March 19, 1976: Conner Conjecture

X mod p acyclic and finite-dimensional.
$G = \mathbb{Z}/p$ acts on X.

$$\longrightarrow H^*_G(X, X^G) \longrightarrow H^*_G(X) \xrightarrow{(1)} H^*_G \otimes H^*(X^G) \longrightarrow \cdots$$
$$H^*_G(X, X^G) \cong H^*_G(X/G, X^G/G)$$

The theorem of PA Smith : $H^*_G(X)[e^{-1}] = H^*_G[e^{-1}] \otimes H^*(X^G)$,

so $H^*(X)$ has dim 1 over $\mathbb{F}_p$ $\Rightarrow$ $H^*_G(X) = H^*_G$ has rank one over H^*_G $\Rightarrow$ $H^*_G(X)[e^{-1}]$ has rank 1 over $H^*_G[e^{-1}]$ $\Rightarrow$ $H^*(X^G)$ has rank 1 $\Rightarrow$ X^G is mod p acyclic.

So X^G is mod p acyclic and the arrow (1) above must be an isomorphism $\Rightarrow$ $H^*_G(X, X^G) = H^*(X/G, X^G) = 0 \Rightarrow$ $H^*(X/G) = H^*(X^G) = \mathbb{F}_p$ $\Rightarrow$ X/G is mod p acyclic

Iterating we see that if G is a p-group and if X is a mod-p acyclic G-space, then X/G is mod p acyclic.

Suppose next that G is finite of order m prime to p. Consider the spectral sequence

$$E_2^{pq} = H^p(X/G,\ Gx \mapsto H^q_{G_x}) \Longrightarrow H^{p+q}_G(X) = H^{p+q}_G = \mathbb{F}_p$$

Because $|G_x| \not\equiv 0$ mod p $\quad H^*_{G_x} = \mathbb{F}_p$, so the spec. seq. degenerates yielding an isom

$$H^*(X/G) \xrightarrow{\sim} H^*_G(X) = \mathbb{F}_p$$

so X/G is mod p acyclic.

Next we need the transfer. Suppose X is a polyhedron on which the finite group G acts. I want to define a map $H_*(X/G) \longrightarrow H_*(X)$. Actually I define a map $C_\cdot(X/G) \longrightarrow C_\cdot(X)$ by sending

$$\sigma \longmapsto \sum_{\tau \in f^{-1}(\sigma)} |G_\tau|\, \tau = \sum_{g \in G} g\tau_0 \qquad \begin{array}{l} f: X \to X/G \\ \text{if } f(\tau_0) = \sigma. \end{array}$$

Note that $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}$, so $C_\cdot(X/G) = C_\cdot(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Thus the map I have defined is simply the canonical map $C_\cdot(X/G) = C_\cdot(X)_G \longrightarrow C_\cdot(X)^G$ called the norm. So the transfer map I want is:

$$H_*(X/G) = H_*\big(C_\cdot(X)_G\big) \longrightarrow H_*\big(C_\cdot(X)^G\big) \to H_*(X)$$

The composition $H_*(X/G) \to H_*(X) \to H_*(X/G)$ is clearly multiplication by $|G|$.

More generally if $H \leq G$, then we have a hom.

$$M_G \longrightarrow M_H \qquad m \longmapsto \sum_{gH \in G/H} gm$$

hence a transfer map $H_*(X/G) \longrightarrow H_*(X/H)$ such that the composition

$$H_*(X/G) \xrightarrow{\text{tr.}} H_*(X/H) \xrightarrow{\text{can.}} H_*(X/G)$$

is multiplication by $[G:H]$.

Dual results hold in cohomology.

<u>Prop</u>: Let G be a finite group and let X be a finite-dimensional mod-p-acyclic G space. Then X/G is also mod-p-acyclic.

<u>Proof</u>: Let G_p be a Sylow p-subgroup of G. We have

$$H^*(X/G) \xrightarrow[\text{obv}]{} H^*(X/G_p) \xrightarrow{\text{tr}} H^*(X/G)$$

whose composition is multiplication by $(G:G_p) \not\equiv 0 \pmod p$. Thus we reduce to the case where G is a p-group.

Assume the prop true when $G = \mathbb{Z}/p\mathbb{Z}$. Choose a cyclic group A of order p in the center of G. By the assumption X/A is mod p acyclic. But

$$X/G = (X/A)/G/A$$

so one can use induction on $|G|$ to get the prop. for G.

If $G = \mathbb{Z}/p$, then PA Smith theorem tells us X^G is mod p acyclic.

$$\begin{array}{ccccccc} \longrightarrow & H^*_G(X, X^G) & \longrightarrow & H^*_G(X) & \longrightarrow & H^*_G(X^G) & \longrightarrow \cdots \\ & \| & & & & \| & \\ & H^*(X/G, X^G/G) & & & & H^*_G \otimes H^*(X^G) & \end{array}$$

Since X^G, X are mod-p-acyclic, $X^G \to X \to pt$ induce isoms. $H^*_G \xrightarrow{\sim} H^*_G(X) \xrightarrow{\sim} H^*_G(X^G)$, so $H^*_G(X, X^G) = H^*(X/G, X^G/G) = 0$. Since $X^G/G = X^G$ is mod-p-acyclic, it follows that X/G is mod-p acyclic. Q.E.D.

Next consider a general compact Lie group G acting on a mod-p-acyclic finite-dimensional space X (say with finitely many orbit types). [redacted] To show X/G is acyclic, it suffices to know the result for [redacted] and N and G/N for some proper normal subgroup N of G. So I can assume G connected.

Suppose $G = S^1$. Because there are only finitely many orbit types, I can [redacted] find a finite subgroup $A \subset S^1$ such that all [redacted] isotropy groups of non-fixpts. are contained in A. Then X/A is [redacted] mod-p-acyclic, so replacing S^1 by $S^1/A \simeq S^1$, one can suppose that [redacted] the action is free outside the fixpt. set.

Then H^*[redacted]$(X/G, X^G/G) = H^*_G(X, X^G)$, and X^G will be acyclic by the [redacted] PA Smith theorem (note $X^G = X^A$ for A = the subgroup of order p in $G = S^1$). Thus $H^*_G(X, X^G) = 0$, so X/G is acyclic.

So we've proved the result for G a torus or extension of a torus by a [redacted] finite group. So we come to the critical case of a connected Lie gp.

I'd like to prove this using Becker-Gottlieb's transfer. So this raises the question of whether one can define a transfer map

$$tr: H^*(H\backslash X) \longrightarrow H^*(G\backslash X)$$

such that the composition

$$H^*(G\backslash X) \xrightarrow[\text{back}]{\text{pull}} H^*(H\backslash X) \longrightarrow H^*(G\backslash X)$$

is multiplication by $\chi(G/H)$.

Note that if $H\backslash X \longrightarrow G\backslash X$ is a fibre bundle with fibre G/H, then the Becker-Gottlieb map is what we want.

Review what happens for G/H finite: The transfer on the level of 0 cycles arises as follows:

$$\begin{tikzcd} H\backslash X \arrow[d, "f"'] & X \arrow[l, "\gamma_H"'] \arrow[dl, "\gamma_G"] \\ G\backslash X & \end{tikzcd}$$

$$\text{transfer of } \gamma_G(x) = \sum_{Hg \in H\backslash G}{}' \gamma_H(gx)$$

note this is independent of the choice of x.

So what this amounts to is that we find a comm. square

$$\begin{tikzcd} \overset{Hg \longmapsto Hgx}{H\backslash G} \arrow[r, "\alpha"'] \arrow[d, "\beta"'] & H\backslash X \arrow[d, "f"] \\ pt \arrow[r, "\gamma_G(x)"'] & G\backslash X \end{tikzcd}$$

then we define $f^* \gamma_G(x)_* = \alpha_* \beta^*$ where β^* is defined à la Becker + Gottlieb.

Concentrate on the case where $H = id$. Suppose that $X \longrightarrow G\backslash X$ has a section. Then we get a square

$$\begin{tikzcd} G \times Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & G\backslash X \end{tikzcd}$$

and we can define the transfer to be

$$H^*(X) \longrightarrow H^*(G \times Y) \xrightarrow[\substack{\text{to } 1 \in G \\ \text{times } \chi(G)}]{\text{pull-back}} H^*(Y) .$$

when G is connected. The same works for a subgroup

$$H^*(H\backslash X) \longrightarrow H^*(H\backslash G\times Y) \xrightarrow[\text{to point times } \chi(H\backslash G)]{\text{pull-back}} H^*(Y) = H^*(G\backslash X)$$

What to do in general?

Let $f : X \longrightarrow G\backslash X$. Since

$$H^*(X) = H^*(G\backslash X, Rf_*(\mathbb{Z})),$$

the problem is to define a map $Rf_*(\mathbb{Z}) \to \mathbb{Z}$. The first case to understand carefully is when G is finite. Put $Y = G\backslash X$ and let $y = f(x)$. Then because f is finite $Rf_*(\mathbb{Z}) = f_*\mathbb{Z}$ and

$$f_*(\mathbb{Z})_y = H^0(f^{-1}(y), \mathbb{Z}) = \prod_{x\in f^{-1}(y)} \mathbb{Z}.$$

We define $f_*(\mathbb{Z})_y \longrightarrow \mathbb{Z}$ by sending $(a_x) \longmapsto \sum_{x\in f^{-1}(y)} |G_x| a_x$.

Why we get a map $f_*(\mathbb{Z}) \to \mathbb{Z}$ in this way? This is a local question on y, and locally we have slices: Fix x_0 over y_0. Then ∃ an invariant nbd of Gx_0 which retracts onto Gx_0, so we get a nbd V of y_0 and a nbd U of x_0 such that

$$\begin{array}{ccccc} G\times^{G_{x_0}} U & = & f^{-1}(V) & \subset & X \\ \downarrow & & \downarrow & & \downarrow \\ G_{x_0}\backslash U & = & V & \subset & Y \end{array}$$

What I must check is that if I have an element $\alpha \in f_*(\mathbb{Z})_{y_0} = \prod_{G x_0} \mathbb{Z}$ and I extend this to a section s near y_0) then the trace of s is locally constant: where

$$tr(s)_y = \sum_{x \in f^{-1}(y)} |G_x| s_x .$$

Better: $s \in \Gamma(f^{-1}(V), \mathbb{Z})$ is a locally constant function, so by shrinking V I can make it constant on each gU. The rest is clear.

Next I have to understand the Becker-Gottlieb transfer, say in sheaf-theoretic terms.

Let $f : X \longrightarrow Y$ be a differentiable fibre bundle with fibre the compact manifold F. I choose a generic section of the tangent bundle T_f along the fibres of f, and let $Z \subset X$ be where this section vanishes.

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Then $\nu_i = i^* T_f$, so we have exact sequences

$$0 \longrightarrow T_Z \longrightarrow i^* T_X \longrightarrow i^* T_f \longrightarrow 0$$

$$0 \longrightarrow i^* T_f \longrightarrow i^* T_X \longrightarrow g^* T_Y \longrightarrow 0$$

whence T_Z and $g^* T_Y$ are stably-isomorphic. This means g is a "framed" map so it induces a Gysin map

$$g_* : h^\cdot(Z) \longrightarrow h^\cdot(Y)$$

in an cohomology theory. Thus we get a map

$$h^\cdot(X) \xrightarrow{i^*} h^\cdot(Z) \xrightarrow{g_*} h^\cdot(Y)$$

which is the Becker-Gottlieb trace. A lot of work has been done to put this in a more natural form.

First part is to obtain directly the S-map $Y \longrightarrow X$ which gives the trace.

Suppose I am dealing with ordinary cohomology

$\alpha \in h^\cdot(X)$
$$g_* i^* \alpha = f_* i_* i^* \alpha = f_*(e(\nu_i)\alpha)$$
$$= f_*(e(\tau_f)\alpha)$$

<u>March 20, 1976:</u>

<u>Remark:</u> Observe that the transfer $H^*(H\backslash X) \to H^*(G\backslash X)$ is induced by a geometric map:

$$\begin{matrix} \text{0-cycles} \\ \text{on } G\backslash X \end{matrix} \longrightarrow \begin{matrix} \text{0-cycles} \\ \text{on } H\backslash X \end{matrix}$$

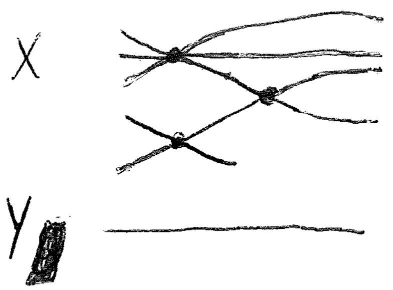
In fact we have a map

$$G\backslash X \longrightarrow SP^{[G:H]}(H\backslash X)$$

$$[Gx] \longmapsto \sum_{g \in H\backslash G} [Hgx] .$$

Do we get any finer structure: Is there a finer theory than cohomology in which one might be able to define the transfer?

More precisely, can one associate a Segal-Anderson style chain. The point is that one has attached to each point x of X an integer, namely the multiplicity of this point $=$ order of G_x.



Can one attach to x the regular representation of G_x? No for the results won't add as we specialize.

<u>Question</u>: Suppose X is a scheme and we consider the map $f: X \longrightarrow G\backslash X = Y$. Does $\mathcal{O}_X$ give a system of vector spaces over the points of X which make an Anderson-Segal chain?

Answer is probably: yes $\Longleftrightarrow$ $\mathcal{O}_X$ is flat over $\mathcal{O}_Y$.

Dold's fixpoint index map.

Let $\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array}$ be a fibre bundle, and let $\Phi: X \longrightarrow X$ be a map over Y. Then we can form the diagram

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Gamma"'] \arrow[d, "f"] & X \times_Y X \\ Y & \end{tikzcd} \qquad \Gamma = (id_X, \Phi).$$

where F is the fixpt set. If Γ, Δ are transversal, one shows $fi: F \longrightarrow Y$ is framed, hence we can define a map

$$(fi)_* i^* : h(X) \longrightarrow h(Y)$$

in any cohomology theory. Thus it should be possible to directly define an S-map $Y \dashrightarrow X$.

First we need to be able to define the Gysin map f_*. So we choose an embedding

$$X \xhookrightarrow{j} Y \times S^N$$

whence f_* is

$$h^{i+d}(X) \underset{\substack{\uparrow \\ \text{Thom isom.}}}{\cong} h^{i+N}(Y\times S^N, Y\times S^N - X) \longrightarrow h^{i+N}(Y\times S^N, Y\times\infty) \cong h^i(Y)$$

where $d = \text{rel. dim } X/Y$.

Suppose X is a closed manifold. Choose an embedding $X \hookrightarrow D^n$ and let N be a tubular nbd. We have a map

$$(1) \qquad S^n = D^n/\partial D^n \longrightarrow X \wedge (N/\partial N)$$

defined as follows. We have the map

$$(N, \partial N) \longrightarrow X \times (N, \partial N)$$

with components the projection of N onto X and the identity. This induces

$$D^n/\partial D^n \longrightarrow N/\partial N \longrightarrow X \wedge (N/\partial N)$$

In addition, we have a map

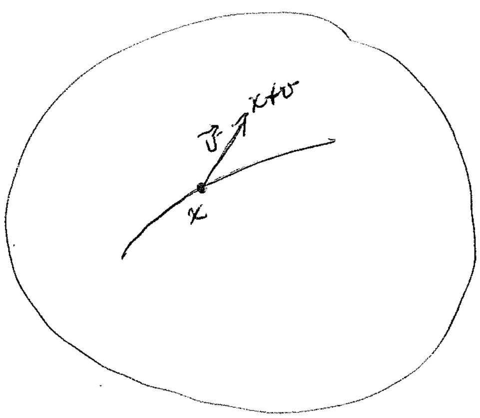
$$(2) \qquad X \wedge (N/\partial N) \longrightarrow S^n$$

defined as follows. Suppose ε chosen so that the distance between X and ∂N is $\geq \varepsilon$. I regard $D^n \subset \mathbb{R}^n$.

Then given $x \in X$ and $n \in N$, I get a vector $n - x \in \mathbb{R}^n$.
Consider the map

$$X \times N \longrightarrow \mathbb{R}^n \longrightarrow D^n_\varepsilon / \partial D^n_\varepsilon = S^n$$
$$(x, n) \longmapsto n - x$$

This sends $X \times \partial N$ to $*$, hence we get the map (2).
Consider the composition $\alpha : S^n \xrightarrow{(1)} X \wedge (N/\partial N) \xrightarrow{(2)} S^n$. Fix a small vector v in $\mathbb{R}^n$. The fibre of (2) over v consists of pairs $(x, x+v)$ in $X \wedge (N, \partial N)$; note that $x \in X \Rightarrow x + v \in N$ because v is small. The fibre of (1) over $(x, x+v)$ is the point $x + v$ precisely, when $x+v$ projects into x, i.e. when v is perpendicular to X at x. Thus the fibre of $(2)(1) = \alpha$ over v can be identified with the zero set of the vector field on X obtained by projecting $x + v$ into the tangent space of X at x. This leads one to suspect that α has degree $\chi(X)$.



Alternate description of (2). One knows such a map comes from a submanifold of codimension n with normal framing. The submanifold is clearly $X \hookrightarrow X \times N$ embedded by the graph of the embedding of $X \hookrightarrow N$. More specifically because N is framed, one gets an open embedding

$$\begin{aligned} X \times D^n_\varepsilon &\hookrightarrow X \times N \\ (x, v) &\longmapsto (x, x+v) \end{aligned}$$

hence a map

$$(3) \qquad X \wedge (N/\partial N) \longrightarrow X \wedge S^n$$

which gives rise to the map (2) when composed with the map $X \to pt$. Composing with (1) I get a map

$$(4) \qquad S^n \longrightarrow X \wedge S^n$$

such that ~~an~~ sending X to a point gives a map $S^n \to S^n$ which should have degree $= \chi(X)$.

Geometric description of (4): Take a point inside of N, say $n \in N$, and write it $n = x + v$ where x is the projection of n on X, and $v \in \mathbb{R}^n$. Then the map $n \mapsto (x, v)$ from N to $X \times \mathbb{R}^n$ carries ∂N into $X \times \mathbb{R}^n - 0$, so we get by the Thom construction a map $S^n = N/\partial N \longrightarrow X \times \mathbb{R}^n / X \times (\mathbb{R}^n - D_\varepsilon) = X \wedge S^n$.

Next suppose X is a fibre bundle over Y with a compact manifold for fibre. Choose an embedding $X \hookrightarrow Y \times \mathbb{R}^n$ and let N be a tubular nbd. Then we have a map:

$$N \longrightarrow X \times_Y N.$$

Note that because X is a fibre bundle over Y, N should be one also, and the retraction of N to X should be fibre-wise (at least stably). Thus we get a map

$$\begin{tikzcd} N \cup_{\partial N} Y \arrow[r] & (X \times_Y N) \cup_{(X \times_Y \partial N)} Y \\ (Y \times D^n) \cup_{Y \times \partial D^n} Y \arrow[u] & \end{tikzcd}$$

which is a fibre-wise version of (1). Similarly there has to be a fibre-wise version of (2).

Suppose X is a closed manifold embedded in $\mathbb{R}^n$. One gets a map $S^n \to X^\nu$. On the other hand ν is a direct summand of the trivial bundle of rank n, so we get a map $X^\nu \hookrightarrow X \wedge S^n$. Combining, we get the map $S^n \to X \wedge S^n$ which sends a vector n near to X into (x, v) where $n = x + v$ and $v \in \nu$. Now when X is a fibre bundle over Y embedded in $Y \times \mathbb{R}^n$, then we can do this fibre by fibre to get a map

$$(Y \times D^n) \cup_{Y \times \partial D^n} Y \longrightarrow (X \times D^n) \cup_{X \times \partial D^n} Y$$

over Y. Thus we get a map $S^n Y \to S^n X$ which is the Gottlieb-Becker transfer.

Check:

$$\begin{tikzcd}
Y \arrow[d, "i'"] & & X \arrow[dl, hook', "j"'] \arrow[d, "i"] \\
S^n Y \arrow[r, "f"'] & X^\nu \arrow[r, hook, "k"'] & S^n X
\end{tikzcd}$$

Let $\alpha \in H^*(X)$. Then because the normal bundle to the embedding of ν in $X \times \mathbb{R}^n$ is the tangent bundle to X lifted to ν, we have

$$k^* i_* \alpha = k^* k_* j_* \alpha = j_* e(\tau_X)\alpha$$

so

$$(i'_*)^{-1} f^* k^* i_* \alpha = \left[(i')_*^{-1} f^* j_*\right] e(\tau_X)\alpha = p_*\left[e(\tau_X)\alpha\right]$$

where $p: X \to Y$ is the structural map.

Suppose next that X is a fibre bundle over Y with fibre a finite complex. There should be no problem making the preceding work. In particular, [redacted] we should get a Becker-Gottlieb map [redacted] $Y \wedge S^n \longrightarrow X \wedge S^n$ "over Y". Can you produce this map by derived category methods?

So what I want is a map [redacted] $Rf_*(\mathbb{Z}_X) \longrightarrow \mathbb{Z}_Y$ in the derived category. In the following I suppress Y. The geometric maps I have defined are as follows

$$S^n \longrightarrow (N/\partial N) \longrightarrow X \wedge (N/\partial N) \longrightarrow X \wedge S^n \longrightarrow S^n$$

The effect on cohomology after shifting by n is

$$\underset{\substack{\uparrow\\ \text{augmentation}}}{k \longleftarrow} H_*(X) \underset{\substack{\uparrow\\ \text{cap-prod.}}}{\longleftarrow} H^*(X) \otimes H_*(X) \underset{\substack{\uparrow\\ \text{some sort of}\\ \text{cap product which}\\ \text{I have to understand.}}}{\longleftarrow} H^*(X) \underset{\substack{\uparrow\\ \text{augm.}}}{\longleftarrow k}$$

The corresponding maps in the derived category are:

$$\mathbb{Z} \longleftarrow Rf_*(f^!\mathbb{Z}) \longleftarrow Rf_*(\mathbb{Z}) \otimes Rf_*(f^!\mathbb{Z}) \longleftarrow Rf_*(\mathbb{Z}) \longleftarrow \mathbb{Z}$$

The outer 2 arrows are obvious [redacted] and I have to understand the middle two arrows. By the duality theory $Rf_*(\mathbb{Z}) \otimes Rf_*(f^!\mathbb{Z})$ has a natural interpretation as "endos. of $Rf_*(\mathbb{Z})$."

March 22, 1976 (Carl is 11)

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

$$\begin{aligned} \mathrm{Hom}_S(f_! F, g_* G) &= \mathrm{Hom}_Y(g^* f_! F, G) \\ &= \mathrm{Hom}_Y(p_{2!}\, p_1^* F, G) \\ &= \mathrm{Hom}_{X\times_S Y}(p_1^* F, p_2^! G) \end{aligned}$$

In this way maps $p_1^* F \to p_2^! G$ over $X \times_S Y$ can be identified with maps $f_! F \to g_* G$ over S. (This is some kind of kernel theorem).

Thus if we apply this to a finite complex X over a point I get

$$\begin{aligned} \mathrm{Hom}_{pt}(Rf_*(\mathbb{Z}), Rf_*(\mathbb{Z})) &= \mathrm{Hom}_{X\times X}(p_1^* \mathbb{Z}_X, p_2^! \mathbb{Z}_X) \\ &= Rf_*(\mathbb{Z}_X) \otimes Rf_*(f^! \mathbb{Z}_X) \end{aligned}$$

Problem: How to define a map

$$Rf_*(\mathbb{Z}) \otimes Rf_*(f^! \mathbb{Z}) \to Rf_*(f^! \mathbb{Z})\ ?$$

We would need a map $Rf_*(\mathbb{Z}) \to \mathbb{Z}$ or equivalently a map $\mathbb{Z} \to Rf_*(f^! \mathbb{Z})$.

Try

$$X \xrightarrow{\Delta} X \times X, \qquad \Delta^*(p_1^* \mathbb{Z}_X \otimes p_2^* f^! \mathbb{Z}) = f^! \mathbb{Z}$$

so the diagonal induces a map

$$\underset{\substack{\parallel \\ p_2^* f^! \mathbb{Z}}}{\mathrm{Hom}_{X\times X}(p_1^* \mathbb{Z}_X, p_2^! \mathbb{Z}_X)} \longrightarrow \mathrm{Hom}_X(\mathbb{Z}_X, f^! \mathbb{Z}) = Rf_*(f^! \mathbb{Z}).$$

so this works.

There should be a canonical class:

$$\gamma \in H^0_\Delta(X \times X, \mathbb{Z} \boxtimes f^!\mathbb{Z})$$

which can be interpreted as a map

$$\Delta_*(\mathbb{Z}_X) \longrightarrow \mathbb{Z} \boxtimes f^!\mathbb{Z}$$

so we get an induced map

$$Rf_*(\mathbb{Z}_X) = R(f \times f)_* \Delta_*(\mathbb{Z}) \longrightarrow R(f\times f)_*(\mathbb{Z} \boxtimes f^!\mathbb{Z})$$

Definiton of γ. Note that $p_1^!\mathbb{Z}_X = \mathbb{Z} \boxtimes f^!\mathbb{Z} = p_2^* f^!\mathbb{Z}$ should be true if X is locally nice.

If so

$$H^0_\Delta(X \times X, p_1^!\mathbb{Z}_X) = \operatorname{Hom}_{X \times X}(\Delta_*\mathbb{Z}_X, p_1^!\mathbb{Z}_X) = \operatorname{Hom}_X(\mathbb{Z}_X, \Delta^! p_1^! \mathbb{Z}_X)$$
$$= \operatorname{Hom}_X(\mathbb{Z}_X, \mathbb{Z}_X) = \mathbb{Z}$$

which gives me the desired element γ.

Summary: Start with the identity

$$p_1^!\mathbb{Z}_X = p_2^* f^! \mathbb{Z}$$

$$\begin{array}{ccc} X \times X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & pt \end{array}$$

which should result from Kunneth. Then one gets maps

$$Rf_*(\mathbb{Z}_X) = R(f\times f)_* \Delta_* \Delta^! p_1^! \mathbb{Z}_X \longrightarrow R(f \times f)_*(p_1^!\mathbb{Z}_X) \longrightarrow R(f\times f)_* \Delta_* \Delta^* p_2^* f^!\mathbb{Z} = Rf_*(f^!\mathbb{Z})$$

where the arrows are induced by the adjunction maps

$$\Delta_*\Delta^! \longrightarrow id \longrightarrow \Delta_*\Delta^*$$

Suppose G is a compact Lie group acting on X, and let H be a subgroup. Let f be the map $X/H \to X/G$. I want to define a transfer map

$$H^*(X/H) \longrightarrow H^*(X/G)$$

and the method I want to use is to define a map in the derived category over X/G:

$$Rf_*(\mathbb{Z}_{X/H}) \longrightarrow \mathbb{Z}_{X/G}.$$

This should be the Becker–Gottlieb transfer when the fibres of f are isom. to G/H, i.e. if the isotropy groups of X act freely on G/H.

Here's an idea for the case of the map $X \to X/G$.

$$\begin{tikzcd}
X \arrow[dr, "j"'] & & & \\
& X\times G \arrow[dr, "q"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& & X\times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
& & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$j(x) = (x, e)$

$q(x,g) = (x, xg)$.

$$Rf_*(f^*\mathbb{Z}) = R(f\times f)_*\, q_* j_* j^! q^! p_1^! \mathbb{Z}_X$$

$$\longrightarrow R((f\times f)q)_*\, q_1^! \mathbb{Z}_X \qquad \text{(adjunction)}$$

$$\longrightarrow R((f\times f)q)_*\, j_* j^* q_1^! \mathbb{Z}_X \qquad \text{(adjunction)}$$

$$= Rf_*(j^* q_1^! \mathbb{Z}_X)$$

$$= Rf_*(j^* pr_2^* \omega_G) \qquad \omega_G = \pi^! \mathbb{Z} \qquad \pi: G \to pt$$

$$= Rf_*(f^*(\omega_G|e)) \qquad ?$$

Another idea: Because one has $f^* R f_* = R p_{1*} p_2^*$ one has

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{aligned} \mathrm{Hom}_Y(Rf_*\mathbb{Z}, Rf_*\mathbb{Z}) &= \mathrm{Hom}_X(f^* Rf_*\mathbb{Z}, \mathbb{Z}) \\ &= \mathrm{Hom}_X(Rp_{1*}p_2^*\mathbb{Z}, \mathbb{Z}) \\ &= \mathrm{Hom}_{X\times_Y X}(p_2^*\mathbb{Z}, p_1^!\mathbb{Z}) \end{aligned}$$

[redacted]

so that $R\Gamma(X\times_Y X, p_1^!\mathbb{Z})$ can be [redacted] viewed as the space of [redacted] endos of $Rf_*(\mathbb{Z})$. To get the trace of an endo, the Lefschetz theorem tells us to take cup product with

$$\Delta_* 1 \in H^0_\Delta(X\times_Y X, p_2^!\mathbb{Z})$$

and then integrate over $X\times_Y X$ using the isom. [redacted] $p_2^!\mathbb{Z} \otimes p_1^!\mathbb{Z} = (f\times f)^!\mathbb{Z}$, which follows from

$$(f\times f)^!\mathbb{Z} = p_1^! f^! \mathbb{Z} = p_1^!\mathbb{Z} \otimes p_1^* f^! \mathbb{Z} = p_1^!\mathbb{Z} \otimes p_2^!\mathbb{Z}$$

(this uses some sort of local niceness properties of f).

So go back to the diagram

$$\begin{tikzcd} X \arrow[dr, "j"] & & \\ & X\times G \arrow[r, "q_2 = \mathrm{mult}"] \arrow[d, "q_1 = pr_1"'] \arrow[dr, "h"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$Rf_*(f^*\mathbb{Z}) \Longrightarrow Rh_* Rj_* j^! q_1^! \mathbb{Z}_X \longrightarrow Rh_*(q_1^!\mathbb{Z}_X)$$

?

March 23, 1976:

Still trying to define a transfer $H^*(X/H) \longrightarrow H^*(X/G)$.

Special case: $H = e$, suppose $X \longrightarrow X/G = Y$ has a section s so that we have a diagram

$$\begin{tikzcd} Y \times G \arrow[r, "u"] \arrow[dr, "pr_2"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Then we can define the transfer

$$H^\cdot(X) \xrightarrow{u^*} H^\cdot(Y \times G) \longrightarrow H^\cdot(Y)$$

Becker-Gottlieb transfer

Similarly I can define

$$H^*(X/H) \longrightarrow H^*(Y \times G/H) \longrightarrow H^*(Y)$$

pull-back $\qquad$ B-G.

whose composition with f^* is multiplication by $\chi(G/H)$.

Remark: Suppose one has a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X'/G \arrow[r, "g"'] & X/G \end{tikzcd}$$

Then $g^* Rf_*(\mathbb{Z}_X) = Rf'_*(\mathbb{Z}_{X'})$, so if we define a map $Rf_*(\mathbb{Z}_X) \longrightarrow \mathbb{Z}_{X/G}$, then we get a similar map for X'. This means I have to understand the universal situation

March 31, 1976

Let X be a complex algebraic variety, say projective, on which a finite group G acts. Let H be a subgroup of G. I know how to define a transfer map

$$f^* : H_*(X/G) \longrightarrow H_*(X/H)$$

on homology. [redacted] Question: Can one define a finer map on Chow groups:

$$f^* : A_\cdot(X/G) \longrightarrow A_\cdot(X/H)$$

Possible procedure. Compatibility with [redacted] G-maps [redacted] forces f^* to be determined by what it does to the fundamental cycle. [redacted] $[X/G]$ of $A_d(X/G)$, $d = \dim X$. Clearly $f^*[X/G] = k\cdot[X/H]$ for some integer k which is generically determined, [redacted] assuming X is irreducible. So it is now clear that one takes the formula

$$y = [Gx] \xmapsto{\ f^*\ } \sum_{Hg \in H\backslash G} [Hgx]$$

[redacted] on generic points in order to define f^* on cycles.

Inductive construction: Let G be a compact Lie group acting on a [redacted] G-cell-complex X, and suppose X/G is triangulated. Try to define a transfer on homology inductively over the skeleta.

points. Let $y = [Gx] \in X/G$. ~~How~~ to make sense out of

$$\sum_{Hg \in H\backslash G} [Hgx]$$

$$\begin{tikzcd} H\backslash G \arrow[r, "u"] \arrow[d, "f'"'] & H\backslash X \arrow[d, "f"] \\ \{y\} \arrow[r] & \text{[redacted]}\ G\backslash X \end{tikzcd}$$

$$f^*[y] = u_* f'^* 1$$

by naturality. But we know that we ought to define $f'^* 1 = \chi(H\backslash G) \in H_0(H\backslash G)$, when say G/H is connected.

I've seen that this method defines a transfer provided that what I map to $G\backslash X$ lifts to $H\backslash X$.

<u>April 1, 1976.</u>

X G-space, G finite, p prime. Suppose for each element θ of order p one has X^θ is mod p acyclic. Does it follow that

$$(*) \qquad \left(\hat{H}^*_{G}\right)_{(p)} \xrightarrow{\sim} \hat{H}^*_G(X)_{(p)} \ ?$$

We can assume G is a p-group by transfer theory. Nothing to prove if $G=1$, so let θ be an element of order p in the center of G. Then X^θ is mod p-acyclic, so PASmith $\Rightarrow$ $X^G = (X^\theta)^{G/\langle\theta\rangle}$ is mod p-acyclic. In this case the isom (*) is clear.

<u>Example</u>: Take X to be (the simplicial complex associated to) the poset of p-subgroups of G which are generated by elements of order p, are are non-trivial. If θ is an element of order p in G, then X^θ is the poset of non-trivial p-subgroups H gen. by elements of order p normalized by θ. Then

$$H \subset \langle H, \theta\rangle \supset \langle\theta\rangle$$

is a contraction of X^θ to a point. It follows that for any p-subgroup $P \neq 1$ of G, X^P is mod p acyclic. Thus if G_p is a Sylow group

$$\bigcup_{1<H\subset G_p} X^H \quad \text{is mod } p \text{ acyclic.}$$

so again I can conclude that $\chi(X) \equiv 1 \pmod{|G_p|}$.

April 2, 1976

G finite group, X a G-polyhedron which is mod-p-acyclic. Claim $\chi(X^H) = 1$ if $H \leq G$ has a normal p-subgroup $P \ni H/P$ is cyclic.

Proof: Smith thm $\Rightarrow X^P$ is $\mathbb{F}_p$-acyclic. Lefschetz thm applied to the map $f: X^P \to X^P$ induced by a generator of H/P says $\chi(X^H) = \chi((X^P)^{H/P}) =$ Lefschetz number of f on $H_*(X^P, \mathbb{Q}) = 1$. (Note $\mathbb{F}_p$-acyclic $\Rightarrow \mathbb{Q}$-acyclic)

$$\therefore \quad [X] \in \bigcap_{\substack{H \leq G \\ H \in \mathcal{G}_p^1}} \operatorname{Ker}\left\{A(G) \xrightarrow{\chi_H} \mathbb{Z}\right\} \qquad \chi_H([Y]) = \chi(Y^H)$$

where $\mathcal{G}_p^1$ denotes groups H having normal p-subgp P $\ni H/P$ is cyclic.

Oliver proves that conversely any element α of $A(G)$ killed by χ_H for all $H \in \operatorname{Sub}(G) \cap \mathcal{G}_p^1$ comes from a mod p-acyclic G-polyhedron.

Special case: Let X_0 be a G-polyhedron such that X_0^H is $\mathbb{F}_p$-acyclic for $1 < H \leq G$. We wish to attach free G-cells to X_0 to obtain a $\mathbb{F}_p$-acyclic acyclic G-polyhedron X. I know we can attach free G-cells to make X_0 an n-complex with homology concentrated in degree n. Then $H_n(X_0) = H_n(X_0, \bigcup_{1<H\leq G} X_0^H)$

is a projective $\mathbb{F}_p[G]$-module, which I want to show is free. One knows two projective $\mathbb{F}_p[G]$-modules with the same Brauer character are isomorphic, so I have to show that $H_n(X)$, has the same Brauer character as a free $\mathbb{F}_p[G]$-module.

Better: Because X^H is $\mathbb{F}_p$-acyclic $\forall H$, $1<H\le G$, I know $\bigcup_{1<H\le G} X^H$ is $\mathbb{F}_p$-acyclic, hence

$$\bar{X} = X/\bigcup X^H$$

has the same $\mathbb{F}_p$-homology as X. But G acts freely on $\bar{X}-\{*\}$, so one knows that because the complexes $H_n(\bar{X},*)[n]$ and $C_\cdot(\bar{X},*)$ are **quasi-isomorphic**, the class of $H_n(\bar{X},*) = \tilde{H}_n(X)$ in $K_0(\mathbb{F}_p[G])$ is free. Now use Krull-Schmidt to conclude $H_n(\bar{X},*)$ is free.

Suppose we start with $\alpha \in A[G]$ killed by χ_H for $H \in \mathcal{G}_p'$, and suppose we have found X_0 such that

$$\chi(X_0^H) = \chi_H\,\alpha \qquad \text{for all } 1<H\le G$$

and such that X_0^H is $\mathbb{F}_p$-acyclic when H is a p-subgp. I suppose G is not in $\mathcal{G}_p'$; otherwise $\alpha=1$.

There is an obstruction in $K_0(\mathbb{F}_p[G])$ to attaching free cell-orbits to X_0 to kill all mod p homology. It's known from the Brauer theory that

$$K_0(\mathbb{F}_p[G]) \hookrightarrow R_{\mathbb{F}_p}(G) \hookrightarrow \mathrm{Mapcent}(G^{(p')}, \mathbb{C}).$$

I can suppose if I want that X_0 is a $(n-1)$-connected complex of dim. n, whence $H_n(X_0, \mathbb{Z}_{(p)})$ is projective over $\mathbb{Z}_{(p)}[G]$.

I have to somehow determine the character of $H_n(X_0, \mathbb{Z}_{(p)})$ on the p'-elements of G. By Lefschetz, the value of the character on g is essentially the Euler char of the fixpoint set:

$$1 - (-1)^n \operatorname{tr} g \text{ on } H_n(X_0, \mathbb{Z}_{(p)}) = \chi(X_0^g)$$

By assumption $\chi(X_0^g) =$

$$= \chi_{\langle g \rangle}(\alpha) = 1 \quad \text{if} \quad g \neq e$$

Brauer character of $H_n(X_0)$ is therefore zero on all non-p-elements of G. Remains to see that rank $H_n(X_0)$ is divisible by $|G|$. Can suppose G is a q-group, q a prime; if $q = p$ OK; if $q \neq p$ the representation $H_n(X_0, \mathbb{Q})$ has to be a rational multiple of the regular repn. But in the reg. repn. the trivial repn. occurs once, so the only multiples of the reg. repn. are integral multiples.

Question: Suppose X_0 is such that X_0^H mod p acyclic for $1<H<G$. Can I then attach free cell-orbits to X_0 to get a mod-p-acyclic X?

Yes, for I have seen that the obstruction to getting such an X lies in $\tilde{K}_0(\mathbb{F}_p[G])$. If $H<G$, then

$$H_n(X) = H_n(X, \bigcup_{1<K\leq H} X^K)$$

has to be stably free over $\mathbb{Z}[H]$, so this obstruction vanishes on any subgroup, and to get it to be zero you only have to get it to vanish on each cyclic subgroup of order prime to p.

April 3, 1976

Start with $GL_n\mathbb{C}/GL_{n-1}\mathbb{C} \sim U_n/U_{n-1} = S^{2n-1}$. Thus a map $X \to S^{2n-1}$ can be identified with a kind of unimodular vector in $\mathbb{C}(X)^n$. From homotopy theory we know that $[X, S^{m-1}]$ is a group provided $\dim(X) = d < 2m-2$. The problem is whether I can construct a group out of the set $F_1(A^n)$ of unimodular vectors in A^n in analogy with this topological example.

First point is to understand the operation in the continuous case. Suppose we have two maps $s_1, s_2 : X \to S^{m-1}$, whence we get $(s_1, s_2): X \to S^{m-1} \times S^{m-1}$. Because $\dim X < 2m-2$ we can suppose the image of (s_1,s_2) misses a point so we can deform (s_1,s_2) into a map $X \to S^{m-1} \vee S^{m-1}$, the wedge being with respect to some basepoint e_m of S^{m-1}. This means $X = A \cup B$ where

$$A = \{x \in X \mid s_1(x) = e\}$$
$$B = \{x \in X \mid s_2(x) = e\}$$

The folding map $S^{m-1} \vee S^{m-1} \to S^{m-1}$ gives us a map $s : X \to S^{m-1}$ such that

$$\begin{aligned} s(x) &= s_1(x) \quad \text{if } x \in B \\ &= s_2(x) \quad \text{if } x \in A \end{aligned}$$

~~Suppose g is a map g: $X \to [0,1]$. Then g~~

April 4, 1976

Recall that if Z is a non-singular divisor in a non-singular variety X, then one has a Gysin sequence in top K-theory

$$\cdots \longrightarrow K^0(X) \longrightarrow K^0(X-Z) \xrightarrow{\delta} K^{-1}(Z) \longrightarrow K^1(X) \longrightarrow \cdots$$

where δ is defined using periodicity. For example,

$$0 \longrightarrow K^0(Z\times\mathbb{C}) \longrightarrow K^0(Z\times\mathbb{C}^*) \longrightarrow K^{-1}(Z) \longrightarrow 0.$$

Now [redacted] one has for X regular

$$\begin{tikzcd} K^0(X) \arrow[r] & K^0(X-Z) \\ K^0_{alg}(X) \arrow[u] \arrow[r, two heads] & K^0_{alg}(X-Z) \arrow[u] \end{tikzcd}$$

so that $\delta : K^0(X-Z) \longrightarrow K^{-1}(Z)$ has to kill all algebraic [redacted] bundles. What happens in dim -1?

$$\begin{tikzcd} 0 \arrow[r] & K^{-1}(Z) \arrow[r] & K^{-1}(Z\times\mathbb{C}^*) \arrow[r, "\delta"] & K^{-2}(Z) \arrow[r] & 0 \\ & & & K^0(Z) \arrow[u, phantom, "\cong" rotate=90] & \\ 0 \arrow[r] & K^{-1}_{alg}(Z) \arrow[uu] \arrow[r] & K^{-1}_{alg}(Z\times\mathbb{G}_m) \arrow[uu] \arrow[r, "\delta"] & K^0_{alg}(Z) \arrow[u] \arrow[r] & 0 \end{tikzcd}$$

What is curious is that for K^{-1}, δ is non-trivial stably.