April 7, 1976

A finite group \( p \) prime number
say \( G \in \mathcal{G}_p \) if \( G \) has a normal \( p \)-subgroup \( P \)
such that \( G/P \) is cyclic. \( \mathcal{G}_p \) closed under subgroups.
If \( G \in \mathcal{G}_p \) and \( X \) is an \( F_p \)-acyclic \( G \)-space, then
\( X^p \) is \( F_p \)-acyclic (Smith), and so
\[
X(G^G \cong X((X^p)^G) = 1 \quad \text{(Lefschetz)}.
\]

Stratification of a \( G \)-space \( X \): Let \( \mathcal{T}_G \) be the
set of isomorphism classes of transitive \( G \)-sets, same
as conjugacy classes of subgroups. Partially order
\( \mathcal{T}_G \) by saying \( X \leq Y \) iff \( \exists G \)-map \( Y \to X \); equivalently
\( G/H \leq G/K \) iff \( K \) conjugate to a subgroup of \( H \). Reason
for this ordering is that a general \( G \)-space \( X \) will
be built up starting from \( X^G \) and ending with
free orbits.

A "family" of subgroups \( F \) in \( G \) is the same as
an open subset of \( \mathcal{T}_G \) (corresponds to an open subset
of \( X \)). A "cofamily" (subgroups closed under enlarging)
corresponds to a family of supports, i.e., a closed
set of \( X \).

Suppose \( Y \) is a \( G \)-space, and let's consider the
problem of embedding $Y$ in an $F_p$-acyclic acyclic $G$-space without changing the fixed point set. (Thus $Y \in CY$ is out).

If $Y$ is $F_p$-acyclic, nothing to do. If $Y$ not $F_p$-acyclic, we might try to attach orbits of type $G$ to $Y$ to get an $F_p$-acyclic $X$. If so, then $G$ acts freely on $X - Y$, so $X^H = Y^H$ for $1 < H \leq G$. It follows that $Y^H$ has to be $F_p$-acyclic for $H$ a $p$-group, and that $X(Y^H) = 1$ if $H \not\in Y^p$.

We consider this special case: for all $1 < H \leq G$ if $H$ is a $p$-group, then $Y^H$ is $F_p$-acyclic, and if $H \in Y^p$, then $X(Y^H) = 1$. By attaching free $G$-orbits to $Y$ we obtain a $G$-space $Y'$ which is an $(n-1)$-connected $n$-complex. Claim $\tilde{H}_n(Y')$ is a projective $F_p[G]$-module. Pf: Let $G_p$ be a Sylow $p$-subgroup of $G$. If $1 < H \leq G_p$, then $Y^H = Y^H$ is $F_p$-acyclic by hypothesis, so

$$\bigcup_{1 < H \leq G_p} Y^H$$

is $F_p$-acyclic.

So

$$\tilde{H}_n(Y') = \tilde{H}_n(Y'/\bigcup Y^H)$$

If $Y_2 = Y'/\bigcup Y^H$, then one has $Y_2$ is a $(n-1)$-connected $n$-complex, so I exact sequence

$$0 \rightarrow \tilde{H}_n(Y_2) \rightarrow C_n(Y_2, \mathbb{Z}) \rightarrow \cdots \rightarrow C_0(Y_2, \mathbb{Z}) \rightarrow 0,$$

and $G_p$ acts freely on $Y_2 - \{x\}$ so $C_i(Y_2, \mathbb{Z})$ are $F_p[G_p]$-free $\Rightarrow \tilde{H}_n(Y_2)$ is $F_p[G_p]$-projective $\Rightarrow \tilde{H}_n(Y')$ is $F_p[G]$-proj.
Claim 1: $\tilde{H}_n(Y)$ is a free $F_p[G]$-module. We will use the known fact that $K_0(F_p[G])$ embeds in $\tilde{R}_p(G)$ which in turn embeds in complex central functions on $G$ via the Brauer character. Let $g \in G$ be of order prime to $p$. Since $\tilde{H}_n(Y)$ lifts to $\tilde{H}_n(Y,Z)$ which is free over $\mathbb{Z}$, the Brauer character of $\tilde{H}_n(Y)$ evaluated on $g$ is the trace of $g$ on $\tilde{H}_n(Y,Z)$ which by Lefschetz and fact $Y \sim V^{s^n}$ is $\pm \chi(Y^g) - 1$.

$$\chi(Y^g) = 1 + (-1)^n + \text{tr}_g \text{ on } \tilde{H}_n(Y,Z)$$

By hypothesis $\chi(Y^g) = 1$ if $g \neq e$. Thus $\tilde{H}_n(Y)$ and $F_p[G]$ have proportional Brauer characters.

It seems necessary now to assume in addition that $\chi(Y) \equiv 1 \pmod{|G|}$, this is necessary that $\chi \chi \text{-acyclic with } X \cap G$-free. If I assume this then $|G|$ divides the rank of $\tilde{H}_n(Y)$, so I then know that $\tilde{H}_n(Y)$ is $F_p[G]$-free.

To finish I need to know the dimension of $\tilde{H}_n(Y)$ is divisible by $|G|$. We know it is divisible by $|G_0|$, since it is free over $F_p[G_0]$. If $g$ is a prime $\neq p$, then the character of $\tilde{H}_n(Y,Z)$ as a rep of $G_0$ vanishes at all elements $\neq e$, hence $\tilde{H}_n(Y,Z) \otimes \mathbb{Q}$ is proportional to $\mathbb{Q}[G]$. 
which means $\tilde{H}_n(Y, Z) \otimes \mathbb{Q}$ is free over $Q[G_\mathfrak{p}]$. (The point is that the trivial repn. occurs only once in the regular repn.) So $|G|$ divides dim $\tilde{H}_n(Y)$, for all $g$, so we win. Therefore the fact involved is:

Assertion: If $M$ is a projective $\mathbb{F}_p[G]$-module whose Brauer character vanishes at all $p'$-elements not the identity, then $M$ is free.

Proof: $M$ free over $\mathbb{F}_p[G_p] \Rightarrow |G_p|$ divides dim $M$.

Over $G_p$, $M$ has the same character as a multiple of $\mathbb{F}_p[G_p]$, hence $M$ is free over $\mathbb{F}_p[G_p]$. Because the trivial repn. occurs only once in $\mathbb{F}_p[G_p]$. Thus $|G|$ divides dim $M$. So $|G|$ divides dim $M$ so $M$ has the same character as an integral multiple of $\mathbb{F}_p[G]$, so $M$ is free over $\mathbb{F}_p[G]$. QED.

Summarizing we have proved:

Proposition: If a $G$-space such that for all $1 \leq H \leq G$

one has

\begin{itemize}
  \item[(a)] $H \in G_p$, \hspace{1cm} $Y^H \mathbb{F}_p$-acyclic
  \item[(b)] $H \in \mathfrak{p}$, \hspace{1cm} $\chi(Y^H) = 1$.
\end{itemize}

Then $Y < X$ with $X \mathbb{F}_p$-acyclic and $X-Y$ $G$-free.
Next suppose we have a $G$-space $Y$ which we want to embed in an acyclic $G$-space, without changing fixpoint set. There is no problem if (a) b) hold for all $1 \leq H \leq G$. So let us assume this is not true and let $H$ be maximal such that either a) or b) fail. We want then to attach $G/H$ orbits to $Y$, so as to remedy the situation.

Suppose $H$ is a $p$-group. Let $N = N_G(H)$. Consider $Y^H$ as a $N/H$-space. Then for all $1 \leq H' \leq N/H$ we have $(Y^H)_{H/H} = Y^H$, so a) and b) hold for $Y^H$ as an $N/H$-space. Then by the prop. I get an $F_p$-acyclic $N/H$-space $Z$ containing $Y^H$ such that $Z - Y^H$ is $N/H$-free. Put

$$Y_1 = (G \times^N Z) \cup_{G \times^N Y^H} Y$$

Then $Y_1 - Y = G \times^N (Z - Y^H)$ consists of $G/H$-orbits, and $Y_1^H = Z$ is $F_p$-acyclic.

Next suppose $H \not\in \mathbb{F}_p^*$, but that $H$ is not a $p$-group. Here I want to embed
$y^H$ in $Z \ni 2 - y^H$ is $N/H$-free such that $X(Z) = 1$. Clearly necessary and sufficient that $X(y^H) = 1 \mod |N/H|$. 

I get stuck at this point, so it is necessary to introduce some extra condition. The point is that $X(\bigcup_{H \leq K \leq N} H^N K^N) = 1 \mod (N/H)$, which is a condition involving subgroups larger than $H$. Oliver's method to get around this point is to suppose given an element $\varphi = [V]$ in $A(G)$ satisfying the Euler conditions:

$$X(V^H) = 1 \quad H \in \mathcal{P}, \quad \forall 1 \leq H \leq G$$

Next one wants to construct an $F_p$-acyclic $X$ with $[X] = [V]$, so one wants the conditions for each $1 \leq H \leq G$

\(\alpha\) \quad $H \in \mathcal{P}$ \quad $X^H$ \quad $F_p$-acyclic

\(\beta\) \quad $X(x^H) = X(V^H)$.

Suppose $Y$ is a $G$-space, and let $\overset{\cdot}{Y}$ be a maximal subgroup not satisfying both $\alpha$ and $\beta$. If $H \in \mathcal{P}$, then we can apply the proposition to the $(N/H)$-space $y^H$ to get a $Y_1$ satisfying $\alpha$ and $\beta$ for $H$ and for those subgroups preceding $H$. If $H \notin \mathcal{P}$, then we want to attach orbits of type $N/H$ to
Let $F \subseteq A(G)$ be the ideal of $LV$. Let $J = LV$. Let $X$ be the family of all subgroups of $V$ that are $G$-spaces. Theorem: Let $J \in A(G)$. Let $J$ be the ideal of $X$. Let $Y$ be the family of all subgroups of $V$. Then $X(Y) = 1$. Furthermore, if $Y$ is a $G$-space, then $X(Y) = 1$. This is possible if $x(y) = x(v(y))$. But...
Then we have $J_p \rightarrow \mathbb{Z}$, $[V] \mapsto x(V^G)$. Let $Y$ be a complex with $x(Y) \in \chi_0(J_p)$, better such that $F[V]_{p-1}$ in $J_p \rightarrow x(Y) = \chi(V^G)$. If $G$ is not a $p$-group, then conditions (a), (b) hold for all $H \in F_p$, so we get an $F$-acyclic space $X$, with $[X] = [V]$ in $A(G)$, such that $X^G = Y$. In general $\chi_0(J_p) = m_p(G)\mathbb{Z}$, where $m_p(G)$ in principle can be determined by doing some algebra in $A(G)$.
Let $X$ be a $G$-complex such that $\forall 1 < H \leq G$, $X^H$ is contractible or empty, and $X^H$ is contractible for each non-zero $p$-subgroup for each $p$ dividing $|G|$. Claim one can add free orbits to $X$ to make it contractible.

We can make $X$ an $(n-1)$-connected $\Lambda$ complex by attaching free $G$-orbits. We need to know that $\tilde{H}_n(X)$ is stably $\mathbb{F}[G]$-free, and we know it is projective because $X^H$ is contractible for each non-zero $p$-subgroup $H$. A theorem of Swan tells us that $\tilde{H}_n(X) \otimes \mathbb{F}_p$ is $\mathbb{F}[G]$-free (same theorem used by Brown: $\tilde{H}_n(X) \otimes \mathbb{Q}[G]$-free if Brauer char. theory). Hence we can attach free $G$-orbits to $X$ to get an $\mathbb{F}_p$-acyclic $X^p$ containing $X$ such that $X^p = X^H$ for $1 \leq H \leq G$.

$X^p: \mathbb{F}_p$-acyclic $\Rightarrow X^p$ acyclic except at a finite set of primes. Recall that the reduced homology of the join $A \star B$ is $\tilde{H}_x(\Lambda(A) \otimes \tilde{H}_x(B))$ shifted up one degree:

$$0 \rightarrow \tilde{H}_x(A \star B) \xrightarrow{2} \tilde{H}_{x-1}(A \star B) \rightarrow \tilde{H}_{x-1}(A) \otimes \tilde{H}_{x-1}(B) \rightarrow 0$$

Thus for some choice of primes $Y = X_{p_1} \star X_{p_2} \star \cdots \star X_{p_k}$ will be contractible. Thus

$$Y^H = X_{p_1}^H \star \cdots \star X_{p_k}^H = X^H \star \cdots \star X^H$$
is contractible at empty when $X^H$ is. Now let $f: X \to Y$ be the inclusion $X \subseteq X^*_p \subseteq X^*_p \times \cdots \times X^*_p = Y$, and let $\tilde{Y} = \text{Cone}(f; X \to Y)$. Then

i) $\tilde{Y}^H \cong pt \quad \forall \; 1 \leq H \leq G$

ii) $\tilde{H}_n^*(\tilde{Y}, Z) = \tilde{H}_n^*(X, Z)$.

From i) we know $\bigcup_{1 \leq H \leq G} \tilde{Y}^H \cong pt$, so $H_{n+1}(\tilde{Y}, Z) = \tilde{H}_n(X, Z)$ is $Z[G]$-stably free. Thus there are no obstructions for attaching $G$-orbits to $X$ to make it contractible.

Suppose now $F$ is a family of subgroups and we want to construct a $G$-space $X$ such that $X^H$ is contractible or empty according to whether $H$ is in $F$ or not. Start with a maximal $H$ in $F$ and with $Y = G/H$. $Y^H = (G/H)^H = NH/H$ is a point because $NH = H$. (Recall that $H \leq K$ and $K/H$ solvable $\Rightarrow$ $K \lhd H$ both in or both outside of $F$.

Thus $H$ maximal in $F \Rightarrow NH = H$.) Further, constructed a $G$-space $X$ with $x_0$ not in $F$ (and $3 \times X^H$ is contractible or empty). Let $H$ be a maximal subgroup in $G$ and $x_0$.

Suppose given a $G$-space $Y$ will all isotropy groups
in $F$, let $H$ be a maximal subgroup in $F$ such that $X^H$ is not contractible. Then for $H < K < NH$ we have $(Y^H)^K/H = Y^K$ is contractible if $K \in F$, $\phi$ if $K \notin F$, so by the preceding stuff, we can attach free $NH/H$ orbits to $Y^H$ to make it contractible. Then we have enlarged $Y$ by $G/H$-orbits so that $X^H$ is contractible without changing other orbit types. It follows that $X$ has isot. $J$ as in $F$, that $X^K = Y^K$ unless $(G/H)^K \neq \phi$ i.e. $K \rightarrowtail H$. (Maybe the good way is to consider the family of $H \trianglerighteq X^K$ for some $K \supseteq H_\phi$, $K \in F$), seems okay.

Should I similarity between Oliver theory + Hatcher theory.
Suppose $X$ is a $G$-space, $\Rightarrow X^H$ is contractible or empty according to whether $H$ is solvable or not. Remove from $X$ the free orbits to obtain a $G$-space $Y = \bigcup_{H \leq G} X^H$.

Let $J$ be the poset of solvable non-trivial subgroups of $G$. Then $V H \in J$ we have a subset $Y^H$ of $Y$ which is contractible.

\[
\begin{align*}
\{ g \mapsto \prod_{H_0 < < H_g} Y^H_g \} & \rightarrow Y \\
\end{align*}
\]

So it seems then that $Y$ is of the homotopy type of the poset $J$. However $Y$ need not be $G$-homotopy equivalent to $J$, for there might be a solvable subgroup whose normalizer is not solvable.

Let $X$ be a $G$-space such that $H \leq G \Rightarrow X^H \simpt$ or $\simpt$. Attaching free $G$-orbits to $X$, I can assume $X$ is an $(n-1)$-connected $n$-complex.

\[ \tilde{H}_n(X \times X) = \tilde{H}_n X \otimes \tilde{H}_n X. \] If $P$ is a projective $\mathbb{Z}[G]$-module is $P \otimes m$ stably-free for some $m$?
\( H \leq G \), \( X \) an \( H \)-space. Then we have Serre's induction process:
\[
\tilde{X} = \text{sections } \{ G \times^H X \to G/H \}^2
\]

Change notation:
\[
X = \text{sections } \{ G \times^{G'} X' \to G/G' \}^2.
\]

Let \( H \leq G \). What is \( X^H \)? It is a product over \( H \backslash G / G' \) of some sort. \( Hg G' / G' \cong \mathbb{1} H / H \sigma g G' \sigma^{-1} \).

\[
X^H = \prod_{H \sigma g G'} (X')^g \text{, } \sigma H \sigma^{-1} \cap G'
\]

So note that this is contractible provided \((X')^g \text{, } \sigma H \sigma^{-1} \cap G'\) is contractible \( \forall g \). Suppose \( X' \) such that \((X')^H \text{, } \sigma H \sigma^{-1} \cap G'\) contractible for all \( 1 \leq H' < G' \) and empty for \( H' = G' \).

Better, suppose \( X' \text{, } H' \) contractible or empty for all \( H' \leq G' \). Then the same is true for \( X \).

Suppose \((X')^H \cong \ast \text{ for } 1 \leq H' < G'\), yet \((X')^G' = \varnothing \).

Then
\[
X^H \cong \ast \text{ if } G' \neq g^{-1} H g \text{ for any } g
\]
\[
= \varnothing \text{ if } G' \leq g^{-1} H g \text{ for some } g.
\]
There might be another approach to Oliver's theorem once the minimal simple groups were understood. The problem is to construct $G$-spaces such that $\forall H$, $1 \leq H \leq G$, $X^H$ is contractible or empty. Call for each such $X$ we get a separating family of subgroups. Separating families are the same as closed subsets in the poset of conjugacy classes of perfect subgroups. Call this poset $I$. Assume inductively that I can find for any $G'$ perfect $\leq G$ a special $G$-space without fixpoints such that any $1 \leq H \leq G'$ has $(x')^H$ contractible. Then inducing $X'$ up to $G$ multiplicatively gives a special $G$-space with $X^H = \emptyset$ iff $G'$ is conjugate to a subgroup of $H$. So this means that for each $x \in I$ I get a special $G$-space associated to the complement of $\{y \geq x\}$, except for $x = [G]$.

$X$, $X_2$ are special $\Rightarrow X_1 \times X_2$ and $X_1 \star X_2$ are special for

$$(X_1 \times X_2)^H = X_1^H \times X_2^H = \begin{cases} \emptyset & \text{if } X_1^H, X_2^H = \emptyset \\ \text{ pt.} & \text{if } X_1^H \otimes X_2^H = \text{ pt.} \end{cases}$$

These give us the usual operations of union $\cup$, intersection $\cap$ for the "supports" in $I$, etc.

So suppose $U$ is a family of open sets in $T$ closed under $\cup$, $\cap$ and containing $\{y \geq x\}$ for all $x$ not largest element of $I$. Let $x_1, \ldots, x_n$ be the
maximal elements of $J$ not the largest. Then if $r > 2$, \( \{y \geq x_1\} \cap \{y \geq x_2\} \neq \emptyset \) would be the largest element of $J$. So there is a problem if $G$ contains a perfect subgroup $G' \leq G$ such that every other perfect subgroup is conjugate to a subgroup of $G'$. For example if $G$ has a minimal simple quotient group $G/N$.

Let $J_p$ be the poset of non-zero $p$-subgroups of $G$. Then for any non-zero $p$-subgroup $H$ we have $J_p^H$ is contractible (Brown). If $g$ is a $p'$-element, I need $X(J_p^g) = 1$ in order to complete $J_p$ to an $\mathbb{F}_p$ acyclic space.

Let $G$ be a perfect group. By Oliver $\exists$ a special $G$-space with $X^H \sim \text{pt}$ for $1 \leq H < G$ and $X^G = \emptyset$. Then consider the non-free part of $X$:

\[ Y = \bigcup_{1 \leq H < G} X^H \]

This has the homotopy type of the poset of proper subgroups of $G$, but not the $G$-homotopy type since $T_G \neq \emptyset$ if $G$ is not simple. If $G$ is simple, then if
If \( K \subseteq J \) is normalized by \( H \), then \( KH < NK \subseteq J \) so \( JH \) is contractible for all \( 1 < H < G \).

April 11, 1976:

**Frobenius thm:** \( H < G \) finite \( \implies H \cap H^x = 1 \) for \( x \not\in H \) \( \Rightarrow N = \{ e \} \cup G - UH^x \) is a subgroup of \( G \).

Such an \( H \) called a Frobenius subgroup.

Let \( X = G/H \). Then \( G \) acts transitively on \( X \) and card \( xg \subseteq \frac{1}{H} \) for \( g \neq e \). Conversely, such an \( X \) is of \( G/H \) where \( H \) is a Frobenius group, or \( H = 1 \).

If \( K \) is a subgroup of \( G \), then each orbit of \( K \) on \( X \) is of the form \( K^x = 1 \) where \( Kn \times Hx^{-1} \) is a Frobenius subgroup of \( K^n \). If \( K \) is nilpotent then the only self-normalizing subgroup of \( K \) is \( K \) itself. Thus

\[ K \text{ nilpotent} \implies Kn \times Hx^{-1} = 1 \text{ or } K. \]

In particular the set \( N \) contains all subgroups of order prime to \( |H| \). (Counting shows \( |N| = 1 + |G| - (G:H)(|H|-1) = 1 = (G:H) \)) so we know \( |N|, |H| \) are rel. prime divisors of \( |G| \).)
Let \( N \) be the poset of subgroups contained in the set \( N \). Consider the poset of cosets of \( N \), which one might denote \( G/N \). Since \( N \) is closed under intersections we have

\[
\begin{array}{ccc}
G/N & \rightarrow & BN & \rightarrow & BG \\
\downarrow & & & & \\
\cup & & & & \\
& K \in N
\end{array}
\]

Assuming Frobenius' theorem, \( N \) has a greatest element, so \( G/N \cong G/N \). Maybe you can directly show that \( H \) acts simply-transitively on \( \pi_0(G/N) \). In any case you have succeeded in geometrically constructing the right representation of \( G \), assuming Frobenius' thm.

Calculate the character \( \chi \) of the representation of \( G \) on \( H_x(G/N, \mathbb{C}) \). If \( h \in H \), and

\[
h (gK_0 < \cdots < gK_m) = (gK_0 < \cdots < gK_m)
\]

then \( hgK_0 = gK_0 \iff g^{-1}hg \in K_0 < N \implies h = e \).

Thus the character vanishes on \( G-N \), because there are no fixed points. If on the other hand \( K \in N \), then \( (G/N)^K \) is the poset consisting of cosets \( gK_0 \) such that \( KgK_0 = gK_0 \).
Feit's calculation: If \( X \) is an irreducible \( \mathbb{F} \)-character of \( H \) of degree \( m_1 \) non-trivial, then because \( H \) is a Frobenius group \( (X-m_1 H)^G = (X-m_1 H)^G \) has the same norm as \( X-m_1 H \), namely \( 1+m^2 \). \( (X-m_1 H)^G = X^G - m(1_H) \) and \( X^G \) does not contain \( 1_G \) (as \( X \neq 1_H \)), and \( (1_H)^G \) contains \( 1_G \) once. Thus \[ (X-m_1 H)^G = \sum a_i X_i - m 1_G \]

\( X_i \) irreducible repn of \( G \neq 1_G \), \( a_i \in \mathbb{Z}, a_i \geq 0 \). So \[ \| (X-m_1 H)^G \| = \sum a_i^2 + m^2 = 1 + m^2 \]

\( \Rightarrow \) exactly one \( a_i = 1 \). \( \therefore \) \( (X-m_1 H)^G = X_i - m 1_G \) and so each non-trivial irreducible repn of \( H \) comes from \( G \).

This shows that \[ (\mathbb{Z}[H] - |H| \cdot \mathbb{Z})^G = \mathbb{Z}[G] - |H| \cdot \mathbb{Z}[G/H] \]
is isomorphic in \( R(G) \) to \( \mathbb{Z}[G/N] - |H| \cdot \mathbb{Z} \), i.e.

\[ \mathbb{Z}[G] = |H| \cdot \mathbb{Z}[G/H] \oplus \mathbb{Z}[G/N] \]

which one can test also by characters.
H acts freely on \( G/N \) \((h \in K = gK \Rightarrow ghg^{-1}eK = h = e)\), so consider \( H \backslash G/N \). I can describe this as the poset formed out of the orbits of the subgroups of \( N \) on \( X = G/H \). It would be nice to show \( X/N \) is contractible. Why connected. I have to show that any two points are connected by a chain:

\[
X_0, \eta_1 \eta_2, \ldots, \eta_k \eta_{k-1} \eta_{k-2} \ldots \eta_1 \eta_0 \]

which \( \eta_i \in N \). So one considers the components of \( X \) defined in this way. Because \( N \) is closed under conjugation, the components are permuted under \( G \). Let us fix \( x_0 = eH \) and let \( S \) be the subgps of \( G \) normalizing the component containing \( x_0 \). Then \( S \) contains \( H \) and all subgps in \( N \), so \( S \) must be all of \( G \) (it contains a typo: subgp for each prime dividing \( |G| \)). One can assume that \( G \) is generated by \( N \).
April 15, 1976

G finite group, H subgroup of G. H is called a Frobenius subgroup if \( H^x = H \) for \( x \notin H \).

Alternative interp. Put \( X = G/H \). Then \( H^x = xHx^{-1} \) is the stabilizer of \( xH \), so \( H \) is a Frobenius group \( \iff \) \( \text{card } (X^g) \leq 1 \) for all \( g \in G \). This condition persists to subgroups \( K \) of \( G \). Thus \( KnH^x \) is a Frobenius subgroup in \( K \) for any \( x \) in \( G \).

Note that if \( H \) is Frob. in \( G \), and \( H \neq 1 \), then \( H \) is its own normalizer, for \( \exists h \neq e \) \( h \in H \) so \( x \notin H \Rightarrow xhx^{-1} \in H^x \) so \( xhx^{-1} \notin H \) otherwise \( xhx^{-1} = e \), which is impossible. Thus \( K \) nilpotent in \( G \Rightarrow KnH^x = K \) or \( 1 \), since \( H' < K \Rightarrow H' \) not its own normalizer.

In particular any Sylow subgroup \( P \) is contained in some \( H^x \) or else intersects each \( H^x \) in \( 1 \), which means it acts freely on \( X \).

Let \( N \) be the subset of \( G \) consisting of the identity and elements without fixpoints on \( X \). We know

\[
G = N + (G:H)(|H|-1)
\]
or
\[
|N| = (G:H)
\]

and we have seen that \( |N|, |H| \) are relatively prime.
factors of \(|G|\). (This is because any Sylow \(p\)-subgroup of \(G\) where \(p\) divides \(|H|\) must intersect \(H_x^*\) non-trivially for some \(x\), hence must be contained in this \(H_x^*\).

Frobenius's theorem says \(N\) is a subgroup, and Thompson's theorem says \(N\) is nilpotent. I want to really understand these theorems.

If \(p\) divides \(|H|\), then \((G:H) \neq 0\) so

\[
\text{res;} \quad H^*(G, F_p) \to H^*(H; F_p)
\]

is injective by transfer. But more is true because

\[
H^*(G, F_p) \to H^*(H; F_p) \to H^*_G((G/H)^2, F_p)
\]

is exact and the \(G\) action on \(X \times X\) is free off the diagonal. Thus one sees that

\[
H^*(G, F_p) \to H^*(H, F_p)
\]

Specifically this works as follows. Given \(\alpha \in H^*(H, F_p)\) induce \(\alpha\) up to \(G\). Then by Mackey formula

\[
\text{Res}_{H \to G} \text{Ind}_{H \to G} (\alpha) = \bigoplus \alpha + \sum_{x \not\in H} \text{Ind}_{1 \to H} \text{Res}_{x \to H} \alpha
\]

\[
= \alpha
\]
Let \( u : H \to A \) be a homomorphism with \( A \) abelian. Then we can induce to \( G \) to get a homomorphism \( G \to A \). Suppose \( u \) is a char. \( X : H \to C^* \). Then \( \text{Ind}_{H \to G} X \) is a \((G:H)\)-dimensional repn. of \( G \). Take its determinant and you get \( X' : G \to C^* \), which restricts to \( X \) on \( H \). Why does \( X' \) vanish on \( N \)? Because any subgroup \( K \) of \( N \) acts freely on \( X \). Hence \( \text{Res}_{K \to G} \text{Ind}_{H \to G} X \) is \( \text{card}(K\backslash G/H) \) copies of the reg. rep of \( K \).

What goes wrong is that the determinant of the regular repn. can be a non-trivial character of \( a \) group. Thus it appears that det of the induced repn. is not the induction we seek.

(When is \( \det \) of \( \square \) regular repn. non-trivial? Fix \( g \). Then \( g \) is a cyclic permutation on \( \langle g \rangle \) so we get

\[
\det (g) = \left( \det g \text{ on } \langle g \rangle \right) \frac{[G:\langle g \rangle]}{[G:G]} = \begin{cases} +1 & \text{order } g \text{ odd} \\ -1 & \text{order } g \text{ even} \end{cases}
\]

Thus the regular repn. has a non-trivial determinant iff the Sylow 2 subgroup is cyclic of even order.)
So in any case one sees that for $\tilde{u}: H \to A$ abelian the induction of $u: \tilde{u}: G \to A$ restricts to $u$ and is trivial on every subgroup of $N$.

Thus we see easily that if $H$ is solvable, then $N$ has to be a group. Use induction on the length of the derived series for $H$.

Frobenius method of proof. Start with an irreducible character $\chi$ of $H$ of degree $m$. Then

$$(\chi - m1_H)^G = \chi^G - mC[G/H]$$

has the same norm as $\chi - m1_H$ because $H$ is Frob. in $G$.

$$\|\chi - m1_H\|^2 = 1 + m^2$$

Since $\chi^G$ doesn't contain $1_G$, $\chi^G - m1_G$ and $C[G/H]$ contains $1_G$ once we have

$$\chi^G - mC[G/H] = \sum_{\chi_i \neq 1} a_i \chi_i - m1_G, \quad a_i \in \mathbb{Z}$$

$$\|\chi^G - m1_G\|^2 = \sum a_i^2 + m^2.$$ 

So

$$\sum a_i^2 = 1, \quad \text{so } \chi^G - mC[G/H] = \chi_i - m1_G \quad \text{where } m = \deg(\chi_i).$$

$\chi_i$ stands for a hom. $G \to GL_m C$.

It remains to see that this homomorphism kills $N$.

But $\chi_i - m1_G = 0$ on $N$, hence $\chi_i = m$ on $N$. 

Now one uses the fact that the value of \( \chi_i \) is a sum of \( m \) roots of unity. Using complex absolute values this can happen only if all roots are \( = 1 \), whence \( N \) has to be killed by \( \chi_i \).

---

Review representations & characters for a finite group \( G \). The group ring \( \mathbb{C}[G] \) can be identified with functions on \( G \)

\[
f \mapsto \sum f(g)g \quad \text{(maybe } f \mapsto \frac{1}{|G|} \sum f(g)g)\]

Then
\[
\sum f_1(g) \sum f_2(g)g = \sum f_1(x)f_2(y)xy
\]
\[
= \sum_{xy = g} \left( \sum f_1(x)f_2(y) \right) g
\]

Hence product in \( \mathbb{C}[G] \) corresponds to convolution of functions

\[
(f_1 \ast f_2)(g) = \sum f_1(x)f_2(y)_{xy = g}
\]

\[
g \sum f(x)x = \sum f(x)gx = \sum f(g^{-1}x)x
\]

Thus the left action of \( G \) on \( \mathbb{C}[G] \) is \( g, f \mapsto f(g^{-1}) \) and the right mult action "" is \( (g, f) \mapsto f \cdot g^\circ \).
As a $G \times G$-module, $C[G]$ is a direct sum of $V_i \otimes V_i^*$ where $V_i$ runs over the different irreducible representations of $G$.

Each irreducible representation $V_i$ of $G$ determines a central idempotent $e_i$ in $C[G]$, which corresponds to a function on $G$ which ought to be the character of the representation.

Suppose $V$ is an irreducible representation of $G$.

$$V \otimes V^* \longrightarrow C^G \quad (v \otimes 1) \mapsto (g \mapsto (g v^\lambda))$$

$$\quad (v \otimes 1) \longmapsto (g \mapsto (g v^\lambda))$$
$$\quad \downarrow$$
$$\quad (g, v \otimes 1) \longmapsto (g \mapsto (g v^\lambda))$$
$$\quad \downarrow$$
$$\quad (g_1^{-1} g_2 v \otimes 1)$$
$$\quad \downarrow$$
$$\quad (g_1^{-1} g_2 v \otimes 1)$$
$$\quad \downarrow$$
$$\quad (g_1^{-1} g_2 v \otimes 1)$$

This shows $\alpha$ is a $G \times G$ map where $G \times G$ acts on $f \in C^G$ by $(g_1, g_2) f = (g \mapsto f(g_1^{-1} g_2))$. Now we have

$$C^G \overset{\beta}{\longrightarrow} C[G] \quad f \longmapsto \int g f(g) g$$

$$\quad (g \mapsto f(g)) \longmapsto \int g f(g) g$$
$$\quad \downarrow$$
$$\quad (g \mapsto f(g_1^{-1} g_2)) \longmapsto \int g f(g_1^{-1} g_2) g = \int g f(g) g_1 g_2^{-1}$$
So $\beta$ is also a $6 \times 6$-map. And we have

$$C[G] \xrightarrow{\gamma} V \otimes V^* = \text{End}(V)$$

$$g \mapsto (v \mapsto g v)$$

Then $\beta \alpha$ is a $6 \times 6$ map from $V \otimes V^*$ to itself, so by Schur's lemma (as $V \otimes V^*$ is irreducible), $\beta \alpha$ must be a multiple of $1$. Thus we have

$$\int (g^{-1} v, j) g x = c \cdot v(x, j) \quad \forall x \in V$$

for any $v \in V$, $j \in V^*$, where $c$ is a scalar to be determined. Rewrite

$$\int \overline{(g, v)} (g x, v) = c \cdot (v, v)(x, j)$$

Now let $v$ run over an orthonormal basis $v_i$ and add up

$$\int \sum_{i} \overline{(g, v_i)} (g x, v_i) = c \sum_{i} (v_i, v_i)(x, j)$$

$$\int (g x, g \lambda) = c \cdot d \cdot (x, j)$$

$$c = \frac{1}{d}$$

where $d = \text{dim} (V)$. 
Therefore one sees that the identity \( \phi : V \otimes V^* \to \sum (g^{-1} \psi_i, \psi_i^*) \)
which goes to the function \( g \mapsto \text{trace } g^{-1} \text{ on } V \), which goes to the element
\[
\frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g \in \mathbb{C}[G]
\]
is \( \frac{1}{d} \) times the central idempotent associated to \( V \).

Good method from Lang's book

\[
e_i = \sum_{\tau \in \mathcal{G}} a_{\tau} \tau
\]

where \( a_{\tau} = \frac{d_i}{|G|} \chi_i(\tau^{-1}) \)

\[
e_i = d_i \int \chi_i(g^{-1}) g
\]

To understand the Frobenius thm., I have to see why \( \chi^G = m \mathbb{C}[G/H] \otimes \chi_i - m 1_G \)
I want to consider the action of a finite group $G$ on a Euclidean space $E$, to consider various $G$-spaces inside of $E$, and the geometry of distances. The basic tools will be an integral lattice inside of $E$, and the metric, so we have the usual machinery from algebraic number theory, a mixture of rigid geometry and integers. This is what character theory also has, so the point is to see if you can get anything new.

Example: $\mathbb{C}[G]$ contains the integral lattice $\mathbb{Z}[G]$.

Suppose $E$ is a Euclidean space on which $G$ acts linearly. Then as a $G$-space $E$ decomposes

$$E = E_1 \times \ldots \times E_k$$

into irreducible representations.

Positive definite function on $G$ is the same thing as a representation of $G$ together with a cyclic vector. Specifically let $V$ be a unitary representation of $G$, and let $\phi_0$ be a non-zero element of $V$. Then we get a map
\[ \mathbb{C}[G] \rightarrow V, \quad g \mapsto gv \]

which is onto if \( v \) is a cyclic vector for \( V \). The inner product on \( V \) lifts to give a (possibly degenerate) inner product on \( \mathbb{C}[G] \).

\[
\| f(g_1)g \|_V^2 = \left( \sum_{g_1} f(g_1)g, \sum_{g_2} f(g_2)g_2v \right)_{g_1} \\
= \sum_{g_1, g_2} f(g_1) \overline{f(g_2)} (g_2^{-1}g, v, v).
\]

The function \( \lambda(g) = (gv, v) \)

is an example of a positive-definite function on \( G \). Positive-definite means simply that the sesqui-linear form \((\star)\) is \( \geq 0 \), i.e., that \( \forall g_1, g_2 \in G \), the matrix \( \lambda(g_1^{-1}g_2) \) is positive semi-definite.

**Example:** Take an irreducible repn. of \( G \times G \) of the form \( W \otimes W^* \) where \( W \) is an irreducible repn. of \( G \).
Suppose \( \lambda(g) = (g_0, \sigma) \) is a positive definite function on \( G \), then
\[
\lambda(g) = \lambda(e) \quad \text{i.e.} \quad (g_0, \sigma) = 0
\]
implies \( g_0 = g_0 - \sigma + \sigma \) is an orth. demp.
so
\[
\|g\|^2 = \|g_0\|^2 = \|g_0 - g\|^2 + \|g\|^2
\]
i.e. \( g_0 = 0 \). Thus \( \{ g \in G \mid \lambda(g) = \lambda(e)^2 \} \) is a subgroup of \( G \); it is the subgroup leaving \( \sigma \) fixed.

For example taking a representation \( W \) of a group \( G_0 \) and letting \( V = W \otimes W^* \), \( G = G_0 \times G_0 \) and \( \sigma = \text{id} = \sum e_i \otimes e_i^* \), then
\[
\lambda(g, g_2) = \left( (g_1 g_2^{-1} \otimes 1) \sigma, \sigma \right)
\]
\[
= \sum_{i,j} \left( g_1 g_2^{-1} e_i \otimes e_i^*, e_j \otimes e_j^* \right)
\]
\[
= \sum_i \left( g_1 g_2^{-1} e_i, e_i^* \right) = \chi(g_1 g_2^{-1})
\]

Then \( \{ (g_1, g_2) \in G_0 \times G_0 \mid \chi(g_1 g_2^{-1}) = \chi(e) = \dim W \} \)
is a subgroup of \( G_0 \times G_0 \) containing \( G_0 \); such subgroups correspond to normal subgroups of \( G_0 \).

**Prop:** \( \lambda(g) \) positive definite on \( G \) \( \Rightarrow \{ g \mid \lambda(g) = \lambda(e) \} \)
is a normal subgroup of \( G \).
Proof: (direct). By definition, for any $g_1, g_2$ in the matrix $\lambda(g_1^{-1}g_2)$ is $\geq 0$. Hence if $g_1, g_2$ are given the matrix

\[
\begin{pmatrix}
\lambda(e) & \lambda(g_1) & \lambda(g_2) \\
\lambda(g_1^{-1}) & \lambda(e) & \lambda(g_1^{-1}g_2) \\
\lambda(g_2^{-1}) & \lambda(g_2^{-1}g_1) & \lambda(e)
\end{pmatrix}
\]

is $\geq 0$. Say $\lambda(e) = \lambda(g_1) = \lambda(g_2) = 1$. Then $\lambda(g_1^{-1}) = \frac{1}{\lambda(g_1)} = 1$, all $\lambda(g_2^{-1}) = 1$. So we get

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & \lambda \\
1 & 1 & 1
\end{pmatrix} \geq 0
\]

which implies first that $1 - |\lambda|^2 \geq 0$, i.e. $|\lambda| \leq 1$. Also the determinant is $\geq 0$, so

\[
\begin{vmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & x-1 & 0 \\
0 & x-1 & 0 & 0
\end{vmatrix} = -(x-1)(x-1) \geq 0
\]

This is possible only if $x=1$.

Unfortunately, pos. def. functions versus representations is a tautology.
Classify positive-definite functions on $G \times G$ right invariant under $\Delta G$. Such a function $\lambda$ is of the form $\lambda(g_1, g_2) = (g_1, g_2) v \cdot v$ where $v$ is some repn. of $G \times G$. Then we've seen that $(g, g) v = v \iff \lambda$ right invariant under $\Delta G$. Better:

\[ \forall g \in G \quad \lambda(g, g) = \lambda(e, e) \implies (g, g) v = v \quad \text{all } g \]

\[ \implies \lambda \text{ is bi-invariant} \]

We should generalize the prop on page 11 to

**Prop:** If $\lambda$ is pos. def. on $G$, then $H = \{ g \mid \lambda(g) = \lambda(e) \}$ is a subgroup of $G$ and $\lambda$ is $H$ bi-invariant.

**Proof:** $\lambda(g) = (g v, v)$. We've seen $\lambda(g) = \lambda(e) \iff g v = v$, so $H = \text{Stabilizer of } G$ is a subgroup. But

\[ \lambda(hg) = (hv, v) = (gv, h^{-1} v) = (g v, v) = \lambda(g) . \]

**QED**

**Prop:** Pos. def. functions on $G \times G$ invariant under $\Delta G$ can be identified with those central functions on $G^2$ which are positive definite linear combinations of characters.
Proof. We know that any pos. def. function is of the form \( \varphi(g_1, g_2) = (g_1g_2v, v) \), where \( v \) is fixed under \( \Delta G \), hence if \( v \) is supposed to be cyclic, the repn. is a quotient of \( C[G \times G]/C[\Delta G] \cong C[G] \) with \( G \) acting by left+right mult. But we know \( C[G] \) is multiplicity 1 so \( v \) must be a multiple of the identity in each irreducible component of \( C[G] \) occurring in \( V \). Best is clear.

The problem is now to start from a Frobenius subgroup and produce a positive-definite function on \( G \) which will give a proper normal subgroup.

Let's see how positive definite translates into for functions on \( G \).

Suppose then \( \varphi(g_1, g_2^{-1}) \) is positive definite on \( G \times G \). This means that if I select \((g_1, \overline{g_1}), \ldots, (g_n, \overline{g_n}) \in G \times G\), the matrix

\[
\varphi((g_i \overline{g_i})(g_j \overline{g_j})^{-1})
\]

is \( > 0 \).

\[
\varphi(g_i \overline{g_i}^{-1} g_j \overline{g_j}^{-1}) = \varphi(g_j \overline{g_j}^{-1} g_i \overline{g_i}^{-1}) = \varphi(g_i \overline{g_i} g_j \overline{g_j})
\]

\[
= \varphi((g_i \overline{g_i} g_j \overline{g_j})^{-1})
\]
where I have used that $\lambda$ is binvariant under $\Delta G \to \mathfrak{g}$.

Therefore $\lambda$ is pos. definite if $\forall g_1, \ldots, g_n \in G$ the matrix $\phi(g_i g_j g_i^{-1})$ is $\succeq 0$.

This means just that $\phi$ is positive definite + central as a function on $G$.

Suppose $H$ is a Frobenius group such that $H \cap G/H$ has 2 elements. This means $H$ acts transitively on the elements of $G/H$ different from $H$. I consider the problem of constructing a positive definite $H$-binvariant function on $G$. The space of $H$-binvariant functions on $G$ is $2$-dimensional.

If $H$ is a Frobenius group, one has

$$\text{Res}_{H \to G} \text{Ind}_{H \to G} \lambda = \lambda + \sum_{H \leq H \neq H} \text{Ind}_{H \to H} \text{Res}_{H \to H} \lambda$$

where this formula takes place in any abelian $\mathbb{F}$-monoid valued functor $F$ in which one has induction. Does this imply $F(G) \to F(H)$ is onto? The answer is yes if $F$ is group-valued. For them we can split $\lambda$ into

$$\lambda - \pi^*(\varepsilon) + \pi^*\varepsilon(\lambda)$$

$$\pi^*\varepsilon(\lambda) = \text{Res}_{H \to e} \text{Res}_{e \to H} \lambda$$
and now it is clear that each piece \( x - \mathcal{T}^*E(x) \), comes from \( G \).

April 20, 1976.

\( H \) Frobenius subgroup of \( G \).

Here is a possible way to construct representations of \( G \) starting from \( H \). Consider a prime \( p \) dividing \( |H| \), and consider the poset of non-trivial \( p \)-subgroups of \( G \); denote this \( S_p(G) \). Obviously

\[
S_p(G) = \bigsqcup_{xH \in G/H} S_p(xHx^{-1})
\]

and the same would hold for any family of subgroups, maybe?

Let \( R \) be a subgroup of \( G \). Assume \( RnN = 1 \).

We know \( RnH \) is Frobenius in \( R \). Assume \( RnH \neq 1 \) and let \( N' \) be the normal subgroup of \( R \) complementary to \( RnH \) for \( n \in R \).

Let \( R \) be a subgroup of \( G \) such that \( RnN = 1 \). Then consider \( X = G/H \) as an \( R \)-space. Assume all proper subgroups of \( R \) have a fixed point on \( X \).
Let $Y$ be the orbit under $R$ of a point of $X^x$ such that some non-trivial element of $R$ fixes $x$. (i.e. $x = gHg^{-1}$ where $R = gHg^{-1}$.) By induction the elements of $R$ not having fixed points on $X$ together with $1$ form a normal subgroup $M < R$ of $R$. If $M \neq 1$, then $M$ has a fixpoint on $X$ by induction, and then $R$ has to preserve this fixpoint set, since $M \triangleleft R$. If $M = 1$, then since we know $M$ acts transitively on $Y$, it follows that $Y$ must be the single point $x$. So we have proved.

**Proof:** If $R$ is a subgroup of $G$ such that $RnN = 1$, then $R$ is contained in a conjugate of $H$.

Suppose $R$ is a group acting on a set $X$ such that $\text{card}(x^R) = 1$ for all $1 \neq r \in R$. Claim that $R$ acts semi-freely on $X$, i.e. if one fixpoint and the action is free on the complement of the fixpt.
April 21, 1976

G, H Frobenius.

Recall

\[ \text{Res}_{H \to G} \left( \text{Ind}_{H \to G} \alpha \right) = \alpha + \sum_{H \times H \neq H} \text{Ind}_{e \to H} \left( \text{Res}_{e \to H} \alpha \right) \]

This shows that \( \text{Res} : A(G) \to A(H) \) is onto. In fact it gives an explicit section as follows. First split

\[ A(H) = A(\mathbb{1}) \oplus \overline{A}(H) \]

\[ \overline{A}(H) = \ker \left( \text{Res} : A(H) \to A(\mathbb{1}) \right) \]

Then \( \text{Ind}_{H \to G} \) is a section of \( A(\mathbb{1}) \to A(H) \).

\[ \text{Res}_{e \to G} \text{Ind}_{H \to G} (\alpha) = \sum_{G/H} \text{Res}_{e \to H} (\alpha) = [G:H] \text{Res}_{e \to H} (\alpha) \]

Question: Is it true that \( \forall \alpha \in \overline{A}(H) \)

\[ \text{Ind}_{H \to G} (\alpha) = \text{Res}_{G \to H} (\alpha) ? \]

Try \( \alpha = [H/H'] - (H:H')1_H \).

\[ \text{Ind}_{H \to G} (\alpha) = [G/H'] - (H:H')[G/H] \]

\[ \text{Res}_{G \to H} (\alpha) = [G/H'N] - (H:H')1_G \]

These are not the same.

Suppose \( \alpha \in R(H). \) Think of \( \alpha \) as a matrix...
Generalized character on $H$ vanishing at $e$. In this case it is clear that
$\text{Ind}_{H \to G}(\chi) = \text{Res}_{G \to H}(\chi)$. More generally suppose
$K$ is a nilpotent subgroup of $G$. Then $K$ is either
contained in $N$, or in a conjugate of $H$.

$$\text{Res}_{N \to G} \text{Ind}_{H \to G}(\chi) = \sum_{N \nmid G/H} \text{Ind}_{N \to G} \text{Res}_{e \to N}(\chi) = 0 \quad \text{if } \chi \in \overline{R}(H).$$
Thus it follows that for $\chi \in \overline{A}(H)$, $\text{Res}_{G \to H}(\chi)$
and $\text{Ind}_{H \to G}(\chi)$ have the same restriction to all
subgroups contained either in $N$ or in a conjugate
of $H$, in particular to all nilpotent subgroups of $G$.

---

**Question:** Can you find a formula for $\text{Res}_{G \to H}(\chi)$ in terms of $\text{Ind}_{H \to G}(\chi)$ and various
connection terms?

Suppose $H$ cyclic of prime order so that there's
only one $\chi$ to consider $[H]|[G] - |H|.1_H$. Then

$$\text{Ind}_{H \to G}(\chi) = [G] - |H|.1_H,$$

$$\text{Res}_{G \to H}(\chi) = [G/N] - |H|.1_G.$$

Obviously not the same because $G$-fixpts are different.
Let $R$ act on a set $X$ such that $1 < H < R \Rightarrow \text{card}(X^H) = 1$. Then the homology $H_0(X)$ should be a stably free $\mathbb{Z}[R]$-module, hence I should be able to complete $X$ to a tree by adding free orbits of $0$ and $1$-simplices. However, Serre has proved that any finite group acting on a tree has a fixpoint.

Check this carefully. I want to attach free orbits of dim $1$ to $X$ to make a contractible graph. So consider $\tilde{H}_0(X)$, which is an integral representation of $G$. If I restrict to a Sylow $p$-subgroup $P$ of $G$, then I know that $\text{card}(X^P) = 1$, hence $\tilde{H}_0(X^P)$ is free over $\mathbb{Z}[P]$. Thus $\tilde{H}_0(X)$ is $\mathbb{Z}[G]$-projective, and so $\tilde{H}_0(X \otimes \mathbb{Q})$ is free over $\mathbb{Q}[G]$. The reason this doesn't work is that not every element of $\tilde{H}_0(X)$ can be represented by a map $S^1 \to X$. 
April 23, 1976:

Künneth property holds for representations and cohomology. Suppose $A$ is an elementary abelian subgroup of $G_1 \times G_2$. It is contained in $A_1 \times A_2$ where $A_i = \text{proj of } A$ in $G_i$, but it is not necessarily equal to $A_1 \times A_2$. However a conjugacy class in $G_1 \times G_2$ is the same thing as a conj class in $G_1$ and one in $G_2$.

\[ \text{Problem: Let } X \text{ be a } G \text{-set such that } \chi^G_H(1) = 1 \text{ for all } 1 < H < G. \text{ Show } X^G = \{1\}, \]

without using the Frobenius theorem.

Different proof of first Sylow theorem.

Use Cauchy thm. that $p$ divides $|G| \Rightarrow G$ contains an element of order $p$. (I direct proof of this using the action of $\mathbb{Z}/p$ on the fibres of $G^p \to G$ over $1$, which is $\{(g_1, \ldots, g_p) \mid g_1g_p = 1\}$. A first element $1 \neq 1$ is an element of order $p$ in $G$.

To use induction on $m$ to show that $p^m | |G| \Rightarrow G$ has a subgroup of order $p^m$. If $Q$ has order $p^{m-1}$, then $(G/Q)^Q = NQ/Q$ has order $\equiv 0 \pmod{p}$, and an element of order $p$ in $NQ/Q$ leads to a subgroup of order $p^m$ containing $Q$.\]
Problem: Let $G$ be a finite group, $A$ be a complete d.v.r. with quotient field of char 0, residue field of char $p$, having enough roots of 1. Then one has an homomorphism of Cartan

$$K_0(P_A(G)) \rightarrow R_A(G)$$

$$K_0(A[G]) \rightarrow K_0(\text{Modf } A[G])$$

which I believe is injective and whose cokernel is killed by a power of $P$. In any case if $P \in P(A[G])$ and if $Q$ is a representation of $G$ over $A$ (free as an $A$-module) then $P \otimes Q$ is in $P(A[G])$ because

$$A[G] \otimes Q = \lambda \ast (A \otimes Q)$$

where $\lambda : L \rightarrow G$. Thus $K_0(A[G])$ is an ideal in $R_A(G)$.

I believe Lusztig shows this ideal is the principal ideal generated by the Steinberg module when $G$ is a Chevalley group. Question: I have seen that the poset of non-trivial $p$-subgroups of $G$ gives an element of $K_0(A[G])$ in fact of $K_0(A[G])$. Can I generalize the Lusztig theorem?
Look carefully at $G = \text{GL}_n(\mathbb{F}_q)$. Let $X$ be the building of $G$, i.e. the poset of proper subspaces of $\mathbb{F}_q^n$. Let $I$ be the poset of $p$-subgroups (non-identity) in $G$, where $q = p^d$. Is there any relation between these two posets?

Let $H$ be a subgroup of $G$. Then $X^H$ is the poset of proper $H$-invariant subspaces of $\mathbb{F}_q^n = V$. $X^H$ is contractible if the socle of $V$ as an $H$-representation, that is, the sum of the irreducible subrepresentations is not all of $V$. In particular, if $H$ is a $p$-group the socle is $V^H$ and this is $\neq V$ if $H \neq 1$. So we see that

$V$ not semi-simple $\iff$ $X^H$ is contractible.

Other case is when $V$ is semi-simple. Then we have an invariant decomposition

$V = V_1 \oplus \cdots \oplus V_m$

where the reps. $V_i$ are disjoint and sums of a single irreducible, i.e. iso-typical. Then an $H$-invariant subspace of $V$ is the same as a family of $H$-invariant subspaces $V_i \subset V_i$.

It should be the case that $X^H$ is of the homotopy type of the join of the posets of $H$-invariant subspaces in each $V_i$, and hence $X^H$ should
be a bouquet of spheres.

Burnside theorem: A is a $p$-subgroup contained in two Sylow groups $P, P'$. A normal in $P$ but not normal in $R$, then there exists $A < H < P$ such that

i) $N_G(H)$ contains a $p'$ element not centralizing $H$

ii) $N_G(H)$ has a Sylow group in which $A$ is normal.

Further if $H < P$ and $|H| > |H'|$ and $H < K < G$ then $A < K$.

Proof: Choose $H$ of max. order such that is the intersection of $N(A)$ with a Sylow $p$-subgroup $S$ in which $A$ is not normal. Because $P$ is a Sylow $p$-subgroup of $N(A)$, we can suppose $H < P$. Let $M = N(H)$.

$\begin{array}{c}
N(A) \\
| \quad \\
\downarrow \quad \\
P \quad \\
| \\
\downarrow \quad \\
\quad H \\
| \\
\downarrow \quad \\
\quad A \\

\end{array}$

$H < P \cap N(H) \\ \leq N(A)$

$p$ a $p$-group

$H < P \cap N(H) \\ \leq N(A)$

by the maximality of $H$. Let $P_j$ be a Sylow subgroup of $N(H)$ containing $P \cap N(H)$. Then $A < P_j$.

Let $K$ be the subgroup of $N(H)$ generated by the $p'$-elements. Since
\[ N(H) = P_1K. \text{ If } K \text{ centralizes } H, \text{ } K \text{ centralizes } A, \text{ so } N(H) = P_1K \text{ would normalize } A, \text{ contradiction. Thus we get a } p' \text{-element normalizing but not centralizing } H, \text{ as the theorem asserts.} \]

April 25, 1976

Let \( H \) be a Frobenius subgroup of \( G \), and \( N \) the kernel. Assume Frobenius theorem known so \( N \) is a subgroup. Let \( P \) be a Sylow subgroup of \( N \). Claim \[ G = N \cdot N_G(P) \]

(Quite generally, this holds for any extension \( N \rightarrow G \rightarrow G/N \) such that all \( S_p \)-subgroups of \( G \) are in \( N \)). It follows that \( N_G(P) \) contains a conjugate of \( H \), hence that there exist Sylow groups of \( N \) invariant under \( H \). Actually Thompson's Thm. says \( N \) has unique Sylow groups.

Next point is that \( H \) has to act first free on \( P \), hence on the subgroup of elements of order \( p \) in the center of \( P \). So we get a representation of \( H \) over \( \mathbb{F}_p \), such that \( h \neq e \Rightarrow \chi(h) \neq 0 \). This should imply the Sylow subgroups of \( H \) are cyclic or generalized quaternion.

(Show it is impossible to have an elementary abelian \( 2 \)-group acting freely on \( V \otimes \mathbb{F}_p \) over \( \mathbb{F}_p \). This means no eigenvalues = 1, but then pass to alg. closure in \( \mathbb{F}_p \).)

Yes.
Observe that $\text{SL}_2(\mathbb{F}_p)$ has all Sylow groups cyclic or generalized quaternion. True for $l = p$. Otherwise one eigenvalue $= 1$ implies both eigenvalues $= 1$, etc.

$$|\text{SL}_2(\mathbb{F}_p)| = \frac{(p^2-1)(p^2-p)}{p-1} = (p+1)(p-1)p.$$ 

If $p \equiv 1 \mod 4$, then the $S_2$ subgroup is $\mathbb{Z}_4 \rtimes (\mathbb{F}_2^\times)^2$

If $p \equiv 3 \mod 4$, it is $\mathbb{Z}_2 \rtimes (\mathbb{F}_p^\times)^2$

$$\text{Ker} \left\{ \mathbb{Z}_2 \rtimes (\mathbb{F}_2^\times)^2 \to \mathbb{F}_p^\times \right\}$$

In both cases the $S_2$ subgroup is generalized quaternion.

Observe any group with only cyclic Sylow groups can't be simple non-abelian. In fact if $p$ is the smallest prime dividing $|G|$, then transfer theory shows that if the Sylow $p$-group is cyclic, then it is in the center of its normalizer: $(p-1)$ is tel. prime to $G$, etc.

Let $\mathcal{E}_p$ be the poset of non-trivial elementary abelian $p$-subgroups of $G$. If $\Theta$ is an element of order $p$ in $G$, then $\mathcal{E}_p$ is the poset of those elementary abelian $p$-subgroups which are normalized.
by $\Theta$, i.e. $\Theta A \Theta^{-1} = A$. Given such an $A$, we can associate the subgroup of elements commuting with $\Theta$, denoted $A^\Theta$. This retracts $A^\Theta$ to the poset $A_p(C_G(\Theta))$. But if $A \in A_p(C_G(\Theta))$, then $\langle A, \Theta \rangle \in A_p(C_G(\Theta))$, so we have the contraction

$$A \leq \langle A, \Theta \rangle = \langle \Theta \rangle.$$  

Assertion: Let $G$ be a finite group, let $A_p(C_p)$ denote the poset of non-trivial elementary abelian $p$-subgroups of $G$. If $P$ is a $p$-subgroup of $G$, then $A_p(C_p)^P$ is contractible.

Proof: We have an inclusion $A_p(C_P(P)) \subseteq A_p(C_p)^P$. If $A \in A_p(C_p)^P$, i.e. $P$ normalizes $A$, then because $P, A$ are $p$-groups, $A^P \cap C_P(P) \neq 1$. So $A \mapsto A^P$ is a map $A_p(C_p)^P \to A_p(C_P(P))$ such that $\pi_i = \text{id}$. Also $\pi_r \leq \text{id}$ for $A^G \subseteq A^G$. So $\pi$ is a homotopy equivalence. Next $A_p(C_P(P))$ is contractible by the core construction for if $B$ is a non-trivial elementary abelian subgroup in the center of $P$ we have

$$A \leq AB \supseteq B$$

so by Brown we get

$$\hat{H}_G^* \cong \hat{H}_G^*(A_p(C_p))$$
For each $P$ in $\mathcal{F}(G)$, the poset $\mathcal{F}_p(G)$ is a homotopy equivalence. The proof is as follows:

It suffices to show that each quotient $i/P$ is contractible. Each $i/P$ is the poset of non-trivial elementary abelian $p$-subgroups of $P$.

For each $B$ in the center of $P$, the quotient $A < AB = B$ so $A_i(P)$ is contractible.

A nice thing about $\mathcal{F}_p(G)$ is that it comes with a filtration by rank. The link is a Tits complex.

Take $G = \text{GL}_n(F)$. Here we have a map from flags to $p$-subgroups given by associating to a flag $0 < W_0 < \cdots < W_k < V$ the subgroup of $G$ normalizing the flag and centralizing the quotients.

$$f: \text{Simp}(	ext{Tits}(V)) \rightarrow \mathcal{F}_p(G)$$

$$\tau < \sigma \Rightarrow f(\tau) \subset f(\sigma)$$

Note that $P \subset f(\sigma) \iff P$ acts trivially on $\text{gr}((\sigma)$.

The problem is whether the poset of flags $\sigma$ such that $P$ acts trivially on $\text{gr}(\sigma)$ is contractible. Call this poset $J.$
Thus I come back to a question raised during our discussion, namely about the poset of chains in $M$ with quotients in the subcategory $B$.

The argument: Put $V = F^n$ and $V/V' = V_\pi$ the largest quotient space on which $P_\pi$ acts trivially. For each $W$, $V' \subset W < V$, let $T_W$ be the closed subset of $T$ consisting of $\tau = w_0 \ldots w_k$ such that $w_k \in W$.

Check $T_W$ is contractible. To any $\tau$ in $T_W$, we can add $W$ thus we get a retraction to flags containing $W$. Case 1: $P$ acts trivially on $W$. Then any $\tau = o < \ldots < W < V$ contains $o < W < V$. Case 2: $P$ acts non-trivially on $W$. In this case, the simplices containing $W$ can be identified with the posets of flags in $W$ such that $P$ acts trivially on $g(W)$. This poset is contractible by induction, so again $T_W$ is contractible.

Now $T = \bigcup T_W$ where each $T_W$ is contractible, $T_W \cap T_{W'} = T_{W_1 \cap W_2}$, and where the poset of $W$ has least element $V'$, $\tau$. $T$ is contractible as was to be shown.

So we have proved.
Another possibility: Instead of just \( p \)-subgroups, I might try the cofibred category whose fibre over \( A \) is \( A \otimes \Omega \). If \( \Theta \) is an element of order \( p \), and \( \Theta \) fixes \( \xi \) in \( A \otimes \Omega \), this means that \( \Theta \) normalizes \( A \). Since \( (A \otimes \Omega)_{\Theta} = (A_{\Theta} \otimes \Omega) \), one sees that \( \xi \) comes from the subgroup \( A_{\Theta} \) in \( C_G(\Theta) \). Unfortunately, if \( \Theta \in A \), it is not the case that \( \xi \) comes from \( \langle \Theta \rangle \otimes \Omega \).

Let \( g \in G \) act on \( A_p(G) \). If \( g \) is not a \( p' \)-element, we can split it \( g = \Theta h \) where \( \Theta h = h \Theta \), \( \Theta \) is a \( p' \)-element \( \neq 1 \), \( h \) is a \( p' \)-element. Then \( gA_{g^{-1}} = A \Rightarrow \Theta A \Theta^{-1} = A \). Since \( \langle \Theta \rangle \triangleleft \langle g \rangle \), \( g \) normalizes \( A_{\Theta} \). Thus if \( \Theta' \) fixes the unique \( p' \)-subgroup of \( \langle g \rangle \), we have \( A \supset A_{\Theta'} \supset A_{\Theta'} \cap A_{\Theta''} \supset \cdots \). Contracting \( A_p(G) \) to a point, hence the character of \( \pi \) the homology of \( A_p(G) \) vanishes at \( g \).

However, any projective \( \mathbb{Z}_p[G] \)-module has this property.

Let \( E \) be a projective \( \mathbb{Z}_p[G] \)-module, and let \( X \) be a finite complex on which \( G \)-acts. Then one has
a triangle of projective complexes
\[ \overline{\mathcal{C}(x)} \otimes E \rightarrow \mathcal{C}(x) \otimes E \rightarrow E \]

which will give us relations in \( K_0(\mathbb{Z}_p[G]) \subset R^+(\mathbb{Z}_p) \).
I wanted to show that the ideal is principal
\[ \overline{\mathcal{C}(p_G)} \] should be enough to show
\[ \mathcal{C}(p_G) \otimes E \] is a multiple of \( \overline{\mathcal{C}(p_G)} \).
\[ \mathcal{C}(p_G) \otimes E \] will be a direct sum of things of the form
\[ \mathbb{Z}[G/N(A)] \otimes E = \text{Ind}_{N(A) \to G}^G \mathbb{R}_0 \]
where \( A \) is a non-trivial elementary abelian subgroup.

Suppose \( G \) has a normal elementary abelian \( p \)-subgroup. Then is \( A_p(G) \) contractible? Call \( A_0 \) this normal elem. abelian subgroup. If \( A \in A_p(G) \) then \( A \) normalizes \( A_0 \), so we have
\[ A \subset A \cdot (A_0)^A \supset (A_0)^A \subset A_0 \]
Unfortunately increasing \( A \) decreases \( (A_0)^A \).

However, suppose \( G \) has a normal \( p \)-subgroup \( Q \), whence it has a normal elementary abelian \( p \)-subgroup, namely the elements of order \( p \) in \( Z(Q) \). Then we can contract \( S_p(G) \) by
\[ P \subset P Q \supset Q \]
This shows that $\mathcal{P}(G)$ is $G$-contractible. It follows that $\mathcal{A}_p(G)$ is contractible.

Direct proof. For each $0 < B < A$, let $T_B$ be the sub-set of $\mathcal{A}_p(G)$ consisting of $A$ centralizing $B$. Then $T_B$ is contractible and

$$T_B \cap T_{B_1} \subseteq T_{B_1 B_2}$$

$$U_{T_B} = \mathcal{A}_p(G)$$

Given $A$ the set of $B$ centralizing by $A$ has a largest element $(A_0)$. \quad \begin{align*} \therefore \mathcal{A}_p(G) \text{ is contractible. Some} \end{align*}

argument shows that $\mathcal{A}_p(G)^H$ is contractible for any subgroup $H$ of $G$, so $\mathcal{A}_p(G)$ is $G$-contractible.

Check that $\iota: \mathcal{A}_p(G)^H \subseteq \mathcal{A}_p(G)^H$ is a homotopy. If $P$ is a $p$-subgroup normal by $H$, $A_0 = \text{order }/p$ elements in center then $\mathcal{A}_p(P)^H = \iota/P$ contracts by $A \subseteq AA_0 = A_0$. \quad \therefore \mathcal{A}_p(G) \subseteq \mathcal{A}_p(G)$ is a $G$ homotopy equivalence.

So over Burnside's theorem again. A normal in some Sylow group (i.e. $G : N(A)$ prime to $p$), but not normal in another $S_p$-group $Q$. Choose $Q$ so that $|Q : N(A)|$ is maximal, put $H = Q \cap N(A)$, choose an $S_p$-subgroup $P < N(A)$ containing $A$. Since $N(H) \cap P > H$, any $S_p$-subgroup $\mathcal{A}_p(G)$ containing
$N(H) \cap P$ must be in $N(A)$. So if $P_1$ is an $S_p$-subgroup of $N(H)$ containing $N(H) \cap P$, $P_1 \subset N(A)$.

If $K$ is the subgroup gen. by the $p'$ elements of $N(H)$, then $N(H) = KP_1$. If $K$ centralizes $H$, then $K \subset N(A)$ and we get $N(H) \subset N(A)$. This contradicts $H < N(H) \cap Q$ and $H = N(A) \cap Q$. Thus there exist $p'$-elements in $N(H)$ which do not centralize $H$.

$G = GL_n(F_q)$. Claim that $Sp(6)$ is homotopy equivalent to $X = Tita(H^6_q)$. For each $p$ group $P \subset Sp(6)$ we associate $x^P$. $PCP' \Rightarrow x^P \subset x^{P'}$. We want to apply the acyclic covering argument:

\[ \therefore \bigcup_{PCP'} x^P' \Rightarrow \bigcup_{P \subset Sp(6)} x^P \rightarrow X \]

\[ \therefore \bigcup_{P \subset Sp(6)} x^P \Rightarrow \bigcup_{P \subset Sp(6)} x^P \]

so we need to know two things:

a) $\forall x \in X$ the poset of $p$-subgroups of $G$ stabilizing $x$ is contractible

b) $x^P$ is contractible for each $P$ in $Sp(6)$.

Proof of a). The stabilizer of $x$ is a parabolic subgroup $Q$ of $G$ such that $Q \neq G$, hence the unipotent radical is a non-trivial normal subgroup of $Q$. This implies $Sp(Q)$
Proof of b): Direct in the case of $G_\nu$. Because $P \neq 1$

is a proper subspace of $V = \mathbb{F}_q^n$, which meets each proper $P$-invariant subspace of $V$. Thus $X_P = W \cap V \subseteq V_P$.

Proof in the case of Chevalley groups: Choose a Borel $B$ of $G$ containing $P$; that is the same as choosing a chamber of $X$ fixed by $P$.

According to Tits if one removes from $X$ the centers of the "opposite" chambers to $B$, i.e. those corresponding to Borels $B'$ of $B' \cap B$ is a torus, then the building has a canonical "geodesic contraction" to the center of $B$, where canonical implies invariance under the $B$-action. So next observe that $X_P$ contains no interior point from a chamber opposite to $B$, because $P$ is a $p$-group $\neq 1$ and a torus $B' \cap B$ has only $p'$-elements. Thus the geodesic contraction furnishes a contraction of $X_P$. 

April 27, 1976

New proof of Tits' theorem. Let $X = \text{Tits}(V)$, let $B$ be a Borel, and let $B_u$ act on $X$. For each $KH \leq B_u$ we can directly see that $X^H$ is contractible by the socle argument. Thus $X$ is homotopic to $X/U^H$. But calculation shows that the only simplices of $X$ with free $B_u$-orbit are the opposite chambers. Thus $X/U^H$ is a bouquet of spheres indexed by the opposite Borels.

Let $H$ be a subgroup of $G$ having a normal $p$-subgroup $1 \neq B < H$. Then $(\mathfrak{I}_p(G))^H$ is contractible, i.e., if $Q \in \mathfrak{I}_p(G)^H$, then $Q \leq QB \geq B$.

Let $\mathfrak{I}_p(G)$ be the poset of subgroups $H$ of $G$ having a non-trivial normal $p$-subgroup. Such an $H$ has a non-trivial normal elementary-abelian $p$-subgroup $B$. If $A \leq \mathfrak{I}_p(H)$, then $A \supseteq C_A(B) \leq \mathfrak{I}_p(C_H(B))$, so $\mathfrak{I}_p(H)$ deforms to $\mathfrak{I}_p(C_H(B))$ which then deforms to a point by the construction $A \leq AB \geq B$. So again $\mathfrak{I}_p(G) \leq \mathfrak{I}_p(G)$ is a homotopy equivalence. Much easier to show that $\mathfrak{I}_p(G) \leq \mathfrak{I}_p(G)$ is a hseg.
April 28, 1976

Let \( G = \text{Gl}_n(\mathbb{F}_q) \), \( X = \text{Tits}(\mathbb{F}_q), \) I have seen that \( \text{Sp}(G) \) is beg to \( X \). I want to see whether \( \text{Sp}(G)^H \) is beg to \( X^H \) for any subgp \( H \) of \( G \).

To each \( P \in \text{Sp}(G)^H \) I associate \( X^{HP} \) which is contractible. In effect if \( K \) is a group with a non-identity \( p \)-subgroup \( P \) then \( X^K \) can be contracted as follows: \( W > W^P < V^P \). To finish I have to see that \( \forall x \in X^H \) the poset of \( P \in \text{Sp}(G)^H \) such that \( x \in X^{HP} \) is contractible. This poset is \( \text{Sp}(G_x)^H \), \( G_x \) has a non-identity normal \( p \)-subgroup \( (G_x)_u = Q \). Then \( P \subset P \cap Q \supset Q \) contracts \( \text{Sp}(G_x)^H \) to a point.

Alperin's thm: One fixes a Sylow \( p \)-subgp \( P \) and then considers the other \( \text{Sp} \)-subgroups. Given a \( \text{Sp} \)-subgp \( Q \) one is going to construct a path from \( Q \to P \) of a special sort such that the size of \( Q \cap P \) increases as one goes along the path. Write \( R \sim Q \) to mean there is such a path. The path is given by \( Q_1, \ldots, Q_n \) \( x_i \in N_G(P \cap Q_i) \) \( x_i \) \( p \)-elt.

\[ P \cap R < P \cap Q, \quad (P \cap R)^{x_i} \prec P \cap Q_{i+1} \]

It seems to be more intricate.
Alperin's Theorem: Let $A, B$ be subsets of the $Sp$-subgroup $P$ which are conjugate in $G$: $A^x = B$. Then one can find $Sp$-subgroups $Q_1, \ldots, Q_m = P$ intersecting $P$ tamely, and $P$-elements $x_i \in N_G(P \cap Q_i)$ such that

$A \subseteq P \cap Q_1$

$A^{x_i}x_i \subseteq P \cap Q_i$ \hspace{1cm} $1 \leq i \leq m$

$x_1 \cdots x_m = x$

$x_i$ is a $p$-element $i < m$

Let's see if I can forget tameness and concentrate instead on the size of intersections.

Gorenstein's generalization involves well-placed tame intersections.

$P$ an $Sp$-subgp of $G$, $H$ any subgroup of $P$.

$W_i(H) = H$ \hspace{1cm} $P_i(H) = N_P(H)$ \hspace{1cm} $N_i(H) = N_G(H)$

$W_2(H) = \varpi P_1(H)$ \hspace{1cm} $P_2(H) = N_P(W_2(H))$ \hspace{1cm} $N_2(H) = N_G(W_1(H))$

$W_3(H) = \varpi P_2(H)$

Call $H$ well-placed if $P_i(H)$ is a $Sp$-subgp of $N_i(H)$ for each $i$. Note that $\varpi P_2(H)$ is char. in $P(H)$ so
\[ \pi_{i+1}(H) = \pi_{i+1}(\pi_i(H)) \geq N(\pi_i(H)) > \pi_i(H) \] if \( \pi_i(H) < \pi_i(H) \). Thus the sequence \( \pi_i(H) \) increases up to \( \pi_i(H) \) and eventually \( \pi_i(H) = \pi_i(J(H) \).

The generalization then says that if \( \pi_i(H) \) is not well-placed tame intersections, one can suppose \( \pi_i(H) \) well-placed tame intersections.

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Application of Alperin's theorem. Let \( x, y \in P \) be conjugate in \( G \). Then by Alperin's theorem we can find tame intersections \( H_i = P \cap Q_1 \), \( 1 \leq i \leq m \)

\[ x = x_0, x_1, \ldots, x_m = y, \quad x_i, x_i \in H_i, \quad x_i y_i = x \]

\( y_i \in N_G(H_i) \)

\[ x^{-1}y = (x_0^{-1}x_1)(x_1^{-1}x_2) \cdots (x_{m-1}^{-1}x_m) = (x_0^{-1}y_1^{-1}x_0y_1) \cdots (x_{m-1}^{-1}y_m^{-1}x_{m-1}y_m) \in \left[ H_1, N_G(H_1) \right] \cdots \left[ H_m, N_G(H_m) \right] \]

Now suppose \( N_G(H_i)/C_G(H_i) \) is a \( p \)-group. Tameness \( \Rightarrow \pi_i(H_i) \) is an \( S_p \)-subgroup of \( N_G(H_i) \). So

\[ \pi_i(H_i) \Rightarrow N_G(H_i)/C_G(H_i) \]

means

\[ N_G(H_i) = (P \cap N_G(H_i))C_G(H_i) = C_G(H_i)(P \cap N_G(H_i)) \]

\[ [H_i, N_G(H_i)] = [H_i, P \cap N_G(H_i)] \quad h^{-1}(xy)^{-1}hxy = h^{-1}y^{-1}hx \]
Thus \([H_i] N_{\mathcal{P} H_i} \subset [H_i] P \subset P'\). It follows that \(P \cap G' \subset P'\), hence \(P \cap G' = P'\). Thus \(P\) has a normal \(p\)-complement.

So what is important, it seems, is the family of \(H \subset P\) such that \(P \cap N_{G}(H)\) is a Sylow \(p\)-subgroup of \(N_{G}(H)\). For example if \(H \trianglelefteq P\) then \(P \cap N_{G}(H) = P\), so this is okay.

Suppose \(G\) has a normal \(p\)-complement \(K\) so that \(G = P \times K\) where \(P\) is a Sylow \(p\)-subgroup. Let \(f: G \to G/K\) be the canonical map. Then

\[
\tilde{f}: S_{p}(G) \to S_{p}(G/K)
\]

is fibred, for if \(Q \in S_{p}(G)\) then \(f(Q) \to \tilde{f}(Q)\) so that subgroups of \(Q\) and \(\tilde{f}(Q)\) are in 1-1 correspondence.

If \(Q \subset P\) what is \(\tilde{f}^{-1}(\tilde{f}(Q))\)? Given \(R \in S_{p}(G)\) with \(f(R) = f(Q)\), then \(R, Q\) are both \(S_{p}\)-subgroups of \(Q \times K\), hence they are conjugate by any element \(g \in K\) of \(Q \times K\): \(k^{-1} g^{-1} Q g k = R \Rightarrow k^{-1} Q k = R\). Thus the fibre of \(\tilde{f}\) over \(\tilde{f}(Q)\) is acted transitively by \(K\). Next note that \(k^{-1} Q k = Q \iff k\) centralizes \(Q\) for \(k^{-1} k = k\) have the same image under \(f\) so they must coincide. Thus

\[
\tilde{f}^{-1}(\tilde{f}(Q)) \cong K/C_{k}(Q) = K/kQ
\]
So we therefore see that $L_p(G)$ is the fibered category over $L_p(P) = L_p(G/K)$ associated to the contravariant functor

$$Q \mapsto K/K^Q.$$ 

If $K^Q = K$, then $Q$ acts trivially on $K$ and conversely.

If $Q = \text{Ker} \{P \to \text{Aut} K\}$,

Then $N(Q) = P \times C_K(Q) = P \times K = G$. Thus $Q_p(G) = 1 \iff P$ acts faithfully on $K \iff K^Q < K$ for $Q \triangleleft P$.

Critical case: Suppose $P$ is an elementary abelian $p$-group acting faithfully on an elementary abelian $l$-group $K$. Is $L_p(G)$ spherical?

Can suppose without changing $L_p(G)$ that $K^p = 1$.

Suppose rank($P$) = 1. Then $Q_p(G)$ has dim. 0.

If rank($P$) = 2, then dim $Q_p(G) = 1$, so we only have to show it is connected. Every component is represented by an element of $K$, and two elements of $K$ are in the same component if they determine the same element of $K/K^Q$ for some $Q \triangleleft P$. Clearly $K$ acts by left mult on $K^Q$ and the action is transitive so $Q_p(G) = K/L$ where $L < K^Q$ for $Q \triangleleft P$. But $L$ because $K$ is a $p'$-group it should be the case that $Q$ has no fixed pts on $K/L \triangleleft K/K^Q$. 

Let $P$ be a fixed $S_p$-subgroup of $G$. Let $Q$ be another $S_p$-subgroup. Can I find a "tame" $H$ in $P$ and an $x \in N_G(H)$ such that

(i) $P \cap Q < H$

(ii) $|P \cap Q^x| > |P \cap Q|.$

Note that (i) $\Rightarrow (P \cap Q)^x < H \subset P \Rightarrow (P \cap Q^x)^x \subset P \cap Q^x$ $\Rightarrow |P \cap Q| \leq |P \cap Q^x|.$ Thus (ii) says the order of the intersection should increase.

Assume this can be done. Then iterating I can construct a sequence of $S_p$-subgroups

$Q, Q^x, Q^{x_1}, Q^{x_1 x_2}, \ldots, Q^{x_1 \cdots x_m} = P$

and tame subgroups of $P$

$H_1, H_2, \ldots, H_{m-1}$

and $x_i \in N_G(H_i)$ such that

$P \cap Q^{x_1 \cdots x_{i-1}} \subset H_i$

$|P \cap Q^{x_1 \cdots x_i}| > |P \cap Q^{x_1 \cdots x_{i-1}}|.$

---

Special case: Suppose you can take $H = P \cap Q, Q^x = P$ Recall that I want $N_G(H)$ to be an $S_p$-subgrp of $N_G(H)$. So if $x \in N_G(H)$ moves $P$ to $Q$, then $N_G(H)$ must be an $S_p$-subgrp of $N_G(H)$. In some sense the tame intersections are like
the walls in the fundamental chamber.

Basic transition is from \( Q \) to \( Q^x \)
where \( x \in N_G(H) \), \( H \) is tame \( \supseteq \) \( P \cap Q \).
If \( H = P \cap Q \) is a tame intersection then

\[ \exists \ x \in N_G(H) \Rightarrow N_p(H) = N_Q(H)^x = Q^x \cap N_G(H), \]
but this doesn’t imply \( P = Q^x \), it seems.

Suppose \( Q \) is immediately related to \( P \).
This means \( \exists \) tame \( H \supseteq P \cap Q \) and \( x \in N_G(H) \Rightarrow Q^x = P^x \).
But then \( H \cap P \Rightarrow H \cap P^x \Rightarrow H \cap P \cap Q \Rightarrow H = P \cap Q \).
Thus \( P \cap Q \) is a tame intersection.

Summary: I consider inside \( P \) those \( H \) such that \( N_p(H) \) is an \( S_p \)-subgroup of \( N_G(H) \).

I write \( Q_1 \rightarrow Q_2 \) to mean \( \exists \) such an \( H \) containing \( P \cap Q \), and an \( x \in N_G(H) \) such that \( Q_1^x = Q_2^x \).

Assertion: \( Q \rightarrow P \) implies \( P \cap Q \) is a tame intersection.

Proof: Let \( H \supseteq P \cap Q \) be such that \( Q^x = P \) for some \( x \in N_G(H) \). Then \( H \cap P \Rightarrow H = H^x \subseteq P^x = Q \Rightarrow H \subseteq P \cap Q \Rightarrow H = P \cap Q \).

Since \( N_G(H) : N_p(H) \neq 0 \) (p) the same is true for \( N_G(H) : N_Q(H) \), so \( H = P \cap Q \) is a tame intersection.
Here's the way to try to understand Alperin's thm. Suppose for every subgroup \( H \) of \( P \) that the restriction of \( x \in H^*(P) \) is invariant under \( N_G(H) \). Try to show then that \( x \) comes from a class in \( G \). We have to prove that for every \( x \in G \), the class \( x \) is equalized by the maps

\[
H^*(P) \xrightarrow{\lambda} H^*(P \cap xPx^{-1})
\]

Put \( Q = xPx^{-1} \). Then we have \( i, j \) are the two maps

\[
\begin{array}{ccc}
H^*(P) & \xrightarrow{\text{res}} & H^*(P_nQ) \\
\Theta & \downarrow & \\
H^*(Q) & \xrightarrow{\text{res}} & \\
\end{array}
\]

where \( \Theta : Q \to P \) is \( \Theta(q) = x^{-1} q x \). How does this depend on \( x \)? If \( n \in N(P) \), then \( x^n P n^{-1} x^{-1} = Q \) and \( (x^n)^{-1} q x^n = n^{-1} \Theta(q) n \). But \( x \) is invariant the action of \( N(P) \) by assumption. Thus the condition that \( x \) is equalized by the arrows \( i, j \) means that after transporting \( x \) to a class on all \( G \)-subgroups, it is compatible with restriction.

Next assume \( x \mid P_nQ \) is invariant under \( N_G(P_nQ) \). If \( Q = xPx^{-1} \) where \( x \in N_G(P_nQ) \), then

\[
\begin{array}{ccc}
\lambda x^{-1} & \to & P_nQ \\
\uparrow & & \uparrow \\
\beta & \to & P_nQ \\
\end{array}
\]

commutes, so the condition is satisfied.
Here is a possible way to view Alperin's theory. The problem is to describe the image of the restriction homomorphism 
\[ \text{res}: \quad H^*(G) \rightarrow H^*(P). \]

One has Eilenberg-Cartan result about stable classes; this means that for each intersection \( P_\alpha \times P_{\beta}^{-1} \), we have to equalize the 2 arrows

\[ H^*(P) \rightarrow H^*(P_\alpha \times P_{\beta}^{-1}) \]

I think what Alperin's result does is to reduce all these equalization conditions to considering just tame \( H \) and the action of \( N_0(G) \). Thus a class of \( H^*(P) \) comes from \( H^*(G) \) iff for all tame \( H \) the restriction of \( \chi \) to \( H^*(H) \) is \( N_0(G) \)-invariant.

Consider \( H \in P \) such that

1. \( N_0(H) \) is an \( S_p \)-subgroup of \( N_0(G) \)
2. \( H \) is the intersection of \( N_0(H) \) and another \( S_p \)-subgroup of \( N_0(G) \).

Then \( H \) is a tame intersection. Proof: Suppose \( Q \) is an \( S_p \)-subgroup of \( N_0(G) \), so \( Q = Q \cap N_0(G) \) for some \( S_p \)-subgroup of \( G \). Then \( H \triangleleft N_0(G) \Rightarrow H \subseteq Q \cap P \triangleleft Q_1 = H \), and \( P = N_0(H) \).
May 1, 1976

Alperin's thm.

Statement of the problem. We know that restriction $H^*(G) \rightarrow H^*(P)$ is injective (mod p coeffs) and that the image consists of classes $x \in H^*(P)$ equalized by the two homomorphisms

$$P \xrightarrow{\Phi_P} P \xrightarrow{\Phi_P} P$$

given by inclusion and $P$-inclusion for any $x \in P$ included in these equalization conditions are of the following types:

which are stable in the following sense. For any subgroup $H$ of $P$ and element $x$ of $G$ such that $x^H x^{-1} \leq P$, the class $x$ is equalized by the two homomorphisms

$$H \rightarrow P \quad h \mapsto h x h^{-1}$$

What Alperin's thm. does is to restrict the number of these equalization conditions to the following types:

(i) $x \in N_G(H)$ (so that $x^H x^{-1} = H$)

(ii) $H$ is a intersection $P \cap Q$ where $Q$ is an $S_p$-subgroup, and this intersection is tame.

So to understand his theorem I have to suppose given $x \in H^*(P)$ such that for certain $H \subset P$ one has $x|H$ is invariant under $N_G(H)$, and then try to prove $x$ stable. Use induction on $P \cap Q$.

If $P \cap Q = P$, then $x \in N_G(P)$ and $x$ is invariant under $x$. Assuming this condition, we know that $x$ invariant under $N_G(P)$.
A determines a definite class $\alpha_\mathfrak{Q} \in H^*(Q)$ for each Sylow group $Q$ of $G$. The problem is to show then that $\alpha_P$ and $\alpha_Q$ have the same restriction to $P^\mathfrak{Q}$.

The next case to consider is where $P^\mathfrak{Q}$ is a maximal Sylow intersection, i.e. $P^\mathfrak{Q} < P^\mathfrak{R} \Rightarrow R = P$ for any $S_p$-subgroup $R$.

**Digression:** What is the homotopy type of the set of $p$-subgroups strictly containing a fixed $p$-group $H$. Put

$$L_H = \{Q \in S_p(G) \mid Q > H\}.$$  

$L_H$ is empty $\iff$ $H$ is a $S_p$-subgroup. Note that $Q > H \Rightarrow N_Q(H) > H$ (normalizer condition).

$$N_Q(H) = N_G(H) \cap Q$$

so if we put

$$L_H^\perp = \{Q \in L_H \mid H < Q\} = \{Q \in S_p(G) \mid H < Q < N_G(H)\}$$
then we have
\[ L^i_H(N_G(H)) \subset L_H(G) \overset{r}{\longrightarrow} L^i_H(G) \]

Q \mapsto Q \cap N_G(H) = N_Q(H).

Then \( \text{tr} = \text{id} \) and \( \text{tr}(Q) = N_Q(H) \subset Q \). So
\[ L^i_H(G) = L_H(N_G(H)) \text{ is reg. to } L_H(G). \]
But
\[ L_H(N_G(H)) = S_p(N_G(H)/H) \]

for any \( H \subset Q \subset N_G(H) \) is in 1-1 corresp. with \( Q/H \subset N_G(H) \).

Suppose \( P \cap Q \) is a max. Sylow intersection
i.e. \( P \cap R \supset P \cap Q \Rightarrow R = P \) for any \( S_p \)-grp. \( R \).
Put \( H = P \cap Q \). So \( P \cap Q \cap R = P \cap R \Rightarrow P \cap Q \cap R \subset P \cap R \Rightarrow \text{either } P \cap R = H \text{ or } R = P. \)
So the set of \( S_p \)-subgps. containing \( H \) looks like:

\[ \begin{align*}
P & \quad \quad Q \\
N_p(H) & \quad \quad R \\
\quad \quad S & \quad \quad \quad \quad H
\end{align*} \]

By maximality, \( P \) is the only \( S_p \)-subgroup of \( G \).
containing $P$. Better: $N_p(H)$ is a $p$-subgroup of $P$ strictly containing $H$, so if $R$ is an $S_p$-subgroup containing $N_p(H)$ one has $R = P$. So if $P_j$ is an $S_p$-subgroup of $N_p(H)$ cont. $N_p(H)$, then $P_j \leq R_j$ so $P_j \leq P$, so $P_j \leq N_p(H)$.

$N_p(H)$ is an $S_p$-subgroup of $N_G(H)$.

To simplify, let me assume that $P \cap Q = H$ is a Sylow intersection with the largest possible order but still $P \neq Q$. In this case the picture of the Sylow graph containing $H$ is

In this case I want to show that $N_G(H)$ transitively permutes the $S_p$-subgroups containing $H$, since $N_G(H) > H$.

Now $Q \cap N_G(H)$ is a $p$-subgroup of $N_G(H)$ so $\exists x \in N_G(H)$ such that $(Q \cap N_G(H))^x = Q^x \cap N_G(H) \subset N_p(H)$

But $Q \cap N_G(H) > H$, so the maximality of $H \Rightarrow Q = P$. 
Proposition: Let $H = P \cap Q$ be a subgroup of $P$ which is maximal with respect to being a Sylow intersection. Claim there exists $x \in N_G(H)$ such that $Q^x = P$.

Proof: $N_P(H) = P \cap N_G(H) > H$ as $H$ is a proper $p'$-subgroup of $P$. Let $P$ be an $S_p$-subgroup of $N_G(H)$ containing $N_P(H)$, and choose an $S_p$-subgroup $R$ of $G$ containing $P$. Then $P \cap Q = H \triangleleft N_P(H) \leq P \cap R$ as $R \leq P$. Thus $P = N_G(H) \cap R = N_P(H)$ is an $S_p$-subgroup of $N_G(H)$.

$N_Q(H)$ is a $p'$-subgroup of $N_G(H)$ hence $Q^x \in N_G(H)$ so that $Q^x \cap N_G(H) = N_Q(H)^x \leq N_P(H)$.

But $N_Q(H) > H$, hence $Q^x$ is an $S_p$-subgroup of $G$ containing $H$ with $Q^x \cap P = N_Q(H)^x > H$, so $Q^x = P$ by maximality. QED.

One can even assume that $x$ is a product of $p'$-elements in $N_G(H)$. In effect, any two Sylow groups $P, Q$ of a group $G$ are contained in the subgroup gen. by $p'$-elements, hence are conjugate in this subgroup.

Note the picture of a maximal Sylow intersection is and $N_G(H)$ permutes the Sylow groups transitively.
Next we want to get the general case.

For each $p$-subgroup $H$ of $G$ let $L_H(G) = \{ H' \mid H'$ is a $p$-group $> H$, $H'$ is properly contained in $H$ $\}$. Recall that we are trying to show that if $H < P \cap Q$, where $P, Q$ are Sylow, then $x_P = x_Q$ when restricted to $H$. Assume this is true for each $H' > H$. Then we have a well-defined function $H' \mapsto \alpha_{H'}$ for all $H'$ in $\mathcal{S}_p(G)$ which properly contain a conjugate of $H$. So there is no problem if $L_H(G)$ is connected.

---

**Digression:** Consider the simplicial complex $K$ whose vertices are the Sylow $p$-groups and whose simplices are subsets whose intersection is non-trivial. This is just the nerve of the covering of $\mathcal{S}_p(G)$ given by the sets $\{ \leq P \}$. Since one has

$$\{ \leq P_1 \} \cap \ldots \cap \{ \leq P_q \} = \{ \leq P_1 \cap \ldots \cap P_q \}$$

the intersections are contractible, so $K$ has the homotopy type of $\mathcal{S}_p(G)$.

Thus we get a deformation of $\mathcal{S}_p(G)$ into the poset consisting of those $p$-subgroups which are intersections of Sylow groups.
I can now prove a version of Alperin's theorem:

**Theorem:** Let $x \in \text{H}^*(P)$ be such that for every tame intersection $H = P \cap Q$, $\text{res}_H^P(x)$ is invariant under $N_G(H)$. Then $x$ comes from $H^*(G)$.

**Proof:** Let $H$ be an $p$-subgroup of $G$. Choose $x \in G$ such that $xHx^{-1} \leq P$. Then we get a homomorphism $i : H \to P$, $h \mapsto xhx^{-1}$, and hence we can pull $x$ back to $H$. Call $H$ good if the class $i^*(x)$ does not depend on $x$, and write $x_H$ for $i^*(x)$. We have to show every $p$-subgroup of $G$ is good. We use decreasing induction on $|H|$. If $H$ is a $p'$-subgroup, then this follows from the fact that $x$ is invariant under $N_G(P)$.

Assume $H$ good for all $|H'| > |H|$, but that $H$ is bad. Then we have two homomorphisms $i_x, i_y : H \to P$ such that $i_x^*(x) \neq i_y^*(x)$. In other words, $H$ is contained in the $p$-subgroups $Q = x^{-1}Px$, $R = y^{-1}Py$ and $Q \neq R$ restrict differently on $H$. Can't have $Q \cap R > H$, by induction.

Let $S$ be an $S_p$-subgroup of $G$ such that $S \cap N_G(H)$ is an $S_p$-subgroup of $N_G(H)$ containing $N_G(Q)$. Since $N_Q(H) > H$, we have $Q \cap S > H$, so by induction $Q \cap S$ have the same restriction to $H$. Also $S \cap R = H$, since $x_S = x_R$ on $H$. Thus replacing $Q$ by $S$ we can suppose $N_Q(H)$ is an $S_p$-subgroup of $N_G(H)$. Similarly we can suppose $N_R(H)$ is an $S_p$-subgroup of $N_G(H)$. Hence $H = Q \cap R$ is a tame intersection. Also $\exists x \in N_G(H) : xN_R(H)x^{-1} = N_Q(H)$. 

so \( Q \times R x^{-1} > H \) and \( \alpha Q / H = \alpha \times R x^{-1} / H = (\alpha R / H) \).

But there is no loss in generality in assuming \( R = P \) and by assumption \( \alpha P / H \) is invariant under \( N_G(H) \), so we get a contradiction. QED.

Really the point of the above proof is that if you have \( H = P \cap Q \) a bylow intersection such that are in different components of the poset of p-groups properly containing \( H \), then you can move \( P, Q \) within these components to a tame intersection. Namely, choose an \( S_p \)-subgp \( R \supseteq R \cap N_G(H) \) is an \( S_p \)-subgp of \( N_G(H) \) containing \( N_G(H) > H \). Then \( R \cap P \supseteq N_p(H) > H \), so \( R \) and \( P \) are in the same component, so still \( R \cap Q = H \). But now \( R \cap N_G(H) \) is an \( S_p \)-subgp of \( N_G(H) \).

**Question:** For what groups \( G \) is \( S_p(G) \) connected?

I want to refine this question. The point is that if \( H \) is maximal bad \( p \)-subgroup, then we’ve defined the function \( \alpha \) on \( L_H(G) \) and it might be constant on bigger chunks than just the components of \( L_H(G) \) because we have put in the relations of conjugacy on larger tame intersections.
A critical $p$-group $H$ is one such that $\pi_0 (\mathcal{L} \langle N_G(H)/H \rangle)$ is not a point.

Recall $\mathcal{L} \langle N_G(H)/H \rangle \cong L_H(G)$. Let $P$ be an $S_p$-subgroup of $G$ containing $H$. Then $P \cap N_G(H) > H$ unless $H = P$. So $L_H(G) = 1 \iff H$ is an $S_p$-subgroup. Suppose $P$ chosen so that $P \cap N_G(H)$ is an $S_p$-subgroup of $N_G(H)$. If $L_H(G)$ has more than one component choose an $S_p$-subgroup $Q$ in another component. $N_Q(H) > H$ so we can find another $S_p$-subgroup $Q'$ of $N_Q(H)$ containing $N_Q(H)$. Then $Q' \cap N_Q(H) > H$, so $Q$ and $Q'$ are in the same component. Then $H = P \cap Q'$ is a tame intersection.

Conclusion: For the Alperin thm. we have only to consider $H \leq P$ such that $N_p(H)$ is an $S_p$-subgroup of $N_G(H)$ and such that $\pi_0 (\mathcal{L} \langle N_G(H)/H \rangle) \neq pt.$
Theorem: Let $\alpha \in H^*(P)$. Assume $\text{res}_{H \to G}(\alpha)$ is invariant under $N_G(H)$ for each $H$, $1 \leq H \leq P$ such that
(i) $N_p(H)$ is an $Sp$-subgroup of $N_G(H)$.
(ii) $\pi_0(D_p(N_G(H)/H)) \neq pt$.
Then $\alpha$ comes from $H^*(G)$.

Proof: (i), (ii) hold for $H = P$, so $\alpha$ is invariant under $N_G(P)$. This implies we can define $\alpha_Q \in H^*(Q)$ for each $Sp$-subgroup $Q$ such that $\alpha_p = \alpha$ and $\alpha_Q$ is compatible with inner automorphisms.

To show $\alpha$ comes from $H^*(G)$, it suffices to prove for any non-identity p-group $H$, that $\alpha_Q|_H = \alpha_Q|_H$ for any two $Sp$-subgroups $Q_1, Q_2$. Choose $H$ maximal so that it does not have this property.

For any p-subgroup $H$ of $G$ I have seen that $Sp(N_G(H)/H)$ is homotopy equivalent to the simplicial complex whose simplices are sets $\{Q_0, \ldots, Q_n\}$ of $Sp$-subgroups of $G$ with $H < Q_0 \cap \cdots \cap Q_n$. (To each $Sp$-subgroup associate the subposet of $F/CQ$. This gives a covering with contractible intersections whose nerve is the simplicial complex).

$Sp(N_G(H)/H)$ is connected, then for every pair $Q_0, Q_1$ of $Sp$-subgroups one has

For every 1-simplex $[Q_0, Q_1]$ of $K(G, H)$ one has
Q \cdot Q' > H, hence \alpha_Q | H = \alpha_{Q'} | H. Thus the fact that 7 2 vertices Q_1, Q_2 of K(G, H) with \alpha_{Q_1} | H \neq \alpha_{Q_2} | H implies that \pi_0 K(G, H) hence \pi_0 \text{Sp}(N_G(H)/H) has at least 2 elements.

Next choose an S_p-subgrp of G such that 
Q \cap N_G(H) = N_Q(H) is an S_p-subgrp containing N_Q(H).

As \nbig Q \nbig_1 (H) > H, and Q \cap Q' \supseteq N_Q(H) it follows that

\alpha_Q | H = \alpha_{Q_1} | H \neq \alpha_{Q_2} | H. Thus we can suppose Q_1 chosen so that N_Q(H) is an S_p-subgroup of N_G(H).

Now by an inner automorphism we can replace Q_1 by P in which case H becomes a subgroup of P with properties i) and ii).

Now choose an \alpha \in N_Q(H) such that \alpha N_Q(H)^x \subset N_P(H). Then H < N_Q(H)^x \subset Q_2 \cap P so

\alpha_P | H = \alpha_{Q_2} | H = (\alpha_{Q_2} | H)^x

However by hypothesis (\alpha^x_P | H)^{-1} = \alpha_P | H. Thus we get \alpha_P | H = \alpha_{Q_2} | H a contradiction. QED

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Example: Suppose H = P \cap Q is a maximal Sylow intersection, i.e. P \cap R > P \cap Q \Rightarrow P = R for any S_p-subgroup R.

Then if we choose R so that R \cap N_G(H) is an S_p-subgroup of N_G(H) containing N_P(H) > H, we have R \cap P > H.
So \( R = P \), i.e. \( N_p(H) \) is an \( S_p \)-subgroup of \( N_G(H) \). If then \( S \) is any \( S_p \)-subgroup of \( G \) containing \( H \), we can choose \( x \) so that \( xN_p(H)x^{-1} \subset N_p(H) \), so

\[(xSx^{-1}) \cap P \supset xN_p(H)^x > H\]

so \( xSx^{-1} = P \). Thus \( N_G(H) \) transitively permutes the \( S_p \)-subgroups containing \( H \), and these Sylow groups are disjoint over \( H \) (i.e. have intersection \( H \)). Look at \( N_G(H)/H \). Any two Sylow groups are disjoint.

**Question:** What are the groups having disjoint \( S_p \)-subgroups?

Such a group has \( S_p(G) \sim \mathbb{Z}/G/N(P) \). Example:

\( \text{GL}_2(F_q) \)
May 2, 1976

Consider the case where $P$ is abelian. If $H = P \cap Q$, then $P, Q$ are both $S_p$-subgroups of $C_G(H)$, hence conjugate under an element of $C_G(H)$. Since $C_G(H)$ acts trivially on $H^*(H)$, the condition on $x$ due to $P, Q$ is vacuous, so $H^*(G) = H^*(N_G(P))$. This shows that we don't yet have fusion in good shapes.

$H$ will be okay if the stabilizer of $x/H \in H^*(H)$ as a subgroup of $N_G(H)$ acts transitively on the components of $\pi_6^{(p)}(N_G(H)/H)$. This stabilizer contains $N_G(H) \cap N_G(P)$ and $H \subseteq G(H)$.

Suppose that $[H_G(H) : H] \equiv 0 \pmod{p}$ and that $H$ is critical. Let $P, Q$ be in different components of $L_H(G)$ such that $N_{pam} H, N_G H$ are $S_p$-subgroups of $N_G(H)$. Then because $H \subseteq G(H)$, its intersection with any $S_p$-subgroup of $N_G(H)$ is an $S_p$-subgroup of $H \subseteq G(H)$.

So then we can conjugate $P \subseteq H \subseteq G(H)$ into $Q \subseteq H \subseteq G(H)$ via an element of $H \subseteq G(H)$. Thus $H$ will be okay.

So if $H$ is a bad $p'$-subgroup, then we see that $H \subseteq G(H)/H = C_G(H)/Z(H)$ must be a $p'$-group, i.e. any $p'$-element of $C_G(H)$ must be in $Z(H)$. In particular, i.e. any $p'$-element central, $H$ must be in $H_3$.
$H$ contains the center of any $S_p$-groups containing it.

**Drum's theorem.** Let $x \in \text{Im}\left\{H^*(N_G(ZP)) \to H^*(P)\right\}$. Since $ZP$ char. in $P$, $N_G(P) \leq N_G(ZP)$ so $x$ is invariant under $N_G(P)$. Let $H$ be critical in $P$. Then we've seen that $x \in 2P < H$. So we get

```
g / \
1 /  \\
N_p(H) ----> N_G(H) ----> 1
  |     |         |     |
U     |         |     |
ZP    ----> N_Q(H) ----> Q
  |   \
ZQ   
```

Assume $G$ is $p$-normal: $ZP < Q$. Then $P, Q$ are $S_p$-subgroups of $N(ZP)$ so $Q = P^x$, $x \in N(ZP)$, so $ZQ = (ZP)^x = ZP$.

In fact one sees directly that because $P, Q \leq N(ZP)$ $x_P$ and $x_Q$ have the same restriction to $H$, since they come from a class in $H^*(N_G(ZP))$.

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In Thompson's approach to normal $p$-complements, the idea somehow is to deduce the conclusion that $G$ has a normal $p$-complement from this assumption on groups $N(H)$ where $H$ is a char. subgroup of $P$. 