<u>January 14, 1975.</u>

Yesterday I found the following idea for describing unimodular subspaces. Let E be a bundle on $\mathbb{P}^1$ of rank n; a unimodular subspace V of $\Gamma(\mathbb{A}^1, E)$ can be identified with a rational map $\mathcal{O} \otimes V \longrightarrow E$ which is an isomorphism off ∞. Thus $\mathcal{O} \otimes V$ is a bundle $E' \subset E \otimes F$ which differs from E only at ∞ and which is isom to $\mathcal{O}^n$; this last condition is equivalent to $\deg(E') = 0$ and $\mathrm{Hom}(\mathcal{O}(1), E') = 0$. So we can get at unimodular subspaces in E by considering the space of all lattices for $\mathcal{O}_\infty$ in $E \otimes F$ of the correct index Λ and looking at the open set where $\mathrm{Hom}(\mathcal{O}(1), E') = 0$.

Go back to $K \in Q_d^{(2)}(\mathbb{A}^1)$. This K comes with an embedding $K \subset \mathcal{O}^2$, which is an isomorphism near ∞. Thus we are concerned with all lattices Λ in F^2 for $\mathcal{O}_\infty$ of index 2. In this problem it is natural to look at those lattices Λ containing $\mathcal{O}_\infty^2$ which is a 2 dimensional variety Y. Thus the variety of unimodular sub-spaces of K I look for is an open set of Y.

Y is hard to understand because it is singular. So why not take a pair of generic points and play a similar game.

So start with a bundle K in $Q_2^{(2)}(\mathbb{P}^1)$ and suppose $\mathcal{O}^2/K$ has support not meeting 2 distinct points a, b of $\mathbb{P}^1$. Then I consider all K' such that $K \subset K'$ and $K'/K \simeq k(a) \oplus k(b)$. The set of K' can be identified with $Y = \mathbb{P}^1 \times \mathbb{P}^1$ in a fashion independent of K. So now I want the open set of Y where $\mathrm{Hom}(\mathcal{O}(1), K') = 0$. $Q_2^{(2)}(\mathbb{P}^1 - \{a, b\}) \times \mathbb{P}^1 \times \mathbb{P}^1$. For each point of $\mathbb{P}^1 \times \mathbb{P}^1$ i.e. each $L \supset \mathcal{O}^2$ such that $\mathrm{Supp}(L/\mathcal{O}^2) = \{a + b\}$ we get a 2-dimensional v.s. $\mathrm{Hom}(\mathcal{O}(1), L) = \mathrm{Hom}(\mathcal{O}(1), L/\mathcal{O}^2)$

If $K = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then what is the open subset of $\mathbb{P}^1 \times \mathbb{P}^1$ which gives $K' \simeq \mathcal{O}^2$. Should be the complement of Δ.

If $K = \mathcal{O} \oplus \mathcal{O}(-2)$, then what is the answer?

$$0 \longrightarrow \mathcal{O} \longrightarrow K \longrightarrow \mathcal{O}(-2) \longrightarrow 0$$

How do we identify the quotients? By duality I want to arrange for $(K')^\vee \subset K^\vee = \mathcal{O} \oplus \mathcal{O}(2)$ such that $(K')^\vee$ has no $\mathcal{O}(1)$ maps.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(2) \arrow[r] & K^\vee \arrow[r] & \mathcal{O} \arrow[r] & 0 \\
& & \cdot \arrow[u, hook, "-a"'] & & \\
& & (K')^\vee \arrow[u, hook, "-b"'] & &
\end{tikzcd}$$

so you want $\mathcal{O}(2)$ to map onto $K^\vee/(K')^\vee$

Reformulation: Let E be a bundle isom to $\mathcal{O}\oplus\mathcal{O}(2)$ or $\mathcal{O}(1)^2$. Then for each pair of lines $L_a \subset E(a) = E/m_aE$, $L_b \subset E/m_bE$ I get a bundle $E' \subset E$ and I want to know when $E' \simeq \mathcal{O}^2$.

Case 1: $E \simeq \mathcal{O}(1)^2$. Then E canon. isom to $\mathcal{O}(1)\otimes V$, $V = \Gamma(E(-1))$, so if we choose a section of $\mathcal{O}(1)$ not zero at a, b then we can identify $E(a)$, $E(b)$, in fact we get a canonical identification of $\mathbb{P}E(a)$ and $\mathbb{P}E(b)$. Here the bad subset of $\mathbb{P}E(a)\times\mathbb{P}E(b)$ is the graph of this identification.

Case 2: $E \simeq \mathcal{O}(2)\oplus\mathcal{O}$. Here one has a canonical exact sequence

$$0 \longrightarrow \mathcal{O}(2) \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

up to scalar mult. on $\mathcal{O}(2) + \mathcal{O}$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(2) \arrow[r] & E \arrow[r] & \mathcal{O} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(1) \arrow[r] & E\times_{E(a)} L_a \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{O} & \\
0 \arrow[r] & \mathcal{O} \arrow[u] \arrow[r] & E\times_{(E(a)\oplus E(b))}(L_a\oplus L_b) \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{O} & \\
 & & E' \arrow[u, phantom, "\shortparallel"] & &
\end{tikzcd}$$

If E' is not to receive non-zero maps from $\mathcal{O}(1)$, then it must be the case that $\mathcal{O}(2)(a) \longrightarrow E(a) \to E(a)/L_a$

be non-zero and similarly for b. Thus the bad set is

$$\mathbb{P}E(a) \times \{*\} \cup \{*\} \times \mathbb{P}E(b) \subset \mathbb{P}E(a) \times \mathbb{P}E(b)$$

where $*$ denotes the canonical line obtained from $\mathcal{O}(2) \subset E$.

However $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta\mathbb{P}^1$ is not contractible. It is the affine bundle over $\mathbb{P}^1$ of all splittings of

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0$$

[redacted]

Observe that the two sets we have described $\mathbb{P}^1 \vee \mathbb{P}^1$, $\Delta\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are divisors, hence determine line bundles. The line bundle is $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ whose sections are $V^* \otimes V^*$ = bilinear forms on V. To get $\Delta\mathbb{P}^1$ choose [redacted] a non-[redacted]degenerate alternating form. To get $\mathbb{P}^1 \vee \mathbb{P}^1$ take the [redacted] product [redacted] $\lambda \otimes \lambda$ with $\lambda \in V^* - 0$. So it seems that E defines a canonical pairing of $E(a)$ and $E(b)$, maybe up to a scalar.

Possibility: $E^\vee = \mathcal{O}(-1)^2$ or $\mathcal{O}(-2) \oplus \mathcal{O}$ so $E^\vee(1) = \mathcal{O}^2$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Thus we get a correspondence

$$\underset{2}{\Gamma(E^\vee(1))} \hookrightarrow \underset{4}{E^\vee(a) \oplus E^\vee(b)}$$

which does not sit entirely in $\check{E}(a)$ or $E^\vee(b)$. Thus

$$\Lambda^2 \Gamma(E^\vee(1)) \subset \check{E}(a) \otimes E^\vee(b)$$

will give us the desired bilinear form.

~~Have not yet succeeded~~

Let $K \subset \mathcal{O}^2$ be of index 2 and $K_\infty = \mathcal{O}^2_\infty$. Let Z be the variety of $\mathcal{O}_\infty$-lattices $L \supset \mathcal{O}^2_\infty$ of index -2 wrt $\mathcal{O}^2_\infty$. Then K together with L give rise to a bundle K' on $\mathbb{P}^1$. K is isomorphic to $\mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. $\mathcal{O}^2 + K' = \mathcal{O}^2 + L$ is isomorphic to either $\mathcal{O}(1)^2$ or $\mathcal{O} \oplus \mathcal{O}(2)$. Thus K' is isomorphic to $\mathcal{O}^2$, $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, or $\mathcal{O}(-2) \oplus \mathcal{O}(2)$. For $K' \simeq \mathcal{O}^2$ it is necessary + sufficient that $H^0(K'(-1)) = 0$. Now as L varies over Z $H^0(K(1))$ has constant rank. Thus from a sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(1)^2 \longrightarrow \mathcal{O}(3) \longrightarrow 0$$

we get

$$0 \longrightarrow K'(-1) \longrightarrow K'(1)^2 \longrightarrow K'(3) \longrightarrow 0$$

so

$$0 \longrightarrow H^0(K'(-1)) \longrightarrow \underset{\text{dim} = 4+4 = 8}{H^0(K'(1))^2} \longrightarrow \underset{\text{dim } 8}{H^0(K'(3))} \longrightarrow H^1(K'(-1)) \longrightarrow 0$$

Consequently on Z we get a map of 8 dimensional bundles and the open set we want is where this map is an isomorphism. (Z is a variety of dim 2)

General case: Let $\mathcal{E}$ "$\{E_s\}$" be a family of vector bundles on $\mathbb{P}^1$ parameterized by a scheme S. Assume each E_s is of degree 0. We can find m so that $H^1(E_s(m)) = 0$ for all s. Then $s \mapsto H^0(E_s(m))$ is a vector bundle of rank $n \cdot (m+1)$ where $n = \operatorname{rank} E_s$. So from a sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(m)^2 \longrightarrow \mathcal{O}(2m-1) \longrightarrow 0$$

we get

$$0 \to H^0(E_s(-1)) \longrightarrow \underset{2n(m-1+1)}{H^0(E_s(m-1))^2} \longrightarrow \underset{n(2m-1+1)}{H^0(E_s(2m-1))} \to H^1(E_s(-1)) \to 0$$

so again the subscheme of S where $E_s \simeq \mathcal{O}^m$ is described by where a homomorphism between vector bundles is an isomorphism.

January 15, 1976:

Problem: Let $\mathcal{E}$ be a rank 2 bundle over $\mathbb{P}^1_S$ such that on each fibre $\mathcal{E}$ is isomorphic to either $\mathcal{O}(1)^2$ or to $\mathcal{O} \oplus \mathcal{O}(2)$. (Then $p_*(\mathcal{E})$ is a vector bundle on S of rank 4. Using a sequence

$$0 \longrightarrow \mathcal{E}(-2) \longrightarrow \mathcal{E}^2 \longrightarrow \mathcal{E}(2) \longrightarrow 0$$

we get the perfect complex

$$p_*(\mathcal{E})^2 \xrightarrow{\ \theta\ } p_*(\mathcal{E}(2))$$

on S whose homology groups are $R^q p_*(\mathcal{E}(-2))$. The open set where $\mathcal{E}$ is isomorphic to $\mathcal{O}(+1)^2$ is where θ is an isomorphism.) The problem is whether, when S is affine, I can find $\mathcal{E}' < \mathcal{E}$ with $\mathcal{E}/\mathcal{E}'$ a length 2 vector bundle over S supported at the ∞ section such that $\mathcal{E}'$ is isomorphic to $\mathcal{O}^2$ on each fibre.

Suppose $\mathcal{E}$ is a rank 2 bundle over $S \times \mathbb{P}^1$ such that on each fibre over S $\mathcal{E}$ is isomorphic to $\mathcal{O}^2$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Then $p_*(\mathcal{E})$ is a bundle of rank 2 over S. Using

$$0 \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{E}^2 \longrightarrow \mathcal{E}(1) \longrightarrow 0$$

we get the map of 4 dimensional bundles on S

$$p_*(\mathcal{E})^2 \longrightarrow p_*(\mathcal{E}(1))$$

which will be an isomorphism exactly where $\mathcal{E} \simeq \mathcal{O}^2$.

The rank of this vector bundle map is always ≥ 3.

Suppose this map is never an isomorphism, i.e. for example deforming $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Start again. $\mathcal{E}$ is a rank 2 vector bundle over $\mathbb{P}^1_S$ isomorphic to $\mathcal{O}^2$ or to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on each fibre. To show that S affine $\Longrightarrow \exists$ exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow M \longrightarrow 0$$

where M is finite flat over S of rank 2 and supported on the ∞ section, such that $\mathcal{E}'$ is isomorphic to $\mathcal{O}(-1)^2$ on the fibres. Thus

$$0 \longrightarrow \underset{\overset{\shortparallel}{0}}{p_*(\mathcal{E}')} \longrightarrow p_* \mathcal{E} \longrightarrow p_* M \longrightarrow \underset{\overset{\shortparallel}{0}}{R^1 p_*(\mathcal{E}')} \longrightarrow 0 \to 0$$

so $p_* \mathcal{E} = p_* M$. Therefore all you have to do to get M as a sheaf is to produce a nilpotent endo of $p_* E$. Therefore the problem is to select in $E \otimes \mathcal{O}_\infty / m_\infty^2$ a submodule complementary to the image of $\Gamma(E)$.

Let's examine the different cases.

First suppose all fibres $E = \mathcal{E}(s)$ are isomorphic to $\mathcal{O}^2$. Then $E \otimes \mathcal{O}_\infty / m_\infty^2 \cong (\mathcal{O}_\infty / m_\infty^2)^2$ and the images of $\Gamma(E)$ is the subspace of constant $\mathbb{C}^2$. I recall showing the possible choices for M are given by 2×2 nilpotent matrices.

Next suppose all fibres isomorphic to $\mathcal{O}(-1)\oplus\mathcal{O}(1)$. We know this happens when the map (canonical)

$$p_*(\mathcal{E})^2 \longrightarrow p_*(\mathcal{E}(1))$$

of 4 diml. bundles on S is nowhere an isom. On S_{red} $p_*(\mathcal{E}(-1))$ will be a line bundle, so over S_{red} we get an exact sequence of

$$0 \longrightarrow \mathcal{O}(1)\otimes L \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1)\otimes L' \longrightarrow 0$$

vector bundles, where L, L' are unique line bundles on S_{red}. Thus $L^2 \to p_*\mathcal{E}$.

<u>January 16, 1976.</u>

Let E be a bundle over $\mathbb{P}^1_A$. Assume there is a nilpotent ideal I in A such that there is an exact sequence

$$0 \longrightarrow \mathcal{O}(1)\otimes L_1 \longrightarrow E/IE \longrightarrow \mathcal{O}(-1)\otimes L_2 \rightarrow 0$$

where L_1, L_2 are line bundles over A/I. $E/E(-1) = E(\infty)$ $= E\otimes_{\mathcal{O}} A(\infty)$ is a rank 2 bundle over A, and one has

$$\bar{E} = E/IE$$

$$0 \longrightarrow L_1 \longrightarrow \bar{E}(\infty) \longrightarrow L_2 \longrightarrow 0$$

an exact sequence of A/I bundles. I am looking for a rank 2 quotient M of $E \otimes \mathcal{O}/m_\infty^2$ such that $\mathrm{Ker}\{\bar{E} \to M\} = \mathcal{O}(-1)^2$ locally.

Question: Suppose one has $E \twoheadrightarrow M$ $\quad M \in \mathrm{Vect}_2(A)$ such that

$$\begin{tikzcd}
\mathcal{O}(1) \otimes L_1 \arrow[r] \arrow[d] & \bar{E} \arrow[r, two heads] & \bar{M} \arrow[dll, "S"'] \\
\mathcal{O}(1)/\mathcal{O}(-1) \otimes L_1 & &
\end{tikzcd}$$

Is it then true that M works?
Yes, because $E' = \mathrm{Ker}\{E \twoheadrightarrow M\}$ is a bundle on $\mathbb{P}^1_A$ such that at each point of A it is isom. to $\mathcal{O}(-1)^2$.

I got an idea: If M exists we know that as a module on $\mathbb{P}^1_A$ it is simply $p_* E$ together with a nilpotent endomorphism ν. But a good choice somehow would have $\nu^2 = 0$

Assume S reduced for the moment, and let Z be the closed subvariety where the fibres are isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Over Z we have the canonical exact sequence

$$0 \longrightarrow L_1 \longrightarrow \bar{E}(\infty) \longrightarrow L_2 \longrightarrow 0.$$

If I can find M I claim that I can extend the line bundles L_1, L_2 to a neighborhood of Z. In effect recall that the set of possible lattices at ∞ is a variety with a unique singular point - most 2 dimensional quotients of $\mathcal{O}^2/m_\infty^2$ are monogenic. The complement of the singular point is an affine bundle over $\mathbb{P}^1$. Over Z, we know that

$$(\mathcal{O}(1) \oplus \mathcal{O}(-1)) \otimes L_1 \xrightarrow{\sim} \bar{M}.$$

hence L_1 has to extend to the open set containing Z on which the lattice chosen at ∞ doesn't meet the singular point.

So given E on $\mathbb{P}^1_S$ we consider p_*E which is a 2-dimensional bundle mapping injectively into $E \otimes \mathcal{O}_\infty/m_\infty^2$. Let U be the open set where

$$(*) \qquad p_* E \longrightarrow E \otimes \mathcal{O}_\infty/m_\infty$$

is an isomorphism (this is the open set where the fibre is $\simeq \mathcal{O}^2$) and let Z be the complement (so that on Z, E is an extension

$$0 \longrightarrow \mathcal{O}(1) \otimes L_1 \longrightarrow \bar{E} \longrightarrow \mathcal{O}(-1) \otimes L_2 \longrightarrow 0$$

where L_1, L_2 are line bundles on Z. Also on Z the map $(*)$ has rank 1, so that over Z $p_* E$ has a canonical sub-line bundle which is

$$p_* \bar{E}(-1) \simeq L_1$$

$$\begin{tikzcd} 0 \arrow[r] & p_* \bar{E}(-1) \arrow[r] \arrow[d, equal] & p_* \bar{E} \arrow[r] \arrow[d, equal] & E \otimes \mathcal{O}_\infty / m_\infty \arrow[r] & L_2 \arrow[r] & 0 \\ & L_1 & L_1 \oplus L_1 & & & \end{tikzcd}$$

Thus over Z, $p_* \bar{E} = p_*(\mathcal{O}(1) \otimes L_1) \simeq L_1 \oplus L_1$.

~~Suppose I succeed in extending the sequence~~

$$\cancel{0 \longrightarrow L_1 \longrightarrow \bar{E} \otimes \mathcal{O}_\infty / m_\infty \longrightarrow L_2 \longrightarrow 0}$$

~~to a neighborhood V of Z. Then~~

January 16, 1976

Let E be a rank 2 vector bundle over $\mathbb{P}^1_S$ whose fibres are isom. to $\mathcal{O}^2$ or $\mathcal{O}(-1)\oplus\mathcal{O}(1)$; I want to find an exact sequence of sheaves on $\mathbb{P}^1_S$

$$0 \longrightarrow E' \longrightarrow E \longrightarrow M \longrightarrow 0$$

where M is finite flat over S of rank 2 and supported on the ∞ section of $\mathbb{P}^1_S$ over S. More precisely I want $m_\infty^2 M = 0$ if this can be arranged.

We consider
The sequence

$$0 \longrightarrow E \xrightarrow{\cdot t^{-1}} E(1) \longrightarrow E\otimes \mathcal{O}_\infty/m_\infty \longrightarrow 0$$

and put $V = E\otimes\mathcal{O}_\infty/m_\infty$. Since the fibres of E over points of S are $\cong$ $\mathcal{O}^2$ or $\mathcal{O}(-1)\oplus\mathcal{O}(1)$, we know that

$$R^1p_*\left(\underline{\mathrm{Hom}}\left(\mathcal{O}\otimes V, E\right)\right) = 0$$

hence $\mathrm{Ext}^1_{\mathbb{P}^1_S}(\mathcal{O}\otimes V, E) = 0$ if S is affine

Therefore we can do the lifting:

$$\begin{tikzcd}
\mathcal{O}\otimes V \arrow[rr] \arrow[d, dashed] & & V \\
E(1) \arrow[urr] & &
\end{tikzcd}$$

In other words I can reduce the family E over $\mathbb{P}^1_S$ to a family of lattices.

To simplify suppose V is trivialized, $V \simeq \mathcal{O}^2_S$ whence $E(1) \supset \mathcal{O}^2_{\mathbb{P}^1_S}$ will be a family of lattices of index -2. This is classified by a map $S \longrightarrow \mathbf{Q}^{(2)}_2(\mathbb{A}^1)$. So I am now looking at the case I have been interested in all along.

Go back to page 56 where S is broken up into $\mathcal{U}$ and Z. In the case of $\mathbf{Q}^{(2)}_2(\mathbb{A}^1)$ one knows $\mathcal{U} \simeq 2\times 2$ matrices and $Z = \mathbb{P}^1 \times \mathbb{A}^1$. To be more specific, let M be a quotient of $\mathbb{C}[z]^2$ of dim. 2. If $\mathbb{C}^2 \xrightarrow{\sim} M$, then mult. by z on M gives us a matrix α and

$$0 \longrightarrow \mathbb{C}[z]^2 \xrightarrow{zI-\alpha} \mathbb{C}[z]^2 \longrightarrow \overset{M}{\overset{\shortparallel}{M_\alpha}} \longrightarrow 0$$

On the other hand if $\mathbb{C}^2 \longrightarrow M$ is not an isom. we get a line L in $\mathbb{C}^2$ and an eigenvalue λ so that

$$0 \longrightarrow \mathbb{C}[z]\otimes L + (z-\lambda)^2\mathbb{C}[z] \longrightarrow \mathbb{C}[z]^2 \longrightarrow M \longrightarrow 0$$

Correction: $\mathbb{Z} = \mathbb{P}^1 \times \mathbb{A}^2$. If $\mathbb{C}^2 \longrightarrow M$ is not an isomorphism, then we get a line L and a degree 2 monic poly $f(z) = (z-\lambda_1)(z-\lambda_2)$ such that

$$M = (\mathbb{C}^2/L)\,[z]\big/ f$$

There is a map $S = Q_2^{(2)}(\mathbb{A}^1) \longrightarrow S^2(\mathbb{A}^1) = \mathbb{A}^2$ which given the support divisor. Fibres of $S \longrightarrow \mathbb{A}^2$ are $\mathbb{P}^1 \times \mathbb{P}^1$ in the generic case, otherwise a one point compactification of an affine line bundle over $\mathbb{P}^1$. Bad set Z is $\mathbb{P}^1 \times \mathbb{A}^2$

$$\begin{tikzcd}
 & S \arrow[d] \\
Z = \mathbb{P}^1 \times \mathbb{A}^2 \arrow[ur] \arrow[r] & \mathbb{A}^2
\end{tikzcd}$$

$\mathcal{U}$ contains the singular points of the fibres. Is S non-singular?

It appears that S is non-singular. For the fibres over multiplicity free divisors are $\mathbb{P}^1 \times \mathbb{P}^1$ hence non-singular. The fibre over the divisor $2a$ has a unique singular point. Thus the singular locus for $S \longrightarrow \mathbb{A}^2$ is a section over $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$. However such singularities occur in the open set $\mathcal{U}$ which is the affine space of 2×2 matrices. The singularities are scalar matrices which are the conical points among matrices having the same eigenvalue

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Given $\quad 0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \longrightarrow M \longrightarrow 0$

with M of length 2 supported at ∞, we know $K \cong \mathcal{O}(-1)^2$ or $\mathcal{O}(-2) \oplus \mathcal{O}$. Thus $H^1(K(1)) = 0$, so

$$0 \longrightarrow \Gamma(K(1)) \longrightarrow \Gamma(\mathcal{O}(1)^2) \longrightarrow \Gamma(M(1)) \longrightarrow 0$$

is exact. Now we have a map of this sequence into stalks

$$\begin{tikzcd}
0 \arrow[r] & K(1)_\infty \arrow[r] & \mathcal{O}(1)^2_\infty \arrow[r] \arrow[d] & M(1)_\infty \arrow[r] \arrow[d, equal] & 0 \\
& & \left(\mathcal{O}(1)/\underset{\overset{\|}{\mathcal{O}(-1)}}{m_\infty^2}\right)^2 \arrow[r] & M(1)_\infty \arrow[r] & 0
\end{tikzcd}$$

Thus it seems I get a map of

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(K(1)) \arrow[r] & \Gamma(\mathcal{O}(1)^2) \arrow[r] \arrow[d] & \Gamma(M(1)) \arrow[r] \arrow[d] & 0 \\
& & \left(\mathcal{O}(1)/\mathcal{O}(-1)\right)^2_\infty \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & M(1)_\infty \arrow[d, phantom, "\cong" rotate=90] & \\
& & \left(\mathcal{O}/\mathcal{O}(-2)\right)^2_\infty \arrow[r] & M_\infty &
\end{tikzcd}$$

Given $0 \longrightarrow K \longrightarrow \mathcal{O}^2 \longrightarrow M \longrightarrow 0$ with length $M = 2$, we know that $K \simeq \mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$ hence $H^1(K(1)) = 0$. Thus we get an exact sequence

$$0 \longrightarrow \Gamma(K(1)) \longrightarrow \Gamma(\mathcal{O}(1))^2 \longrightarrow \Gamma(M(1)) \longrightarrow 0$$

which gives us a map from $Q_2^{(2)}(\mathbb{P}^1)$ to $\mathrm{Grass}_2(\Gamma(\mathcal{O}(1))^2)$. In $Q_2^{(2)}(\mathbb{P}^1)$ we have the open set U where $\Gamma(\mathcal{O}^2) \xrightarrow{\sim} \Gamma(M)$, or equivalently $K \simeq \mathcal{O}(-1)^2$. We have previously identified U with an open subset of a Grassmannian. Specifically on U $0 \longrightarrow K \longrightarrow \mathcal{O}^2 \longrightarrow M \longrightarrow 0$ is the canonical resolution of M. Thus I looked at

$$\Gamma(K(1)) \hookrightarrow \Gamma(\mathcal{O}(1))^2$$

to get the desired embedding into the Grassmannian.

I conjecture that $Q_2^{(2)}(\mathbb{P}^1)$ is the blowup of $\mathrm{Grass}_2(\Gamma(\mathcal{O}(1))^2)$ with respect to some subvariety. So recall that if I give

$$W \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V$$

both of dimension 2, then I only land in U when $\alpha - t\beta$ is generically injective. If this is not the case then I get

$$\begin{tikzcd} \mathcal{O}(-1) \otimes W \arrow[rr] \arrow[dr] & & \mathcal{O} \otimes V \\ & \underline{\underline{V}} \arrow[ur, hook] & \end{tikzcd}$$

where I is a bundle of rank 1 (α, β not both 0 as $W \hookrightarrow V \times V$). Either $I = \mathcal{O}$ in which case α, β have a common line of V for image, or $I = \mathcal{O}(-1)$ in which case α, β have a common line in W for kernel which is impossible. Thus $I = \mathcal{O}$ and we get a line $L \subset V$ together with an isomorphism $W \xrightarrow{\sim} L \times L \subset V \times V$. So ~~inside of~~ $\mathrm{Grass}_2(V^2)$ we have identified a copy of $\mathbb{P}V$

$$\begin{array}{ccc} \mathbb{P}V & \hookrightarrow & \mathrm{Grass}_2(V^2) \\ L & & L \oplus L \end{array}$$

and $\mathcal{U}$ is the complement of this subvariety. Next recall that Z has been identified with $\mathbb{P}^1 \times \mathbb{P}^2$.

So I have to get more functorial.

$$0 \longrightarrow K \longrightarrow \mathcal{O} \otimes V \longrightarrow M \longrightarrow 0$$

and $\mathbb{P}^1 = \mathbb{P}(W^*)$ so that $\Gamma(\mathcal{O}(1)) = W^*$.

$$0 \longrightarrow \Gamma(K(1)) \longrightarrow \begin{array}{c} \Gamma(\mathcal{O}(1)) \otimes V \\ \parallel \\ W^* \otimes V \\ \parallel \\ \mathrm{Hom}(W, V) \end{array} \longrightarrow \Gamma(M(1)) \longrightarrow 0$$

$$\begin{array}{ccc} \mathbb{P}V & \hookrightarrow & \mathrm{Grass}_2(\mathrm{Hom}(W, V)) \\ \cup\!\!\!\!\;\shortmid & & \cup\!\!\!\!\;\shortmid \\ L & \longmapsto & \mathrm{Hom}(W, L) \end{array}$$

So the next thing is

$$Z = \mathbb{P}V \times \mathrm{Sym}^2(\mathbb{P}W)$$

A pair of points in $\mathbb{P}W$, is a divisor of degree 2; it determines a section of $\mathcal{O}(2)$ up to scalars, so

$$\mathrm{Sym}^2(\mathbb{P}W) = \mathbb{P}\big(\Gamma(\mathcal{O}(2))\big) = \mathbb{P}\big(\mathrm{Sym}^2(\underset{\overset{\shortparallel}{W^*}}{\Gamma(\mathcal{O}(1))})\big)$$

Note $\mathbb{P}W = \mathbb{P}W^*$ $\qquad = \mathbb{P}(S^2(W^*))$

What is the normal bundle of the embedding $\mathbb{P}V \hookrightarrow \mathrm{Grass}_2(\mathrm{Hom}(W,V))$. Tangent space to the Grassmannian at $\mathrm{Hom}(W,L) \subset \mathrm{Hom}(W,V)$ is

$$\mathrm{Hom}\left(W^* \otimes L,\ W^* \otimes V/L\right) = \mathrm{End}(W^*) \otimes \mathrm{Hom}(L, V/L)$$

Tangent space to $\mathbb{P}V$ at L is $\mathrm{Hom}(L, V/L)$, which probably gets embedded via $id \in \mathrm{End}(W^*)$. Thus normal bundle is

$$\left(\mathrm{End}(W^*)/id\right) \otimes \underbrace{\mathrm{Hom}\left(\mathcal{O}(-1), V\otimes\mathcal{O}/\mathcal{O}(-1)\right)}_{\overset{\shortparallel}{\Lambda^2 V \otimes \mathcal{O}(2)}}$$

$$0 \longrightarrow \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow \Lambda^2 V \otimes \mathcal{O}(2) \longrightarrow 0$$

So we seem to get for the normal bundle of the embedding, 3 copies of $\mathcal{O}(2)$. So the conjecture seems to be sound.

<u>January 17, 1976.</u>

Are the schemes $\mathrm{Quot}^n_d(C)$ non-singular for a curve? We know this is true for $n=1$ for then

$$\mathrm{Quot}^n_d(C) = \mathrm{Sym}_d(C).$$

Try the infinitesimal criterion for smoothness. Suppose

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ S_0 \arrow[r, hook] & S \end{tikzcd}$$

S_0 closed subscheme defined by an ideal I of square 0. Say $S = \mathrm{Spec}\, A$. Let us be given an exact sequence

$$\begin{array}{c} 0 \longrightarrow K_0 \longrightarrow \mathcal{O}^n_{X_0} \longrightarrow M_0 \longrightarrow 0 \\ \qquad\qquad \| \\ \qquad\qquad A/I \otimes_A \mathcal{O}^n_X \end{array}$$

with M_0 flat over A/I I want to extend this to a quotient M of $\mathcal{O}^n_X$ flat over A. Then $IM = I \otimes_A M = I \otimes_{A_0} M_0$.

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r] & \mathcal{O}^n_{X_0} \arrow[r] & M_0 \arrow[r] & 0 \\ 0 \arrow[r] & (K) \arrow[r] & \mathcal{O}^n_X \arrow[r] \arrow[u] & (M) \arrow[r] & 0 \\ 0 \arrow[r] & I \otimes_{A_0} K_0 \arrow[r] & I \otimes_{A_0} \mathcal{O}^n_{X_0} \arrow[r] \arrow[u] & I \otimes_{A_0} M_0 \arrow[r] & 0 \end{tikzcd}$$

Let $J = \mathrm{Ker}\{\mathcal{O}_X^n \to M_0\} = K_0 \times_{\mathcal{O}_{X_0}^n} \mathcal{O}_X^n$. $\quad J' = J/I\otimes_{A_0} K_0$.

Then

$$\begin{array}{ccccccccc} & & I\otimes_{A_0} M_0 & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & I\otimes_{A_0}\mathcal{O}_{X_0}^n & \longrightarrow & J & \longrightarrow & K_0 & \longrightarrow & 0 \\ & & \cup & & & & \| & & \\ 0 & \longrightarrow & I\otimes_{A_0} K_0 & \longrightarrow & (K) & \longrightarrow & K_0 & & \\ & & \uparrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

so ~~(K)~~ the K we seek will be the kernel of the dotted arrow in

$$\begin{array}{ccc} I\otimes_{A_0}\mathcal{O}_{X_0}^n & \longrightarrow & J \\ \downarrow & \swarrow & \\ I\otimes_{A_0} M_0 & & \end{array}$$

Therefore the set of M we are after is in 1-1 correspondence with splittings of the sequence

$$0 \longrightarrow I\otimes_{A_0} M_0 \longrightarrow J' \longrightarrow K_0 \longrightarrow 0 .$$

Is J' an A_0 module, i.e. is $I\cdot J \subset I\otimes_{A_0} K_0$.

$$0 \longrightarrow J \to \mathcal{O}_X^n \longrightarrow M_0 \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_1^A(A/I, M) \longrightarrow J/IJ \longrightarrow \mathcal{O}_{X_0}^n \longrightarrow M_0 \longrightarrow 0$$

$\wr\|$ because M flat

$I/I^2 \otimes M_0$

In $\mathcal{O}_X^n$ we have a filtration

$$\begin{tikzcd} H \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr"] & I\mathcal{O}_X^n \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr"] & J \arrow[r, phantom, "\subset"] & \mathcal{O}_X^n \\ I\otimes K_0 \arrow[r, phantom, "\subset"] & I\otimes \mathcal{O}_{X_0}^n & & \end{tikzcd}$$

~~Consider the map $I\otimes J \to I\otimes \mathcal{O}_X^n \to I\otimes M_0$~~ I wish to prove that $IJ \subseteq H$, so I consider the map

$$I\otimes J \longrightarrow IJ \longrightarrow I\mathcal{O}_X^n/H$$

and try to show it is zero. This

$$\begin{tikzcd} I\otimes J \arrow[r] \arrow[d, phantom, "\parallel"] \arrow[dd, bend right] & I\otimes \mathcal{O}_X^n \arrow[d, phantom, "\parallel"] & & \\ I\otimes_{A_0} J/IJ \arrow[r] & I\otimes_{A_0}\mathcal{O}_{X_0}^n \arrow[r] \arrow[d, "\wr"] & I\otimes_{A_0} M_0 \arrow[r] & 0 \\ IJ \arrow[r, phantom, "\subset"] & I\mathcal{O}_X^n & & \end{tikzcd}$$

Point: $I\mathcal{O}_X^n \simeq I\otimes \mathcal{O}_X^n \longrightarrow I\otimes_A M_0$ has kernel H.
$\therefore\ IJ = H$.

Thus we see that J' is an $\mathcal{O}_{X_0}$-module and so the obstruction to lifting M_0 to M

lies in the group

$$\mathrm{Ext}^1_{X_0}(K_0, I \otimes_{A_0} M_0)$$

Assume now that X/S is smooth of relative dim 1. Then because M_0 is flat over A_0, K_0 is locally free on X_0, so [redacted]

$$\mathrm{Ext}^1_{X_0}(K_0, I \otimes_{A_0} M) = H^1(X_0, \underline{\mathrm{Hom}}(K_0, I \otimes_{A_0} M_0)).$$

If further M_0 is finite [redacted] over A_0, then $\underline{\mathrm{Hom}}(K_0, I \otimes_{A_0} M_0)$ has support a [redacted] subscheme Z [redacted] finite over S, therefore affine, so this group vanishes.

Thus $\mathrm{Quot}^n_d(C)$ is [redacted] non-singular and [redacted] projective. The tangent space at a point $0 \to K \to \mathcal{O}_C^n \to M \to 0$ is the set of liftings of this sequence over the dual numbers $\mathbb{C}[\varepsilon] = \mathbb{C} + \varepsilon\mathbb{C}$. It should be the group

$$\mathrm{Hom}_C(K, M) = H^0(C, \underbrace{K^\vee \otimes M}_{\text{length } nd})$$

which has dimension nd. Check this by calculating the dimension of the fibres of

$$Q^{(n)}_d(C) \longrightarrow \mathrm{Sym}^d(C)$$

over a generic divisor. The fibre over generic divisor $P_1 + \cdots + P_d$ is $\simeq (\mathbb{P}(\mathbb{C}^n))^d$ which has dimension $d(n-1)$. Add to $\dim \mathrm{Sym}^d(C) = d$ to get $\dim Q^n_d(C) = dn$.

We've described $Q_2^{(2)}(\mathbb{P}^1)$ in terms of the Grassmannian $\mathrm{Grass}_2(W^*\otimes V)$ blown-up along $\mathbb{P}V$. U is the complement of $\mathbb{P}V$, and Z is the blown-up divisor.

We saw that on Z the canonical bundle K has a canonical filtration

$$0 \longrightarrow L_1 \otimes \mathcal{O} \longrightarrow \bar{K} \longrightarrow L_2 \otimes \mathcal{O}(-2) \longrightarrow 0$$

In this case $Z = \mathbb{P}_1 V \times \mathbb{P}(S_2 W^*)$. Specifically given a point in $\mathbb{P}_1 V$ and a $f \in \Gamma(\mathcal{O}(2)) - \{0\}$.

$$0 \longrightarrow L \longrightarrow V \longrightarrow L' \longrightarrow 0$$

one has
$$K = \mathcal{O}\otimes L + f^{-1}\cdot\mathcal{O}^2 \subset \mathcal{O}^2$$

$$L_1 = \mathcal{O}_{\mathbb{P}V}(-1) \qquad L_2 = \mathcal{O}_{\mathbb{P}V}(1) \qquad \text{lifted}$$

This means that K

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$$S = Q_2^{(2)}(\mathbb{A}^1).$$

$$0 \longrightarrow K \longrightarrow \mathcal{O}^2 = \mathcal{O}\otimes V \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow \Gamma(K(1)) \longrightarrow \Gamma(\mathcal{O}(1))^2 \longrightarrow \Gamma(M(1)) \longrightarrow 0$$
$$\Gamma(\mathcal{O}(1))^2 = W^*\otimes V$$

Assume this is a point of U:

$$0 - \overset{\circ}{\Gamma}(K) \longrightarrow \Gamma(\mathcal{O}^2) \xrightarrow{\sim} \Gamma(M).$$

Let S be an affine scheme $/k$ and suppose given over $\mathbb{P}^1_S$ an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{M}$ is finite and flat over S of length 2. This means that we have given a family of vector bundles $K \subset \mathcal{O}^2$ of index 2 parameterized by S. If necessary, I will assume that the support of $\mathcal{M}$ does not meet the ∞ section of $\mathbb{P}^1_S/S$.

Let U be the open set of S over which the fibres K of $\mathcal{K}$ are isom. to $\mathcal{O}(-1)^2$. Thus U is where the map

$$(*) \qquad \mathcal{O}_S^2 \longrightarrow p_* \mathcal{M}$$

is an isomorphism. Let Z be the complement of U. Define $\mathcal{O}_Z$ to be the quotient of $\mathcal{O}_S$ by the ideal generated by the determinant of $(*)$. (This is the obvious way to make Z into a closed subscheme of S). Over Z then we have a line bundles $\mathcal{L}_1 = p_*(\mathcal{K} \otimes \mathcal{O}_Z)$, $\mathcal{L}_2 = \mathcal{O}_S^2/\mathcal{L}_1$, $\mathcal{L}_3 = R^1 p_*(\mathcal{K} \otimes \mathcal{O}_Z)$ and exact sequences

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{O}_Z^2 \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow p_* \mathcal{M} \longrightarrow \mathcal{L}_3 \longrightarrow 0$$

Also we have a canonical exact sequence

$$0 \longrightarrow p_* \mathcal{K}(1) \longrightarrow p_* \mathcal{O}(1)^2 \longrightarrow p_* \mathcal{M}(1) \longrightarrow 0$$

of bundles over S. On $\mathbb{P}^1_Z$ we have a canonical sequence

$$0 \longrightarrow \mathcal{L}_1 \otimes \mathcal{O} \longrightarrow \mathcal{K}_Z \longrightarrow \mathcal{L}_3 \otimes \mathcal{O}(-2) \longrightarrow 0$$

so $p_* \mathcal{K}_Z(1) \simeq \mathcal{L}_1 \otimes p_* \mathcal{O}(1)$. Put $V = k^2$ and think of $\mathcal{O}^2$ as $\mathcal{O} \otimes V$. Then we have exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O} \otimes V \longrightarrow \mathcal{M} \longrightarrow 0$$

$$\mathcal{O}_S \otimes V \longrightarrow p_* \mathcal{M}$$

$$(\Rightarrow) \qquad 0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{O}_Z \otimes V \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

$$* \qquad 0 \longrightarrow p_* \mathcal{K}(1) \longrightarrow p_* \mathcal{O}(1) \otimes V \longrightarrow p_* \mathcal{M}(1) \longrightarrow 0$$

$$p_* \mathcal{K}_Z(1) = p_* \mathcal{O}(1) \otimes \mathcal{L}_1 \subset p_* \mathcal{O}(1) \otimes V.$$

What I want to see is whether the sequence $(\Rightarrow)$ on Z extends to a nbd. of Z in S. What is true is that if I double the sequence by tensoring with $p_* \mathcal{O}(1) \simeq \mathcal{O}_S^2$, then it becomes the sequence $*$.

~~what I do is to project~~

$$\cancel{p_* \mathcal{K}(1) \longrightarrow p_* \mathcal{O}(1) \otimes V} \qquad ?$$

If I can find $\mathcal{K} \subset \mathcal{K}'$ of index 2 with all fibres of $\mathcal{K}$ isomorphic to $\mathcal{O}^2$ and $\mathcal{K}'/\mathcal{K}$ supported at ∞, then I can argue as follows. On Z. $K \simeq \mathcal{O} \oplus \mathcal{O}(-2)$, hence K'/K as an $\mathcal{O}_\infty/m_\infty^2$ module is monogenic. This would have to be true in some nbd of Z, also. But then in some nbd of Z we would have a filtration

$$K \subset K'' \subset K'.$$

with $K''/K,\ K'/K'' \simeq \mathcal{O}/m_\infty$,
$m_\infty = z^{-1}\mathcal{O}(-1)$.
This means that

$$K/m_\infty K = \mathcal{O}/m_\infty \otimes V = V$$

has a canonical line in it namely $m_\infty K''/m_\infty K$.
But on Z one has

$$0 \longrightarrow \mathcal{O}\otimes L_1 \longrightarrow K \longrightarrow \mathcal{O}(-2)\otimes L_2 \longrightarrow 0$$

$$\cap$$

$$0 \longrightarrow \mathcal{O}\otimes L_1 \longrightarrow K'' \longrightarrow \mathcal{O}(-1)\otimes L_2 \longrightarrow 0$$

$$\cap$$

$$0 \longrightarrow \mathcal{O}\otimes L_1 \longrightarrow K' \longrightarrow \mathcal{O}\otimes L_2 \longrightarrow 0$$

Thus $K/m_\infty K'' \cong L_1$. It seems over Z we have to split the sequence $0 \to L_1 \to V \to L_2 \to 0$.

January 19, 1976

Start again with a scheme S/k and an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbb{P}^1_S} \otimes V \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{M}$ is finite flat over S of rank 2 and has support off ∞; $V = k^2$. Assume there exists

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}' \longrightarrow \mathcal{N} \longrightarrow 0$$

such that $\mathcal{N}$ is finite flat over S of rank 2 and killed by m_∞^2; ~~let Z be the closed subset of S where~~ also $\mathcal{K}' \simeq \mathcal{O}^2$ on the fibres. Z is again the closed subset of S where $K \simeq \mathcal{O} \oplus \mathcal{O}(-2)$ on the fibres. On Z we get a canonical exact sequence

$$0 \longrightarrow L_1 \longrightarrow V \longrightarrow L_2 \longrightarrow 0$$

where $L_1 = \Gamma(K)$. At a point of z, $K = \mathcal{O} \oplus \mathcal{O}(-2)$, $K' = \mathcal{O} \oplus \mathcal{O}$ hence $\mathcal{N}$ has to be $\mathcal{O}/z^{-2}\mathcal{O}(2)$. Thus N is monogenic on Z, and this should also hold in some neighborhood W of Z in S. Then we should get a $\mathcal{K}''$:

$$\mathcal{K} \subset \mathcal{K}'' \subset \mathcal{K}'$$

on W such that

$$K''/K \simeq K'/K'' \simeq N/m_\infty N$$

is a line bundle on W. I want to carefully examine the map $m_\infty \mathcal{K}'' \subset \mathcal{K} \hookrightarrow \mathcal{O} \otimes V \xrightarrow{(\infty)} V$.

The image of this map is a line in V which I claim is $\simeq L_2$ over Z. To see this note that over Z we have a canonical sequence

$$0 \longrightarrow \mathcal{O}\otimes L_1 \longrightarrow K \longrightarrow \mathcal{O}(-2)\otimes L_3 \longrightarrow 0$$

for some line bundle L_3 on Z. $L_3 = L_2$ because look at ∞ fibre. Also we have a canonical sequence

$$0 \longrightarrow \mathcal{O}\otimes L_1'' \longrightarrow K'' \longrightarrow \mathcal{O}(-1)\otimes L_3'' \longrightarrow 0.$$

Comparing one sees that $L_1'' = L_1$. Also using the inclusion $m_\infty K''$ one gets

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1)\otimes L_1 & \longrightarrow & m_\infty K'' & \longrightarrow & \mathcal{O}(-2)\otimes L_3'' & \longrightarrow & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \longrightarrow & \mathcal{O}\otimes L_1 & \longrightarrow & K & \longrightarrow & \mathcal{O}(-2)\otimes L_2 & \longrightarrow & 0 \end{array}$$

so therefore on taking the fibre over ∞ one sees $\mathcal{O}(-1)\otimes L_1$ maps to 0 in V, hence the image is a line in V mapped isom. onto L_2.

Conclusion: If we do find the sequence

$$(*) \qquad 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}' \longrightarrow \mathcal{N} \longrightarrow 0$$

over S, then we get a ~~splitting~~ of the sequence

$$0 \longrightarrow L_1 \longrightarrow V \longrightarrow L_2 \longrightarrow 0$$

over Z and an extension of this sequence to a nbd. W of Z in S.

Let us now consider the universal situation where $Z = \mathbb{P}V \times \mathbb{A}^2$, $\mathbb{A}^2$ being degree 2 divisors on $\mathbb{A}^1$. Thus given $L_1 \subset V$ and monic degree 2 poly $f = z^2 + a_1 z + a_2$ one has

$$K = \mathcal{O}\otimes L_1 + f\mathcal{O}(-2)\otimes V \subset \mathcal{O}\otimes V$$

Now according to the above if I hope to find the sequence $(*)$ I have to split the sequence

$$0 \to L_1 \to V \to L_2 \to 0$$

over Z and extend it to a nbd. W of Z in S, which means that I extend the map $Z \to \mathbb{P}V$ to W.

So let me consider a quotient M of $\mathcal{O}\otimes V$ near to $\mathcal{O}\otimes V / \mathcal{O}\otimes L_1 + f\mathcal{O}(-2)\otimes V = \mathcal{O}/f\mathcal{O}(-2)\otimes L_2$

If f has distinct roots we know that the fibre of $Q_2^2(\mathbb{A}^1)$ over f is $\mathbb{P}V \times \mathbb{P}V$ and the Z part is just the diagonal. Thus I want to retract a nbd of $\Delta\mathbb{P}V$ in $\mathbb{P}V \times \mathbb{P}V$ back to the diagonal. This should be done equivariantly so as to allow the roots of f to coallesce. ~~Observe this can be done if ? is invertible, because~~

Question: Given $S \xrightarrow{f} \mathbb{P}V \times \mathbb{P}V$ with S affine does there exist a nbd W of $f^{-1}(\Delta\mathbb{P}V) = Z$ and a diagram

$$\begin{tikzcd} W \arrow[r, dashed, "g"] & \mathbb{P}V \\ Z \arrow[u, phantom, "\cup"] \arrow[ur, "f"'] & \end{tikzcd} \qquad g = pr_1 f \qquad ?$$

January 20, 1978

S scheme$/k$, on $\mathbb{P}^1_S = S \times_k \mathbb{P}^1$ I am given an exact seq

(1) $$0 \longrightarrow K \longrightarrow \mathcal{O} \otimes_k V \longrightarrow M \longrightarrow 0 \qquad V = k^2$$

where M is finite flat of rank 2 over S. U is the open subscheme where $V \xrightarrow{\sim} p_* M$. $S = Z \amalg U$ is the flattening stratification for the map $V \longrightarrow p_* M$. Over Z we have a canonical seq.

(2) $$0 \longrightarrow L_1 \longrightarrow V \longrightarrow L_2 \longrightarrow 0$$

where $L_1 = p_*(K \otimes \mathcal{O}_Z)$. In the universal situation $Z = \mathbb{P}V \times \mathbb{P}(S_2 W^*)$ and

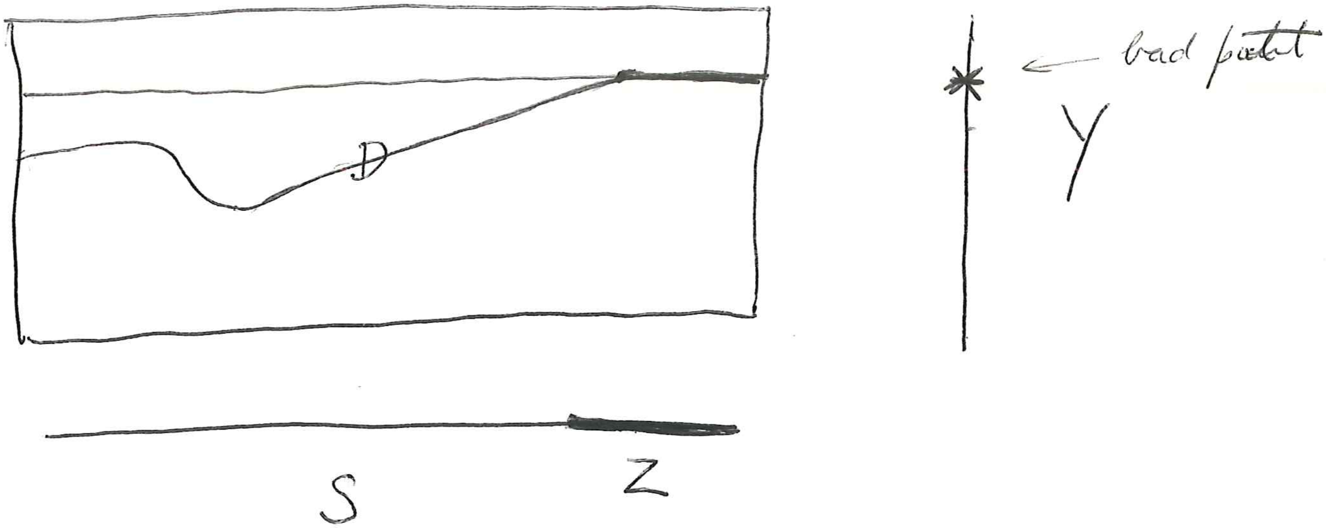
$$M = \mathcal{O}/f\mathcal{O}(-2) \otimes L_2 \qquad K = \mathcal{O} \otimes L_1 + f\mathcal{O}(-2) \otimes V.$$

where $f \in (S_2 W^*) - 0$ is a section of $\mathcal{O}(2)$. OKAY. **Better** to keep the support different from ∞.

I saw that if I could produce a unimodular subspace $H \subset \Gamma(K(2))$ over all of S, then I get a splitting of (2) and an extension to some nbd. of Z in S.

Result: Suppose S is a polyhedron.

The universal family of K is $S = Q_2^{(2)}(\mathbb{A}^1)$ which is a non-singular variety. Let Y be the space of index -2 lattices containing $\mathcal{O}_\infty^2$. Then on $S \times Y$ we have a Cartier divisor D whose complement is the variety of unimodular subspaces in the [redacted] universal family [redacted] K.



So I know that $S\times Y - D$ has contractible fibres. Topologically I can produce a section of $S\times Y - D$ over S as follows. [redacted] $Z\times Y - D_Z \to Z$ is an affine space bundle, so I can find a section here, i.e. a map $Z \to Y$ whose graph misses D. This map extends to a nbd. of Z in S: $W \xrightarrow{s} Y$ and shrinking this nbd. W, we can assume it still misses D. Now over $S-U$ one has the bad point section and the radial retraction of the fibres of $S\times Y - D$. So if we select a function $f: S \to [0,1]$ with $f=1$ on some nbd of Z and $f=0$ [redacted] on some nbd of $S-W$, then we can use f to damp s to the bad point section off some nbd. of $S-W$ and then extend in the trivial way.

~~Can we proceed this way in general? If I have a family of bundles on P^1 parameterized by a space T, then twisting we can arrange~~

Let there be given a family of bundles over $\mathbb{P}^1$ parameterized by a polyhedron T with trivialization $\mathcal{O}^n \longrightarrow E$ at ∞. This family should be induced by a map $T \longrightarrow Q_d^{(n)}(\mathbb{A}^1) = S$ where the variety S is non-singular. Let Y be the space of lattices in $\mathcal{O}_\infty^n$ of index d. Over $S \times Y$ there should be a canonical family of vector bundles on $\mathbb{P}^1$ of degree 0 and for each $s \in S$ there should be lattices in y such that the bundle corresp. to (s, y) is isom. to $\mathcal{O}^n$. Thus on $S \times Y$ there should be a Cartier divisor D which intersects each $s \times Y$ nicely. I know that the fibres of the map $S \times Y - D \to S$ are contractible affine varieties. To actually lift a map $T \to S$ into $S \times Y - D$ would seem to require using the natural stratification of Y. Proof is necessary.

Let M be an $A[T]$-module and $Af_1 + Af_2 = A$. Suppose I can find map

$$u_i : A_{f_i}[T]^2 \xrightarrow{\sim} M_{f_i}$$

agreeing with a given isomorphism modulo T

$$v : A^2 \xrightarrow{\sim} M/TM$$

Can I find an isom $u : A[T]^2 \to M$ compatible with u.

We get a cocycle over $A_{f_1 f_2}$. This is a 2×2

matrix
$$\theta(T) = I + a_1 T + \cdots + a_n T^n$$
where $a_i \in M_2(A_{f_1 f_2})$ such that there exists
$$\varphi(T) = I + b_1 T + \cdots + b_m T^m$$
with $\theta(T)\varphi(T) = I = \varphi(T)\theta(T)$. The problem is to factor θ into $\theta_1\theta_2$ where θ_i extends to A_{f_i} invertibly.

Idea I had is to take
$$\theta_2 = \theta(gT) = I + ga_1 T + g^2 a_2 T^2 + \cdots + g^n a_n T^n$$
$$\theta_1 = \theta(T)\varphi(gT)$$
$$= (1 + a_1 T + \cdots + a_n T^n)(1 + g b_1 T + \cdots + g^m b_m T^m)$$

Here $g^{\in A}$ is to be chosen such that

$g \in Af_1^N$ $\qquad$ N large enough so that $g^\nu a_\nu \in A_{f_2}$, $g^\nu b_\nu \in A_{f_2}$

(Assume f_1, f_2 non-zero divisors)
so $A \hookrightarrow A_{f_i} \hookrightarrow A_{f_1 f_2}$

$$\theta_1(T) - I = \theta_1(T)\varphi(gT) - \theta(gT)\varphi(gT)$$
$$= ((1-g)a_1 T + (1-g^2)a_2 T^2 + \cdots + (1-g^n) a_n T^n)\varphi(gT)$$
$$= (1-g)\left(a_1 T + \cdots + \left(\frac{1-g^n}{1-g}\right) a_n T^n\right)\varphi(gT)$$

Thus you want $1-g \in Af_2^N$ $\quad$ N so large that $(1-g)a_i b_j \in A_{f_1}$. This should be possible.

January 21, 1976

M an $A[T]$-module equipped with an isomorphism $A^2 \xrightarrow{\sim} M/TM$ and locally on Spec A this isomorphism can be lifted to an isomorphism $A_f[T]^2 \xrightarrow{\sim} M_f$. Can one find a global lifting? Thus we can find a covering of Spec A by open sets $U_i = \mathrm{Spec}(A_{f_i})$ $i=1,\dots,r$ and for each i,j we get a 2×2 matrix polynomial

$$\theta_{ij}(T) = 1 + a_1^{ij} T + \cdots + a_n^{ij} T^n$$

whose coefficients are in $M_2(A_{f_i f_j})$ and whose inverse matrix is of the same form

$$\theta_{ij}^{-1}(T) = 1 + b_1^{ij} T + \cdots + b_n^{ij} T^n.$$

Thus on Spec A we have a sheaf of groups G such that $G(U)$ = the group of invertible matrix polys $\theta(T) = 1 + a_1 T + \cdots + a_n T^n$ (same n) with $a_\nu \in \Gamma(U, \mathcal{O}_U)$. We have a 1-cocycle on $\{U_i\}$ with values in G which we want to make a 1-coboundary.

Case of Spec $A = U_1 \cup U_2$: First suppose f_1, f_2 are non-zero divisors to simplify, so that $A \subset A_{f_i} \subset A_{f_1 f_2}$. Consider $\theta_{12}(gT) = 1 + a_1^{12} g T + \cdots + a_n^{12} g^n T^n$ whose inverse is $\theta_{21}(gT) = 1 + a_1^{21} g T + \cdots + a_n^{21} g^n T^n$. If $g \in A f_1^N$ for N large, then the coefficients $a_\nu^{ij} g^\nu$ lie in A_{f_2} hence we get a section $\varphi_2 \in G(U_2)$ such that $\varphi_2 | U_1 \cap U_2 = \theta_{12}(gT)$

Question: Let $\theta(T) = I + a_1 T + \cdots + a_n T^n \in GL_n(A_f[T])'$.
Is it true that $\theta(gT) \in \operatorname{Im}\{GL_n(A[T])' \longrightarrow GL_n(A_f[T])'\}$ for any $g \in Af^N$ with N sufficiently large.

Let $\theta(T)^{-1} = I + b_1 T + \cdots + b_m T^m$. If N is sufficiently large, there exist elements $\tilde{a}_i, \tilde{b}_i \in M_n(A)$ such that $(f^N)^i a_i = \rho(\tilde{a}_i)$, $(f^N)^i b_i = \rho(\tilde{b}_i)$ where $\rho : A \longrightarrow A_f$ is the canonical map. Put

$$\varphi(T) = 1 + \tilde{a}_1 T + \cdots + \tilde{a}_n T^n$$
$$\psi(T) = 1 + \tilde{b}_1 T + \cdots + \tilde{b}_m T^m$$

Then
$$\rho(\varphi(T)\psi(T)) = (1 + a_1 f^N T + \cdots + a_n f^{nN} T)(1 + b_1 f^N T + \cdots)$$
$$= \theta(f^N T)\,\theta^{-1}(f^N T) = 1.$$

hence the pos. degree coefficients of $\varphi(T)\psi(T)$ lie in the kernel of ρ. Thus $\varphi(f^{N'}T)\psi(f^{N'}T) = 1$ so replacing $\tilde{a}_i$ by $(f^{N'})^i \tilde{a}_i$ and N by $N + N'$, we can arrange that $\varphi(T)\psi(T) = 1$. Thus the question has the answer Yes.

Case of $\operatorname{Spec} A = U_1 \cup U_2$. To show ~~[illegible]~~ any $\theta(T) \in GL_n(A_{f_1 f_2}[T])' = G(U_1 \cap U_2)$ is the product $\theta(T) = \varphi_1(T)|_{U_1 \cap U_2} \cdot \varphi_2(T)|_{U_1 \cap U_2}$ where $\varphi_i \in G(U_i)$. Let $g \in A$ to be chosen later. Write

$$\theta(T) = \theta(T)\theta(gT)^{-1} \cdot \theta(gT)$$

We've seen that if $g \in Af_1^N$ for N suff. large, then $\exists\, \varphi_2 \in G(U_2)$ such that $\varphi(T)|_{U_1 \cap U_2} = \theta(gT)$. To show that if $1 - g \in Af_2^N$ and again N is suff. large, then $\theta(T)\theta(gT)^{-1}$ extends to an element of $G(U_1)$.

$$\theta(gT)\,\theta(T)^{-1} = 1 + [\theta(gT)-\theta(T)]\,\theta(T)^{-1}$$
$$= 1 + [a_1(g-1)T + \cdots + a_n(g^n-1)T^n]\,\theta(T)^{-1}$$
$$= 1 + \sum_{1\le i\le n,\ 1\le j\le m} a_i(g^i-1)\,b_j\,T^{i+j}$$

Thus if N is large enough so that $f_2^N a_i b_j$ $\in \operatorname{Im}\{A_{f_1} \to A_{f_1 f_2}\}$ this matrix extends.

Also

$$\theta(T)\,\theta(gT)^{-1} = 1 + [\theta(T)-\theta(gT)]\,\theta(gT)^{-1}$$
$$= 1 + \sum_{\substack{1\le i\le m \\ 1\le j\le m}} a_i(1-g^i)\,g^j b_j$$

will extend. The product of the 2 extension will be 1 modulo elements killed by f_2^e some e. So if g is congruent to 1 modulo an even higher power of f_2 one wins.

How should ~~this~~ proof be interpreted in terms of torsors? Somehow by using

January 22, 1976

$\operatorname{Spec} A = U_1 \cup U_2 \cup U_3$. $\therefore$ $\mathrm{Zero}(f_1) \subset U_2 \cup U_3$. We have this G-torsor over A given by the cocycle θ_{ij}. I would like to trivialize this torsor over $U_2 \cup U_3$ using the case of 2 open sets, however $U_2 \cup U_3$ is not affine which produces some problems where?

$$\theta_{23}(T) = \theta_{23}(T)\,\theta_{23}(gT)^{-1} \cdot \theta_{23}(gT)$$

where g is to be chosen so that $\theta_{23}(gT) \in G(U_2 \cap U_3)$ extends to $G(U_3)$, (this can be arranged by $g \equiv 0\ (f_2^N)$), and $\theta_{23}(T)\theta_{23}(gT)^{-1}$ extends to $G(U_2)$, (this can be arranged by $g \equiv 1\ (f_3^N)$). Here $g \in \Gamma(U_2 \cup U_3, \mathcal{O}) = A_{f_2} \times_{A_{f_2 f_3}} A_{f_3}$. Does such a g exists?

Seems unlikely unless it comes from A. If not it would have singularities along a divisor and no divisor would be contained in the complement of $U_2 \cap U_3$. Therefore we must seek for $g \in A$. Impossible since there are points where f_2, f_3 vanish simultaneously.

~~Instead the situation is topological. We have a torsor over a 2 simplex with setting given over the open stars of the vertices. Also we have the structural group G of the torsor is contractible.~~

Since Spec $A = U_1 \cup U_2 \cup U_3$ $\quad U_i = \operatorname{Spec} A_{f_i}$
I know that $Af_1 + Af_2 + Af_3 = A$, hence $\exists\, \rho_i \in A$ such that $\rho_1 f_1 + \cdots + \rho_3 f_3 = 1$. So I can assume if I want that $f_1 + f_2 + f_3 = 1$. Then

$$\{f_1 \neq 1\} \subset \{f_2 \neq 0\} \cup \{f_3 \neq 0\}$$

In other words Spec$(A_{(f_1-1)})$ is an open affine containing $\{f_1 = 0\}$ and contained in $U_2 \cup U_3$. Consequently I know I can produce a section of the torsor over $\{f_1 \neq 1\}$. Since I already have a section over $\{f_1 \neq 0\}$, I thus get a section in general.

In general you proceed by induction. Suppose Spec $A = U_{f_1} \cup \cdots \cup U_{f_n}$ and the torsor is trivial over each U_{f_i}. I can suppose $f_1 + \cdots + f_n = 1$. Then

$$\operatorname{Spec} A_{(1-f_1)} \subset U_{f_2} \cup \cdots \cup U_{f_n}.$$

$$\operatorname{Spec} A_{(1-f_1)} = U_{(1-f_1)f_2} \cup \cdots \cup U_{(1-f_1)f_n}$$

so by induction the torsor is trivial over U_{1-f_1}. Since Spec $A = U_{f_1} \cup U_{1-f_1}$ one wins.

For any scheme S let $G(S) = GL_n(\Gamma(S,\mathcal{O}_S)[T])'$ be the group whose elements are matrix polys.

$$\theta = I + a_1 T + \cdots + a_m T^m$$

with a_i a $n \times n$ matrix over $\Gamma(S,\mathcal{O}_S)$ such that θ is invertible (det θ is a unit in $\Gamma(S,\mathcal{O}_S)[T]^*$). Then G is a sheaf for the Zariski topology.

<u>Theorem</u>: If S is affine, $H^1(S,G) = 0$. (Zariski cohomology).

<u>Proof</u>: We must show every torsor P over S under G has a section. We can find a finite covering of S by open sets U_{f_i}, $f_i \in \Gamma(S,\mathcal{O}_S) = A$ over which P has a section. Then $A = \sum_i A f_i$ so replacing f_i by $g_i f_i$ we reduce to

<u>Lemma</u>: If P is a G-torsor over Spec A which is trivial over Spec A_{f_i}, $i = 0, \ldots, q$ where $f_i \in A$ are $f_0 + \cdots + f_q = 1$, then P is trivial.

Use induction on q. Trivial for $q=0$. Assume true for $q=2$, we prove it in general. $f_0 + \cdots + f_q = 1$

$$\Longrightarrow \quad U_{1-f_0} \subset U_{f_1} \cup \cdots \cup U_{f_q}$$

$$\Longrightarrow \quad U_{1-f_0} = U_{(1-f_0)f_1} \cup \cdots \cup U_{(1-f_0)f_q}$$

But U_{1-f_0} is affine, so the induction hypothesis implies P is trivial over U_{1-f_0}. Thus P is trivial over U_{f_0} and U_{1-f_0}, so by the case $q=2$, P is trivial.

Proof for $q=2$. The torsor P is given by a cocycle $\theta \in G(U_0 \cap U_1)$. We factor θ as follows

$$\theta(T) = \left(\theta(fT)\,\theta(T)^{-1}\right)\theta(fT)$$

where f is an element of A to be chosen. We will show that if N is suff. large then $\theta(fT)$ extends to an element of $G(U_1)$ provided $f \in Af_0^N$, and $\theta(fT)\theta(T)^{-1}$ extends to an element of $G(U_0)$ provided $1-f \in Af_1^N$. Since $Af_0^N + Af_1^N = A$, we have $A \twoheadrightarrow A/Af_0^N \times A/Af_1^N$ hence such an element f exists, and so the cocycle θ is a coboundary.

Suppose $\theta(T) = 1 + a_1 T + \cdots + a_p T^p$ where $a_i \in M_n(A_{f_0 f_1})$.

Then $\theta(fT) = 1 + a_1 f T + \cdots + a_p f^p T^p$ so provided N is suff large all the matrices $a_i f^i$ extend to A_{f_1}, hence we can find a matrix-polynomial $\varphi(T) = 1 + b_1 T + \cdots + b_p T^p$ such that $\varphi(T)$ restricts to $\theta(fT)$. Then $\det(\varphi(T)) = 1 + c_1 T + \cdots + c_r T^r$ restricts to $\det \theta(fT)$ which is invertible, i.e. all its positive degree coefficients are nilpotent. It follows that the pos. degree coefficients of $\varphi(T)$ are nilpotent if multiplied by a power of f, so $\varphi(f^m T)$ is invertible

Then $\theta(hf_0^r T) = 1 + a_1 h f_0^r T + \cdots + a_p h^p f_0^{rp} T^p$. If r is sufficiently large $a_i (f_0^r)^i$ extends to $b_i \in A_{f_1}$, hence $\varphi(T) = 1 + b_1 T + \cdots + b_p T^p$ is a matrix poly. over A_{f_1} such that $\varphi(hT) = \theta(hf_0^r T)$ for all $h \in A$. Now $\det \varphi(T)$ restricts to $\det \theta(f_0^r T)$ which is invertible in

$A_{f_0 f_1}[T]$, which means all its pos. degree coeffs. are nilpotent. It follows that all pos. degree coeffs. of det $\varphi(T)$ are nilpotent after being multiplied by f_0^s for some s. Thus ~~$\varphi(f_0^s T) \in G(U_1)$~~ $\varphi(f_0^s T)$ is invertible for s large, so $\theta(hf_0^{r+s}T)$ extends to the element $\varphi(hf_0^sT)$ of $G(U_1)$, which was to be proved.

Suppose $\theta(T) = 1 + a_1T + \dots + a_pT^p$ and $\theta(T)^{-1} = 1 + b_1T + \dots + b_qT^q$.

Then
$$\theta(fT)\theta(T)^{-1} = 1 + [\theta(fT) - \theta(T)]\theta(T)^{-1}$$
$$= 1 + \sum_{\substack{1 \le i \le p \\ 0 \le j \le q}} a_i(f^i - 1)b_jT^{i+j}$$

If $f = 1 + hf_1^r$, with r large enough so that $f_1^r a_i b_j$ extend to c_{ij} in A_{f_0}, then we get a matrix polynomial
$$\psi_h(T) = 1 + \sum hc_{ij}\left[\frac{(1+hf_1^r)^i - 1}{hf_1^r}\right]T^{i+j}$$
over A_{f_0} such that
$$\theta((1+hf_1^r)T)\theta(T)^{-1} = \text{image of } \psi_h(T) \text{ in } A_{f_0f_1}$$

Since det $\psi_h(T)$ restricts to a unit in $A_{f_0f_1}$, all its pos. degree coeffs. are nilpotent after multiplying a power of f_1. Thus if $h \in Af_1^s$ for large enough s, $\psi_h(T)$ will be invertible, which was to be proved.

Suppose $\theta(T) = 1 + a_1 T + \cdots + a_p T^p$, $\theta(T)^{-1} = 1 + b_1 T + \cdots + b_q T^q$.
Let X be an indeterminate.

$$\theta((1+Xf_1^r)T)\,\theta(T)^{-1} = 1 + \left[\theta((1+Xf_1^r)T) - \theta(T)\right]\theta(T)^{-1}$$

$$= 1 + \sum_{\substack{1\le i\le p\\ 0\le j\le q}} X f_1^r a_i b_j \left(\frac{(1+Xf_1^r)^i - 1}{Xf_1^r}\right) T^{i+j}$$

For r large, $\exists$ $c_{ij} \in A_{f_0}$ with image of c_{ij} in $A_{f_0 f_1}$ equal to $f_1^r a_i b_j$. Put

$$\psi_X(T) = \sum_{\substack{1\le i\le p\\ 0\le j\le q}} c_{ij} \frac{(1+Xf_1^r)^i - 1}{Xf_1^r} T^{i+j}$$

so that $\theta((1+Xf_1^r)T)\,\theta(T)^{-1} = 1 + X\psi_X(T)$ rest. to $U_0 \cap U_1$.

This is invertible, so $\det(1+X\psi_X(T))$ rest. to $U_0\cap U_1$ is a unit. $\det(1+X\psi_X(T)) = 1 + c_1(X)XT + \cdots + c_n(X)XT^n$
This means the $c_i(X) \in A_{f_0}[X]$ are nilpotent in $A_{f_0 f_1}[X]$, hence that $f_1^s c_i(X)$ are nilpotent in $A_{f_0}[X]$. Thus if $h \in Af_1^s$ one has that $hc_i(X)$ hence $hc_i(h)$ is nilpotent, which implies $1 + h\psi_h(T)$ is invertible. Thus

$$\theta((1+hf_1^r)T)\,\theta(T)^{-1}$$

extends to $1 + h\psi_h(T) \in G(U_0)$ for $h \in Af_1^s$ as was to be proved.

B is an algebra over A

$\theta(T) = 1 + a_1 T + \cdots + a_p T^p \in B_f[T]^*$

1) Claim $\exists N$, $\varphi(T) \in B[T]^*$ const coeff. $= 1$

such that $\theta(T) = \rho(\varphi(u))\big|_{u = \frac{T}{f^N}}$

where $\rho : B[u] \longrightarrow B_f[u]$ is the canonical map.

Proof: Let $\theta(T)^{-1} = 1 + a_1' T + \cdots + a_q' T^q$. Choose r $\ni$ $a_i (f^r)^i$, $a_i'(f^r)^i$ lift to elements b_i, b_i' of B.

Put $\varphi(T) = 1 + \sum b_i T^i$ $\quad \varphi'(T) = 1 + \sum b_i' T^i$. Then

$$\rho\big(\varphi(T)\varphi'(T)\big) = \theta(f^r T)\,\theta(f^r T)^{-1} = 1$$

so the pos. degree terms of $\varphi\varphi'$ are killed by f^s some s. Thus $(\varphi\varphi')(f^s T) = 1$. Similarly $(\varphi'\varphi)(f^s T) = 1$, so $\varphi_0(T) = \varphi(f^s T) \in B[T]^*$ ...

and $\theta(f^{s+r} T) = \underline{\varphi(f^s T)} = \varphi_0(T)$.

2) Claim $\exists M$, $\psi(T) \in B[T]^*$ ... such that

$$g_1, g_2 \in Af^M \implies \theta(g_1 T)\,\theta(g_2 T)^{-1} = \rho\psi(T)$$

Proof: Let $\theta(f^N T) = (\rho\varphi)(T)$ as above. We have

$$\varphi(X)\varphi(Y)^{-1} =$$

$$1 + [\varphi(X) - \varphi(Y)]\varphi(Y)^{-1} = 1 + (X-Y)\psi_0(X,Y) + \cdots + (X-Y)\psi_m(X,Y)$$

where $\psi_i \in B[X,Y]$ is homogeneous of degree i.

Put $\left(X = \frac{g_1 T}{f^N}, Y = \frac{g_2 T}{f^N}\right.$

$$\psi(T) = 1 + \frac{g_1 - g_2}{f^N} T \psi_0 + \frac{g_1 - g_2}{f^{2N}} T \psi_1(g_1 T, g_2 T)$$

$$+ \cdots + \frac{g_1 - g_2}{f^{(m+1)N}} T \psi_m(g_1 T, g_2 T)$$

This is an element of $(1 + TB[T])^*$ since it's inverse is essentially of same form:

$$\varphi(Y)\varphi(X)^{-1} = 1 + (\cancel{X} - \cancel{X})\psi_0(Y,X) + \cdots + (Y - X)\psi_m(Y,X)$$

$$\psi(T)^{-1} = 1 + \frac{g_2 - g_1}{f^N} T \psi_0 + \cdots + \frac{g_2 - g_1}{f^{N(m+1)}} T \psi_m(g_2 T, g_1 T)$$

But it's also clear that

$$\rho\psi(T) = 1 + (g_1 - g_2)\frac{T}{f^N}\rho\psi_0\left(\frac{g_1 T}{f^N}, \frac{g_2 T}{f^N}\right) + \cdots + \left(\frac{g_1 T}{f^N} - \frac{g_2 T}{f^N}\right)\rho\psi_m\left(\frac{g_1 T}{f^N}, \frac{g_2 T}{f^N}\right)$$

$$= \rho\left[\varphi(X)\varphi(Y)^{-1}\right]\Big|_{X = g_1 T}$$

$$\theta(T) = (\rho\varphi)\left(\frac{T}{f^N}\right)$$

$$\theta(g_1 T)\,\theta(g_2 T)^{-1} = \cancel{\rho[\varphi(X)\varphi(Y)^{-1}]_{X=g}} \; (\rho\varphi)\left(\frac{g_1 T}{f^N}\right)(\rho\varphi)\left(\frac{g_2 T}{f^N}\right)$$

$$= \rho\left(\varphi(X)\varphi(Y)^{-1}\right)\Big|_{X = \frac{g_1 T}{f^N},\, Y = \frac{g_2 T}{f^N}}$$

$$= \left[1 + (X - Y)\rho\psi_0 + \cdots + (X - Y)\rho\psi_m(X,Y)\right]_{X = \frac{g_1 T}{f^N},\, Y = \frac{g_2 T}{f^N}}$$

January 27, 1976

Line bundles on $A[t]$.

If A is a field k and L is an invertible $k[t]$-module, we know that the set of extensions of L to a line bundle over $\mathbb{P}^1_k$ is canonically isom. to $\mathbb{Z}$. This is because such extensions are the same thing as $\mathcal{O}_\infty$-lattices in $L \otimes_{k[t]} k(t) \simeq k(t)$ = quotient field of the discrete valuation ring $\mathcal{O}_\infty$. Thus we see in any case that the possible extensions form a $\mathbb{Z}$-torsor which we can trivialize using the degree.

Classify extensions of L to a coherent sheaf on $\mathbb{P}^1_A$ (A supposed noetherian). These are the same as finitely generated $A[t^{-1}]$-modules M with

$$A[t,t^{-1}] \otimes_{A[t^{-1}]} M \simeq A[t,t^{-1}] \otimes_{A[t]} L$$

If M has no elements killed by t^{-1}, then M can be identified with an $A[t^{-1}]$-lattice inside $A[t,t^{-1}] \otimes_{A[t]} L$.

It seems therefore we ought to look at the following more general situation. We have a scheme X/S smooth of relative dimension 1 with

a section
$$\begin{tikzcd} & X \arrow[d, "p"] \\ S \arrow[ur, "s"] \arrow[r, phantom, "="] & S \end{tikzcd}$$

Then we have a line bundle L over $X-S$ which we want to extend to X. Assume S, X affine.

~~Consider Z = image of ∞ section of $\mathbb{P}^1_A/A$. Z defined by $t^{-1}=0$. Let U be an open set containing Z. Is it true that $\mathbb{A}^1 \cup U = \mathbb{P}^1_A$?~~

Assume A local with max. ideal m and residue field k. Let $f \in A[z]$ be such that $f(0) \notin m$, i.e. $f \notin zA[z] + m = \mathrm{Ker}\{A[z] \twoheadrightarrow A \twoheadrightarrow k\}$. Claim that

$$\mathrm{Spec}\, A[z, z^{-1}] \cup \mathrm{Spec}\, A[z]_f = \mathrm{Spec}\, A[z].$$

This is clear because $(A[z]z + A[z]f)/A[z]z$ is an ideal in $A[z]/A[z]z \simeq A$ not contained in m, hence it is all of A since A is local. Therefore for any $A[z]$-module M we have an exact sequence

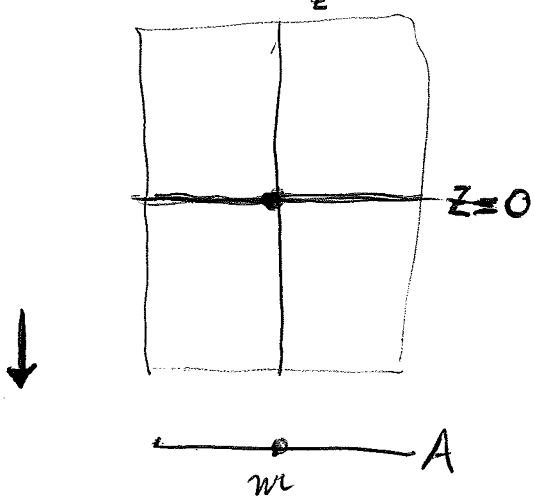
$$0 \longrightarrow M \longrightarrow M_z \times M_f \longrightarrow M_{zf} \longrightarrow 0$$

i.e.

$$0 \to \Gamma(\mathrm{Spec}\, A[z], \tilde{M}) \to \Gamma(\mathrm{Spec}\, A[z]_z, \tilde{M}) \times \Gamma(\mathrm{Spec}\, A[z]_f, \tilde{M}) \to \Gamma(\mathrm{Spec}\, A[z]_{zf}, \tilde{M}) \to \underset{\substack{\parallel \\ 0}}{H^1} \to 0$$

This exact sequence will hold as we take the limit over all $f \notin \mathfrak{m} + A[z]z = \mathfrak{n}$. Thus to give an $A[z]$-module M is the same as giving an $A[z,z^{-1}]$-module L, an $A[z]_{\mathfrak{n}}$ module Λ, and an isomorphism

$$A[z]_{z,\mathfrak{n}} \otimes_{A[z]_z} L \xrightarrow{\sim} A[z]_{z,\mathfrak{n}} \otimes_{A[z]_{\mathfrak{n}}} \Lambda$$



So we reach the following problem. Suppose given a line bundle L over $A[z,z^{-1}]$ whence I get an invertible $B[z^{-1}]$ module $B \otimes_{A[z,z^{-1}]} L$ where $B =$ the local ring $A[z]_{\mathfrak{m}}$. The problem is now to extend this $B[z^{-1}]$ module $B \otimes_{A[z,z^{-1}]} L$ to an invertible B-module Λ.

Concentrate on the following special case to see what's happening. Take A to be a discrete valuation ring. In this case $B = A[z]_{\mathfrak{m}+(z)}$ is a regular local ring of dimension 2.

Then one knows that $B[z^{-1}]$ is regular of dimension 1 and all its [redacted] non-zero prime ideals are principal. Thus $B[z^{-1}]$ is a [redacted] principal ideal domain, so Λ exists, because $B \otimes_{A[z,z^{-1}]} L$ is a vector bundle over $B[z^{-1}]$ hence trivial. Combined with your localization result this would yield a new proof of the Sheshadri thm.

<u>Question</u>: Let X be a regular affine scheme, U an open affine (one knows $X-U$ is a divisor). [redacted] Does every vector bundle on U extend to a vector bundle on X? Same question but for X a reg. local ring?

True that $K_0(X) \twoheadrightarrow K_0(U)$. Hence $\mathrm{Pic}(X) \twoheadrightarrow \mathrm{Pic}(U)$. Note [redacted] because regular local rings are factorial, one has on a regular scheme an exact sequence of sheaves

$$0 \to \mathcal{O}_X^* \longrightarrow \bigoplus_{x \in X^0} k(x)^* \longrightarrow \bigoplus_{x \in X^1} \mathbb{Z} \longrightarrow 0$$

which is a resolution by flask sheaves. Thus

$$H^q(X, \mathcal{O}_X^*) = 0 \qquad q \geq 2$$

$$H^1(X, \mathcal{O}_X^*) = \text{divisor classes on } X$$

This shows again that $\mathrm{Pic}(X) \twoheadrightarrow \mathrm{Pic}(U)$ if U open in X.

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28 January 1976:

Problem: Let $X = \mathrm{Spec}\, A$, let z be a non-zero divisor in A, let N be a vector bundle over $A_z = A[z^{-1}]$. Assume I can locally extend N to a vector bundle M on X; this means I can find an open covering X_{f_i} of X and vector bundles M_i on X_{f_i} such that $M_i[z^{-1}] \simeq N_{f_i}$. Can I then find a global extension M?

[redacted] Since I can always arrange the covering X_{f_i} to be such that M_i is free over A_{f_i}, the hypothesis on [redacted] N amounts to the assertion that N_{f_i} is free over $A_{f_i}[z^{-1}]$ for some covering $X = \bigcup X_{f_i}$. However in the form stated one has only to prove it [redacted] for a covering $X =$ [redacted] $X_f \cup X_{1-f}$.

[redacted] Suppose $X = X_{f_0} \cup X_{f_1}$ where we have found bundles M_i over A_{f_i} such that $M_i[z^{-1}] \xrightarrow{\sim} N_{f_i}$. In order to piece together M_0 and M_1 together we would need to know that $(M_0)_{f_1}$ and $(M_1)_{f_0}$ coincide (they are $A_{f_0 f_1}$-[redacted] lattices inside $N_{f_0 f_1}$. So this leads me to examine the case of lattices.

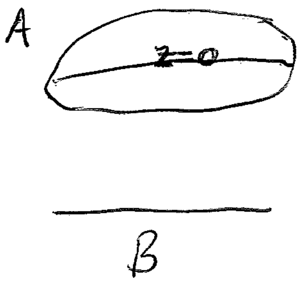
Situation: A is a ring, z a non-zero divisor in A, N bundle over $A[z^{-1}]$. I am interested in all A-bundles $M \subset N$ such that $M[z^{-1}] = N$, assuming this set is non-empty, say $N \cong A[z^{-1}]^n$.

Given $M, M' \in Lat(N)$ replacing M' by $z^N M'$ we can assume that $M' \subset M$. Then

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

so M/M' is of projective dimension 1 and it is killed by a power of z.

I am willing to assume that A is an algebra over $B = A/zA$, so that we have the good geometric situation



Note that one has

$$M' \subset M \subset z^{-t} M'$$

so that

$$0 \longrightarrow M/M' \longrightarrow z^{-t}M'/M' \longrightarrow z^{-t}M'/M \longrightarrow 0$$

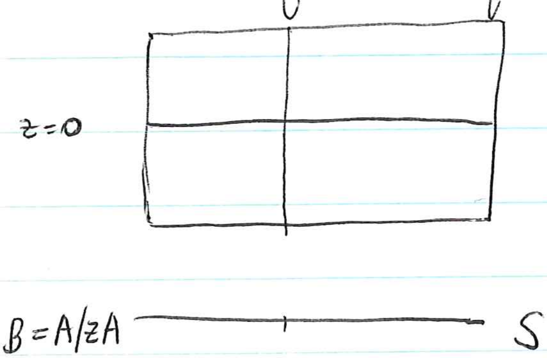
$z^{-t}M'/M'$: proj. over B

$z^{-t}M'/M$: proj. dim 1 over A, hence over B

thus M/M' is always flat over B.

Basic structure of M/M' is therefore that it is a vector bundle over B equipped with a nilpotent endomorphism, and moreover that it is a quotient of $(B[T]/(T^m))^r$ for some [redacted] m where r = rank of N. If I assume A local, so that $M \simeq A^r$, then any [redacted] such [redacted] bundle with nilpotent endom. occurs.

Think of the following picture. N is a given vector bundle over $A[z^{-1}]$. Locally over $S = \operatorname{Spec} B$ I can extend N. I want to know if it is possible to make the extension globally.



Consider this as a topological question – [redacted] We have over each point of S, the space [redacted] of lattices which has the homotopy type [redacted] ΩU_n. Thus I seem to have a locally trivial fibre bundle with fibre [redacted] of the homotopy type ΩU_n. It would seem unlikely for there to be a section.

The situation I am concerned with is the localization $B[z] \subset B[z, z^{-1}]$ where z is an indeterminate [redacted]. I start with a vector bundle N for $B[z, z^{-1}]$ i.e. a torsor $H^1(\operatorname{Spec} B[z, z^{-1}], \underset{\sim}{GL_n})$ which I want to extend to $B[z]$.

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29 Jan.

[redacted]

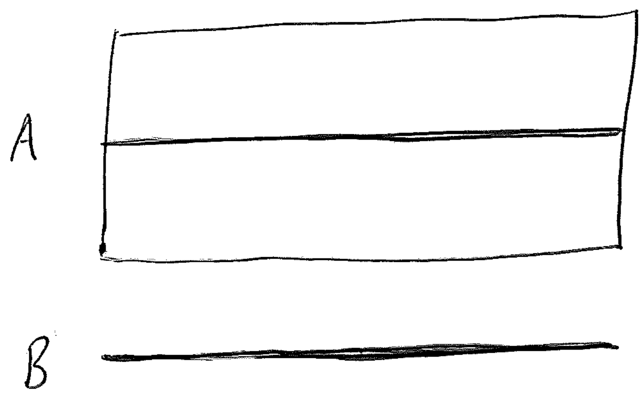
Review the setup: Assume I am given a vector bundle N over $B[z, z^{-1}]$ and I am trying to extend it to a vector bundle M over $B[z]$. [redacted] I will assume that M exists locally over Spec B. [redacted]. So the next point is to [redacted] find the obstruction to a [redacted] global extension.

So suppose we have $Bf_0 + Bf_1 = 1$ and we have found M_i over $B_{f_i}[z]$ with $M_i[z^{-1}] \xrightarrow{\sim} N_{f_i}$. [redacted] The problem now is to see if I can [redacted] modify M_i so as to get them to agree over $B_{f_0 f_1}$.

Look carefully at the case of a line bundle N.

[redacted]

Over $B_{f_0 f_1}[z]$ I have two line bundles $(M_0)_{f_1}$ and $(M_1)_{f_0}$ which becomes isomorphic after localizing wrt z. Assume $(M_0)_{f_1} \subset (M_1)_{f_0}$.



$$A/zA \xrightarrow{\sim} B$$

$\operatorname{Spec}(B) = \bigcup_{i=0}^{8} \operatorname{Spec}(B_{f_i})$ and we give bundles M_i over A_{f_i} together with isomorphisms

$$(M_i)_{f_j, z} \simeq (M_j)_{f_i, z}$$

satisfying the cocycle condition. Thus there is a bundle N over A_z such that $(M_i)_z = N_{f_i}$ for each i.

So think topologically. At each point b of B I am looking for a lattice $M(b)$ inside $N(b)$, which will vary nicely as b moves.

Work topologically:

Think of B as being functions on some space.

Suppose S is a the zero submanifold of a nice section of a line bundle L over X. With topological K-theory one has

$$\longrightarrow K^0(S) \xrightarrow{i_*} K^0(X) \xrightarrow{j^*} K^0(X-S) \xrightarrow{\partial} K^1(S) \longrightarrow \cdots$$

and it should be possible to find examples where $K^0(X) \longrightarrow K^0(X-S)$ is not onto. If X is a line bundle L over S, then $i_* : K^q(S) \longrightarrow K^q(X)$ is multiplication by $c_1(L)$ over S. Hence you want $\operatorname{Pic}(S) \neq 0$ and $K^1(S) \neq 0$, i.e. $H^1(S,\mathbb{Z})$, $H^2(S,\mathbb{Z})$ both $\neq 0$. Can take S to be a complete curve not $\mathbb{P}^1$.

(better take $S = \mathbb{G}_m$ embedded in $X = \mathbb{A}^2$).

Unstable analysis of the preceding example: Suppose I give a bundle N over $X-S$, where S is a complex divisor in X. The problem of extending N to X as a topological vector bundle one can to look at in a tubular nbd U of S. The problem is that one is given a bundle on ∂U which is a circle bundle over S, which one wants to extend. The set of extensions form a torsor over S for the group ΩU_n $\quad n = \text{rank } N$ from the homotopy viewpoint. Since $B(\Omega U_n) = U_n$ one should have a map $S \to U_n$ defined at least up to homotopy. Actually this isn't accurate, but it would be if N were trivial, and if the normal bundle of S in X were trivial. Precisely N is a torsor over S for the sheaf

$$GL_n\Big(\bigoplus_n L^{\otimes n}\Big)$$

so if L is trivial $\quad \bigoplus_n L^{\otimes n} = \bigoplus_n \mathcal{O}_S z^n$

$$= \mathcal{O}_S \otimes \mathbb{C}[z, z^{-1}].$$

And

$$\begin{array}{l} 1 \longrightarrow GL_n(\mathbb{C}[z,z^{-1}])' \to GL_n\,\mathbb{C}[z,z^{-1}] \longrightarrow GL_n \longrightarrow 0 \\ \qquad\quad \wr \\ \qquad\ \Omega U_n \end{array}$$

Also I could look at

$$\to H^0(S,\mathbb{Z}) \to H^2(X,\mathbb{Z}) \to H^2(X-S,\mathbb{Z}) \to H^1(S,\mathbb{Z}) \to H^3(X,\mathbb{Z}) \to \cdots$$

which shows that topologically even line bundles should present an obstruction. The problem is to understand why this obstruction is zero in the algebraic context.

Possible problem: Take a bundle N on $B[T]$ which is trivialized near $T=0$. Then try to extend this bundle to a bundle on $\mathbb{P}^1$. Topologically I ought to get somekind of obstruction which might be a kind of map $\mathrm{Spec}(B) \longrightarrow GL_n$. Try to define this.

Obstruction in the case of line bundles. Suppose N is a line bundle over $B[z, z^{-1}]$ which locally extends to $B[z]$, locally over B that is. First try to understand all possible invertible extensions of N when such extensions exists. Then we can suppose $N = B[z, z^{-1}]$ and we look at all invertible modules M between $z^m B[z]$ and $z^{-m} B[z]$ for m sufficiently large.

Instead look at A and try to classify all invertible A-modules M between $z^m A$, $z^{-m} A$. This is the same as looking for invertible ideals M with $z^m A \subset M \subset A$ some m. And this is the same as if I replace A by $\hat{A} = \varprojlim A/(z^n)A$, (by Karoubi etc.). But $\hat{A}$ is local if A/zA is, so $\hat{M}$ will be principal, and so we have a non-zero divisor which is a factor of z^m in $\hat{A}$ which generates $\hat{M}$.