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Tate cohomology

Let $G$ be a finite group. Let $P_\ast$ be a resolution of $\mathbb{Z}$ by free \mathbb{Z}[G]\text{-modules. Then} \n
$$P^n = \text{Hom}_{\mathbb{Z}}(P_n, \mathbb{Z}[G]) = \text{Hom}_{\mathbb{Z}}(P_n, \mathbb{Z})$$

is a resolution of $\mathbb{Z}$ to the right as $H^i(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$

$$\cdots \rightarrow P_i \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

$$0 \rightarrow \mathbb{Z} \rightarrow P_0^\vee \rightarrow P_1^\vee \rightarrow \cdots$$

Splice these together to get an acyclic complex of free \mathbb{Z}[G]\text{-modules}

$$\check{W}_G : \rightarrow P_1 \rightarrow P_0 \rightarrow P_0^\vee \rightarrow P_1^\vee \rightarrow \cdots$$

Then for any complex $M$ of \mathbb{Z}[G]\text{-modules one puts}

$$\hat{H}^i(G, M) = H^i\left(\check{W}_G \otimes_G M\right)$$

$$H^i(G, M) = H^i\left(P_n^\vee \otimes_G M\right) = H^i(\text{Hom}_G(P_n, M))$$

$$H_i(G, M) = H_i\left(P \otimes_G M\right)$$

whence we get an exact sequence

$$\cdots \rightarrow H_i(G, M) \rightarrow H^i(G, M) \rightarrow \hat{H}^i(G, M) \rightarrow H_{i-1}(G, M) \rightarrow \cdots$$
... Magal is also acyclic.

$M \cong \mathbb{Q} \oplus H^1(M) \cong \mathbb{Q} \oplus H^1(M)$

where

$\text{Hom}_G(\mathbb{Q}, M) \cong \bigoplus_{b \in H^1(M)} \mathbb{Q}$

* This holds only if $M$ is acyclic (i.e. in $D^1$).

\[ \text{Hom}_G(\mathbb{Q}, M) = \bigoplus_{b \in H^1(M)} \mathbb{Q} \]
Note: \( \hat{W}_G \) is a complex of f.t. free \( \mathbb{Z}[G] \)-mods. and

\[
\hat{H}^i(G, M) = \lim_{\longrightarrow} H^i((\hat{W}_G)_{sp} \otimes_G M) = \lim_{\longrightarrow} H^i(\text{Hom}_G((\hat{W}_G)_{sp}^\vee, M))
\]

Put \( K(p) = (\hat{W}_G)_{sp}^\vee \).

Let \( X \) be a \( G \)-space and let \( F \) be a complex of \( G \)-sheaves on \( X \) bdd. below \( (in C^+) \). Then we define

\[
\hat{H}^i_G(X, F) = \hat{H}^i_G(R\Gamma(X, F)) = \lim_{\longrightarrow} H^i(\text{Hom}_G(K(p), R\Gamma(X, F))) = \lim_{\longrightarrow} H^i(G, R\Gamma(X, \text{Hom}(K(p), F))) = \lim_{\longrightarrow} H^i_G(X, \text{Hom}(K(p), F))
\]

So it's clear from this formula that one is going to have all the good properties. Note that \( \{K(p)\} \) is an inverse system of complexes bdd. above.
A = \mathbb{Z}/p\mathbb{Z}$. Let $X$ be a smooth manifold on which $A$ acts. $X^A$ is a submanifold and the normal bundle of $X^A$ in $X$ has an $A$-action, so it breaks up according to the irreducible representations of $A$ over $\mathbb{R}$. If $p$ is odd, this means that the normal bundle can be given a complex structure, hence we will have a Thom isomorphism

$$H^*_A(X, X - X^A) \cong H^*_{A^d}(X^A)$$

where $d = \text{codim}_X X^A$. Same is true for $p=2$.

$$\to H^*_{A} (x^A) \to H^*_{A} (x) \to H^*_{A} (X - X^A) \xrightarrow{\partial}$$

$$\left[ H^*_A \otimes H^*_A \right]^{x - d} \xrightarrow{\partial} H^*_{A} (X - X^A/A)$$

Now the composite $H^*_{A} (x^A) \to H^*_{A} (x) \to H^*_{A} (x^A)$ is multiplication by the Euler class of the normal bundle of $X^A$ in $X$, and calculation shows this Euler class is a non-zero divisor. Thus the structure of $H^*_{A} (x)$ is given by an exact sequence.

$$0 \to \left[ H^*_A \otimes H^*_A \right]^{x - d} \to H^*_{A} (x) \to H^* (X - X^A/A) \to 0$$

This sequence is not a homotopy invariant of $X$ because we could multiply $X$ by a representation of $A_A$ and change $d$. 

This page contains a mathematical proof involving algebraic topology and the Thom isomorphism. It discusses the action of the group $A = \mathbb{Z}/p\mathbb{Z}$ on a smooth manifold $X$, and how the normal bundle of the submanifold $X^A$ in $X$ can be given a complex structure. The text outlines the steps of a Thom isomorphism and related exact sequences, which are fundamental in understanding the topological invariants of these spaces.
Quality theorem:

\[ A = k[T_0, \ldots, T_r] \quad m = (T_0, \ldots, T_r) \subset A \]

\[ M = \bigoplus_{n \in \mathbb{Z}} M_n \] is a graded f.t. \( A \)-module

\[ Y = \text{Spec}(A) - m \subset \text{Spec}(A) \]

\[ \xymatrix{ p & Y \ar[l] \ar[d]^p \ar@/_{0.5cm}/[ld] \ar@{}[ll]_{p^*} \ar[r]^p \ar[d]^p \ar@/_{0.5cm}/[rd] \ar@{}[ll]_{F(n)} \ar[r] & \mathbb{P}^n } \]

If \( \mathcal{F} \) is the sheaf assoc. to \( M \) on \( \mathbb{P}^n \) we have

\[ p^*F = j^*M \]

\[ p_*p^*F = \bigoplus_{n \in \mathbb{Z}} F(n) \]

so

\[ H^i(\mathbb{P}^n, \bigoplus_{n \in \mathbb{Z}} F(n)) = H^i(Y, p^*F) \quad \text{p affine} \]

\[ = H^i(Y, j^*M) \]

But we have

\[ 0 \to H^0_m(M) \to M \to \bigoplus H^0(Y, j^*M) \to H^1_m(M) \to 0 \]

\[ H^i(Y, j^*M) = H^{i+1}_m(M) \quad i \geq 1. \]

Therefore one gets

\[ H^i(\mathbb{P}^n, \bigoplus_{n \in \mathbb{Z}} F(n)) = H^{i+1}_m(M) \quad i \geq 1 \]

\[ 0 \to H^0_m(M) \to M \to H^0(\mathbb{P}^n, \bigoplus_{n \in \mathbb{Z}} F(n)) \to H^1_m(M) \to 0 \]
Grothendieck duality thm.

\[ H^i(\mathbb{P}^n, F(n))^\vee = \bigwedge^r \operatorname{Ext}^{r-i}_{\mathbb{P}^n}(F(n), O(-r-1)) \]

\[ E_2 = H^p(\mathbb{P}^n, \operatorname{Ext}^{q}(F, O(-r-1-n))) \]

Thus for \( n < 0 \) one has

\[ H^i(\mathbb{P}^n, F(n))^\vee = H^0(\mathbb{P}^n, \operatorname{Ext}^{q}(F, O)(-r-1-n)). \]

Consequently, \( H^{i+1}_m(M) \) finite length \( \iff \ operatorname{Ext}^{r-i}(F, O) = 0 \) (for \( i > 1 \)).

Observe that I have two procedures for killing off free orbit types:

\[ H_G^i(X, F) \rightarrow H_G^i(X, F) \]

\[ H_G^*(X, F) \rightarrow H_G^*(X, F)[c^{-1}] \]

where \( c \) is the Euler class of some representation. Can I relate these two procedures? Both arise from pro-objects in the derived category of \( G \)-modules.

For example, suppose \( c \in H^2_G \) is represented...
by an extension

\[ 0 \rightarrow \Lambda \rightarrow \bigoplus X_i \rightarrow X_0 \rightarrow \Lambda \rightarrow 0 \]

Let \( P_i \) be a free \( \mathbf{A}^1 \)-resolution of \( \Lambda \). An element of \( H^0_\mathcal{O}(M) \) is rep. by

\[ \rightarrow P_{i+2d} \rightarrow \cdots \cdots \rightarrow P_0 \]

\[ \begin{array}{c}
\uparrow \\
M
\end{array} \]

Multiplying by \( e \) gives the element rep. by

\[ \rightarrow P_{i+2d} \rightarrow \cdots \cdots \rightarrow P_0 \]

\[ \begin{array}{c}
\uparrow \\
M
\end{array} \]

Thus we can arrange an inverse system of complexes

\[ K(d) : \quad P_{i+2d} \rightarrow \cdots \cdots \rightarrow P_0 \]

\[ \begin{array}{c}
\text{deg } 1 \\
\text{deg } 2d
\end{array} \]

and

\[ (H^\mathcal{O}_e(M)[e^{-1}]^i = \lim_{d \rightarrow \infty} H^i (\text{Hom}_\mathcal{O}(K(d), M)) \]

For \( G \)-cyclic, it is clear that

\[ H^\mathcal{O}_e(M) = (H^\mathcal{O}_e(M)[e^{-1}])^\wedge. \]
Question: What sort of modified cohomologies can be constructed in this manner?

Suppose $\tilde{H}^*(X,F)$ is a modified cohomology theory, say $H^*(G,M)$ is. Then we can consider those subgroups $H$ such that

$$N \rightarrow \tilde{H}^*(G, \text{Ind}_{H \rightarrow G}(N)) = 0$$

If $F$ is the family of these subgroups, then $F$ satisfies:

$$H' \in H \in F \Rightarrow H' \in F$$

$$H \in F \Rightarrow gHg^{-1} \in F.$$  

So the question is whether I can construct a modified cohomology associated to such a family $F$.

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Relative homological algebra: Suppose $A$ is a $k$-algebra, $i : k \rightarrow A$ the structural homomorphism. Then for each $A$-module $M$, I have a "standard" resolution

$$0 \rightarrow A \otimes_k M \rightarrow A \otimes_k M \rightarrow M \rightarrow 0$$

with the following properties:

i) $i^*$ of the resolution splits

ii) the $k$-modules $A \otimes_k N$ with $N$ a $k$-module are relatively projective, i.e. lifting for exact sequences split
over $k$.

One knows that such a relative-projective resolution of $M$ is unique up to homotopy, and one can define relative Ext's.

So suppose a subgroup $H$ of $G$ is given. Then we construct a relative injective resolution

$$0 \to \Lambda \to I_0^I \to I_1^I \to I_2^I \to \cdots$$

and form the following inverse system of complexes

$$K(0): \Lambda \to I_0^I$$

$$K(1): \Lambda \to I_0^I \to I_1^I$$

$$K(2): \Lambda \to I_0^I \to I_1^I \to I_2^I$$

and take

$$\lim_{\longrightarrow}^\kappa \mathbb{E}xt^i_G(K(k), M) = \mathbb{H}^i_G(G, M)$$

Now if $N \to Q^*$ is an injective resolution of an $H$-module $N$, then $\Lambda \otimes N \to \Lambda \otimes Q^*$ is an injective resolution of the $G$-module $\Lambda \otimes N$. So

$$\mathbb{E}xt^i_G(K(k), \Lambda \otimes N) = H^i_G(\mathbb{H}om_G(K(k), \Lambda \otimes Q^*))$$

$$= H^i_G(\mathbb{H}om_H(K(k), Q^*))$$
\[ = \text{Ext}^i_{H} (K(k), N) = \text{Ext}^{i+k}_{H} (H^k K(k), N) \]

But as we go from \( k \) to \( k+1 \) the map is zero.

\[ 0 \to N \to \cdots \to H^k K(k) \to 0 \]

\[ \overset{i}{\sim} \]

\[ o \to H^k K(k) \to \text{I}^{k+1} \to H^{k+1} (K(k+1)) \to 0 \]

\[ \text{splits.} \]

Let \( C \) be the category of finite \( G \)-sets, \( G \)-finite and let \( R \) be a trivial in \( C \). One has then Beilinson cohomology for any \( G \)-module \( F \)

\[ H^p (R, F) = \overset{\text{lim}}{\longrightarrow} F \]

and one has a spectral sequence

\[ E_2^{pq} = H^p (R, H^q (F)) \Rightarrow H^{p+q} (G, F) \]

Tate coh is outside of this theory because one wants to ignore the free \( G \)-sets, which are in \( R \). Somehow what's going on is that we have some sort of...
Cohomological localization process which replaces a sheaf $F$ by an inductive system of complexes:

$$k \mapsto R\mathrm{Hom}(K(k), F).$$

Brown's theorem: Let $G$ be a finite group and let $J$ be the poset of non-trivial subgroups of $G$. Then for any $G$-module $M$

$$\hat{A}^i(G, M) \rightarrow \hat{A}^i_G(J, M)$$

isomorphism when localized at $p$ (this means on the $p$-primary components since both sides are torsion).

Proof: Suppose $\hat{G}$ is a $G$-space and we wish to show:

$$f^* : H_0(X, M)_p \rightarrow H_0(Y, M)_p$$

is an isomorphism. Then by transfer theory it is enough to do this for a Sylow subgroup of $G$.

Another point: To calculate $H^*_G(X, M)$ we can consider covering by fixpt. sets.
Idea: \( G \) finite group. Let \( C(G) \) denote the category of transitive \( G \)-sets. In \( C(G) \) we have interesting objects. Take the case where \( G \) is cyclic of prime order \( p \). Then \( C(G) \) has 2 objects; it is the cone on the category \( G \). If \( X \) is a \( G \)-space it divides up into 2 strata\[ X = X^G \cup (X - X^G)\]

where \( X^G \) is closed in \( X \). Let \( F \) be a \( G \)-sheaf on \( X \); we have a local cohomology sequence
\[
\cdots \rightarrow H^i_G(X, X - X^G; F) \rightarrow H^i_G(X, F) \rightarrow H^i_G(X^G, F) \rightarrow \cdots
\]

Go back to Tate cohomology. One replaces \( M \in D^+(G\text{-mod}) \) by \( \{\text{Hom}(K(k), M)\} \) where \( K(k) \) is the inverse system of truncations of the Tate co.

Then \( \text{Hom}(K(k), M) \) is in \( D^+(G\text{-mod}) \) and it is coh. trivial in each dimension.

To construct \( K(k) \) one starts with
\[
\cdots \rightarrow \mathbb{Z} \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}
\]
\[
\mathbb{Z} \rightarrow P_0^r \rightarrow P_1^r \rightarrow \cdots
\]
and then $K(k)$ is the complex

$$
\cdots \to P_{i} \to P_{0} \to P_{-1} \to \cdots \to P_{-k} \to 0 \to 0
$$

degree $-k$.

This is a complex of free f.t. $\mathbb{Z}[G]$-modules split over $\mathbb{Z}$. Note

$$
K(k) \text{ homot. equiv to } K(k)[k]
$$

$$
\text{Hom}(K(k), M) \xrightarrow{\text{equiv}} \mathbb{Z} \text{ to } H^{k}(K(k))[k] \xrightarrow{\text{equiv}} [-k] \otimes M
$$

But

$$
0 \to H^{k}(K(k)) \to P_{k} \to \cdots
$$

Thus if we let $Z_{k}$ be defined by

$$
0 \to Z_{k} \to P_{k} \to \cdots \to P_{0} \to \mathbb{Z} \to 0
$$

we have a quasi

$$
\text{Hom}(K(k), M) \leftarrow Z_{k}[-k] \otimes M.
$$

So now we can see explicitly what the replacement $M \to \{\text{Hom}(K(k), M)\}$ consists of. One embeds $M$ into the complex

$$
0 \to M \to \cdots
$$

$$
0 \to P_{0} \otimes M \to \cdots \to P_{-1} \otimes M \to M \to 0
$$

$$
0 \to Z_{k} \otimes M \to 0
$$
Note that the cofibre of the map

\[ M[0] \longrightarrow Z[H] \otimes M \]

thus constructed is built up out of \( Z[G] \otimes N \) modules, that is, modules whose Tate cohomology is trivial. Therefore what it looks like I am after is a "largest" complex with trivial Tate cohomology mapping to \( M \).

"Trivial Tate cohomology" means roughly "built up out of the chains complexes of free \( G \)-spaces."

Now in general I am trying to construct a cohomology theory which will ignore modules of the form \( \text{Ind}_{H \rightarrow G}(N) \) for all subgroups in a certain class \( T \) closed under subgroups and conjugates.
Let $H$ be a subgroup of $G$, $i: H \to G$ the inclusion, and let $M$ be a $G$-module. One has a canonical isomorphism

$$i^! : M \cong \frac{Z[G]}{Z[H]} \otimes M \cong \frac{i^! \otimes M}{Z}$$

for there is a natural imprimitivity system on the right.

Let us form the complex

$$(*) \quad \xrightarrow{\mathbb{Z}} (i^! \otimes \mathbb{Z}) \quad \xrightarrow{\mathbb{Z}} (i^! \otimes \mathbb{Z}) \quad \xrightarrow{\mathbb{Z}} 0$$

and denote by $J(n)$ the truncated subcomplex which is zero in degrees $< n$ and the same in degrees $\geq -n$.

Suppose $M$ is a bounded complex of $G$-modules. We form the ind-complex

$$n \mapsto J(n) \otimes M$$

Note that $J(n)$ is free over $\mathbb{Z}$ in each dimension.

**Prop.** If $M$ is of the form $i^! N$, then the maps $J(n) \otimes M \to J(n+1) \otimes M$ are null-homotopic.

**Proof:** $J(n) \otimes i^! N = i^! (i^* J(n) \otimes N)$ and one knows that $i^*$ of the standard complex $(*)$ has a
contracting homotopy. This implies trivially that 
\[ i^* J(n) \to i^* \bar{G}(n+1) \] is null-homotopic.

Suppose we have an exact sequence of complexes
\[ (**) \quad 0 \to M' \to M \to M'' \to 0 \]
of \( G \)-modules,
\[ 0 \to J(n) \otimes M' \to J(n) \otimes M \to J(n) \otimes M'' \to 0 \]

Supposing that the ind objects \( [J(n) \otimes M'] \) and \( [J(n) \otimes M''] \) are zero in the category of complexes modulo homotopy, I can't conclude the same for \( [J(n) \otimes M] \) unless I know that the exact sequence (**) is split locally. So we should work in the derived category.

Now suppose \( M \) is such that \( M \to J(n) \otimes M \) is the zero map in the derived category. Then \( M \) is a retract of the fibre of this map, which is the total complex associated to the double complex
\[ \cdots \to (i_1^*)^k M \to \cdots \to i_1^* M \to \cdots \]

Thus \( M \) is a retract of a complex built up out of
completes of the form \( \mathbb{I}^N \).

Prop: The following conditions are equivalent for a bold complex of \( G \)-modules:

i) The inductive system \( \{ T(n) \otimes M \} \) is essentially zero in \( D^b(G\text{-mod}) \).

ii) \( \exists n \) such that \( M \rightarrow T(n) \otimes M \) is the zero map in \( D^b(G) \).

iii) \( M \) is a retract of a complex \( M' \) having a finite filtration

\[
0 \subset F_0 M' \subset \cdots \subset F_r M' = M'
\]

such that \( F_i M'/F_{i-1} M' \) is of the form \( \mathbb{I}^N \).

Proof: Clear from the preceding.

Remark: The above ought to hold for \( D^+(G) \), and maybe \( D(G) \).

Condition iii) above shows the construction given does not depend on the choice of \( T(n) \). In fact, we get similar results by taking any resolution of \( \mathbb{Z} \) by modules of the form \( \mathbb{I}^N \) which splits when \( \mathbb{I}^N \) is applied.
Suppose now we have a complex $F$ in the class described in the proposition, and a map $F \to M$. Then for $n$ large:

$$
\begin{array}{ccc}
F & \to & M \\
\downarrow & & \downarrow \\
J(n) \otimes F & \to & J(n) \otimes M \\
\end{array}
$$

so $F$ factors through the fibers of $M \to J(n) \otimes M$, which I should denote $J(n) \otimes M$. Let

$$
0 \to \lim_{\to} [F, J(n) \otimes M] \to [F, M] \to 0
$$

Thus we have a universal property for the ind system $\{J(n) \otimes M\}$.

So now define

$$
\tilde{H}^i(G, M) = \lim_{\to} H^i(G, J(n) \otimes M)
$$

This is the "localized" cohomology of $M$ with

$$
i : H \to G.$$

Variants: Let $S$ be a $G$-set, and denote by $S_G$ the cofibrand category over $G$ associated to $S$. We have a functor

$$
S_G \to \mathcal{M}_G = \mathcal{G}
$$
and adjoint functors

\[
\text{Funct}(S_G, \text{Ab}) \xrightarrow{i_!} \text{Funct}(\text{pt}_G, \text{Ab}) \xleftarrow{i^*}
\]

with \(i_!\) exact because \(i\) is cofibred with discrete fibre:

\[
(i_! F)(Y) = \lim_{X \in i^{-1}(Y)} F(X) = \lim_{X \in i^{-1}(Y)} F(X)
\]

The preceding construction ought to generalize easily.

Suppose now that \(X\) is a \(G\)-space, say a \(G\)-polyhedron. Let \(C(X)\) be the group of chains of \(X\). Then one defines

\[
H^i_G(X, M) = H^i_G(\text{Hom}(C(X), M))
\]

The localized cohomology is defined to be

\[
\tilde{H}^i_G(X, M) = \lim_{\rightarrow n} H^i_G(J(n) \otimes C^i(X, M))
\]

Summary: For each family \(F\) of subgroups of \(G\) closed under conjugation and subgroups, I have a universal ind-object in the derived category \(\Gamma_F(M) \to M\).
It has the universal property that each member of $\Gamma_{\mathcal{F}}(M)$ is built out of modules of the form $\mathbb{Z}[\mathbb{F}] \otimes \mathbb{Z}[H] M$ with $H \subseteq \mathcal{F}$ and that any such complex mapping to $M$ uniquely factors through $\Gamma_{\mathcal{F}}(M)$.

We seem to also have a pro-object:

$$0 \rightarrow M \rightarrow \Lambda_\ast \otimes M \rightarrow \cdots \rightarrow (\Lambda_\ast)^{k} M$$

is the term of the pro-object. Something like the "completion" of $M$.

So because every object $M$ of $D^b(G)$ has the "ind-filtration" $\Gamma_{\mathcal{F}}(M)$ we should get some sort of "localized" cohomology with respect to a subgroup $H$ such that

$$\Gamma_{\leq H < H}(M)$$

picks up the orbit type of $G/H$ of $M$ if say $M = C_\ast(X)$. 
Let $f : X \to Y$ be a map of $G$-spaces, and suppose we want to show:

$$f^* : H^*_G(Y) \to H^*_G(X)$$

is an isomorphism. Let us consider the category $\mathcal{C}$ of $G$-spaces $Z$ such that

$$H^*_G(Y \times Z) \to H^*_G(X \times Z)$$

is an isomorphism. Assume that $\mathcal{C}$ contains all $Z$ of the form $G/A$ where $A$ is an abelian subgroup of $G$. By Küneth, if $Z \in \mathcal{C}$ and if $S$ is a trivial $G$-space, then $Z \times S \in \mathcal{C}$.

$$H^*_G(Y \times Z \times S) = H^*_G((PG \times^G (Y \times Z \times S))$$

$$= H^*_G([PG \times^G (Y \times Z)] \times S)$$

I now want to show that $\mathcal{C}$ contains all $G$-spaces with abelian isotropy groups. Try arguing by induction in the number of isotropy groups. Better to look at the map $Z \to Z/G$

$$PG \times^G (Y \times Z) \to Z/G$$

$$U \quad U$$

$$PG^G_z Y \quad PG \times^G (Y \times G_z) \to G/G_z$$

$$PG \times^G z Y$$
so I should get a spectral sequence

\[ E^{p,i}_{2} = H^{p}(Z/G, G_{z} \rightarrow H^{q}_{G_{z}}(Y)) \Rightarrow H^{p+i}_{G}(Y \times Z) \]

and so therefore \( Z \in \mathbb{C} \) if all isotropy groups of \( \mathbb{C} \) are abelian. Now take a faithful representation \( V \) of \( G \), whence the flag manifold of \( V \) has only abelian isotropy groups. This shows \( PV \in \mathbb{C} \). But recall that one has an exact sequence

\[ 0 \rightarrow H^{*}_{G}(Y) \rightarrow H^{*}_{G}(Y \times PV) \rightarrow H^{*}_{G}(Y \times (PV)^{2}) \]

hence it seems we can prove the following

**Theorem:** Let \( f: X \rightarrow Y \) be a map of \( G \)-spaces such that for all abelian subgroups \( A \) of \( G \) one has that \( H^{*}_{A}(Y) \rightarrow H^{*}_{A}(X) \). Then \( \mathbb{C} \) for all \( G \)-spaces \( Z \), one has \( H^{*}_{G}(X \times Z) \rightarrow H^{*}_{G}(X \times Z) \).

(ridiculous: If \( f^{*}: H^{*}(Y) \rightarrow H^{*}(X) \), then \( H^{*}_{G}(Y) \rightarrow H^{*}_{G}(X) \) by the spec.)

\[ \mathbb{C} \] it should be possible to get down to an elementary abelian \( p \)-group \[ \mathbb{C} \] if I consider cohomology modulo \( p \).
Conjecture: Let $A$ be the family of elementary abelian $p$-groups in $G$ and let $M$ be a bounded complex of $\mathbb{Z}/p[\mathbb{G}]$-modules. Then $\Gamma_{a}(M)$ is isomorphic to $M$.

Idea of the proof: Let $V$ be a faithful representation of $G$ and let $X$ be the flag manifold of $V$. Then $M$ should be a retract of $C(X, M)$, and $C(X, M)$ should involve only elem. Fab. $\square$ subgroups.

Question: Does there exist a contractible space $X$ having elementary $p$-abelian isotropy groups; $X$ should be a polyhedron and contractible should perhaps be replaced by acyclic mod $p$. 
Suppose $\Gamma$ is a group having a subgroup $\Gamma'$ of finite index with $\text{cd}(\Gamma) < \infty$. One says then that $\text{vcd}(\Gamma) < \infty$. By Wall's finite-dimensional CW theory, which is a $\Gamma'$-CW which is a finite-dimensional CW complex on which $\Gamma'$ acts freely. Now let $\mathbf{x}$ be the multiplicative induction of $\mathbf{x}$ from $\Gamma'$ to $\Gamma$:

$$\mathbf{x} = \text{sections of } \Gamma \times \Gamma' \to \Gamma/\Gamma'$$

$\mathbf{x}$ is a finite-dimensional CW complex on which $\Gamma$ acts; the isotropy groups of $\mathbf{x}$ are finite subgroups, for each finite subgroup $H$ of $\Gamma$, $\mathbf{x}^H$ is contractible.

Let $S$ be the poset of finite subgroups of $\Gamma$.

We have

$$\mathbf{x} = \mathbf{x}_\text{free} \cup \bigcup_{H \in S} \mathbf{x}^H$$

Since $\mathbf{x}^H$ is contractible for each $H$ in $S$, I know that $\mathbf{x} = \bigcup \mathbf{x}^H$ has the homotopy type of the simplicial complex $K(S)$ associated to $S$:

$$\rightarrow \mathbb{I} \mathbf{x}^H_0 \rightarrow \mathbb{I} \mathbf{x}^H_0 \to \bigcup \mathbf{x}^H_0$$

$$\rightarrow \mathbb{I} \mathbf{x}^\text{pt}_0 \to \bigcup \mathbf{x}^\text{pt}_0$$
So let there be an exact sequence

\[ \cdots \to H^*_\Gamma (X \times X') \to H^*_\Gamma (X) \to H^*_\Gamma (X') \to \cdots \]

\[(*)\]

\[ H^*(X/\Gamma \times X'/\Gamma) \to H^*_\Gamma (X) \to H^*_\Gamma (X') \to H^*_\Gamma (S) \]

and an isomorphism on the Tate cohomology

\[ \hat{H}^*_\Gamma \cong \hat{H}^*_\Gamma (S). \]

Moreover, Brown's machinery is sufficient to establish under suitable finite\footnote{Conditions that the exact sequence (*) leads to a formula for $X$:

\[ \chi(\Gamma) \equiv \chi_\Gamma (S) \pmod{\mathbb{Z}} \]}

conditions that the exact sequence $(\ast)$ leads to a formula for $X$:

In the case where $\Gamma$ is arithmetic, one knows one should take $X$ to be the symmetric space with its corners added.

\[ (p\text{-version}): \]

This time suppose $\Gamma$ is normal in $\Gamma'$ and let $G_p$ be such that $G_p/\Gamma' = \text{defl. p-subgroup of } G/\Gamma'$. Then one looks only at the $G_p$-action of $G_p$ on $X$. All the isotropy groups inject into $G_p^{1/}$, hence they are
Question: Can one always assign an Euler characteristic in \( \mathbb{Q}/\mathbb{Z} \) to Tate cohomology?

\((p)\)-version. Let \( S_p \) be the set of non-trivial \( p \)-subgroups of \( \Gamma \), and put

\[ X_{(p)} = \bigcup_{H \in S_p} X^H \]

Then the isotropy groups of points in \( X - X_{(p)} \) are \( p' \)-groups, so

\[ H^*(X, X_{(p)})_{(p)} \cong H^*(X/\Gamma, X_{(p)}^\prime/\Gamma)_{(p)} \]

is bounded. Thus we get

\[ \hat{H}_p = \hat{H}_p^*(X_{(p)}) \]

which isn't very useful, because \( X_{(p)} \) doesn't have its isotropy groups in \( S_p \).

Suppose \( \Gamma' \triangleleft \Gamma \) and let \( \Gamma/\Gamma' \) be a non-
\( p \)-subgroup of \( \Gamma/\Gamma' \). Then all isotropy groups of
\( \Gamma_p \) on \( X \) are \( p \)-groups, so \( \Gamma_p \) acts freely on \( X - X'_{(p)} \), so

\[ \hat{H}_p^* \cong H_p^*(X_{(p)}) \]
Theorem: \((\hat{\Gamma}^*)_{(p)} \rightarrow \hat{\Gamma}^*(S_p)_{(p)}\).

Proof: By transfer theory it suffices to show
\[
\hat{\Gamma}^*_{(p)} \rightarrow \hat{\Gamma}^*(S_p)_{(p)}
\]
and by the non-\(p\)-results it suffices to show that for any finite non-trivial \(H\) in \(\Gamma_p\) that \(K_{S_p}^H\) is contractible, which is true because \(H\) is a \(p\)-group and if \(H\) normalizes a \(p\)-group \(P\), then \(HP\) is a \(p\)-group.

Theorem: \(\chi(\Gamma) - \chi_{\Gamma_p}(S_p) \in \mathbb{Z}_{(p)}\), integers localized at \(p\).

Proof: \(\chi(\Gamma) = \frac{1}{(\Gamma:\Gamma_p)} \chi(\Gamma_p)\)
\(\chi_{\Gamma_p}(S_p) = \frac{1}{(\Gamma:\Gamma_p)} \chi_{\Gamma_p}(S_p)\)
and since \((\Gamma:\Gamma_p)\) is a \(p\)-unit it suffices to show
\(\chi(\Gamma_p) - \chi_{\Gamma_p}(S_p) \in \mathbb{Z}\).

But this follows from the non-\(p\)-results.

If \(\Gamma\) finite we get \(\frac{1 - \chi(S_p)}{\Gamma_p} \in \mathbb{Z}\).
For example, suppose one is in the case of periodic cohomology, i.e. all Sylow $p$-subgroups cyclic or generalised quaternion. Then each $H$ in $S_p$ contains a unique cyclic subgroup of order $p$, so $K(S_p)$ is homotopy equivalent to the set of cyclic subgroups of order $p$. $X(S_p) = (G: N)$ where $N$ is the normalizer of some cyclic subgroup $A$ of order $p$.

Let $G_p$ act on $G/N$. Let $H$ be the stabilizer in $G_p$ of a cyclic group $B$ of order $p$. Then $B H$ is a $p$-group whose unique order $p$ subgroup is $B$, so $B < H < G_p$, so $B$ is the unique order $p$-subgroup in $G_p$. Thus $G_p$ acts freely on all the other order $p$ subgroups. This proves $|G_p|$ divides $1 - X(S_p)$.

Exactly what is happening? A finite group acts on $S_p$, so one has $X_p(S_p) = \frac{X(S_p)}{|\Gamma|}$.

But one knows that for every $e \neq H < F_p$, that $(S_p)^H$ is contractible, so $X_p(S_p) \equiv X \left( \bigcup_{H \in \text{Hes}(F_p)} S_p^H \right) \pmod{Z}$.
So what's happening is this: Look at $\Gamma_p$ acting on the simplicial complex $K(S_p)$. We know that the non-free part is $\bigcup_{H \leq S_p} K(S_p)^H$ and that $K(S_p)^{\text{H}}$ is pt, so that the non-free part has the homotopy type of $K(S(\Gamma_p))$ which is contractible. Thus

$$\chi(S_p) = \chi(S_p, S_p \text{ free})$$

and the latter is divisible by $|\Gamma_p|$. 

**Question:** What is the smallest class $C$ of subgroups of $G$ containing the elementary abelian groups such that if $A$ is elementary abelian and $A$ normalizes $C$ in $C$, then $AC \in C$?
Question: Let $G$ be a finite group. Can I find a $G$-polyhedron $X$ such that i) the isotropy groups of $X$ are abelian; ii) every abelian subgroup $H$, $xH$ is acyclic?

A bigger question is what sort of $G$-spaces can be found with some sort of acyclic properties.

Fact: Suppose $X$ is a space which is the union of subspaces $X_i$, i.e. $I$ where $I$ is a poset. Assume each $X_i$ is contractible and that for each $x \in X$, $\{ i | x \in X_i \}$ is contractible. Then $X$ is a classifying space for $I$. 
Relative cohomology.

Let $H$ be a subgroup of $G$, let $M$ and $M'$ be $G$-modules. One has the concept of a relative-projective $(\mathbb{Z}[G], \mathbb{Z}[H])$ resolution of $M$. It is a sequence of $G$-modules

$$
\cdots \to P_1 \to P_0 \to M \to 0
$$

which splits over $H$ and where each $P_i$ is relatively-projective, i.e., a retract of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ for some $H$-module $N$ (can take $N = P_0$). Such a resolution is unique up to homotopy; the standard example is to take

$$
\cdots \to \mathbb{Z}[G/H]^2 \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0
$$

and to tensor with $M$. Next one defines relative Ext's by

$$
\text{Ext}^i_{\mathbb{Z}[G], \mathbb{Z}[H]}(M, M') = H^i \left[ \text{Hom}_{\mathbb{Z}[G]}(P_i, M') \right].
$$

Taking $M = \mathbb{Z}$, we get relative cohomology groups

$$
H^i(G/H; M') = H^i \text{Hom} \left\{ \nu \mapsto H^0(G/H^{i+1}, M') \right\}
$$

Special case: $H \triangleleft G$. Then $H$ acts trivially on $G/H$ so

$$
H^0_G((G/H)^{i+1}; M') = H^0_{G/H}((G/H)^{i+1}; M')
$$
and thus

\[ H^i(G/H; M') = H^i(G/H, M' H) \]

More generally

\[ \text{Ext}^i_{\mathbb{Z}[G], \mathbb{Z}[H]}(M, M') = H^i(G/H, \mathbb{H}_{\mathbb{Z}[H]}(M, M')) \]

This shows that \( H \)-split short exact sequences will give rise to long exact sequences of \( \text{Ext} \).

Here's another interpretation of relative cohomology:

The semi-simplicial \( G \)-set

\[ \cdots \to (G/H)^2 \to G/H \]

is the "nerve" of the covering \( G/H \to \text{pt} \) in the category of \( G \)-sets, hence one has a spectral sequence

\[ E^{pq}_2 = H^p(\nu : (G/H)^n H; M')) \Rightarrow H^p(G; M') \]

so

\[ E^0_2 = H^p(G/H; M') \]

i.e. the relative cohomology is the cohomology of the base for this spectral sequence.

So one sees the relative cohomology \( H^i(G/H; M') \) is not simply related to the localized cohomology constructed previously:
\[ \varprojlim H^i(G, J(n) \otimes M) \]
\[ H^i(G/H, MH) \longrightarrow H^i(G, M) \]
\[ \varprojlim H^i(G, J(n) \otimes M) \]
\[ \overline{J}(n) : \ldots \longrightarrow \mathbb{Z}[G/H]^n \longrightarrow \ldots \longrightarrow \mathbb{Z}[G/H] \longrightarrow 0 \]

It is clear that we have
\[ H^0(G, M) \longrightarrow \varprojlim H^0(G, J(n) \otimes M) \]
\[ H^1(H, M) \longrightarrow \varprojlim H^1(G/H, J(n) \otimes M) = 0 \]

which says nothing.

It's clear one has when \( H \triangleleft G \)
\[ \varprojlim H^i(G/H, J(n) \otimes M)^H_H = \varprojlim H^i(G/H, J(n) \otimes M^H) \]
\[ \hat{H}^i(G/H, M^H) \]

Thus we get a commutative square:
\[ H^i(G/H, MH) \longrightarrow H^i(G, M) \]
\[ \hat{H}^i(G/H, MH) \longrightarrow \varprojlim H^i(G, J(n) \otimes M) \]
How do we get the spectral sequence for the extension \( H \to G \to G/H \)? We take the complex

\[
P_1 : \quad Z[\mathbb{G}/H]^2 \to Z[\mathbb{G}/H]
\]

and form

\[
\text{Hom}(P_1, M) = \quad \text{Hom}(P_1, M) \quad \to \quad (\text{Hom}(P_1, M))^2 \quad \to \quad ...
\]

\[
\mathbb{Z}[\mathbb{G}/H] \otimes M \quad \mathbb{Z}[\mathbb{G}/H]^2 \otimes M
\]

if \((G:H) < \infty\).

Now filter this via a standard way to get the spectral sequence. Better: Form the descent spectral sequence using \( J(H) \otimes M \).

\[
E_2^{pq} = H^p \left( \nu \mapsto H^q_{\mathbb{G}} \left( (G/H)^{\nu+1}; J(H) \otimes M \right) \right) \Rightarrow H^{p+q}(J(H) \otimes M)
\]

So for a normal subgroup \( H \) we get the spectral sequence

\[
E_2^{pq} = H^p(G/H; H^q(H, M)) \Rightarrow \lim_{\nu} H^{p+q}(G; J_H(\nu) \otimes M)
\]
February 28, 1976

Change the theorem of page 22:

Theorem: Let \( \mathcal{M} \in D^+(G\text{-mod}) \) and suppose \( \hat{H}_A^*(\mathcal{M}) = 0 \) for all abelian subgroups \( A \leq G \).

Then \( \hat{H}_G^*(\mathcal{M}) = 0 \).

Proof: \( E^\pi_2 = H^p(G,F, Gx \mapsto \hat{H}_G^*(\mathcal{M})) \Rightarrow \hat{H}_G^{p+1}(F, \mathcal{M}) \)

and \( \hat{H}_G^*(\mathcal{M}) \hookrightarrow \hat{H}_G^*(F, \mathcal{M}) \). Here \( F \) is the flag manifold of a faithful representation of \( G \).

Example: Let \( G \) be a finite simple non-abelian group; suppose \( G \) minimal simple, i.e. all proper subgroups are solvable. Let \( T \) be the poset of proper subgroups of \( G \).

Lemma: If \( 0 < H < G \), then \( T^H \) is contractible.

Proof: If \( K \in T^H \), then \( K \) is normalized by \( H \), so \( KH \) is a subgroup containing \( K, H \); but \( KH < G \) otherwise \( K \) would be normal in \( G \). Thus \( K < KH \leq H \) so \( T^H \) is contractible by the cone construction.

Fix \( 0 < H < G \). Then

\[ T = \bigcup_{0 < H' < H} T^{H'} \]

and I know the latter is contractible. Thus
because $H$ acts freely on the complement we have

$$X(T) \equiv 1 \mod (|H|).$$

since $H$ is arbitrary we could take $H$ to be each of the Sylow subgroups of $G$, hence we conclude

$$X(T) \equiv 1 \mod (|G|).$$

More generally, given any group $G$ I can consider the set $T_G$ of non-zero solvable subgroups. The same argument shows that for any solvable subgroup $H \in T_G$, $T_G^H$ is contractible and again

$$U_{G} \sim \bigcup_{0 < H' \leq H} \{H' | 0 < H' \leq H \}$$

hence again we have that $X(T_G) \equiv 1 \mod |G_p|$ for all $p$, hence $X(T_G) \equiv 1 \mod (|G|)$ for any finite group.
February 29, 1976

Burmaide ring.

Let $G$ be a finite group. The Burmaide ring $A(G)$ of $G$ is the Grothendieck group of the category of finite $G$-sets. It is a free $\mathbb{Z}$-module with basis $[G/H]$, where $H$ runs over representatives for the different conjugacy classes of subgroups.

Let $J$ denote the prepot of transitive $G$-sets; say $[G/H] < [G/K]$ if $J$ maps $G/K \to G/H$, i.e., if $K$ is conjugate to a subgroup of $H$. Thus when we specialize $G/K$ to $G/H$ the isotropy group increases. The largest member $G$ is the free orbit class.

For each subgroup $H$ of $G$ we get a homomorphism

$$\lambda_H : A(G) \to \mathbb{Z},$$

$$\lambda_H [X] = \text{card } (X^H).$$

This last formula shows $\lambda_H$ depends only on the conjugacy class of $H$.

Grothendieck idea: Let $X$ be a $G$-polyhedra, and let $A_G(X)$ be the Grothendieck group of the category of constructible $G$-sheaves of sets on $X$.?
If \( X \) is a \( G \)-polyhedron, then we can associate a class \( \chi_X \in A(G) \) as follows:

\[
\chi_X = \sum (-1)^p [x_p]
\]

where \( x_p \) is the set of \( p \)-simplices of \( X \). Note:

\[
\lambda_H (\chi_X) = \sum (-1)^p \lambda_H [x_p] = \sum (-1)^p [x_p^H] = \mathbb{Z} \chi (X^H)
\]

This \( \chi_X \) keeps track of the Euler characteristics of the different first subspaces.

**Proof:**

\[ A(G) \xrightarrow{\{ \lambda_H \}} \prod_{H \in J} \mathbb{Z} \] is injective.

**Proof:** Observe for \( H, K \in J \):

\[
\lambda_H (G/K) = \text{Card } \text{Hom}_G (G/H, G/K)
\]

\[
= \begin{cases} 
(N(H): H) & K \mathrel{\text{=H}} \text{H} \\
0 & \text{otherwise}
\end{cases}
\]

Choose a linear ordering of \( J \) compatible with the natural ordering. Then one sees the matrix

\[
H, K \mapsto \lambda_H (G/K)
\]

is upper triangular for this ordering with non-zero diagonal entries.
Prop. 2: \[ \text{Spec} \left( \frac{T \otimes T}{J} \right) \to \text{Spec} \ A(G) \]

By Chevalley, the image is closed; it is dense because of prop. 1.

Fix a prime \( p \). Then we know that
\[ \text{Spec} \left( \frac{T \otimes T}{J} / p \right) \to \text{Spec} \ A(G) \otimes \mathbb{Z} / p \]
is surjective. Observe that \( N(K)/K \) acts freely on the right of \( G/K \), hence also on the right of \( G/K^H \). Thus \( \lambda_H^{p} (G/K) \equiv 0 \mod (N(K)/K) \) for all \( H \), which shows that \( \lambda_H \mod p \) kill \( G/K \) when \( (N(K)/K) \equiv 0 \mod p \).

Prop. 3: \( N(K)/K \equiv 0 \mod p \iff [G/K] \text{ nilpotent in } A(G)/p. \)

Let \( J_p \subset J \) consist of \( [G/K] \equiv N(K)/K \equiv 0 \mod p \), and let \( J'_p \) be the complement. From the formula
\[ \lambda_H^{p} (G/K) = \begin{cases} (N(H) : H) & \text{if } H = K \\ 0 & \text{unless } H \to K \end{cases} \]
one sees that \( \lambda_H^{p} \) as \( H, K \) run over \( J'_p \) we get a triangular matrix with invertible diagonal entries \( \mod p \). Hence

Prop. 4: \( (A(G)/p)_{\text{red}} \to (\mathbb{Z}/p)^{J'_p}. \)
Prop 5: If $H \triangleleft H'$ and $H'/H$ is a $p$-group,
then $\lambda_H \equiv \lambda_{H'} \pmod{p}$.

Proof: 
$$\chi(xH') = \chi((xH)H'/H')$$
$$= \chi(xH).$$

So given a subgroup $H$, let $P/H$ be a Sylow $p$-subgroup of $N(H)/H$, whence $\lambda_p \equiv \lambda_H$ by the above. So it's clear that starting from $\Phi$ we can construct a chain
$$H = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_n$$
where $H_i/H_{i-1}$ is a $p$-group, and where $N(H_u)/H_u$ is
prime to $p$. Thus we see that $\lambda_H \equiv \lambda_{H_i}$ where
$H' \in J'$. Notice also that starting from $\Phi$, we have a
largest quotient group $K/K'$ which is a $p$-group. Take $K$ to be $H_n$
as above, and let $i$ be least such that $H_i \supset K'$
but $H_{i-1} \not\supset K'$. Then put $K'' = K' \cap H_{i-1}$
$$H_{i-1} \supset H_i \supset K'' \supset K'$$
Now $K''$ might not be normal in $K$, but if $g \in K$,
then \( K'' \cap gK''g^{-1} \) is normal in \( K' \); the intersection of normal subgroups of \( p \)-power index is again a normal subgroup of \( p \)-power index. Thus it is clear that \( K'' \) must be \( K' \) because of the minimality of \( K' \). \[ K' \leq H. \]

**Prop 6:** To each subgroup \( H \) of \( G \), \( \exists ! K \in \mathcal{J}_p \) such that \( K' \leq H \leq K \) where \( K/K' \) is the largest \( p \)-quotient group of \( K \).

For the uniqueness, see page 43.

**Prop 7:** If \( G \) is simple and \( \neq 1 \), then \( \text{Spec}(A(G)) \) is not connected.

**Proof:** We have a surjection

\[
\text{Spec}(\mathbb{Z}^J) \rightarrow \text{Spec} A(G)
\]

\[ \text{Spec} \mathbb{Z} \times \mathcal{J} \]

which is an isomorphism over all primes not dividing \( |G| \). For \( p \mid |G| \), we have to pinch together the layers belonging to the different groups in \( J' \).
If $G$ is simple, better if $H^1(G, \mathbb{Z}) = 0$, then consider the element $[G^\prime / G]$ of $\mathcal{F}$ corresponding to the subgroup $G$. It belongs to $\mathcal{F}_p$ for all $p$, and yet it does not get pinched to any other subgroup. Hence the map $\text{Spec}(\mathbb{Z}) \rightarrow \text{Spec} A(G)$ corresponding to this subgroup is a connected component. \[ \text{QED.} \]

Observe that every group has a maximal solvable quotient group. For if $G/K_1$, $G/K_2$ are solvable, then $G/K_1 \cap K_2 \rightarrow G/K_1 \times G/K_2$ and any subgroup of a solvable group is solvable.

Next suppose $H$ is a subgroup of $G$. If $N(H)/H \neq 1$ then $\exists \ H < H_1$ with $H_1/H$ abelian and non-trivial. Thus we get a chain $H = H_0 < H_1 < \cdots < H_n$ such that $N H_n = H_n$. Put $K = H_n$ and let $K/K'$ be the maximal solvable quotient group of $K$.

Note that $K' = (K', K')$. If $H_i$ contains $K'$, then $K'/K' \cap H_{i-1} \rightarrow H_i/H_{i-1}$ abelian.

so $K' \cap H_{i-1} = K'$ i.e. $K' \subset H_{i-1}$. Thus $H \supset K'$. 


Let $H$ be a subgroup of $G$. There exists a subgroup $K$ such that $K = N(K)$ and $D_\infty(K) \subseteq H \cdot K$. 

Revise:

Prop. 7: For any subgroups $H$ of $G$, there is a unique subgroup $K$ of $H$ such that:

i) $K \subseteq H$ and $H/K$ is solvable (resp. a $p$-group)
ii) $K$ has no solvable quotients (resp. no $p$-group quotients)
iii) $H'(K, \mathbb{Z}) = 0$ (resp. $H'(K, \mathbb{Z}/p) = 0$).

Namely, you take $H/K$ largest quotient group of $H$ such that $H/K$ is solvable (resp. a $p$-group), i.e. $K = D_\infty(H)$. Condition i) implies $K = D_\infty(H)$ so $K$ is unique.

Uniqueness in prop 6: Let $K' < H \cdot K$ where $K/K'$ is the maximal $p$-group quotient of $K$ and where $N(K):K$ is prime to $p$, $N(K) < N(K')$, because $K'$ is char. in $K$. Can find a $P/K'$ Sylow subgroup of $N(K)/K'$ containing $K/K'$. Then $K=P$ because otherwise $K \not\subseteq K'' < P$ contradiction. Such a $P$ is unique up to $^*$ conjugacy.

Dress them: $\text{To } \{\text{Spec } A(G)\} \leadsto \text{Conjugacy classes of subgps } K \text{ with } H'(K, \mathbb{Z}) = 0$.
Proof: Call the set on the right $J^*$. Map

$J \rightarrow J^*$

by sending

Thus we get a map

$\text{Spec}(\mathbb{Z}) \times J \rightarrow \text{Spec}(\mathbb{Z}) \times J^*$

which I claim factors as indicated, and moreover that the fibres over $J^*$ of $\text{Spec} A(G)$ are connected. Suppose then that two elements of $J$ are connected at $p$.

It is enough to show that if $H/K$ is a $p$-group, then $D_\infty(K) = D_\infty(H)$ and this is clear. To see that the fibres are connected, it suffices to show that we can get from the section of $\text{Spec} A(G)/\text{Spec} \mathbb{Z}$ assoc. to $H$ to the section assoc. to $D_\infty(H)$ by means of a sequence of connections at various primes. Thus I have only to show that a sequence

$D_\infty(H) = H_n \lhd H_{n-1} \lhd \cdots \lhd H_0 = H$

with $H_i/H_{i+1}$ a $p$-group for different $p$. This is obvious as $H/D_\infty(H)$ is solvable.

\[ \text{Cov: } \text{Spec}(A(G)) \text{ connected } \Leftrightarrow G \text{ solvable} \]
Suppose $G$-minimal simple. Then $\text{Spec} \, A(G)$ has two components. The fixpt map

$$\lambda_G : A(G) \to \mathbb{Z}$$

is a projection onto a direct factor. This means there is an idempotent $e$ such that

$$\begin{cases}
\lambda_G(e) = 1 \\
\lambda_H(e) = 0 & \text{all } 0 \leq H < G
\end{cases}$$

In fact idempotence follows from these formulas and the fact that $A(G) \to \mathbb{Z}^\pi$. Working with $1-e \text{ then we can find a } G\text{-space } X\text{ such that}

$$\begin{cases}
X(x^H) = 1 & 0 \leq H < G \\
= 0 & H = G
\end{cases}$$

But in fact I have seen that the poset $T$ of proper subgroups of $G$ is a $G$-space such that

$$\begin{cases}
T_H \sim pt \\
T_G = \emptyset
\end{cases}$$

and I know that $X(T) = 1 \pmod{|G|}$. Since

$$X((G/e)^H) = \begin{cases}
0 & 0 < H < G \\
|G| & H = G
\end{cases}$$

we can therefore add $\text{ }$ free gadgets to $T$ to
get a $G$-space $X$ with (1). Maybe Oliver proves we can embed $T$ in a contractible $G$-space $X$ such that $T - X$ is $G$-free.

Question: Can we generalize this construction to construct the different idempotents in $A(G)$?

For example, let's assume $G$ arbitrary and put $T_G = \text{non-zero solvable subgroups of } G$. If $H$ is a subgroup non-solvable, I don't know anything about $(T_G)^H$.

Following Oliver, we define an ideal $\Delta(G) \subset A(G)$ as follows: It consists of $[X] - 1$ where $X$ is contractible. Note that if $X$ is equivariantly contractible, then $X(X^H) = 1$ for all $H$ so $[X] = 1$. Note from

$$X \rightarrow \text{Con}(X) \rightarrow \Sigma(X)$$

that

$$[X] + [\Sigma(X)] = 2$$

Better

$$[\Sigma X] - 1 = [\Sigma X, \mu t] = [C X, X] = 1 - [X]$$

e.g. if $X = \emptyset$, $\Sigma X = S^0$; $X = S^n$, $\Sigma X = S^{n+1}$
\[ [\Sigma X] - 1 = -([X] - 1) \]

Now \( \Delta(G) \) clearly closed under + ; replacing \( X \) by \( \Sigma^2 X \) one can suppose \( X^G \neq \emptyset \), whence wedge gives the desired operation.

Better: \( \Delta(G) \) consists of all elements of the form \([X] - [X']\), where \( X, X' \) are \( G \)-spaces such that \( \exists \) \( G \)-map \( f: X \rightarrow X' \) which is a homotopy equivalence. Clearly this forms an ideal, and one has

\[ [X] - [X'] = [\text{Core}(f)] - [1] \]

so that this definition is the same as Oliver's.

**Problem:** Describe \( \text{Spec} \{ A(G)/\Delta(G) \} \subset \text{Spec} A(G) \).

P.A. Smith thm. If \( X \) is acyclic mod. \( p \), then so is \( X^p \) where \( P \) is a \( p \)-group.

Let \( \Delta_p(G) \) consist of \([X] - [X']\) where \( X, X' \) are \( G \)-spaces such that \( \exists \) a \( G \)-map \( f: X \rightarrow X' \) which is a \( H^*(\mathbb{Z}/p) \)-isomorphism. Again elements are of the form \([X] - 1\) where \( X \) is a \( G \)-space which is acyclic modulo \( p \). Notice that if \( X \) is acyclic mod. \( p \),
then from the exact sequence
\[ \cdots \to H_0(X, \mathbb{Z}) \to \tilde{H}_0(X, \mathbb{Z}/p) \to 0 \]
and the fact that \( X \) is a finite complex, we conclude that \( H_0(X, \mathbb{Z}) \) is finite and of order prime to \( p \).
In particular, \( H_0(X, \mathbb{Q}) = 0 \) so \( x(X) = 1 \). So from the Smith theorem if \( P \) is a \( p \)-subgroup of \( G \) then for \( X \) mod \( p \) acyclic we have
\[ X \text{ acyclic mod } p \implies x^p \text{ acyclic mod } p \implies x(x^p) = 1. \]
Therefore
\[
\lambda_p(\Delta_p(G)) = 0, \text{ for all } p\text{-groups } P.
\]

**Corollary:** \( \Delta_p(G) = 0 \) if \( G \) is a \( p \)-group,
(hence \( \Delta(G) = 0 \) also as \( \Delta(G) \subseteq \Delta_p(G) \)).

**Oliver theorem:** Suppose \( F \) is a family of subgroups containing the \( p \)-subgroups of \( G \) for all primes \( p \),
let \( X \) be a finite \( G \)-complex, and assume \( \exists \, \xi \in \Delta(G) \)
such that
\[
\lambda_H([X] - 1) = \lambda_H(\xi) \quad \forall \, H \in F.
\]
Then \( \exists \) contractible finite \( G \)-complex \( Y \supseteq X \) such that
\[ [Y] - 1 = \xi \]
and all isotropy groups of \( Y \setminus X \) are in \( F \).

In other words, if you take \( F \) to be the all \( p \)-subgroups.
for different \( p \), then you can prescribe the set
\[
\mathcal{I}(x) = \{ y \in Y \mid C_x \text{ not a } p\text{-group} \}
\]
for a contractible \( Y \), arbitrary except for Euler characteristic considerations.

Conjecture: \( \text{Spec } \{ A(G)/A_p(G) \} = \text{Spec } \mathbb{Z} \times \{ \text{conjugacy classes of } p\text{-subgroups} \} \) pinched together

Thus, \( \bigwedge A_p(G)^* = \bigcap_{p \text{-subgp}} \ker(A_p) \).

Idea: Can you carry out the following program?

a) \( G \) minimal simple \( \Rightarrow \) \( \exists \) \( G \)-space \( X \) \( \exists \) \( \bigcup \{ X^H_{\text{pt}} \mid H \in G \} \)

This implies that \( \Delta(G) \) contains \( [X]-1 \) which satisfies
\[
\lambda_H([X]+1) = \begin{cases} 0 & 0 \leq H \leq G \\ 1 & H = G \end{cases}
\]

b) For \( G \) odd this is impossible, because \( \Delta(G) \)
has a certain structure.

If \( X^H \) is contractible or \( \emptyset \) for each \( H \in G \), then the family of \( H \) such that \( X^H_{\text{pt}} \) corresponds to a division of
Spec \{A(G)\}: i.e., if \(K/H\) is solvable, then either \(H, K\) are both in \(F\) or both or not. Oliver's calls such an \(F\) separating and be proves:

Oliver's theorem 1: If \(F\) is a separating family, then \(F\) smooth action of \(G\) on a disk \(D\) such that

\[D^H = \emptyset, \quad H \not\in F,\]

\[D^H = \boxed{\text{disk, } H \in F}\]

Notice that if \(X\) is an acyclic module \(p\)- then so is \(X^P\) for any \(p\)-group \(P\) in \(G\) in particular \(X^P = \emptyset\) for any \(p\)-subgroups. Hence there doesn't exist a contractible G-space having only only elementary abelian isotropy groups.
March 3, 1976

Let $G$ be a finite group. Do we get an analogue of Whitehead simple homotopy theory by using $G$-complexes and replacing simple homotopy equivalence with $G$-homotopy equivalence?

The objects should be contractible (finite) $G$-complexes, the morphisms $G$-homotopy equivalences.

Oliver's construction of attaching $G$-cells:

$$G/H \times S^{i-1} \xrightarrow{f} X$$

$$G/H \times D^i \rightarrow Y$$

cocart.

$f$ is determined by a map $S^{i-1} \rightarrow X^H$. One can think of any $G$-polyhedron as being built up in this way. So there's a natural notion of $G$-complex: A CW-complex on which $G$ acts cellularly.

It's clear that if $y \in Y$ is fixed by $K$, then either $y \in X^K$ or $y \in (G/H \times (D^i - S^{i-1}))^K = (G/H)^K \times e^i$.

Thus we see that

$$(G/H)^K \times S^{i-1} \rightarrow X^K$$

$$G/H \times D^i \rightarrow Y^K$$

is also cocartesian. So we get in homology.
\[ H_t(y^k, x^k) = \begin{cases} 0 & t \neq i \\ \mathbb{Z}[[G/H]^k] & t = i \end{cases} \]

\[ \to H_i(y) \to \mathbb{Z}[[G/H]^k] \xrightarrow{\partial} H_{i-1}(x^k) \to H_{i-1}(y) \to 0 \]

Here \( i \geq 1 \) say. This shows that this cell-attaching process decreases \( H_{i-1}(x^k) \) and keeps \( H_k(x^k) \) the same for \( t < i-1 \).

Suppose we look at the question of whether we can attach free \( G \)-cells to get an embedding \( X \) into a contractible complex. So we can certainly attach cells so as to embed \( X \subset X' \) such that \( \tilde{H}_k(X') = 0 \), \( k \neq n \), where \( n = \text{max} \{ \dim x_j \mathbb{A} \} \) and \( X' \) simply-connected. From then on any further attaching of free cells just adds free \( G \)-modules to \( H_n(X') \). So we need to know \( H_n(X') \) is stably-free in order to get a contractible \( Y \). Thus

\[ 0 \to \tilde{C}(x) \to \tilde{C}(x') \to C(x', x) \to 0 \]

and \( C(x', x) \) is made up of free \( \mathbb{Z}[G] \)-modules, this invariant we get depends only on \( C(x) \).

By the P.A. Smith theorem if \( Y \) is to exist, then for each prime \( p \) and \( p \)-subgroup \( Q > 1 \) of \( G \) we...
must have \( x^G \mod p \) acyclic. Conversely if this condition holds for a prime \( p \) dividing \( |G| \), let \( G_p \) be a Sylow subgroup. Then

\[
\bigcup_{q \in \mathbb{Q}^{G_p}} \langle x^q \rangle
\]

is \( \mod p \) acyclic in \( X \), so as \( G_p \)-modules

\[
\tilde{C}_*(X) \sim \tilde{C}_*(X, U x^Q)_{(p)}.
\]

Better

\[
\tilde{C}_*(X')_{(p)} \sim \tilde{C}_*(X', U x'^Q)_{\text{tors}} \quad \text{or} \quad x'^Q = x^Q
\]

and the last group is a complex of free \( \mathbb{Z}[G_p] \)-modules.

\[ \tilde{H}_n(X')_{(p)} \text{ is } \mathbb{Z}[G_p] \text{-projective, hence } \mathbb{Z}[G] \text{-projective by transfer theory.} \]

Thus doing this for all \( p \) we see \( \tilde{H}_n(X) \) is \( \mathbb{Z}[G] \)-projective. So

**Prop.1:** Let \( X \) be a \( G \)-complex such that for each non-trivial \( p \)-subgroup \( Q \) of \( G \) and any prime \( p \) we have \( X^Q \) is \( \mod p \) acyclic. Then there is an element of \( H_0(\mathbb{Z}[G]) \) which is an obstruction to embedding \( X \) into a contractible \( G \)-complex \( Y \) such that \( X - Y \) is \( G \)-free. This obstruction is the class of the chain complex \( \tilde{C}_*(X; \mathbb{Z}) \) which is a perfect \( \mathbb{Z}[G] \)-module complex.
Prop. 2. If \( X^H \) is acyclic for \( 0 < H \leq G \), then the class of \( \tilde{C}_i(X, Z) \) in \( K_i(\mathbb{Z}[G]) \) is zero.

Proof: I know \( U \times H \) is acyclic, hence from \( 0 < H \leq G \)

\[
0 \rightarrow \tilde{C}_i(U \times H) \rightarrow \tilde{C}_i(X) \rightarrow \tilde{C}_i(X, U \times H) \rightarrow 0
\]

This follows \( G \)-free

Suppose one considers the set of all \( G \)-complexes \( X \) such that \( X^Q \) is mod \( p \) acyclic if \( Q \) is a non-trivial \( p \)-subgroup of \( G \) \((\text{all } p)\). Restrict attention to ones with basepoint. Then one has operations of addition (wedge) negative (suspension).

Question: Let \( X \) be a \( G \)-complex. Can one find an embedding of \( X \) into a contractible \( G \)-complex \( Y \) such that all isotropy groups of \( Y - X \) are \( p \)-subgroups \((\text{for different } p)\)? Any family containing any family containing

Let \( F \) be the family of \( p \)-subgroups of \( G \) for the different primes \( p \). What Oliver shows is that if \( F \) is contractible, \( G \)-space \( Z \) with \( X(xH) = X(zH) \) for all \( H \in F \), then such a \( Y \) exists.
Good question: Let $\mathcal{F}$ be the family of $p$-subgroups of $G$ for all $p$, let $X$ be a $G$-complex. Does there exist a contractible $G$-complex $Y$ containing $X$ such that $Y \setminus X$ has isotropy groups in $\mathcal{F}$?

Recall $\Delta = \{ [Y] - 1 \in A(G) \mid Y \sim pt \}$, and that

$$\Delta \subseteq \text{Ker} \left\{ A(G) \to \prod_{Q \in \mathcal{F}} \mathbb{Z} \right\}$$

$$\varphi = \{ x_Q \}$$

$$A_{\mathcal{F}}(G) = \prod_{Q \in \mathcal{F}} \mathbb{Z}$$

$$\varphi \left( 1_Q \right) = [G/Q]$$

Because $\mathcal{F}$ is a family one has an ideal $A_{\mathcal{F}}(G)$ generated by $[G/Q]$ with $Q \in \mathcal{F}$, i.e.

$$A_{\mathcal{F}}(G) = \{ [Z] \mid Z \text{ has isotropy groups in } \mathcal{F} \}$$

Now if the question above has answer yes, then

$$A(G) = \Delta + A_{\mathcal{F}}(G)$$

and this sum has to be direct since

$$A_{\mathcal{F}}(G) = \text{Ker} \left\{ A(G) \to \prod_{Q \in \mathcal{F}} \mathbb{Z} \right\}$$

$$\Delta \cap A_{\mathcal{F}}(G) \subseteq \text{Ker} \left\{ A(G) \to \prod_{Q \in \mathcal{F}} \mathbb{Z} \times \prod_{Q \in \mathcal{F}} \mathbb{Z} \right\} = 0.$$
So we would get an idempotent in $A(G)$ generating $\Delta$. In fact if we take $X$ to be empty, then we get a contractible complex $Y$ with all isotropy groups in $F$. This means that $F$ is separating hence that all solvable subgroups of $G$ are in $F$, which is possible only if $G$ is a $p$-group.

So let's see what goes wrong. Let $\mathbb{X}$ be a $G$ complex, and let's try to enlarge it using cells $G/H \times \mathbb{E}^i$, where $H \leq F$. I can assume that the homology of $X$ is concentrated in dimension $n$ if I want. Consider a maximal $Q$ in $F$, such that $X^Q$ is not mod $p$ acyclic where $p$ is the prime associated to $Q$. Then $N(Q)$ acts on $X^Q$, so the thing to try maybe is to show that we can add things to $X^Q$.

Suppose that $X^Q$ were mod $p$ acyclic for all $Q$ in $F$ with $Q \neq 1$. Then I have an obstruction in $K_2(Z[\Gamma])$. How to remove?

Here is Oliver's basic induction step:

Suppose $f : X \to Y$ is a mod $p$ homology isomorphism where $Y$ has a basepoint $y$ and all isotropy groups in $Y - y$ are $p$-groups. Take a maximal isotropy group $H$ of $Y - y$ and attach cells $G/H \times \mathbb{E}^i$ to $X^H = GX^H$ (note $X^H \to Y^H$ is a $H_\pi(Z[p])$ coin by the Smith theory).
and attach the same cells to $Y^{(H)}$ to get $f: X_1 \to Y_1$ a $H_\ast(x, \mathbb{Z}/p)$ isomorphism such that $H_\ast(Y^H, \mathbb{Z}/p)$ is concentrated in one dimension. Since $N(H)/H$ acts freely on $Y^H - Y$, this forces $H_\ast((x^H, \mathbb{Z}/p) \cong H_\ast(Y^H, \mathbb{Z}/p)$ to be trivial over $N(H)/H$, hence one can attach maps $G/H \times e^i$ to $X_1^{(H)}$ and $Y_1^{(H)}$ to get another $\mathbb{Z}/p$-homology isomorphism $f: X_2 \to Y_2$ such that $X_2^H$ and $Y_2^H$ are $\mathbb{Z}/p$-acyclic. Then

$$Y_2^{(H)} = G \times_{N(H)} Y^H / G \times [y^H]$$

is $\mathbb{Z}/p$-acyclic, so we can replace $Y_2$ by $Y_2 / Y_2^{(H)}$ to get a $Y_3$ with fewer orbit types than $Y$. Now use induction and you end up with an $X \in X^\ast$ with $X^\ast = \mathbb{Z}/p$-acyclic and all isotropy sets of $X^\ast - X$ equal to isotropy groups of $Y - Y$. 
Problem: Find a periodic $K$-theory $\hat{K}_*$ together with a map $K_* \to \hat{K}_*$ which is an isomorphism in high degrees. What does the fibre theory look like? Some kind of $K$-homology?

Suppose one considers a finite field $\mathbb{F}_q$. Then a "periodic" version can exist only if $q$ is inverted. This is an opinion based on the long exact sequence

$$\cdots \to K_* (\mathbb{F}_q) \to K_* (\mathbb{H}) \to K_* (\mathbb{Q}) \to \cdots$$

It is clearly necessary that we invert $q$ in order to obtain Adams operations in negative degrees. Calculate

$$\hat{K}_* (\mathbb{F}_q) = K_* (\mathbb{F}_q) \quad i > 0$$

$$= \mathbb{Z} [\frac{1}{q}] \quad i = 0, -1$$

$$= K_* (\mathbb{F}_q) \quad i < 0, -1$$

$$\to \mathbb{Z} [\frac{1}{q}] \to \mathbb{Z} [\frac{1}{q}] \to \hat{K}_* (\mathbb{F}_q)$$

$$\to 0 \to 0 \to \hat{K}_* (\mathbb{F}_q)$$

$$\to \mathbb{Z} [\frac{1}{q}] \to \mathbb{Z} [\frac{1}{q}] \to K_* (\mathbb{F}_q)$$
This is what one would expect from a Atiyah-Hirze spectral sequence.

Try to construct $K$ the way one does with Tate cohomology as some sort of inductive limit.

**Lundell construction**. Bott maps:

$$\Sigma: U_n \rightarrow \text{Grass}_n(C^{2n}) \rightarrow \Sigma U_{2n}$$

This gives a map $\Sigma^2 U_n \rightarrow U_{2n}$. Lundell shows this map factors through a map $\Sigma^2 U_n \rightarrow U_{n+1}$.

Thus he gets a commutative diagram.
One has
\[ K^8(X) = \lim_{n \to \infty} [\Sigma^{-g+2n}X, BU] \]
\[ = \lim_{n \to \infty} [\Sigma^{-g+2n}X, U] \]
Define \[ K^8_L(X) = \lim_{n \to \infty} [\Sigma^{-g+2n}X, U_L] \]
and note that if \( X = pt \), then for \( g \geq 0 \), it is an isomorphism:
\[ K^8_L(pt) \cong K^8(pt) \quad g \geq 0 \]

For \( g = 0 \), we get \( \lim_{n \to \infty} \tilde{\pi}_{2n}^0(U_n) = \mathbb{Z} \)

\( g = +1 \), we get \( \lim_{n \to \infty} \tilde{\pi}_{2n}^1(U_n) = \lim_{n \to \infty} \mathbb{Z}/n! = \mathbb{Q} \mathbb{Z} \)?

So we have to review:
\[ \tilde{\pi}_{2n+1}(U_{n+1}) \to \tilde{\pi}_{2n+1}(S^{2n+1}) \to \tilde{\pi}_{2n}(U_n) \to \tilde{\pi}_{2n}(U_{n+1}) \to \tilde{\pi}_{2n}(S^{2n+1}) \]
\( g = 0 \) \( \mathbb{Z} \)
\( g = +1 \) \( \mathbb{Z} \)
\( g = +2 \) \( 0 \)

and we know \( U_{n+1} \sim S^4 \times S^3 x \cdots x S^{2n+1} \)
so we see that \( \tilde{\pi}_{2n}(U_n) \) is cyclic.

We can interpret elements of \( \tilde{\pi}_{2n+1}(U_{n+1}) = \tilde{\pi}_{2n+2}(BU_{n+1}) \)
as $(n+1)$-dimensional bundles over $S^{2n+2}$. The map $\pi_{2n+1}(U_{n+1}) \to \pi_{2n+1}(S^{2n+1})$ is probably the Euler class of this bundle. Bott proved the Euler class was $n!$, so

$$\pi_{2n}(U_n) = \mathbb{Z}/n!.$$ 

Now according to Baez, the homotopy groups of $K^0$ are

$$\begin{array}{cccc}
\mathbb{Q}/\mathbb{Z} & 0 & \mathbb{Q}/\mathbb{Z} & 0 \\
\mathbb{Q}/\mathbb{Z} & 0 & \mathbb{Q}/\mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q}
\end{array}$$

and that the cofibre is the $K$-theory associated to $K^0 \otimes \mathbb{Q}$, i.e.,

$$F^0(X) = \bigoplus_{i=0}^{\infty} H^{2i}(X, \mathbb{Q})$$

and

$$F^{2i}(X) = \bigoplus_{i=0}^{\infty} H^{2i}(X, \mathbb{Q})$$

Serre's example: First let us consider the universal example of a bundle $E$ of rank $n$ such that $E+1$ is trivialized. Thus I wish to consider over $A$ a ring $A$ the set of pairs $(s, p)$ where
Locally for the Zariski topology on $A$, Ker $p$ is $≈ A^n$, so $GL_{n+1}$ acts transitively on the set of pairs $(s, p)$ locally. Thus the universal example occurs over $GL_{n+1}/GL_n$ which has coordinate ring $\mathbb{Z}[X_1,...,X_{n+1}, Y_1,...,Y_{n+1}]/(\sum_i Y_i = 1)$.

By topology, $GL_{n+1}/GL_n \sim U_{n+1}/U_n = S^{2n+1}$.

Furthermore, $GL_{n+1} \to GL_{n+1}/GL_n$ is the projection map $U_{n+1} \to S^{2n+1}$.

This is known to have a section for $n > 2$. The principal bundle $U_n \to U_{n+1} \to S^{2n+1}$ is classified by the generator of $\pi_{2n}(U_n) = \mathbb{Z}/n!$.

---

I can map $\pi_{2n}(U_n) \to \pi_{2n+2}(U_{n+1})$ how?
Let $F$ be a family of subgroups of a finite group $G$. Let $S$ be a finite $G$-set such that $F$ is the set of isotropy groups of $S$.

Let us consider the functor

$$i : (S, G) \rightarrow (\text{pt}, G).$$

Then we have a standard resolution

$$\rightarrow (\mathbb{Z}^* \mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0.$$

Let $T_S(n)$ denote the truncated complex

$$0 \rightarrow (\mathbb{Z}^* \mathbb{Z}, \mathbb{Z}) \rightarrow \ldots \rightarrow i_1^* \mathbb{Z} \rightarrow 0$$

and let $\overline{T}_S(n)$ denote the complex

$$0 \rightarrow (\mathbb{Z}^* \mathbb{Z}, \mathbb{Z}) \rightarrow \ldots \rightarrow i_1^* \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

so that we have an exact sequence of complexes

$$0 \rightarrow \mathbb{Z} \rightarrow \overline{T}_S(n) \rightarrow \Sigma \overline{T}_S(n) \rightarrow 0$$

Let $M$ be a complex of $G$-modules which is bounded below (cochain like). Since we have the formula

$$(i_1^* \mathcal{M}) = i_1^* \mathbb{Z} \otimes \mathcal{M}$$

for all $G$-modules $\mathcal{M}$, it is clear that the complex

$$\overline{T}_S(n) \otimes M \rightarrow M \rightarrow \overline{T}_S(n) \otimes M \rightarrow \Sigma \overline{T}_S \otimes M$$

one sees $M$ is a retract of $\overline{T}_S(n) \otimes M$.

**Prop. 2:** Let $M$ be a complex satisfying the equivalent condition of Prop. 1. Then for any $M$,

$$[M, M] = [\mathcal{M}, \overline{T}_S(n) \otimes \mathcal{M}].$$
complex $J_5(n) \otimes M$ has a filtration with quotients of the form $i/N$.

**Proposition 1. TFAE**

(i) The ind-object $m \mapsto \tilde{J}_5(n) \otimes M$ in $D^+(G)$ is essentially zero.

(ii) $M$ is a retract of a complex which has a filtration with quotients of the form $i/N$.

(iii) $J_5(n) \otimes M \rightarrow M$ has a section.

$(ii) \Rightarrow (i)$ suffices to show that $\tilde{J}_5(n) \otimes i/N = i_*(i^*\tilde{J}_5(n) \otimes N)$ is essentially zero, which is clear because $i^*\tilde{J}_5(n)$ is homotopy-equivalent to a complex concentrated in degree $n$.

$(i) \Rightarrow (ii)$ If $M \rightarrow \tilde{J}_5(n) \otimes M$ is the zero map, then from the triangle

$$
\tilde{J}_5(n) \otimes M \rightarrow M \rightarrow \tilde{J}_5(n) \otimes M \rightarrow \Sigma \tilde{J}_5(n) \otimes M
$$

one sees $M$ is a retract of $\tilde{J}_5(n) \otimes M$.

**Prop. 2.** Let $M$ be a complex satisfying the equivalent condition of Prop. 1. Then for any $M'$

$$
[M, M'] = \lim \sup [M, \tilde{J}_5(n) \otimes M']
$$

**Proof:** It suffices
to show \( \varinjlim_n [M, \tilde{J}_S(n) \otimes M'] = 0 \), and to
do this for \( M = \frac{i}{N} \). Since
\[
[i : \frac{i}{N}, \tilde{J}_S(n) \otimes M'] = [N, i^* (\tilde{J}_S(n) \otimes M')]
\]
and \( i^* \tilde{J}_S(n) \) is essentially zero one wins.

So what we've done is to define in
Prop.
1 a subcategory \( D^+_S(G) \) closed under
extensions and to show that for any \( M \)
there is an ind-object \( n \mapsto \tilde{J}_S(n) \otimes M \) in \( D^+_S(G) \) which is universal for maps
of objects of \( D^+_S(G) \) to \( M \).

Let me now wor.