(\mathrm{Ind}_{H\to G}\,\alpha\right) = \alpha + \sum_{HxH\neq H} \mathrm{Ind}_{e\to H}\left(\mathrm{Res}_{e\to H}\,\alpha\right).$$

This shows that $\mathrm{Res}: A(G) \to A(H)$ is onto. In fact it gives an explicit section as follows. First split

$$A(H) = A(1) \oplus \bar{A}(H)$$

$$\bar{A}(H) = \mathrm{Ker}\left\{\mathrm{Res}: A(H) \to A(1)\right\}$$

Then $\mathrm{Ind}_{H\to G}$ is a section of $\bar{A}(G) \to \bar{A}(H)$,

$$\mathrm{Res}_{e\to G}\,\mathrm{Ind}_{H\to G}(\alpha) = \sum_{G/H} \mathrm{Res}_{e\to H}(\alpha) = [G:H]\,\mathrm{Res}_{e\to H}(\alpha)$$

<u>Question:</u> Is it true that $\forall \alpha \in \bar{A}(H)$

$$\mathrm{Ind}_{H\to G}(\alpha) = \mathrm{Res}_{G\to H}(\alpha) \ ?$$

Try $\alpha = [H/H'] - (H:H')1_H$.

$$\mathrm{Ind}_{H\to G}(\alpha) = [G/H'] - (H:H')[G/H]$$

$$\mathrm{Res}_{G\to H}(\alpha) = [G/H'N] - (H:H')\,1_G$$

So these are not the same.

Suppose $\alpha \in \bar{R}(H)$. Think of α as a generalized character on H vanishing at e. In this case it is clear that $\mathrm{Ind}_{H\to G}(\alpha) = \mathrm{Res}_{G\to H}(\alpha)$. More generally suppose K is a nilpotent subgroup of G. Then K is either contained in N, or in a conjugate of H.

$$\mathrm{Res}_{N\to G}\,\mathrm{Ind}_{H\to G}(\alpha) = \sum_{N\backslash G/H} \cdots = \mathrm{Ind}_{e\to N}\,\mathrm{Res}_{e\to H}(\alpha)$$

$$= 0 \qquad \text{if} \quad \alpha\in \bar{R}(H).$$

Thus it follows that for $\alpha \in \bar{A}(H)$, $\mathrm{Res}_{G\to H}(\alpha)$ and $\mathrm{Ind}_{H\to G}(\alpha)$ have the same restriction to all subgroups contained either in N or in a conjugate of H, in particular to all nilpotent subgroups of G.

<u>Question</u>: Can you find a formula for $\mathrm{Res}_{G\to H}(\alpha)$ in terms of $\mathrm{Ind}_{H\to G}(\alpha)$ and various correction terms?

Suppose H cyclic of prime order so that there's only one α to consider $[H] - |H|\cdot 1_H$. Then

$$\mathrm{Ind}_{H\to G}(\alpha) = [G] - |H|\,[G/H]$$
$$\mathrm{Res}_{G\to H}(\alpha) = [G/N] - |H|\cdot 1_G$$

Obviously not the same because G-fixpts are different.

Let R act on a set X such that $1 \lneq H < R \Longrightarrow \mathrm{card}(X^H) = 1$. Then the reduced [redacted] homology $\tilde{H}_0(X)$ should be a stably [redacted] free $\mathbb{Z}[R]$-module (by Oliver), hence I should be able to complete X [redacted] to a tree by adding free orbits of 0 and 1-simplices. However Serre has proved that any finite group acting on a tree has a fixpoint.

NO

[redacted] Check this carefully. I want to attach [redacted] free orbits of dim 1 to X to make a contractible graph. So consider $\tilde{H}_0(X)$ which is an integral representation of G. If I [redacted] restrict to a Sylow p-subgroup P of [redacted] G, then I know that card $(X^P) = 1$, hence $\tilde{H}_0(X)$ is free over $\mathbb{Z}[P]$. Thus $\tilde{H}_0(X)$ is $\mathbb{Z}[G]$-projective, and so $\tilde{H}_0(X, \mathbb{Q})$ is free over $\mathbb{Q}[G]$. The reason this doesn't ~~work~~ work is that not every element of $\tilde{H}_0(X)$ can be represented by a map $S^0 \longrightarrow X$.

April 23, 1976:

Künneth property holds for complex representations and cohomology. Suppose A is an elementary abelian subgroup of $G_1 \times G_2$. It is contained in $A_1 \times A_2$ where $A_i =$ proj of A in G_i, but it is not necessarily equal to $A_1 \times A_2$. However a conjugacy class in $G_1 \times G_2$ is the same thing as a conj class in G_1 and one in G_2.

Problem: Let X be a G-set such that $\mathrm{card}(X^H) = 1$ for all $1 < H < G$. Show $X^G = pt$, without using the Frobenius theorem.

Different proof of first Sylow theorem. Use Cauchy thm. that p divides $|G| \Rightarrow G$ contains an element of order p. ($\exists$ direct proof of this using the action of $\mathbb{Z}/p$ on the fibre of $G^p \to G$ over 1, which is $\{(g_1, \cdots, g_p) \mid g_1 \cdots g_p = 1\}$. A fixpt $\neq 1$ is an element of order p in G).

So use induction on m to show that $p^m \mid |G| \Rightarrow G$ has a subgroup of order p^m. For if Q has order p^{m-1}, then $(G/Q)^Q = NQ/Q$ has order $\equiv 0 \pmod p$, and an element of order p in NQ/Q leads to a subgroup of order p^m containing Q.

<u>Problem</u>: Let G be a finite group, A be a complete d.v.r. with quotient field of char 0, residue field of char. p, having enough roots of 1. Then one has an homomorphism "of Cartan"

$$\begin{array}{ccc} K_0(\mathcal{P}_A(G)) & \longrightarrow & R_A(G) \\ \| & & \| \\ K_0(A[G]) & \longrightarrow & K_0(\mathrm{Modf}\, A[G]) \end{array}$$

which I believe is injective, and whose cokernel is killed by a power of p. In any case if $P \in \mathcal{P}(A[G])$ and if Q is a representation of G over A (Q free as an A-module) then $P \otimes Q$ is in $\mathcal{P}(A[G])$ because

$$A[G] \otimes Q = i_* 1 \otimes Q = i_*(i^* Q)$$

where $i : 1 \to G$. Thus $K_0(A[G])$ is an ideal in $R_A(G)$.

I believe Lusztig shows this ideal is the principal ideal generated by the Steinberg module when G is a Chevalley group. <u>Question</u>: I have seen that the poset of non-trivial p-subgroups of G gives an element of $\tilde{K}_0(A[G])$ in fact of $K_0(A[G])$. Can I generalize the Lusztig theorem?

Look carefully at $G = GL_n(\mathbb{F}_q)$. Let X be the building of G, i.e. the poset of [redacted] proper subspaces of $\mathbb{F}_q^n$. Let J be the poset of p-subgroups (non-identity) in G, where $q = p^d$. Is there any relation between these two posets?

Let H be a subgroup of G. Then X^H is the poset of [redacted] proper H-invariant subspaces of $\mathbb{F}_q^n = V$. X^H is contractible if the socle of V as an H-representation, that is, the sum of the irreducible subrepresentations is not all of V. In particular, if H is a p-group the socle is V^H and this is $< V$ if [redacted] $H \neq 1$. [redacted] So we see that

$$V \text{ not semi-simple } /H \implies X^H \text{ is contractible.}$$

Other case is when V is semi-simple. Then we have an invariant decomposition

$$V = V_1 \oplus \cdots \oplus V_m$$

where the reps. V_i are disjoint and sums of a single irreducible, i.e. iso-typical. Then an H-[redacted] invariant subspace of V is the same as [redacted] a family of H-invariant subspaces $W_i \subset V_i$.

It should be the case that X^H is [redacted] of the homotopy type of the join of the posets of H-invariant subspaces in each V_i, and hence X^H [redacted] should be a [redacted] bouquet of spheres.

[redacted] Burnside theorem: A is a p-subgroup contained in two Sylow groups P, R, A normal in P but not normal in R, then $\exists$ $A \leq H \leq P$ such that
i) [redacted] $N_G(H)$ contains a p' element not centralizing H
ii) $N_G(H)$ has a Sylow group in which A is normal.
Further if $H_1 \subset P \ni |H_1| > |H|$, and $H_1 \subset K \subset G$ then $A \triangleleft K$.

Proof: Choose H of max. order such that is the intersection of $N(A)$ with a Sylow p-subgroup S in which A is not normal. [redacted] Because P is a Sylow p-subgp of $N(A)$, we can suppose $H \subset P$. Let $M = N(H)$.

$$\begin{tikzcd} & N(A) \arrow[d, dash] \\ S \arrow[dr, dash] & P \arrow[d, dash, "P \cap N(H)"] \\ & H \arrow[d, dash] \\ & A \end{tikzcd}$$

$H < P \cap N(H)$
$\leq N(A)$

P a p-group
$H < P$

$\therefore$ A normal in any Sylow p-subgroup [redacted] of G containing $P \cap N(H)$

$H < S \cap N(H) \nsubseteq N(A)$ [redacted] by the maximality of H. Let P_1 be a Sylow subgroup of $N(H)$ containing $P \cap N(H)$. [redacted]. Then $A \triangleleft P_1$. Let K be the subgp of $N(H)$ gen. by the p'-elements. Since

$N(H) = P_1 K$. If K centralizes H, K centralizes A, so $N(H) = P_1 K$ would normalize A, contradiction. Thus we get a p'-element normalizing but not centralizing H, as the theorem asserts.

<u>April 25, 1976</u>

Let H be a Frob. subgroup of G, and N the kernel. Assume Frobenius theorem known so N is a subgroup. Let P be a Sylow subgroup of N. Claim

$$G = N \cdot N_G(P)$$

(Quite generally this holds for an extension $N \to G \to G/N$ such that all S_p-subgps of G are in N). It follows that $N_G(P)$ contains a conjugate of H, hence that there exist Sylow groups of N invariant under H. Actually Thompson's thm. says N has unique Sylow groups.

Next point is that H has to act fixpt free on P, hence on the subgroup of elements of order p in the center of P. So we get a representation V of H over $\mathbb{F}_p$ such that $h \neq e \Rightarrow V^h = 0$. This should imply the Sylow subgroups of H are cyclic or generalized quaternion. Yes. (Show it is impossible to have an elementary abelian l group of rank ≥ 2 acting freely on $V - \{0\}$ over $\mathbb{F}_p$. This means no eigenvalues $= 1$, but then pass to alg. closure $\overline{\mathbb{F}_p}$.)

Observe that $SL_2(\mathbb{F}_p)$ has all Sylow groups cyclic or gen. quaternion. True for $\ell = p$. Otherwise one eigenvalue $= 1$ implies both eigenvalues $= 1$, etc.

$$|SL_2(\mathbb{F}_p)| = \frac{(p^2-1)(p^2-p)}{p-1} = (p+1)(p-1)p.$$

If $p \equiv 1 \mod 4$, then the S_2 subgroup is $\mathbb{Z}_4 \underset{\mathbb{Z}_2}{\ltimes} (\mathbb{F}_q^*)_2$

If $p \equiv 3 \mod 4$, it is

$$\operatorname{Ker}\left\{\mathbb{Z}_2 \ltimes (\mathbb{F}_q^*)^2_{(2)} \to \mathbb{F}_q^*\right\}$$

$$\operatorname{Ker}\left\{\operatorname{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) \ltimes \left(\mathbb{F}_{p^2}^*\right) \longrightarrow \mathbb{F}_p^*\right\}_{(2)}$$

In both cases the S_2 subgroup is generalized quat.

Observe any group with only cyclic Sylow groups can't be simple non-abelian. In fact if p is the smallest prime dividing $|G|$, then transfer theory shows that if the Sylow p-group is cyclic, then it is in the center of its normalizer: $(p-1)$ is rel. prime to G, etc.

Let $\mathcal{A}_p$ be the poset of non-trivial elementary abelian p-subgroups of G. If θ is an element of order p in G, then $\mathcal{A}_p^\theta$ is the poset of those elementary abelian p-subgroups which are normalized by θ, i.e. $\theta A \theta^{-1} = A$. Given such an A, we can associate the subgroup of elements commuting with θ, denoted A^θ. This retracts $\mathcal{a}_p^\theta$ to the poset $\mathcal{a}_p(C_G(\theta))$. But if $A \in \mathcal{a}_p(C_G(\theta))$, then $\langle A, \theta \rangle \in \mathcal{a}_p(C_G(\theta))$ so we have the contraction

$$A \subseteq \langle A, \theta \rangle \supseteq \langle \theta \rangle .$$

<u>Assertion:</u> Let G be a finite group, let $\mathcal{a}_p(G)$ denote the poset of non-trivial elementary abelian p subgroups of G. If P is a p-subgroup of G, then $\mathcal{a}_p(G)^P$ is contractible.

<u>Proof</u>: We have an inclusion $\mathcal{a}_p(C_G(P)) \overset{i}{\subset} \mathcal{a}_p(G)^P$. If $A \in \mathcal{a}_p(G)^P$ i.e. P normalizes A, then because P, A are p-groups, $A^P = A \cap C_G(P) \neq 1$. So $A \longmapsto A^P$ is a map $r : \mathcal{a}_p(G)^P \longrightarrow \mathcal{a}_p(C_G(P))$ such that $ri = id$. Also $ir \leqslant id$ for $A^G \subseteq A^G$. So i is a homotopy equivalence. Next $\mathcal{a}_p(C_G(P))$ is contractible by the cone construction for if B is a non-trivial elementary abelian subgroup in the center of P we have

$$A \subseteq AB \supseteq B$$

So by Brown we get

$$\hat{H}_G^* \xrightarrow{\sim} \hat{H}_G^*(\mathcal{a}_p(G))$$

$\mathcal{S}_p(G)$ = poset of non-trivial p-subgps of G

<u>Proposition</u>: $i : \mathcal{A}_p(G) \subset \mathcal{S}_p(G)$ is a homotopy equivalence

<u>Proof</u>: It suffices to show i/P contractible for each P in $\mathcal{S}_p(G)$. But i/P is the poset of non-trivial elementary abelian p-subgroups of P, i.e. $\mathcal{A}_p(P)$. If B = elements of order $|p$ in center of P, then $A \leq AB \geq B$ so $\mathcal{A}_p(P)$ is contractible.

Nice thing about $\mathcal{A}_p(G)$ is that it comes with a filtration by rank. The links are Tits complexes.

Take $G = GL_n(\mathbb{F}_q)$. Here we have a map from flags to p-subgroups given by associating to a flag $\sigma : 0 < W_0 < \cdots < W_k < V$ the subgroup of G normalizing the flag and centralizing the quotients

$$f : \mathrm{Simp}(\mathrm{Tits}(V)) \longrightarrow \mathcal{S}_p(G)$$

$$\tau \subset \sigma \implies f(\tau) \subset f(\sigma)$$

Note that $P \subset f(\sigma) \iff P$ acts trivially on $gr(\sigma)$.

So the problem is whether the poset of flags σ such that P acts trivially on $gr(\sigma)$ is contractible. Call this poset J.

Thus I come back to a question raised during devissage, namely about the poset of chains in M with quotients in the subcategory $\mathcal{B}$.

The argument: Put $V = \mathbb{F}_q^n$ and $V/V' = V_P$ = largest quotient space on which P acts trivially. For each W, $V' \subset W < V$, let J_W be the closed subset of J consisting of $\sigma = W_0 < \cdots < W_k$ such that $W_k \subset W$.

Check J_W is contractible. To any σ in J_W we can add W, thus we get a retraction to flags containing W. Case 1: P acts trivially on W. Then any σ $0 < \cdots < W < V$ contains $0 < W < V$. Case 2: P acts non-trivially on W. In this case the simplices containing W can be identified with the poset of flags τ in W such that P acts trivially on $\mathrm{gr}(\tau)$. This poset is contractible by induction, so again J_W is contractible

Now $J = \bigcup_{V' \subset W < V} J_W$ where each J_W is contractible, $J_{W_1} \cap J_{W_2} = J_{W_1 \cap W_2}$, and where the poset of W has least element V'. $\therefore$ J is contractible as was to be shown.

So we have proved.

<u>Prop</u>: For $G = GL_n(\mathbb{F}_q)$, $\mathcal{A}_p(G)$, $\mathcal{S}_p(G)$ are heq to $\mathrm{Tits}(\mathbb{F}_q^n)$.

Another possibility: Instead of just elementary abelian p-subgroups I might try the cofibred category whose fibre over A is $A \otimes \Omega$. If θ is an element of order p, and θ fixes ξ in $A\otimes\Omega$, this means that θ normalizes A. Since $(A\otimes\Omega)^\theta = (A^\theta \otimes \Omega)$, one sees that ξ comes from the subgroup A^θ in $C_G(\theta)$. Unfortunately, if $\theta \in A$, it is not the case that ξ comes from $\langle\theta\rangle\otimes\Omega$.

Let $g \in G$ act on $\mathcal{A}_p(G)$. If g is not a p' element, we can split it $g = \theta h$ where $\theta h = h\theta$, θ is a p-elemt $\neq 1$, h is a p'-element. Then $gAg^{-1} = A \Rightarrow \theta A\theta^{-1} = A$. Since $\langle\theta\rangle \triangleleft \langle g\rangle$, g normalizes A^θ. Thus if θ' gen. the unique order p subgp of $\langle g\rangle$, we have $A \supset A^{\theta'} \subset A^{\theta'}\langle\theta'\rangle \supset \langle\theta'\rangle$ contracting $\mathcal{A}_p(G)^g$ to a point. Hence the character of the homology of $\mathcal{A}_p(G)$ vanishes at g.

However any projective $\mathbb{Z}_{(p)}[G]$-module has this property.

Let E be a projective $\mathbb{Z}_{(p)}[G]$-module, and let X be a finite simplicial complex on which G acts. Then one has a triangle of projective complexes

$$\bar{C}(X) \otimes E \longrightarrow C(X) \otimes E \longrightarrow E$$

which will give us relations in $K_0(\mathbb{Z}_{(p)}[G]) \subset R_{\mathbb{Z}_{(p)}}(G)$. I wanted to show that the ideal is principal generated by $[\bar{C}(\mathcal{A}_p(G))]$ Should be enough to show $C(\mathcal{A}_p(G)) \otimes E$ is a multiple of $[\bar{C}(\mathcal{A}_p(G))]$.

$C(\mathcal{A}_p(G)) \otimes E$ will be a direct sum of things of the form

$$\mathbb{Z}[G/N(A)] \otimes E = \operatorname{Ind}_{N(A) \to G} \operatorname{Res}_{N(A) \to G} E$$

where A is a non-trivial elementary abelian subgroups.

Suppose G has a normal elementary abelian p-subgroup. Then is $\mathcal{A}_p(G)$ contractible? Call A_0 this normal elem. abelian subgroup. If $A \in \mathcal{A}_p(G)$, then A normalizes A_0, so we have

$$A \subset A \cdot (A_0)^A \supset (A_0)^A \subset A_0$$

Unfortunately increasing A decreases $(A_0)^A$.

However, suppose G has a normal p-subgroup Q, whence it has a normal elementary abelian p-subgroup, namely the elements of order p in $Z(Q)$. Then we can contract $\mathcal{S}_p(G)$ by

$$P \subset PQ \supset Q$$

This shows that $\mathcal{S}_p(G)$ is G-contractible. It follows that $\mathcal{A}_p(G)$ is contractible.

Direct proof: For each $0 < B \subset A_0$ let T_B be the sub-set of $\mathcal{A}_p(G)$ consisting of A centralizing B. Then T_B is contractible and

$$T_{B_1} \cap T_{B_2} \subset T_{B_1 B_2}$$

$$\bigcup_{0<B\subset A_0} T_B = \mathcal{A}_p(G)$$

Given A the set of B centralized by A has a largest element $(A_0)^A$. $\therefore$ $\mathcal{A}_p(G)$ is contractible. Same argument shows that $\mathcal{A}_p(G)^H$ is contractible for any subgroup H of G, so $\mathcal{A}_p(G)$ is G-contractible

Check that $i: \mathcal{A}_p(G)^H \subset \mathcal{S}_p(G)^H$ is a heq. If P is a p-subgp norm. by H, A_0 = order/p elemts in center then $\mathcal{A}_p(P)^H = i/P$ contracts by $A \subset AA_0 \supset A_0$. $\therefore$ $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a <u>G homotopy equivalence</u>

Go over Burnside's theorem again: A normal in some Sylow group (i.e. $G:N(A)$ prime to p), but not normal in another S_p-group Q. Choose Q so that $|Q \cap N(A)|$ is maximal, put $H = Q \cap N(A)$, choose an S_p subgp $P < N(A)$ containing A. Since $N(H) \cap P > H$, any S_p-subgp of G containing

$N(H) \cap P$ must be in $N(A)$. So if P_1 is an S_p-subgroup of $N(H)$ containing $N(H) \cap P$, then $P_1 \subset N(A)$. If K is the subgroup gen. by the p' elements of $N(H)$, then $N(H) = KP_1$. If K centralizes H, then $K \subset N(A)$ and we get $N(H) \subset N(A)$. This contradicts $H < N(H) \cap Q$ and $H = N(A) \cap Q$. Thus there exist p'-elements in $N(H)$ which do not centralize H.

$G = GL_n(\mathbb{F}_q)$. Claim that $\mathcal{S}_p(G)$ is homotopy equivalent to $X = \text{Tits}(\mathbb{F}_q^n)$. For each p group $P \in \mathcal{S}_p(G)$ we associate X^P. $P \subset P' \Rightarrow X^P \supset X^{P'}$. We want to apply the acyclic covering argument:

$$\begin{array}{ccccccc}
\Rrightarrow & \coprod_{P \subset P'} X^{P'} & \rightrightarrows & \coprod_{P \in \mathcal{S}_p(G)} X^P & \longrightarrow & X \\
& \downarrow & & \downarrow & & \\
\Rrightarrow & \coprod_{P \subset P'} pt & \rightrightarrows & \coprod_{P} pt & &
\end{array}$$

so we need to know two things:

a) $\forall x \in X$ the poset of p-subgroups of G stabilizing x is contractible

b) X^P is contractible for each P in $\mathcal{S}_p(G)$.

Proof of a). The stabilizer of x is a parabolic subgroup Q of G such that $Q \neq G$, hence the unipotent radical is a non-trivial normal subgroup of Q. This implies $\mathcal{S}_p(Q)$

= poset of p-subgroups of G stabilizing x is contractible.

Proof of b): Direct in the case of GL_n. Because $P \neq 1$ V^P is a proper subspace of $V = \mathbb{F}_q^n$ which meets each proper P-invariant subspace of V. Thus X^P = simp. complex assoc. to the poset of proper P-invariant subspaces is "conically" contractible: $W \geq W \cap V^P \leq V^P$.

Proof in the case of Chevalley groups: Choose a Borel B of G containing P; that is the same as choosing a chambre C of X fixed by P. According to Tits if one removes from X the centers of the "opposite" chambres to B, i.e. those corresponding to Borels $B' \ni B' \cap B$ is a torus, then the building has a canonical "geodesic contraction" to the center of C, where canonical implies invariance under the B-action. So next observe that X^P contains no interior point from a chambre opposite to B, because P is a p-group $\neq 1$ and a torus $B' \cap B$ has ~~only~~ p'-elements. Thus the geodesic contraction furnishes a contraction of X^P.

April 27, 1976

New proof of Tits' theorem. Let $X = \mathrm{Tits}(V)$, let B be a Borel, and let B_u act on X. For each $1 < H \leq B_u$ we can directly see that X^H is contractible by the socle argument. Thus X is heq to $X / \bigcup_{1<H\leq B_u} X^H$. But calculation shows that the only simplices of X with free B_u-orbit are the opposite chambers. Thus $X/\bigcup X^H$ is a bouquet of spheres indexed by the opposite Borels.

Let H be a subgroup of G having a normal p-subgroup $1 \neq B \triangleleft H$. Then $(\mathcal{S}_p(G))^H$ is contractible, i.e. if $Q \in \mathcal{S}_p(G)^H$, then $Q \leq QB \geq B$.

Let $\tilde{\mathcal{S}}_p(G)$ be the poset of subgroups H of G having a non-trivial normal p-subgroup. Such an H has a non-trivial normal elementary-abelian p-subgrp B. If $A \in \mathcal{A}_p(H)$, then $A \geq C_A(B) \in \mathcal{A}_p(C_H(B))$, so $\mathcal{A}_p(H)$ deforms to $\mathcal{A}_p(C_H(B))$ which then deforms to a point by cone construction $A \leq AB \geq B$. So again $\mathcal{A}_p(G) \subset \tilde{\mathcal{S}}_p(G)$ is a homotopy equivalence. Much easier to show that $\mathcal{S}_p(G) \subset \tilde{\mathcal{S}}_p(G)$ is a heq.

April 28, 1976

Let $G = GL_n(\mathbb{F}_q)$, $X = \text{Tits}(\mathbb{F}_q^n)$. I have seen that $\mathcal{S}_p(G)$ is heq to X. I want to see whether $\mathcal{S}_p(G)^H$ is heq to X^H for any subgp H of G.

To each $P \in \mathcal{S}_p(G)^H$ I associate [redacted] X^{HP} which is contractible. In effect if K is a group with a non-identity normal p-subgroup P, then X^K can be contracted as follows: $W \supset W^P \subset V^P$. To finish I have to see that [redacted] $\forall x \in X^H$ the poset of $P \in \mathcal{S}_p(G)^H$ such that $x \in X^{HP}$ is contractible. [redacted] This poset is $\mathcal{S}_p(G_x)^H$. [redacted] G_x has a non-identity normal p-subgroup $(G_x)_u = Q$. Then $P \subset PQ \supset Q$ contracts $\mathcal{S}_p(G_x)$ to a point.

Alperin's thm: One fixes a Sylow p-subgp P and then considers the [redacted] other S_p-subgroups. [redacted] Given a S_p-subgp Q one is going to construct a path from Q to P of a special sort such that the [redacted] size of $Q \cap P$ increases as one goes along the path. Write [redacted] $R \sim Q$ to mean there is such a path. The path [redacted] is given by $Q_1, \ldots, Q_n$ $\quad x_i \in N_G(P \cap Q_i)$ $\quad x_i$ p-elt

$$P \cap R \subset P \cap Q_1 \qquad (P \cap R)^{x_1 \cdots x_i} \subset P \cap Q_{i+1}$$

It seems to be more intricate.

Alperin's Theorem: Let A, B be subsets of the S_p-subgroup P which are conjugate in G: $A^x = B$. Then one can find S_p-subgroups $Q_1, \ldots, Q_m = P$ intersecting P tamely, and elements $x_i \in N_G(P \cap Q_i)$ such that

$$A \subset P \cap Q_1$$
$$A^{x_1 \cdots x_i} \subset P \cap Q_i \qquad 1 \le i < m \quad .$$
$$x_1 \cdots x_m = x$$
$$x_i \text{ is a } p\text{-element} \quad i < m$$

Let's see if I can forget tameness, and concentrate instead on size of intersections

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Gorenstein's generalization involves well-placed tame intersections.

P an S_p-subgp of G, H any subgroup of P.

$$W_1(H) = H \qquad P_1(H) = N_P(H) \qquad N_1(H) = N_G(H)$$
$$W_2(H) = ZJ\,P_1(H) \qquad P_2(H) = N_P(W_2(H)) \qquad N_2(H) = N_G(W_1(H))$$
$$W_3(H) = ZJ\,P_2(H)$$

Call H well-placed if $P_i(H)$ is a S_p-subgp of $N_i(H)$ for each i. Note that $W_{i+1}(H) = ZJP_i(H)$ is char. in $P_i(H)$ so

$P_{i+1}(H) = N_P(W_{i+1}(H)) \supseteq N_P(P_i(H)) > P_i(H)$ if $P_i(H) < P$. Thus the sequence $P_i(H)$ increases up to P and eventually $W_i(H) = Z\,J(P)$.

The generalization then says that ~~fusion of subsets~~ in Alperin's thm. one can suppose $P \cap Q_i$ well-placed tame intersections.

Application of Alperin's theorem. Let $x, y \in P$ be conjugate in G. Then by Alperin's theorem we can find tame intersections $H_i = P \cap Q_i$ $\quad 1 \le i \le m$

$x = x_0, x_1, \dots, x_m = y, \quad x_{i-1}, x_i \in H_i, \quad x_{i-1}^{y_i} = x_i$

$y_i \in N_G(H_i)$

$$\begin{aligned} x^{-1}y &= (x_0^{-1}x_1)(x_1^{-1}x_2)\cdots(x_{m-1}^{-1}x_m) \\ &= (x_0^{-1}y_1^{-1}x_0y_1)\cdots\cdots(x_{m-1}^{-1}y_m^{-1}x_{m-1}y_m) \\ &\in [H_1, N_GH_1]\cdots\cdots[H_m, N_GH_m] \end{aligned}$$

Now suppose N_GH_i/C_GH_i is a p-group. Tameness $\Rightarrow N_PH_i$ is an S_p-subgp of N_GH_i. So

$$N_PH_i \twoheadrightarrow N_GH_i/C_GH_i$$

means

$$N_GH_i = (P\cap N_GH_i)C_GH_i = C_GH_i(P\cap N_GH_i)$$

$[H_i, N_GH_i] = [H_i, P\cap N_GH_i] \qquad h^{-1}(zy)^{-1}hzy = h^{-1}y^{-1}hy$

Thus $[H_i, N_G H_i] \subset [H_i, P] \subset P'$. It follows that $P \cap G' \subset P'$, hence $P \cap G' = P'$. Thus P has a normal p-complement.

So what is important, it seems, is the family of $H \subset P$ such that $P \cap N_G(H)$ is a Sylow p-subgroup of $N_G(H)$. For example if $H \triangleleft P$, then $P \cap N_G(H) = P$, so this is OKAY.

Suppose G has a normal p-complement K so that $G = P \ltimes K$ where P is a Sylow p-subgroup. Let $f: G \longrightarrow G/K$ be the canonical map. Then

$$\tilde{f}: \mathcal{S}_p(G) \longrightarrow \mathcal{S}_p(G/K)$$

is fibred, for if $Q \in \mathcal{S}_p(G)$ then $f: Q \xrightarrow{\sim} \tilde{f}(Q)$ so that subgroups of Q and $\tilde{f}(Q)$ are in 1-1 correspondence.

If $Q \subset P$ what is $\tilde{f}^{-1}\tilde{f}(Q)$? Given $R \in \mathcal{S}_p(G)$ with $f(R) = f(Q)$, then R, Q are both S_p-subgroups of $Q \ltimes K$, hence they are conjugate by and element gk of $Q \ltimes K$: $k^{-1}g^{-1}Qgk = R \Rightarrow k^{-1}Qk = R$. Thus the fibre of $\tilde{f}$ over $\tilde{f}(Q)$ is acted on transitively by K. Next note that $k^{-1}Qk = Q \iff k$ centralizes Q for $k^{-1}qk, q$ have the same image under f so they must coincide. Thus

$$\tilde{f}^{-1}(\tilde{f}Q) \cong K/C_K(Q) = K/K^Q$$

So we therefore see that $\mathcal{S}_p(G)$ is the fibred category over $\mathcal{S}_p(P) = \mathcal{S}_p(G/K)$ associated to the contravariant functor

$$Q \longmapsto K/K^Q .$$

If $K^Q = K$, then Q acts trivially on K and conversely. If $Q = \mathrm{Ker}\,\{P \to \mathrm{Aut}\,K\}$ Then $N_G(Q) = P \ltimes C_K(Q) = P \ltimes K = G$. Thus $O_p(G) = 1$ $\iff$ P acts faithfully on $K \iff K^Q < K$ for $1 < Q \subseteq P$.

Critical case: Suppose P is an elementary abelian p-group acting faithfully on an elementary abelian ℓ-group K. Is $\mathcal{S}_p(G)$ spherical?

Can suppose without changing $\mathcal{S}_p(G)$ that $K^P = 1$.

Suppose $\mathrm{rank}(P) = 1$. Then $\mathcal{A}_p(G)$ has dim. 0
If $\mathrm{rank}(P) = 2$, then $\dim \mathcal{A}_p(G) = 1$, so we only have to show it is connected. Every component is represented by an element of K, and two elements of K are in the same component if they determine the same element of K/K^Q for some $1 < Q \leq P$. Clearly K acts by left mult on $\pi_0 K$ and the action is transitive so $\pi_0 K \cong K/L$ where $L \supset K^Q$ for $1 < Q \leq P$. But because K is a p'-group it should be the case that Q has no fixpts on $K/L \twoheadleftarrow K/K^Q$.

April 30, 1976. Alperin's theorem.

Let P be a fixed S_p-subgroup of G. Let Q be another S_p-subgroup. Can I find a "tame" H in P and an $x \in N_G(H)$ such that

(i) $P \cap Q \subset H$

(ii) $|P \cap Q^x| > |P \cap Q|$.

Note that i) $\Rightarrow$ $(P \cap Q)^x \subset H \subset P \Rightarrow (P \cap Q)^x \subset P \cap Q^x$ $\Rightarrow |P \cap Q| \leq |P \cap Q^x|$. Thus ii) says the order of the intersection should increase.

Assume this can be done. Then iterating I can construct a sequence of S_p-subgroups

$$Q, Q^{x_1}, Q^{x_1 x_2}, \ldots, Q^{x_1 \cdots x_m} = P$$

and tame subgroups of P

$$H_1, H_2, \ldots, H_{m-1}$$

and $x_i \in N_G(H_i)$ such that

$$P \cap Q^{x_1 \cdots x_{i-1}} \subset H_i$$

$$|P \cap Q^{x_1 \cdots x_i}| > |P \cap Q^{x_1 \cdots x_{i-1}}|.$$

Special case: Suppose you can take $H = P \cap Q$, $Q^x = P$. Recall that I want $N_P(H)$ to be an S_p-subgp of $N_G(H)$. So if $x \in N_G(H)$ moves P to Q, then $N_Q(H)$ must be an S_p-subgp of $N_G(H)$.

In some sense the tame intersections are like the walls in the fundamental chambre.

Basic transition is from Q to Q^x where $x \in N_G(H)$, H is tame $\supset P \cap Q$. If $H = P \cap Q$ is a tame intersection then $\exists\, x \in N_G(H) \ni N_P(H) = N_Q(H)^x = Q^x \cap N_G(H)$, but this doesn't imply $P = Q^x$, it seems

Suppose Q is immediately related to P. This means $\exists$ tame $H \supset P \cap Q$ and $x \in N_G(H) \ni Q = P^x$. But then $H \subset P \Rightarrow H \subset P^x \Rightarrow H \subset P \cap Q$ so $H = P \cap Q$. Thus $P \cap Q$ is a tame intersection.

<u>Summary</u>: I consider inside P those H such that $N_P(H)$ is an S_p-subgp of $N_G(H)$. I ~~consider transitions from~~ write $Q_1 \rightsquigarrow Q_2$ to mean $\exists$ such an H containing $P \cap Q_1$ and an $x \in N_G(H)$ such that $Q_1^x = Q_2$.
<u>Assertion</u>: $Q \rightsquigarrow P$ implies $P \cap Q$ is a tame intersection.

<u>Proof</u>: Let $H \supset P \cap Q$ be such that $Q^x = P$ for some $x \in N_G(H)$. Then $H \subset P \Rightarrow H = H^x \subset P^x = Q \Rightarrow H \subseteq P \cap Q \Rightarrow H = P \cap Q$. Since $N_G(H) : N_P(H) \not\equiv 0 \ (p)$ the same is true for $N_G(H) : N_Q(H)$, so $H = P \cap Q$ is a tame intersection.

Here's the way to try to understand Alperin's thm. Suppose for every subgroup H of P that the restriction of $\alpha \in H^*(P)$ is invariant under $N_G(H)$. Try to show then that α comes from a class in G. We have to prove that for every $x \in G$, the class α is equalized by the maps

$$H^*(P) \underset{j}{\overset{i}{\rightrightarrows}} H^*(P \cap xPx^{-1}).$$

Put $Q = xPx^{-1}$. Then we have i, j are the two maps

$$\begin{tikzcd}
H^*(P) \arrow[dr, "res"] \arrow[d, "\theta^*"', "\wr"] & \\
H^*(Q) \arrow[r, "res"'] & H^*(P \cap Q)
\end{tikzcd}$$

where $\theta : Q \longrightarrow P$ is $\theta(q) = x^{-1}qx$. How does this depend on x? If $n \in N_G(P)$, then $xnPn^{-1}x^{-1} = Q$ and $(xn)^{-1} q\, xn = n^{-1}\theta(q)\, n$. But α is invariant the action of $N(P)$ by assumption. Thus the condition that α is equalized by the arrows i, j means that after transporting α to a class on all S_p-subgroups, it is compatible with restriction.

Next assume $\alpha | P \cap Q$ is invariant under $N_G(P \cap Q)$. If $Q = x^{-1}Px$ where $x \in N_G(P \cap Q)$, then

$$\begin{array}{cccc}
xqx^{-1} & P & \supset & P \cap Q \\
\uparrow & \uparrow & & \uparrow \\
q & Q & \supset & P \cap Q
\end{array}$$

commutes, so the equalization condition is satisfied.

Here is a possible way to view Alperin's theory. The problem is to describe the image of the restriction homom.

$$\text{res}: H^*(G) \hookrightarrow H^*(P).$$

One has Eilenberg-Cartan result about stable classes: this means that for each intersection $P \cap xPx^{-1}$ we have to equalize the 2 arrows

$$H^*(P) \rightrightarrows H^*(P \cap xPx^{-1})$$

I think what Alperin's result does is to reduce all these equalization conditions to considering just tame H and the action of $N_G(H)$. Thus a class α of $H^*(P)$ comes from $H^*(G)$ iff for all tame H the restriction of α to $H^*(H)$ is $N_G(H)$-invariant.

Consider H in P such that

i) $N_P(H)$ is an S_p-subgp of $N_G(H)$

ii) H is the intersection of $N_p(H)$ and another S_p-subgroup of $N_G(H)$.

Then H is a tame intersection. *Proof*: Suppose Q_1 is an S_p-subgp of $N_G(H)$, so $Q_1 = Q \cap N_G(H)$ for some S_p-subgp Q of G. Then $H \triangleleft N_G(H) \Rightarrow H \subset Q_1$

$$H \subset P \cap Q \cap N_G(H) = P_1 \cap Q_1 = H \qquad P_1 = N_p(H).$$

May 1, 1976 Alperin's thm.

Statement of the problem. We know that restriction $H^*(G) \longrightarrow H^*(P)$ is injective (mod p coeffs) and that the image consists of classes $\alpha \in H^*(P)$

~~equalized by the two homomorphisms~~

$$\cancel{P \cap x^{-1}Px \rightrightarrows P}$$

~~given by inclusion and $p \mapsto xpx^{-1}$ for any $x \in G$. Included in these equalization conditions are ones of the following type~~

which are stable in the following sense. For any subgroup H of P and element x of G such that $xHx^{-1} \subset P$, the class α is equalized by the two homomorphisms

$$H \rightrightarrows P \qquad h \mapsto \begin{matrix} h \\ xhx^{-1} \end{matrix}$$

What Alperin's thm. does is to restrict the number of these equalization conditions to the following types: (i) $x \in N_G(H)$ (so that $xHx^{-1} = H$) (ii) H is a intersection $P \cap Q$ where Q is an S_p-subgroup, and this intersection is tame.

So to understand his theorem I have to suppose given $\alpha \in H^*(P)$ such that for certain $H \subset P$ one has $\alpha|H$ is invariant under $N_G(H)$, and then try to prove α stable. Use decreasing induction on $P \cap Q$.

If $P \cap Q = P$, then $x \in N_G(P)$ and α is invariant under x. Assuming this condition (α invariant under $N_G(P)$), we know that

α determines a definite class $\alpha_Q \in H^*(Q)$ for each Sylow group Q of G. The problem is to show then that α_P and α_Q have the same restriction to $P \cap Q$.

The next case to consider is where $P \cap Q$ is a maximal Sylow intersection, i.e. $P \cap Q < P \cap R \Rightarrow R = P$ for any S_p-subgroup R.

<u>Digression:</u> What is the homotopy type of the set of p-subgroups strictly containing a fixed p-group H. Put

$$L_H = \{ Q \in \mathcal{S}_p(G) \mid Q > H \}.$$

L_H is empty $\Longleftrightarrow$ H is a S_p-subgroup. Note that

$Q > H \Rightarrow N_Q(H) > H$ normalizer condition.

$$N_Q(H) = N_G(H) \cap Q$$

So [redacted] if we put

[redacted]

$$L_H^1 = \{ Q \in L_H \mid H \triangleleft Q \} = \{ Q \in \mathcal{S}_p(G) \mid H < Q \subset N_G(H) \}$$

then we have

$$L'_H(G) = L_H(N_G(H)) \overset{i}{\subset} L_H(G) \xrightarrow{r} L'_H(G)$$

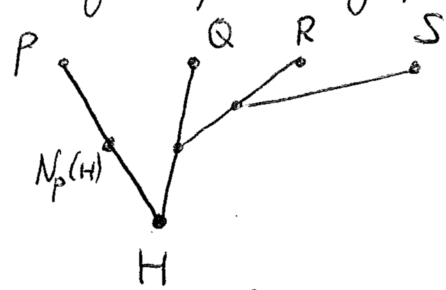
$$Q \longmapsto Q \cap N_G(H) = N_Q(H).$$

Then $ri = id$ and $ir(Q) = N_Q(H) \subset Q$. So $L'_H(G) = L_H(N_G(H))$ is heq. to $L_H(G)$. But

$$L_H(N_G(H)) = S_p(N_G(H)/H)$$

for any $H < Q \subset N_G(H)$ is in 1-1 corresp. with $Q/H \subset N_G(H)$.

Suppose $P \cap Q$ is a max. Sylow intersection i.e. $P \cap R > P \cap Q \Rightarrow R = P$ for any S_p-gp. R. Put $H = P \cap Q$. ~~[illegible]~~ So $P \cap Q \subset R \Rightarrow$ $P \cap Q = P \cap Q \cap R \subset P \cap R \Rightarrow$ either $P \cap R = H$ or $R = P$. So the set of S_p-subgrps containing H looks:

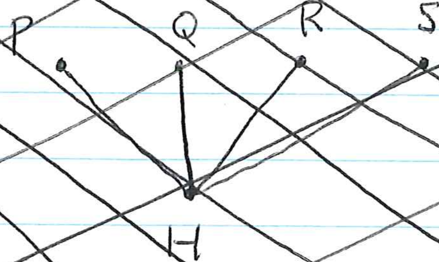


~~[illegible]~~

$N_p(H) > H$; let P_1 be an S_p-subgroup of $N_G(H)$ containing $N_p(H)$. By maximality P is the only S_p-subgroup of G

containing P_1. Better: $N_P(H)$ is a p-subgroup of P strictly containing H, so if R is an S_p-subgroup containing $N_P(H)$ one has $R = P$. So if P_1 is an S_p-subgroup of $N_G(H)$ cont. $N_P(H)$, then $P_1 \subset$ some R, so $P_1 \subset P$, so $P_1 \subset N_P(H)$.
$\therefore$ $N_P(H)$ is an S_p-subgroup of $N_G(H)$.

~~To simplify let me assume that $P \cap Q = H$ is a Sylow intersection with the largest possible order but still $P \neq Q$. In this case the picture of the Sylow gps containing H is~~



~~In this case I want to show that $N_G(H)$ transitively permutes the S_p-subgps containing H. Since $N_Q(H) > H$~~

Now $Q \cap N_G(H)$ is a p-subgp of $N_G(H)$ so $\exists\, x \in N_G(H)$ such that

$$(Q \cap N_G(H))^x = Q^x \cap N_G(H) \subset N_P(H)$$

But $Q \cap N_G(H) > H$, so the maximality of $H \Longrightarrow Q^x = P$.

~~Let $H = P \cap Q$ be a maximal Sylow intersection~~

Proposition: Let $H = P \cap Q$ be a subgroup of P which is maximal with respect to being a Sylow intersection. Claim $\exists\ x \in N_G(H)$ such that $Q^x = P$.

Proof: $N_p(H) = P \cap N_G(H) > H$ as H is a proper p-subgroup of P. Let P_1 be an S_p-subgp of $N_G(H)$ containing $N_p(H)$, and choose an S_p-subgp R of G containing P_1. Then $P \cap Q = H < N_p(H) \subset P \cap R$ maximality assumption on H, $R = P$. Thus $P_1 = N_G(H) \cap R = N_p(H)$ is an S_p-subgp of $N_G(H)$.

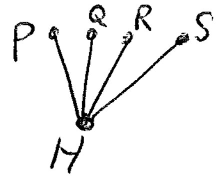
$N_Q(H)$ is a p-subgroup of $N_G(H)$ hence $\exists x \in N_G(H)$ so that

$$Q^x \cap N_G(H) = N_Q(H)^x \subset N_p(H)$$

But $N_Q(H) > H$, hence Q^x is a S_p-subgp of G containing H with $Q^x \cap P \supset N_Q(H)^x > H$, so $Q^x = P$ by maximality. QED.

One can even assume that x is a product of p-elements in $N_G(H)$. In effect, any two Sylow groups P, Q of a group G are contained in the subgroup gen. by p-elements, hence are conjugate in this subgroup.

Note the picture of a maximal Sylow intersection is



and $N_G(H)$ permutes the Sylow groups transitively.

Next we want to get the general case.

For each p subgroup H of G let $L_H(G) = \{H' \mid H' \text{ is a } p\text{-group} > H\}$. Recall that we are trying to show that if $H \subset P \cap Q$, where P, Q are Sylow, then $\alpha_P = \alpha_Q$ when restricted to H. Assume this is true for each $H' > H$. Then we have a well-defined function $H' \mapsto \alpha_{H'}$ for all H' in $\mathcal{S}_p(G)$ which properly contain a conjugate of H. So there is no problem if $L_H(G)$ is connected

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Digression: Consider the simplicial complex K whose vertices are the Sylow p-groups and whose simplices are subsets whose intersection is non-trivial. This is just the nerve of the covering of $\mathcal{S}_p(G)$ given by the sets $\{\leq P\}$. Since one has

$$\{\leq P_1\} \cap \cdots \cap \{\leq P_q\} = \{\leq P_1 \cap \cdots \cap P_q\}$$

the intersections are contractible, so K has the homotopy type of $\mathcal{S}_p(G)$.

Thus we get a deformation of $\mathcal{S}_p(G)$ into the poset consisting of those p-subgroups which are intersections of Sylow groups.

I can now prove a version of Alperin's theorem:

<u>Theorem</u>: Let $\alpha \in H^*(P)$ be such that for every tame intersection $H = P \cap Q$, $res_{H \to P}(\alpha)$ is invariant under $N_G(H)$. Then α comes from $H^*(G)$.

<u>Proof</u>: Let H be an p-subgroup of G. Choose $x \in G$ such that $xHx^{-1} < P$. Then we get a homo $i_x : H \longrightarrow P$, $h \mapsto xhx^{-1}$, and hence we can pull α back to H. Call H good if the class $i_x^*(\alpha)$ does not depend on x, and write α_H for $i_x^*(\alpha)$. We have to show every p-subgroup H of G is good. We use decreasing induction on $|H|$.

If H is a S_p-subgroup, then this follows from the fact that α is invariant under $N_G(P)$.

Assume H' good for all $|H'| > |H|$, but that H is bad. Then we have two homos. $i_x, i_y : H \longrightarrow P$ such that $i_x^*(\alpha) \neq i_y^*(\alpha)$. In other words H is contained in the S_p-subgroups $Q = x^{-1}Px$, $R = y^{-1}Py$ and α_Q, α_P restrict differently on H. Can't have $Q \cap R > H$ by induction. Let S be an S_p-subgp of G such that $S \cap N_G(H)$ is an S_p-subgp of $N_G(H)$ containing $N_Q(H)$. Since $N_Q(H) > H$, we have $Q \cap S > H$ so by induction α_Q, α_S have the same restriction to H. Also $S \cap R = H$ since $\alpha_S \neq \alpha_R$ on H. Thus replacing Q by S we can suppose $N_Q(H)$ is an S_p-subgroup of $N_G(H)$. Similarly we can suppose $N_R(H)$ is an S_p-subgrp of $N_G(H)$. $\therefore$ $H = Q \cap R$ is a tame intersection. Also $\exists\ x \in N_G(H) \ni xN_R(H)x^{-1} = N_Q(H)$.

so $Q \cap xRx^{-1} > H$ and [redacted] $\alpha_Q|H = \alpha_{xRx^{-1}}|H = (\alpha_R|H)^x$
But there is no loss in generality in assuming $R = P$, and by assumption $\alpha_P|H$ is invariant under $N_G(H)$, so we get a <u>contradiction</u>. QED.

Really the point of the above proof is that if you have $H = P \cap Q$ a Sylow intersection such that P, Q are in different components of the poset of p-groups properly containing H, then you can [redacted] move P, Q within these components to a tame intersection. Namely, choose an S_p-subgp $R \ni$ [redacted] $R \cap N_G(H)$ is an S_p subgp of $N_G(H)$ containing $N_Q(H) > H$. Then $R \cap P \supset N_P(H) > H$, so R and [redacted] P are in the same component, so [redacted] still $R \cap Q = H$. But now $R \cap N_G(H)$ is an S_p-subgp of $N_G(H)$.

<u>Question</u>: For what groups G is $\mathcal{S}_p(G)$ connected?

I want to refine this question. The point is that if H is maximal bad p-subgroup, then we've defined the function α on $L_H(G)$ and it might be constant on bigger chunks [redacted] than just the components of [redacted] $L_H(G)$ because we have put in the relations of conjugacy on [redacted] larger tame intersections.

<u>Def:</u> A "critical" p-group H is one such that $\pi_0\left(S_p(N_G(H)/H)\right)$ is not a point.

~~Prop: H a critical p-subgroup of G, P an S_p-subgroup containing H. Then $\exists$ S_p-subgp Q such that $H = P \cap Q$ is a tame inter~~

Recall $S_p(N_G(H)/H) \sim L_H(G)$. Let P be an S_p-subgroup of G containing H. Then $P \cap N_G(H) > H$ unless $H = P$. So $L_H(G) = \phi \Longleftrightarrow$ H is an S_p-subgroup. Suppose P chosen so that $P \cap N_G(H)$ is an S_p-subgp of $N_G(H)$. If $L_H(G)$ has more than one component choose an S_p-subgp Q in another component. $N_Q(H) > H$ so we can find another S_p-subgp $Q' \ni N_{Q'}(H)$ is an S_p-subgp of $N_G(H)$ containing $N_Q(H)$. Then $Q' \cap Q \supset N_Q(H) > H$, so Q and Q' are in the same component. Then $H = P \cap Q'$ is a tame intersection.

<u>Conclusion:</u> For the Alperin thm. we have only to consider $H \subset P$ such that $N_p(H)$ is an S_p-subgroup of $N_G(H)$ and such that $\pi_0 S_p(N_G(H)/H) \neq pt.$

Check:

Theorem: Let $\alpha \in H^*(P)$. Assume $\mathrm{res}_{H\to G}(\alpha)$ is invariant under $N_G(H)$ for each H, $1 < H \subset P$ such that
(i) $N_P(H)$ is an S_p-subgroup of $N_G(H)$.
(ii) $\pi_0(\mathcal{S}_p(N_G(H)/H)) \neq pt$.
Then α comes from $H^*(G)$.

Proof: (i), (ii) hold for $H = P$, so α is invariant under $N_G(P)$. This implies we can define $\alpha_Q \in H^*(Q)$ for each S_p-subgroup Q such that $\alpha_P = \alpha$ and such that $\{\alpha_Q\}$ is compatible with inner automs.

To show α comes from $H^*(G)$, it suffices to prove for any non-identity p-group H, that $\alpha_{Q_1}|H = \alpha_{Q_2}|H$ for any two S_p-subgroups Q_1, Q_2 of G containing H. Choose H maximal so that it does not have this property. ~~and let [illegible] the poset of p-subgroups of G strictly containing H.~~

For any p-subgroup H of G I have seen that $\mathcal{S}_p(N_G(H)/H)$ is homotopy equivalent to the simplicial complex $K(G,H)$ whose simplices are sets $\{Q_0, \ldots, Q_n\}$ of S_p-subgroups of G with $H < Q_0 \cap \cdots \cap Q_n$. (To each S_p-subgroup associate the subposet of F/H with $F \subset Q$. This give a covering with contractible intersections whose nerve is the simplicial complex).

~~If $K(G,H)$ is connected, then for every pair Q_0, Q_1 of S_p-subgroups cont~~

For every 1-simplex $\{Q_0, Q_1\}$ of $K(G,H)$ one has

$Q_2 \cap Q_2' > H$, hence $\alpha_{Q_2}|H = \alpha_{Q_2'}|H$. Thus the fact that $\exists$ 2 vertices Q_1, Q_2 of $K(G,H)$ with $\alpha_{Q_1}|H \neq \alpha_{Q_2}|H$ implies that $\pi_0 K(G,H)$ hence $\pi_0 \mathcal{S}_p(N_G(H)/H)$ has at least 2 elements.

Next choose an S_p-subgp Q of G such that $Q \cap N_G(H) = N_Q(H)$ is an S_p-subgp containing $N_{Q_1}(H)$. As $N_{Q_1}(H) > H$, and $Q \cap Q_1 \supset N_{Q_1}(H)$ it follows that $\alpha_Q|H = \alpha_{Q_1}|H \neq \alpha_{Q_2}|H$. Thus we can suppose Q_1 chosen so that $N_{Q_1}(H)$ is an S_p-subgroup of $N_G(H)$. ~~Similarly we can do the same for Q_2.~~ Now by an inner autom. we can replace Q_1 by P, in which case H becomes a subgroup of P with properties i) and ii).

Now choose an $x \in N_G(H)$ such that $N_{Q_2}(H)^x \subset N_P(H)$. Then $H < N_{Q_2}(H)^x \subset Q_2^x \cap P$ so

$$\alpha_P|H = \alpha_{Q_2^x}|H = (\alpha_{Q_2}|H)^x$$

However by hypothesis $(\alpha_P|H)^{x^{-1}} = \alpha_P(H)$. Thus we get $\alpha_P|H = \alpha_{Q_2}|H$ a contradiction. QED

<u>Example</u>: Suppose $H = P \cap Q$ is a maximal Sylow intersection, i.e. $P \cap R > P \cap Q \Rightarrow P = R$ for any S_p-subgroup R.

Then if we choose R so that $R \cap N_G(H)$ is an S_p-subgroup of $N_G(H)$ containing $N_P(H) > H$, we have $R \cap P > H$

so $R = P$, i.e $N_P(H)$ is an S_p-subgp of $N_G(H)$. ~~Choosing $x \in N_G(H)$ so that xN~~ If then S is any S_p-subgp of G containing H, we can choose x so that $x N_S(H) x^{-1} \subset N_P(H)$, so

$$(xSx^{-1}) \cap P \supset x N_S(H) x^{-1} \supset H$$

so $xSx^{-1} = P$. Thus $N_G(H)$ transitively permutes the S_p-subgroups containing H, and these Sylow groups are disjoint over H (i.e. have intersection H.) Look at $N_G(H)/H$. Any two Sylow groups are disjoint.

<u>Question:</u> What are the groups having disjoint S_p-subgroups?

Such a group has $\mathcal{S}_p(G) \sim G/N(P)$. <u>Example:</u> $GL_2(\mathbb{F}_q)$

May 2, 1976

Consider the case where P is abelian. If $H = P \cap Q$, then P, Q are both S_p-subgroups of $C_G(H)$, hence conjugate under an element of $C_G(H)$. Since $C_G(H)$ acts trivially on $H^*(H)$ the condition on α due to P, Q is vacuous, so $H^*(G) = H^*(N_G(P))$. This shows that we don't yet have fusion in good shape.

H will be OKAY if the stabilizer of $\alpha/H \in H^*(H)$ as a subgroup of $N_G(H)$ acts transitively on the components of $\pi_0 \mathcal{S}_p(N_G(H)/H)$. This stabilizer contains $N_G(H) \cap N_G(P)$ and $HC_G(H)$.

Suppose that $[HC_G(H) : H] \equiv 0 \ (p)$ and that H is critical. Let P, Q be in different components of $L_H(G)$ such that $N_P H$, $N_Q H$ are S_p-subgroups of $N_G(H)$. Then because $HC_G(H) \triangleleft N_G H$ its intersection with any S_p-subgroup of $N_G H$ is an S_p-subgp of $HC_G(H)$.

$$\begin{tikzcd}
N_P(H) \arrow[rr, dash] \arrow[d, dash] & & N_G(H) \arrow[dd, dash] \\
P \cap HC_G(H) \arrow[r, dash] \arrow[d, dash, "\neq"'] & HC_G(H) \arrow[ur, phantom, "\triangleleft"] \arrow[d, dash] & \\
H \arrow[r, dash, "\neq"'] & Q \cap HC_G(H) \arrow[r, dash] & N_Q(H)
\end{tikzcd}$$

So then we can conjugate $P \cap HC_G(H)$ into $Q \cap HC_G(H)$ via an element of $HC_G(H)$. Thus H will be OKAY.

So if H is a bad p-subgroup, then we see that $HC_G(H)/H \simeq C_G(H)/Z(H)$ must be a p'-group, i.e. any p-element of $C_G(H)$ must be in $Z(H)$. In particular

i.e. any p element central. H must be in H_i

H contains the center of any S_p-group containing it.

Grün's theorem. Let $\alpha \in \operatorname{Im}\{H^*(N_G(ZP)) \longrightarrow H^*(P)\}$. Since ZP char. in P, $N_G(P) \subset N_G(ZP)$ so α_P is invariant under $N_G(P)$. Let H be critical in P. Then we've seen that w.m.a. $ZP \subset H$. So we get

$$\begin{tikzcd}
 & P \arrow[d, dash] & & \\
 & N_P(H) \arrow[r, dash] \arrow[d, dash] & N_G(H) \arrow[d, dash] & \\
 & H \arrow[r, dash] & N_Q(H) \arrow[r] & Q \\
 ZP \arrow[ur, phantom, "\subset"] & ZQ \arrow[u, phantom, "\cup"] & &
\end{tikzcd}$$

Assume G is p-normal: $ZP \lhd Q$. Then P, Q are S_p-subgps of $N(ZP)$ so $Q = P^x$, $x \in N(ZP)$, so $ZQ = (ZP)^x = ZP$. In fact one sees directly that because $P, Q \subset N(ZP)$ α_P and α_Q have the same restriction to H, since they come from a class in $H^*(N_G ZP)$.

In Thompson's approach to normal p-complements the idea somehow is to deduce the conclusion that G has a normal p-complement from this assumption on groups $N(H)$ where H is a char. subgroup of P.