Jan. 1976

Vector bundles on $\mathbb{P}^1_A$ and $\mathbb{A}^1_A$.

Let M be a vector bundle over $A[T]$. I want to know when there exists an isom.

$$A[T]\otimes_A (M/TM) \simeq M$$

reducing to the obvious canonical isomorphism modulo T. Put $V = M/TM$ and let $\alpha : M \to V$ be the canonical map. I am looking for a section $s : V \to M$ of α, which is an A-module morphism, such that the induced map $A[T]\otimes_A V \to M$ is an isomorphism.

Assume that for each $\mathfrak{p}$ in Spec A I can find $f \in A - \mathfrak{p}$ such that there exists a section s of α_f inducing an isomorphism $A_f[T]\otimes V_f \xrightarrow{\sim} M_f$. I want to prove a global section s exists.

More general formulation: For each scheme S over Spec A let $P(S)$ be the set of $\mathcal{O}_S$-module sections s of $\alpha_S : \mathcal{O}_S\otimes_A M \to \mathcal{O}_S \otimes_A V$ which induce isomorphisms

$$\cancel{\mathcal{O}_S\otimes_A (A[T]\otimes_A E) \to}$$

$$\mathcal{O}_S[T]\otimes_A V \xrightarrow{\sim} \mathcal{O}_S\otimes_A M$$

and let $G(S)$ be the group of automorphisms of $\mathcal{O}_S[T]\otimes_A V$ as $\mathcal{O}_S[T]$-module which induce the identity on $\mathcal{O}_S\otimes_A V = \mathcal{O}_S[T]\otimes_A V / T\,\mathcal{O}_S[T]\otimes_A V$. Then

P is formally principally homogeneous under G (as presheaves over Sch/A), i.e. $P(S) \neq \phi \Rightarrow P(S)$ is a torsor for the group $G(S)$.

<u>Assertion to be proved</u>: <u>Assume</u> $P(U_{f_i}) \neq \phi$ $i = 0, \ldots, q$ <u>where</u> $\operatorname{Spec} A = U_{f_0} \cup \cdots \cup U_{f_q}$, $U_{f_i} = \operatorname{Spec}(A_{f_i})$. <u>Then</u> $P(\operatorname{Spec} A) \neq \phi$.

We prove this by induction on q. Suppose $q = 1$. Let $s_i \in P(U_{f_i})$ and let $\theta \in G(U_{f_0} \cap U_{f_1})$ be the unique element such that $\theta s_0 = s_1$ on $U_{f_0} \cap U_{f_1} = U_{f_0 f_1}$. We have to show there exists $\varphi_i \in G(U_{f_i})$ $i = 0, 1$ such that $\theta = \varphi_1^{-1} \varphi_0$ on $U_{f_0 f_1}$.

Identify $G(S)$: Let $R = \operatorname{End}_A(E)$. A map of $\mathcal{O}_S[T]$-mod $\mathcal{O}_S[T] \otimes_A E \longrightarrow \mathcal{O}_S[T] \otimes_A E$ is the same thing as a A-module map $E \longrightarrow \Gamma(S, \mathcal{O}_S[T] \otimes_A E) =$

$$\bigoplus_{\nu \geq 0} T^\nu \operatorname{Hom}(E, B \otimes_A E) = \bigoplus_{\nu \geq 0} T^\nu B \otimes_A \operatorname{Hom}_A(E, E) = B \otimes_A R[T],$$

where $B = \Gamma(S, \mathcal{O}_S)$ and S is assumed quasi-compact. Thus an element $\theta \in G(S)$ can be identified with a polynomial

$$\theta(T) = 1 + c_1 T + \cdots + c_p T^p$$

where $c_i \in B \otimes_A R$ which is invertible, i.e. such that $\exists$ any poly. $\theta'(T)$ of the same form such that $\theta\theta' = \theta'\theta = 1$.

So when $S = U_{f_0 f_1}, B = A_{f_0 f_1}$ we have

$$\theta(T) = 1 + a_1 T + \cdots + a_p T^p$$
$$\theta(T)^{-1} = 1 + b_1 T + \cdots + b_p T^p$$

with $a_i, b_i \in R_{f_0 f_1}$. As $\operatorname{Spec} A = U_{f_0} \cup U_{f_1}$ we have $Af_0^N + Af_1^N = A$ for any N, hence for any N there exists an element g of A with $g \in Af_1^N$ and $1-g \in Af_0^N$. By the following lemma the elt. $\theta(gT)$ of $G(U_{f_0 f_1})$ extends to an element φ_0 of $G(U_{f_0})$ if N is suff. large, and the elt. $\theta(gT)\theta(T)^{-1}$ extends to an element φ_1 of $G(U_{f_1})$. Then in $G(U_{f_0 f_1})$ we have

$$\varphi_1^{-1}\varphi_0 = \left(\theta(gT)\theta(T)^{-1}\right)^{-1}\theta(gT) = \theta(T)$$

as desired.

<u>Lemma</u>: Let $\theta(T) \in G(\operatorname{Spec} A_f)$. There exists an integer N such that for any $g_1, g_2 \in A$ such that $g_1 - g_2 \in Af^N$, we have that $\theta(g_1 T)\theta(g_2 T)^{-1}$ extends to an element of $G(\operatorname{Spec} A)$.

<u>Proof</u>: Let
$$\theta(T) = 1 + a_1 T + \cdots + a_p T^p$$
$$\theta^{-1}(T) = 1 + b_1 T + \cdots + b_p T^p$$

$a_i, b_i \in R_f$. (Here R is an alg. over A not nec. comm.)
Let Y, Z be indeterminates.

$$\theta((Y+f^N Z)T)\,\theta(YT)^{-1} = 1 + \left[\theta((Y+f^N Z)T) - \theta(YT)\right]\theta(YT)^{-1}$$

$$= 1 + \sum_{\substack{1\le i\le p \\ 0\le j\le p}} \left[(Y+f^N Z)^i - Y^i\right] Y^j a_i b_j T^{i+j}$$

$$= 1 + ZT \sum_{\substack{1\le i\le p \\ 0\le j\le p}} f^N a_i b_j \left[\sum_{\nu=0}^{i-1} (Y+f^N Z)^\nu Y^{i-1-\nu}\right] Y^j T^{i+j-1}$$

For N suff. large, $\exists\ c_{ij} \in R$ such that $f^r a_i b_j = \rho(c_{ij})$ where $\rho: R \to R_f$ is the canonical localization homomorphism. ~~Thus we see~~ $\exists\, r, \psi(Y,Z,T) \in R[Y,Z,T]$ ~~such that~~

$$\theta((Y+f^r Z)T)\,\theta(YT)^{-1} = \rho(\psi(Y,Z,T))$$

$$\psi(Y,0,T) = 1$$

~~Replacing Y by $Y+f^rZ$ and Z by $-Z$ we get~~

$$\begin{cases} \theta(YT)\,\theta((Y+f^r Z)T)^{-1} = \rho(\psi'(Y,Z,T)) \\ \psi'(Y,0,T) = 1 \end{cases}$$

~~for~~ $\psi'(Y,Z,T) = \psi(Y+f^rZ, -Z, T)$. ~~It follows that~~ $\rho(\psi\psi') = 1$, ~~hence~~ $\psi\psi' = 1 + Z\sigma$ ~~where~~ $\rho(\sigma) = 0$, ~~hence~~ $f^s\sigma = 0$ ~~for some~~ s. ~~Thus if~~

~~Hence~~ $\exists\ r$ ~~and~~ $\psi(Y,Z,T) \in R[Y,Z,T]$ ~~such that~~

$$\theta((Y+f^rZ)T)\,\theta(YT)^{-1} = \rho(1 + Z\psi(Y,Z,T)).$$

~~Replacing~~ Y ~~by~~ $Y+f^rZ$ ~~and~~ Z ~~by~~ $-Z$ ~~we get~~

$$\theta(YT)\,\theta((Y+f^rZ)T)^{-1} = \rho(1 + Z\psi'(Y,Z,T))$$

~~It follows that~~ $(1+Z\psi)(1+Z\psi')$ ~~and~~ $(1+Z\psi')(1+Z\psi)$

Hence there exists an integer r and a $\psi \in 1+ZT\,R[Y,Z,T]$ such that

$$\Theta((Y+f^rZ)T)\,\Theta(YT)^{-1} = \rho(\psi)$$

~~$\psi \in 1+Z\,B[Y,Z,T]$~~

Replacing Y by $Y+f^rZ$ and Z by $-Z$, we get a $\psi' \in 1+ZT\,R[Y,Z,T]$ such that

$$\Theta(YT)\,\Theta((Y+f^rZ)T)^{-1} = \rho(\psi').$$

It follows that $\psi\psi'-1$ and $\psi'\psi-1$ are polys in $ZT\,R[Y,Z,T]$ killed by ρ, hence their coefficients are killed by f^s for s suff. large. This means that for any $z \in f^s A$ $\psi(Y,z,T)$ will have inverse $\psi'(Y,z,T)$. Thus if $g_1 - g_2 = hf^{r+s}$, we can take $z = hf^s$ $y = g_2$ and we get

$$\Theta(g_1T)\,\Theta(g_2T)^{-1} = \rho(\psi(y,z,T))$$

where $\psi(y,z,T) \in (1+TR)^*$, proving the lemma.

Formulate lemma: R be an A alg. not nec. comm.

$$G(B) = (1+T(B\otimes_A R)[T])^*$$

How the lemma implies factoring. Apply the lemma to A_{f_0} rep. by A, $f = f_1$, $R = \mathrm{End}_A(V)_{f_0}$ $g-0 \in A_{f_0} f_1^N$ to get $\Theta(gT)$ extends to an element of $G(A_{f_0})$. Similarly $g-1 \in A_{f_1} f_0^N$ and the lemma implies $\Theta(gT)\,\Theta(T)^{-1}$ comes from A_{f_1}.

Now suppose $\operatorname{Spec} A = U_{f_0} \cup \dots \cup U_{f_q}$ i.e. $A = Af_0 + \dots + Af_q$ and proceed by induction on q. Since $\exists g_i \in A$ such that $1 = \sum g_i f_i$ we can, replacing f_i by $g_i f_i$, assume that $\sum f_i = 1$. Then

$$U_{1-f_q} \subset U_{f_0} \cup \dots \cup U_{f_{q-1}}$$

so if we apply the induction hypothesis to the ring A_{1-f_q} whose spectrum is covered by $U_{(1-f_q)f_0} \cup \dots \cup U_{(1-f_q)f_{q-1}}$ we see $P(A_{1-f_q}) \neq \emptyset$. Then applying the case $q = 1$ to $\operatorname{Spec} A = U_{f_q} \cup U_{1-f_q}$ we conclude that $P(A) \neq \emptyset$ as was to be shown.

Assertion: If B_i is a filtered inductive system of A-algebras, then $\varinjlim P(B_i) \xrightarrow{\sim} P(\varinjlim B_i)$.

Put $B = \varinjlim B_i$ and let $s : B[T] \otimes_A V \xrightarrow{\sim} B \otimes_A M$ be an element of $P(B)$. ~~s is determined by its restriction to E since E is finitely presented over A~~

~~$\operatorname{Hom}_A(E, B \otimes_A M) = \varinjlim \operatorname{Hom}_A(E, B_i \otimes_A M)$~~

~~so s comes from an $s_i : E \to B_i \otimes_A M$. Enlarging i we can suppose~~ Because $A[T] \otimes_A V$ and M are finitely presented $A[T]$-modules one has

$$\operatorname{Hom}_{B[T]}(B[T] \otimes_A V, B \otimes_A M) = \operatorname{Hom}_{A[T]}(A[T] \otimes_A V, B \otimes_A M)$$

$$= \varinjlim \operatorname{Hom}_{A[T]}(A[T]\otimes_A V, B_i \otimes_A M)$$

$$= \varinjlim \operatorname{Hom}_{B_i[T]}(B_i[T]\otimes_A V, B_i \otimes_A M).$$

etc., so this isomorphism s comes from an isomorphism $s_i : B_i[T]\otimes_A V \xrightarrow{\sim} B_i \otimes_A M$ for some i. Since s reduced modulo T coincides with the given isom $B\otimes_A V \xrightarrow{\sim} B\otimes_A M/TM$, this must also be true for s_i if we enlarge i. Q.E.D.

Thus $P(A_\mathfrak{p}) = \varinjlim_{f\notin\mathfrak{p}} P(A_f)$ and so we see that $P(A_\mathfrak{p}) \neq \emptyset \Rightarrow P(A_f) \neq 0$ for some $f \notin \mathfrak{p}$. This means that if $P(A_\mathfrak{m}) \neq \emptyset$ for all maximal ideals $\mathfrak{m}$, then $P(U_{f_i}) \neq \emptyset$ for U_{f_i} covering Spec A, hence by the previous assertion $P(A) \neq \emptyset$.

Let E be a vector bundle over $\mathbb{P}^1_A$, and M its restriction to $A[T]$. I propose to show that there exists an isomorphism

$$A[T] \otimes_A V \xrightarrow{\sim} M$$

reducing modulo T to a given isom. $V \xrightarrow{\sim} M/TM$. By the preceding this question is local on A; I can assume A is a local ring if I want.

To simplify suppose A is an algebra over a field k. Let E be a vector bundle of rank n over $\mathbb{P}^1_A$. Let $V_0 = E/zE(-1)$ $V_\infty = E/z^{-1}E(-1)$ be the fibres over the 0 and ∞ section of $\mathbb{P}^1_A$.

$$0 \longrightarrow E(-1) \longrightarrow E \longrightarrow V_\infty \longrightarrow 0$$

Twisting E sufficiently we can, without changing E restricted to $\mathbb{A}^1_A$, assume $H^1(E(-1)) = 0$, hence

$$0 \longrightarrow \Gamma(E(-1)) \longrightarrow \Gamma(E) \longrightarrow V_\infty \longrightarrow 0$$

As V_∞ is a projective A-module, we can split this sequence and so find a map

$$\mathcal{O} \otimes V_\infty \longrightarrow E$$

which is an isomorphism near the ∞ section.

Replacing E by $E(m) = E \otimes \mathcal{O}(1)^{\otimes m}$ does not change the restriction to $\mathbb{A}^1_A$. Thus we may suppose E is regular, i.e. $R^1p_*(E(-1)) = 0$. In this case the restriction of E to $\mathbb{P}^1_{k(p)}$ for any $p \in \operatorname{Spec} A$ is isom. to $\mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n)$ where $0 \leq p_1 \leq \cdots \leq p_n$. Let d be the degree of E. ~~we are going to argue by~~

<u>Assertion</u>: If A is local and E is a regular vector bundle on $\mathbb{P}^1_A$ of rank n, then $M = \Gamma(\mathbb{A}^1_A, E)$ is isomorphic to $A[T]^n$.

<u>Proof</u>: I argue by induction on degree$(E) = d$. If $d = 0$, then the canonical map

$$\mathcal{O} \otimes \Gamma(E) \longrightarrow E$$

is an isomorphism. (E regular $\Rightarrow \Gamma(E)$ is a bundle and this map is onto. But rank $\Gamma(E)$ = rank(E).)

Suppose $d \geq 1$. The restriction $E/T_0E(-1)$ of E to the ∞ section of $\mathbb{P}^1_A$ over A is a rank n vector bundle over A. Let Z be the projective bundle of hyperplanes in $E/T_0E(-1)$; $Z \simeq \mathbb{P}^{n-1}_A$. ~~Over $Z \times_A \mathbb{P}^1_A$ one has a canonical exact sequence~~

$$\cancel{0 \to E' \to \mathcal{O}_Z \boxtimes E \longrightarrow \mathcal{O}_Z(+1) \boxtimes \mathcal{O}_{\mathbb{P}^1_A} \longrightarrow 0}$$

Over $Z \times_A \mathbb{P}^1_A = \mathbb{P}^1_Z$ there is a canonical exact sequence

$$0 \longrightarrow E' \longrightarrow E_Z \longrightarrow i_{\infty *}(L) \longrightarrow 0$$

where L is the canonical line bundle on Z. (Recall over Z one has a canonical quotient line bundle $\mathcal{O}_Z \otimes (E/T_0 E(-1)) \twoheadrightarrow L$) where E' is a vector bundle of degree 1 less and where $i_\infty : Z \to \mathbb{P}^1_Z$ is the ∞ section.

<u>Fact</u>: The subset of Z where E' is regular is open. More precisely the functor on Schemes/Z to sets given by

$$F(S) = \begin{cases} \phi & \text{if } E'_S \text{ is not regular on } S \\ \{\phi\} & \text{if } E'_S \text{ is regular on } \mathbb{P}^1_S \end{cases}$$

is represented by an open subscheme U of Z.

Because $\mathbb{P}^1_Z$ is of rel. dim 1 over Z and hence $R^q p_* \equiv 0$ for $q \geq 2$, one knows that forming $S \mapsto R^1 p_{S*}(E'_S)$ is compatible with base change: $R^1 p_{S*}(E'_S) = \mathcal{O}_S \otimes_Z R^1 p_*(E')$. But where $R^1 p_*(E')$ vanishes is an open subset of Z.
QED.

So let U be this open subset of $Z \simeq \mathbb{P}^{n-1}_A$. If we choose an isomorphism of

$$k \otimes_A E \simeq \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n) \qquad 0 \leq p_1 \leq \cdots \leq p_n$$

then if $d = p_1 + \cdots + p_n > 0$, we can find a ~~point in~~ one dimensional quotient of $k \otimes (E/T_0 E(-1))$

such that the corresponding bundle is $\simeq \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n - 1)$ which is regular. Thus U contains a rational point over the closed point of A.

Finally, because $Z \simeq \mathbb{P}^n_A$ and A is local, this section of Z over the closed point of A extends to a section s of Z over A. [Any unimodular vector in k^n lifts to a unimodular vector in A^n]. This section s is entirely contained in U, because $s^{-1}(U)$ is an open set containing the closed point, hence as A is local it is all of Spec A.

Therefore we have constructed inside of E a bundle E' of degree $d-1$ which is also regular, and such that E and E' have the same restriction to $\mathbb{A}^1_A$. By induction hypothesis this restriction is trivial.

January 3, 1976

1

Let C be a $\overset{\text{(alg)}}{\text{Riemann}}$ surface not necessarily complete. Let F be the field of meromorphic functions on C. We can make a space out of rank n subbundles of F^n over C. Denote this space $L_n(C)$. We have a map

$$L_n(C) \longrightarrow \mathrm{Map}_{pt}(C \cup \{\infty\}, BU_n)$$

which comes from considering the bundle E with its trivialization (canonical) outside a finite set. The conjecture is that this map is a homotopy equivalence.

Space $L_n(C)$ can be defined for any Riemann surface? Its elements are pairs (E, u) where E is a rank n vector bundle and $u: E \longrightarrow \mathcal{O}^n$ is a "rational" map. This data is glueable and so once we define $L_n(C)$ for the disk, we can extend it by fibre products. Curious, but what this amounts to is different from the preceding, because we allow singularities of u to move to ∞. So it seems that I have two sorts of spaces both with the same points, a point being a restricted choice of $\mathcal{O}_p$-lattice for each point P in C, where restricted means the choice agrees with $\mathcal{O}_p^n$ for almost all P.

So I have to worry about what happens in the disk. <u>Question</u>: One has a map from pairs (E, u) to divisors. Is this proper? Answer: **NO** for at a point P the lattice determines the index of the corresponding divisor.

The problem is how to define the topology on the space $L_n(C)$. If I don't allow singularities to go to the boundary, then this space is naturally an inductive limit of compact spaces.

Idea seems to be to allow holomorphic things. So the **non-compact gadget** has singularities tending toward infinity, Mittag-Leffler style. So I ought to be able to define $L_n(C)$ as a projective limit over compact things.

Look at divisors on C. These determine line bundles over C. With the correct topology on divisors ~~it~~ probably will be true that I get the space of maps from C to BU_1.

Jan. 5, 1976

Let C be a complete curve and let $\mathrm{Div}\, C$ be the space of divisors on C. I know

$$\pi_q(\mathrm{Div}\, C) = H_q(C, \mathbb{Z}).$$

Let $\mathrm{Div}^0 C$ be the divisors of degree 0 and let J be the Jacobian of C. Then one has a map (surjective)

$$\mathrm{Div}^0 C \twoheadrightarrow J$$

whose kernel is of the homotopy type $K(\mathbb{Z}, 2)$. But this kernel can also be described as the quotient $F^*/\mathbb{C}^*$ where F is the function field of C.

<u>Conjecture</u>: F^* is contractible in a suitably defined natural topology. Summary:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & F^* & \longrightarrow & F^*/\mathbb{C}^* & \longrightarrow & 0 \\
 & & \wr & & \wr & & \wr & & \\
 & & S^1 & & pt & & K(\mathbb{Z},2) & & \\
 & & \| & & & & & & \\
 & & K(\mathbb{Z},1) & & & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^*/\mathbb{C}^* & \longrightarrow & \mathrm{Div}^0 C & \longrightarrow & J & \longrightarrow & 0 \\
 & & & & \wr & & \wr & & \\
 & & & & (BU_1{}^C)^0 & & K(\mathbb{Z}^{2g},1) & &
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Div}^0 C & \longrightarrow & \mathrm{Div}\, C & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0 \\
 & & & & \wr & & & & \\
 & & & & (BU_1)^C & & & &
\end{array}$$

Next consider $L_n(C)$ the space of rank n subbundles in F^n. Conjecturally we have

$$L_n(C) \sim BU_n^C$$

hence we have a tower

$$\begin{tikzcd} BU_n^C \arrow[r] & BU_n^{C-\infty} \arrow[r] & BU_n \\ \Omega U_n \arrow[u] & (U_n)^{2g} \arrow[u] & \end{tikzcd}$$

corresponding to the "skeleta filtration of C".

~~Notice Because $L_n(C) \sim BU_n^C$ it follows~~

Assume there exists a "space" $\underset{\sim}{Vect_n}(C)$ of algebraic vector bundles over C. Note that if we believe F^* is contractible, then from the fibration

$$F^* \longrightarrow Div(C) \longrightarrow \underset{\sim}{Pic}(C)$$

$\underset{\sim}{Vect_1}(C)$ should be homotopy equivalent to $Div(C) = L_1(C)$.

Thus it is clear that $\underset{\sim}{Pic}(C)$ is the "space" resulting from letting F^* act on $Div(C)$, or what's the same, the "space" belonging to the complex $F^* \longrightarrow Div(C)$

of topological abelian groups.

<u>Conjecture</u>: $GL_n(F)$ is contractible for each n in a suitable topology. Question: Is F contractible as a topological field?

Concrete interpretation of the above conjecture. Let $\{E_t\}$ be a family of algebraic vector bundles over C parameterized by a space T. Then it is always possible to find a family of rational maps $u_t : E_t \dashleftarrow \mathcal{O}^n$. <u>Proof</u>: Can replace E_t by $E_t(m)$ and so assume that E_t is generated by global sections. Take the case $n = 1$. Then $\Gamma(C, E_t)$ is a vector space of a given dimension, and $t \mapsto \Gamma(C, E_t)$ is a vector bundle. Select over T a section s which vanishes nowhere, this is possible if $\dim \Gamma(C, E_t) > \dim T$ which can be arranged. Then $s_t : \mathcal{O} \longrightarrow E_t$ is non-zero, hence gives a rational map $s_t : \mathcal{O} \dashrightarrow E_t$ for each t. In the general case we have to be careful to select n-sections in $\Gamma(C, E_t)$ which are generically independent.

To consider the following problem. Let E be a vector bundle over C which is sufficiently "ample". To calculate the codimension of those subspaces $W \subset \Gamma(C, E)$ which are generically independent,

i.e. such that $\mathcal{O}_C \otimes W \hookrightarrow E$. Here $\dim W = n$.
[redacted].

Given

$$0 \longrightarrow R \longrightarrow \mathcal{O} \otimes V \longrightarrow E \longrightarrow 0$$

with $V \hookrightarrow H^0(E)$, let $v_1 \in V$, $v_1 \neq 0$. Then

$$\Updownarrow$$
$$H^0(R) = 0$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & \mathcal{O} \arrow[r, "\sim"] \arrow[d] & \mathcal{O}v_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & \mathcal{O} \otimes V \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] & \mathcal{O} \otimes V/k \cdot v_1 \arrow[r] \arrow[d] & E/\mathcal{O}v_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

so $H^0(R) = 0 \Longrightarrow V/kv_1 \hookrightarrow H^0(E/\mathcal{O}v_1)$.
Unfortunately $E/\mathcal{O}v_1$ may have torsion. [redacted]

Here's how to proceed. Suppose $\dim T = d$. Choose $d+1$ points $P_0 \dots P_d$ in C and twist E enough so that

$$V = \Gamma(C, E) \twoheadrightarrow E(P_0) \times \dots \times E(P_d)$$

Then those $W \in \mathrm{Grass}_n(V)$ which fails to project isomorphically onto $\hat{E}(P_0)$ form a hypersurface, so those failing to project isom. onto some $\hat{E}(P_i)$ lie on an intersection of $(d+1)$ hypersurfaces.

Reminiscent with Siegel formula.

Review stuff on ζ function.

$$\zeta(s) = \sum_{D \geq 0} \frac{1}{N(D)^s} \qquad N(D) = q^{\deg D}$$

$$= \sum_{D \geq 0} z^{\deg D} \qquad z = q^{-s}$$

$$= \sum_{L \in \operatorname{Pic} C} \sum_{D \mapsto L} z^{\deg D}$$

$$= \sum_{L \in \operatorname{Pic} C} z^{\deg L} \frac{q^{H^0(L)} - 1}{q - 1}$$

Analogous 2-dimensional ζ is

$$\sum_{\mathcal{O}^2 \subset E \subset F^2} \frac{1}{\operatorname{card}(E/\mathcal{O}^2)^s} = \sum_{E \supset \mathcal{O}^2} z^{\operatorname{length}(E/\mathcal{O}^2)}$$

$$= \sum_{E \supset \mathcal{O}^2} z^{\deg(E)}$$

$\deg(E) = \operatorname{length}(E/\mathcal{O}^2)$
index of E wrt. $\mathcal{O}^2$.

$$= \sum_{\substack{E \in \text{iso classes} \\ \text{of rank 2} \\ \text{bundles}}} z^{\deg(E)} \cdot \frac{\operatorname{card}\{\operatorname{Inj}(\mathcal{O}^2 \hookrightarrow E)\}}{\operatorname{card}\{\operatorname{Aut}(E)\}}$$

Problem: Understand the variety consisting of all n-diml subspaces W of $H^0(E)$ such that $\mathcal{O}\otimes W \to E$ is injective.

For each $x \in C$ one has the open set U_x in $\mathrm{Grass}_n(V)$, $V = H^0(E)$, consisting of W such that $W \to E(x)$ is an isomorphism. We seek those W belonging to some U_x:

$$W \in \bigcup_{x\in C} U_x$$

Because open sets satisfy the ascending chain condition one knows

$$\bigcup_{x\in C} U_x = U_{x_1} \cup \cdots \cup U_{x_g}$$

for some finite subset $\{x_1, \ldots, x_g\}$ of C.

Another possiblity would be to look at derivatives at at point. Suppose we look at

$$E/m_x^2 E \simeq \left(\mathcal{O}_x/m_x^2\right)^2$$

Two elements here give a 2×2 matrix in $\mathcal{O}_x/m_x^2$ which we can ask to be $\neq 0$. Given $W \subset (\mathcal{O}_x/m_x^2)^2$ with non-zero determinant, just what is the codimension of the complement of this set.

~~[illegible]~~ Fix $x_1, \dots, x_q$ distinct points in C and assume $V = H^0(E) \twoheadrightarrow E(x_1) \times \dots \times E(x_q)$. Let us try to calculate the ~~bad~~ $W \in \mathrm{Grass}_n(V)$ which are bad at $x_1, \dots, x_q$. This means that there is a hyperplane $Z_i \subset E(x_i)$ ~~such that~~ for each $i = 1, \dots, q$ such that

$$W \subset \bigcap_i ev_{x_i}^{-1}(Z_i) = \text{Inverse image of } Z_1 \times \dots \times Z_q \text{ in } V$$

So $Z_1, \dots, Z_q$ fixed this inverse image is of codim q in V, so the possible W contained in the inverse image form a variety of dim

$$\dim \{W \mid W \subset f^{-1}(Z_1 \times \dots \times Z_q)\} = n(\dim V - n - q)$$

Add $q(n-1)$ for the possible $(Z_1, \dots, Z_q)$ and we get

$$\dim \{\text{bad } W\} = n(\dim V - n - q) + q(n-1)$$

$$= n(\dim V - n) - q$$

$$\dim \{\mathrm{Grass}_n V\} = n(\dim V - n)$$

So we do get codimension q as I thought.

Try a similar ~~[illegible]~~ calculation with $E/m_x^q E$. Here one has to consider all "hyperplanes" in $E/m_x^q E$, which by duality are all unimodular vectors modulo scalars. The dimension of the space of unimodular vectors is $n \cdot q$, the dimension of $(\mathcal{O}_x/m_x^q)^*$ is q, so the "hyperplanes" form a space of dimension $(n-1)q$. Given a hyperplane $Z \subset E/m_x^q E$ we want to compute how many n-planes W are to be found inside of Z. Now $W \cap mZ$ will contain at least a line, so the dimension of these lines is

$$\dim(mZ) - 1 = (q-1)(n-1) - 1$$

Once the line is chosen the ways of extending to an n-dimensional space are

$$(n-1)\big(\dim(Z) - 1 - (n-1)\big) = (n-1)\big((n-1)q - n\big)$$

Thus it seems that the bad W in $E/m_x^q E$ form a variety of dimension $\leq$

$$(n-1)q + (q-1)(n-1) - 1 + (n-1)\big((n-1)q - n\big).$$

$$(n-1)\big[(2q-1) + (n-1)q - n\big] - 1$$

$$(n-1)\big[nq + q - 1 - n\big] - 1 = (n-1)(n+1)(q-1) - 1$$
$$= (n^2-1)(q-1) - 1$$

But the dimension of all W in $E/m_x^q E$ is

$$n(nq - n) = n^2(q-1)$$

Thus the codimension of the set of bad W is $(q-1)+1 = q$ which agrees with the calculation on page 9.

So we expect $\mathrm{Inj}\{\mathcal{O}^n \hookrightarrow E\}/GL_n \hookrightarrow \mathrm{Grass}_n(H^0E)$ to have its complement of codimension

$$\geq \frac{\dim H^0(E)}{n}$$

$$\dim H^0(E) - \dim H^1(E) = \deg E + n(1-g)$$

<u>January 6, 1976</u>

Review. C complete nonsing curve over $\mathbb{C}$. We are interested in the "space" of algebraic vector bundles over C of rank n. The meaning of the word "space" is not clear. The rough idea is that for each finite complex T one considers families of alg. bundles over C parameterized by T and one forms homotopy classes of these. Then one wants a space to represent these families.

Now I have seen that given a family $\{E_t\}_{t\in T}$ then by twisting sufficiently $E_t(m\cdot\infty)$ I can find an injective map

$$\mathcal{O}^n \hookrightarrow E_t(n\infty).$$

whence I have found a lifting of the family to the space of bundles contained in F^n.

A consequence of this is that we can always deform the poles of a family $(E_t, \mathcal{O}^n \dashrightarrow E_t)$ to a fixed basepoint ∞. For example suppose I have [redacted] $L \subset F$ and a $f \in F^*$. Then the lattices L, fL are homotopic. To see this note that

$$f\mathcal{O} \cap \mathcal{O} \subset \mathcal{O}$$

zeroes of f

$$\cup$$

$$f\mathcal{O}$$

poles of f.

$(f) = f^{-1}(0) - f^{-1}(\infty)$. Now $f^{-1}(t)$ as t goes from 0 to ∞ gives a homotopy between zeroes and poles. Shift from 0 to 1 via

$$z \longmapsto z + t \qquad 0 \le t \le 1$$

then from 1 to ∞ via

$$z \longmapsto \frac{1}{\frac{1}{z} + t} = \frac{z}{1 + tz} \qquad 0 \le t \le 1$$

etc.

Thus if I have $L \subset F$ and if I find a non-zero section $\mathcal{O} \hookrightarrow L$ of F I get a homotopy of L to a Λ containing $\mathcal{O}$.

[redacted]

Consider the space S of non-zero rational sections of a line bundle [redacted] L. [redacted] This space is isomorphic to F^*. Given $\alpha : T' \longrightarrow S$, [redacted] where $T' \subset T$, I want to extend α to T. Can one homotop α so that all poles of $\alpha(t)$ are at one point ∞? [redacted]

Assume that I can find a family $\Lambda_t \subset$ [redacted] L of lattices such that

$\Lambda_t \subset \alpha(t)$ [redacted] $L \qquad \forall t \in T'$.

Thus Λ_t acts as a [redacted] bound for the family of poles of $\alpha(t)$. ??

$\alpha : t \mapsto f_t$,

Let [redacted] $T' \longrightarrow F^*$ be a family of rational non-zero functions, where T' is a subcomplex of T. I want to show α can be extended to T. I will assume I can find a family of ideals

$$\Lambda_t \subset \mathcal{O} \cap f_t \mathcal{O}$$

Thus Λ_t is a positive divisor which bounds the zero divisor of f_t. For m large enough I know that the bundle $t \mapsto \Gamma(C, \Lambda_t(m\infty))$ on T' has a nowhere-vanishing section s_t. Moreover (recall $\Lambda_t \subset \mathcal{O}$) s_t can be extended to a non-zero section of $\mathcal{O}(m\infty)$ for all $t \in T$. Thus we get a family $s_t \in \Gamma(\mathcal{O}(m\infty))$ for $t \in T$ with $s_t \neq 0$ for all t such that $s_t \in \Gamma(\Lambda_t(m\infty)) \subset \Gamma(\mathcal{O}(m\infty))$ for all $t \in T'$. We have

$$\mathcal{O}s_t(-m\infty) \subset \Lambda_t$$

so we can replace Λ_t by $\mathcal{O}s_t(-m\infty)$ in which can we have extended our choice of Λ_t to all of T. Also

$$\mathcal{O}s_t(-m\infty) \subset f_t\mathcal{O}$$

so

$$\begin{array}{c} \mathcal{O}(-m\infty) \subset s_t^{-1} f_t \mathcal{O} \\ \cap \\ \mathcal{O} \end{array}$$

so, as s_t extends to T, to see if f_t does we can suppose $\Lambda_t = \mathcal{O}(-m\infty)$ for all t. Thus we have reduced to the case where the poles of f_t are at ∞ and of order $\leq m$ for all t. Thus f_t is the same thing as a non-zero section of $\mathcal{O}(m\infty)$, hence by enlarging m we know we can extend it to all of T.

The preceding explains the contractibility of F^* (at least the weak contractibility). The proof works without essential change for $GL_n(F)$.

<u>Summary</u>: I have a bit better understanding of the contractibility of $GL_n(F)$. Therefore I understand somewhat why the space $L_n(C)$ has the same homotopy as $\underline{Vect}_n(C)$ (the space classifying families of rank n vector bundles over C.) What remains now is to understand at least conjecturally the K-theoretic implications.

Consider the localization sequence

$$\longrightarrow K_i C \longrightarrow K_i F \xrightarrow{\partial} \bigoplus_{P \in C} K_{i-1}(k(P)) \longrightarrow$$

$$\bigoplus_{P \in C} K_{i-1}(k(P)) = K_{i-1}(k) \otimes \mathrm{Div}\, C.$$

In the topological analogue of this spectral sequence $K_* F$ will be $\mathbb{Z}$. So we find that $$K_i^{top} C = K_i^{top}\left(\begin{array}{c}\text{torsion sheaves}\\ \text{on } C\end{array}\right) \oplus \begin{cases} \mathbb{Z} & i=0 \\ 0 & i\neq 0 \end{cases}$$

These appear to be consistent with all earlier conjectures.

To prove the conjectures. Is it possible to piece together locally?

You might try to proceed globally. Start with a top vector bundle E over C, i.e. a map $C \to BU_n$. You need now a procedure to convert this to an algebraic vector bundle.

Look at $\mathbb{P}^1$. Standard method is to use the fact that if I remove ∞ from $\mathbb{P}^1$ the bundle becomes contractible. Thus topologically the bundle is specified by giving a map $S^1 \to GL_n$. (basept. pres)

$$\Omega U_n \longrightarrow BU_n^{\mathbb{P}^1} \longrightarrow BU_n$$

Next one uses the fact that algebraic maps $S^1 \to U_n$ are dense in ΩU_n.

What does approximation consist of? Under what **general** circumstances could I approximate continuous by algebraic maps.

Consider the general ~~[illegible]~~ situation. Let $E \longrightarrow B$ be a space over B.

$$\begin{tikzcd} X \arrow[d] \\ Y \arrow[d, "f"] \\ S \end{tikzcd} \qquad f_* X = \prod_{Y/S} X$$

$$\mathrm{Hom}_{/S}(Z, f_* X) = \mathrm{Hom}_{/Y}(Z \times_S Y, X)$$

Thus $\quad (f_* X)_s = \prod_{y \in f^{-1}\{s\}} X_y$

One gets $\Gamma(X/Y)$ by taking $S = pt$.

So suppose E is a space over B and we have some candidate Z for $\Gamma(E/B)$, i.e. Z comes with a map u

$$\begin{tikzcd} & E \arrow[d] \\ Z \times B \arrow[ur, dashed, "u"] \arrow[r, "pr_2"'] & B \end{tikzcd}$$

When can I conclude that $Z \longrightarrow \Gamma(E/B)$ is a homotopy equivalence? One method is to arrive at $\Gamma(E/B)$ by localization on B. If $B = U \cup V$, then

$$\begin{tikzcd} \Gamma(E/B) \arrow[r] \arrow[d] & \Gamma(E_V/V) \arrow[d] \\ \Gamma(E_U/U) \arrow[r] & \Gamma(E_{U\cap V}/U\cap V) \end{tikzcd}$$

is cartesian, possibly homot. cart. If Z localizes also, one might use this method.

This doesn't work for $\underset{\mathbb{Z}}{\underset{\shortparallel}{\Omega U_n^{alg}}} \subset \underset{U_n^{(S^1, pt)}}{\underset{\shortparallel}{\Omega U_n}}$.

Because algebraic maps $S^1 \to U_n$ don't localize over S^1. [redacted] OK.

[redacted]

Note that iso classes of vector bundles on $\mathbb{P}^1$ differ from iso classes of [redacted] topological bundles on $\mathbb{P}^1$. This amounts to the fact that to modify a function $S^1 \to GL_n$ by holom. maps $\{|z| \leq 1\} \to GL_n$ and $\{|z| \geq 1\} \to GL_n$ is not the same thing as modifying it by continuous maps. Example $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ can be deformed into $\mathcal{O}^2$ hence it is topologically trivial.

The [redacted] fact that ΩU_n is rep. by invertible Laurent poly matrices is equivalent to the fact that any family of algebraic vector[redacted] bundles $\{E_t\}$ [redacted] on $\mathbb{P}^1$ can be trivialized over $\mathbb{A}^1$. Is there a direct method of proving this?

If C is a non-compact Riemann surface, then any holom. line bundle over C is trivial. This follows by the Grauert theorem. In more elementary terms, one can choose a non-vanishing section, then the line bundle becomes a divisor, and one can by Mittag-Leffler construct a meromorphic function associated to this divisor. (This is really quite different from trivializing an algebraic bundle over $\mathbb{A}^1$).

Over $\mathbb{P}^1$ let E be a bundle. Then I can consider the set of unimodular subspaces in E. Such a thing is a $\mathbb{C}$ subspace $V \subset \Gamma(\mathbb{A}^1, E) \ni$ $\mathcal{O} \otimes V \longrightarrow E(N \cdot \infty)$ is an isomorphism off ∞. Because E is trivial over $\mathbb{A}^1$, the set of such unimodular subspaces V can be identified with $Gl_n(k[t])/Gl_n(k)$, hence it should form a contractible space.

Suppose that E is a bundle over $\mathbb{P}^1$. How do I show it becomes trivial over $\mathbb{A}^1$? Select a sub-line bundle L in E. L has a fairly canonical trivialization off ∞. Thus if I can produce a full flag in E I get the required trivialization.

General question: Given a family of vector bundles over $\mathbb{P}^1$ it is possible to trivialize the family over $\mathbb{P}^1 - \infty$? This question can be asked both in the topological and scheme contexts. In fact I think one has a proof in the topological context by the building theory.

Begin with the family E_t and try to find a map

(*) $$\mathcal{O} \otimes E_t(0) \longrightarrow E_t$$

which is an isomorphism over $0 \in \mathbb{P}^1$. This can be put as a lifting problem

$$\begin{tikzcd} & \mathcal{O} \otimes E_t(0) \arrow[dl, dashed] \arrow[d] \\ E_t \arrow[r] & E_t(0) \end{tikzcd}$$

and can be solved by replacing E_t with $E_t \otimes \mathcal{O}(m\infty)$ for m sufficiently large. Because T is compact this map (*) is an isom. over a nbd of 0 which I can assume to be $|z| \leq 1+\varepsilon$.

Here's the problem: All you can show by this method is that any family E_t is homotopic to one which can be trivialized over $\mathbb{A}^1$. There might be a way of showing the space of unimodular subspaces is contractible.

The problem: I know that one can vary bundles over $\mathbb{P}^1$ so that they are not isomorphic. Thus a family of bundles on $\mathbb{P}^1$ is not usually locally trivial. So it is not obvious that such a family when restricted to $\mathbb{A}^1$ is locally trivial.

So return again to the case of a family $\{E_t\}$ of bundles on $\mathbb{P}^1$. Assume $E_t(0) \simeq \mathbb{C}^n$, i.e. we give $\mathcal{O}^n \hookrightarrow E_t$ isomorphism around zero. Then I want to modify this so as to make an isomorphism over all of $\mathbb{P}^1$.

Heuristic idea: Because $\mathcal{O}^n \hookrightarrow E_t$ is an isomorphism near zero, we can find a disk D independent of t on which it is an isomorphism. Then E_t is the bundle specified by a scattering matrix, which is a matrix of rational functions non-singular on ∂D. Now let $\mathcal{G}$ be the group of these rational functions matrices, and $\mathcal{G}^+$ the subgroup non-singular inside D and $\mathcal{G}^-$ those non-singular outside D with possible singularity at ∞. We know that

$$\mathcal{G}^+ \times^{GL_n(\mathbb{C}[z])} \mathcal{G}^- \xrightarrow{\sim} \mathcal{G}$$

so we ought to be able to lifting the clutching function for E_t to a product $g^+ g^-$ where $g^+(0)=1$. If this can be done then over $\mathbb{A}^1$ the cocycle defining E_t becomes a boundary, so E_t restricted to $\mathbb{A}^1$ is trivial.

Summary: I do not yet know that any family E_t of bundles on $\mathbb{P}^1$ becomes trivial over $\mathbb{A}^1$. I have some hope that the work on Laurent loops will shed light on this problem. How?

So I will consider outgoing subspaces in $L^2(S^1)^n$ which correspond to rational scattering matrices. This means they are commensurable with $\mathcal{O}^n = H^2(S^1)^n$. The idea is to somehow let act on these outgoing spaces those rational matrices which are nonsingular outside $|z|=1$ excluding ∞.

$\mathcal{G}$ rational matrices regular on S^1. $\mathcal{G}^+$ those regular inside D. $\mathcal{G}^-$ those regular outside $|z|=1$ excluding ∞. Then $\mathcal{G}/\mathcal{G}^+$ = outgoing subspaces of $L^2(S^1)^n$ commensurable with $H^2(S^1)^n$, and $\mathcal{G}^-$ acts transitively on this thing, and the stabilizer of $H^2(S^1)^n$ is $\mathcal{G}^+ \cap \mathcal{G}^- = GL_n(\mathbb{C}[z])$. We can cut $\mathcal{G}^+$ down to those matrices $= 1$ at 0, so as to make $\mathcal{G}^+ \cap \mathcal{G}^-$ contractible.

<u>Idea</u>: Given D a lattice in $\mathbb{C}[z]^n$ with support inside $|z|=1$, we can choose the orthogonal complement of zD in D for the metric defined by integration over $|z|=r$ and then

take the limit as $r \to \infty$. Do we get a unimodular subspace for D?

<u>Example</u>: let $D = \mathbb{C}[z](z-\alpha)$. Then the generator for D for $|z| = 1$ is

$$\frac{z-\alpha}{1-\bar{\alpha} z}$$

Note: this ~~[illegible]~~ has absolute value one on ~~[illegible]~~ $|z| = 1$, and $1-\bar{\alpha} z$ is non-singular on $|z| \leq 1$, for it vanishes when $z = \frac{1}{\bar{\alpha}}$ which is outside $|z| = 1$.

The generator for D using $|z| = r$ is

$$\frac{z-\alpha}{r - \frac{\bar{\alpha}}{r} z} \qquad \text{no limit as } r \to \infty$$

$$\left(z\bar{z} = r^2, \qquad \left| \frac{z-\alpha}{\frac{r^2}{z} - \bar{\alpha}} \right| = \underline{1} \right.$$

$$\left. \underline{1} = \left| \frac{z-\alpha}{\frac{r^2}{z} - \bar{\alpha}} \right| = |z| \left| \frac{z-\alpha}{r^2 - \bar{\alpha} z} \right| = r \left| \frac{z-\alpha}{r^2 - \bar{\alpha} z} \right| = \left| \frac{z-\alpha}{r - \frac{\bar{\alpha}}{r} z} \right| \right)$$

so the idea doesn't work.

January 9, 1976

I consider the old problem of lattices in $\mathbb{C}[z]^n$. Specifically I want to consider the space X consisting of all [redacted] $\mathbb{C}[z]$-submodules Λ in $\mathbb{C}[z]^n$ [redacted] of a given codimension. Thus I am considering all quotients of $\mathbb{C}[z]^n$ of a given length, so X is some kind of Quot scheme.

I know that any Λ in X is [redacted] isomorphic to $\mathbb{C}[z]^n$, hence to $\mathbb{C}[z] \otimes \Lambda/z\Lambda$. The question is [redacted] how to choose a global isomorphism between the two.

Purely algebraic question: Let X be the scheme of quotients of $\mathcal{O}^n_{\mathbb{A}^1}$ of a fixed length. [redacted] Over $X \times \mathbb{A}^1$ one has ~~two~~ vector bundles. Are these isomorphic?

Work over $\mathbb{P}^1$ first. If E is a quotient of $\mathcal{O}^2$ of length ρ one has then $\Gamma(E)$ is of dimension ρ.

$0 \longrightarrow K(E)$ [redacted] $\longrightarrow \mathcal{O}^2 \longrightarrow E \longrightarrow 0$

$$0 \longrightarrow \mathcal{O}(-1) \otimes \Gamma_1(E) \longrightarrow \mathcal{O} \otimes \Gamma(E) \longrightarrow E \longrightarrow 0$$

Now we are restricting the support of E to be away from infinity, whence E itself is just the vector [redacted] space $\Gamma(E)$ with the endo given by multiplying by z. Therefore we may describe an element of X as consisting of a vector space V with endo θ and two elements v_1, v_2 which span it, all this up to isomorphism.

What are then the two bundles over X. One is K:

$$0 \longrightarrow K \longrightarrow \mathbb{C}[z]^2 \longrightarrow E \longrightarrow 0$$

and the other is $K/zK \otimes \mathbb{C}[z]$. Maybe it would help to assume that E has no support at 0, i.e. that θ is an isomorphism, for then K/zK is canonically isomorphic to $\mathbb{C}^2$.

Question: Is the bundle $\mathcal{E}$ on X with fibres E trivial or non-trivial? Is the bundle $\mathcal{K}/z\mathcal{K}$ with fibres K/zK trivial over X?

So again I consider the space of all Λ inside $\mathbb{C}[z]^2$ of codimension ρ. I will restrict the support to be inside of $|z| < 1$, so as to have a scattering matrix to describe Λ in a 1-1 fashion.

Then I have a map

$$\{\Lambda\} \longrightarrow S_\rho(D)$$

~~which is not very nice exactly~~ given by support. Are the fibres of the same dimension?

Take the fibre over O. In this case I want the dimension of the set of Λ in $\mathbb{C}[[z]]^n$ of length ρ. This sort of thing you computed when you worked over a finite field and worked out the local zeta factor:

$$\sum_{\Lambda < \mathcal{O}^n} \frac{1}{\operatorname{card}(\mathcal{O}^n/\Lambda)^s}$$

To compute those Λ one intersects

$$0 < \Lambda \cap \mathcal{O}e_1 < \Lambda \cap (\mathcal{O}e_1 + \mathcal{O}e_2) < \cdots < \Lambda \cap (\mathcal{O}^n) = \Lambda.$$

$$\pi^{a_1}\mathcal{O}e_1 \qquad \pi^{a_2}$$

so you get a unique basis for Λ of the form

$$x_1 = \pi^{a_1} e_1$$

$$x_2 = f_{12} e_1 + \pi^{a_2} e_2 \qquad f_{12} \text{ unique mod } \pi^{a_1}\mathcal{O}$$

$$x_3 = f_{13} e_1 + f_{23} e_2 + \pi^{a_3} e_3 \qquad \begin{array}{l} f_{13} \text{ unique mod } \pi^{a_1}\mathcal{O} \\ f_{23} \text{ ——— } \pi^{a_2}\mathcal{O} \end{array}$$

And so one sees that for given $a_1, a_2, \ldots, a_n \geq 0$ one has $q^0\, q^{a_1}\, q^{a_1+a_2} \cdots q^{a_1+\cdots+a_{n-1}}$ for the number.

Thus

$$\sum_{a_1,\dots,a_n \geq 0} \frac{q^{a_1}\, q^{a_1+a_2} \cdots q^{a_1+\cdots+a_{n-1}}}{\left(q^{a_1+\cdots+a_n}\right)^s}$$

is the local ζ factor

$$= \sum_{a_1 \geq 0} \frac{1}{(q^{a_1})^{s-n+1}} \cdots \sum_{a_n \geq 0} \frac{1}{(q^{a_n})^{s}}$$

$$= \frac{1}{1-\frac{q^{n-1}}{q^{s}}} \cdots \frac{1}{1-\frac{1}{q^{s}}}$$

$$= \frac{1}{1-q^{n-1}z} \cdots\cdots \frac{1}{1-z}$$

Now I want the coefficient of z^ρ which is the number of lattices with index ρ.

$n=2$ $\qquad \frac{1}{1-qz}\,\frac{1}{1-z} = (1+z+z^2+\dots)(1+qz+q^2z^2+\dots)$

$\rho = 2$ $\qquad$ number is $1+q+q^2$ $\qquad$ dim. 2

Thus if you take a degree 2 divisor without multiplicity the fibre is all pairs ???

a, b distinct. To find Λ of codim 2 in $\mathbb{C}[z]^2$ whose support is a, b. This means that as

$$\mathbb{C}[z]^2 / (z-a)(z-b)\mathbb{C}[z]^2 = \left(\mathbb{C}[z]/z-a\right)^2 \oplus \left(\mathbb{C}[z]/(z-b)\right)^2$$

that Λ amounts to a line in the first factor and a line in the second factor. Total of 2 dimension. On the other hand if $a=b=0$, then the number of lattices of codim 2 in $(\mathbb{C}[z]/z^2)^2$ includes at least the direct factors

$$\dim \mathbb{P}_1(\mathbb{C}[z]/z^2) = 4-2 = 2$$

$n=2$ $\quad$ dim ρ $\quad$ same as for generic divisor.

$n=n$ $\quad$ general case then the dimension is just the coff. $q^{(n-1)\rho} z^{\rho}$, so the dimension is $\rho(n-1)$ which is the same as for the generic divisor.

Example: The fibre over a generic divisor is a product of projective spaces. I can view it as the obvious compactification of the set of n-independent lines. Think of semi-simple elements of GL_n or better regular elements of $M_n(\mathbb{C})$.

Look at the bundle $\mathcal{E}$ in the case $n=2$, $\rho=1$. Thus I am looking at all dimension 1 quotients of $\mathbb{C}[z]^2$.

An element of X is just a one dimensional quotient of $\mathbb{C}^2$ plus a number λ.

$$X = \mathbb{P}_1\mathbb{C} \times \mathbb{C}.$$

The bundle $\mathcal{E}$ on X is just $\mathcal{O}(1)$ pulled up from $\mathbb{P}_1$. Thus $\mathcal{E}$ is not trivial.

Next point is compute the bundle $\mathcal{K}$ and whether it is essentially trivializable. Given E, λ we have

$$K = \mathrm{Ker}\{\mathbb{C}[z]^2 \longrightarrow E\}$$

fits into an exact sequence canonical

$$0 \longrightarrow \mathbb{C}[z] \otimes E' \longrightarrow K \longrightarrow \mathbb{C}[z] \longrightarrow 0$$

$$K = \mathbb{C}[z]^2 (z - \lambda) + \mathbb{C}[z] \otimes E'$$

where $E' = \mathrm{Ker}\{\mathbb{C}^2 \longrightarrow E\}$. Then

$$0 \longrightarrow E' \longrightarrow K/zK \longrightarrow \mathbb{C} \longrightarrow 0$$

so $\mathcal{K}/z\mathcal{K}$ is an extension of $\mathcal{O}$ by $\mathcal{O}(-1)$ on $\mathbb{P}^1 \times \mathbb{C}$. Such extensions split.

So
$$\mathcal{K}/z\mathcal{K} \simeq \mathcal{O} \oplus \mathcal{O}(-1) \quad \text{lifted from } \mathbb{P}^1$$

However $\mathcal{K}$ which is a bundle on $X \times \mathbb{A}^1$ is probably a non-trivial extension of $\mathcal{O}$ by $\mathcal{O}(-1)$?

? ?

$$0 \longrightarrow E' \longrightarrow \mathbb{C}^2 \longrightarrow E \longrightarrow 0$$

$$K = \mathrm{Ker}\,\{\mathbb{C}[z]^2 \longrightarrow E\} \qquad z \longmapsto \lambda$$

$$= \mathbb{C}[z]^2(z-\lambda) + \mathbb{C}[z]\otimes E'$$

Thus we have a canonical exact sequence

$$0 \longrightarrow \mathbb{C}[z]\otimes E' \longrightarrow K \longrightarrow \mathbb{C}[z]\otimes E \longrightarrow 0$$

$$\mathbb{C}[z]\otimes E = \mathbb{C}[z]^2(z-\lambda)/\mathbb{C}[z]\otimes E'\cdot(z-\lambda)$$

which means that over $X = \mathbb{P}^1\times\mathbb{C}$ we have an exact sequence

$$0 \longrightarrow \mathbb{C}[z]\otimes\mathcal{O}(-1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{C}[z]\otimes\mathcal{O}(1) \longrightarrow 0.$$

which probably doesn't split. Now however one has

$$K/zK \simeq E' \oplus z\mathbb{C}^2/(z-\lambda)E'$$

Over $\lambda = 0$ at least this splits into

$$E' \oplus zE \simeq E' \oplus E$$

so that for $\lambda=0$, $\mathcal{K}/z\mathcal{K} \simeq \mathcal{O}(1)\oplus\mathcal{O}(-1)$. So it should be the case that $\mathcal{K}$ is not isom to $\mathcal{K}/z\mathcal{K}\otimes\mathbb{C}[z]$ globally although this should hold locally on X.

Consider all quotients of $\mathcal{O}^2_{\mathbb{P}_1}$ of length d. Call this space $Q^{(2)}_d$. Then we have a map

$$Q_d^{(2)} \longrightarrow S_d(\mathbb{P}_1) = \mathbb{P}_d$$

given by the determinant. Over a generic divisor of $S_d(\mathbb{P}_1)$ this map has fibre isomorphic to $(\mathbb{P}_1)^d$. To see this suppose we start with the divisor $a_1 + \cdots + a_d$ where the a_i are distinct points of $\mathbb{P}_1$ all different from ∞. The fibre consists of all quotients E of $\mathbb{C}[z]^2$ of length d having support exactly at the points $a_1, \dots, a_d$. Then

$$E = E_1 \oplus \cdots \oplus E_d$$

where each E_j is a one-dimensional quotient of $\mathbb{C}^2$ and z acts by multiplying by a_j on E_j. Thus the fibre over $a_1 + \cdots + a_d$ can be identified with $(\mathbb{P}_1)^d$. On the other hand the fibre over 0 consists of all d dimensional quotients of $\mathbb{C}[z]^2$ of length d on which z is nilpotent. We have seen this is a CW complex with one $2j$ cell for each $j = 0, \dots, d$. Note that it is an approximation to $\Omega SU_2 = \Omega S^3$ whose homology ring is a polynomial ring on one generator of degree 2. It's cohomology ring is thus a truncated divided power algebra, hence this fibre has singularities.

January 10, 1975

Review: Let $Q_d^{(2)}(\mathbb{A}^1)$ be the space of quotients E of $\mathbb{C}[z]^2$ having length d. Over $Q_d^{(2)}$ I have a family $\mathcal{K}$ of $\mathbb{C}[z]$ modules whose fibre at E is K:

$$0 \longrightarrow K \longrightarrow \mathbb{C}[z]^2 \longrightarrow E \longrightarrow 0$$

Let's restrict to those E whose support doesn't contain zero. Then we get a canonical isom.

$$\gamma_0 : K/zK \xrightarrow{\sim} \mathbb{C}^2$$

so that over $Q_d^{(2)}(\mathbb{G}_m)$ we have $\mathcal{K}/z\mathcal{K} \xrightarrow{\sim} \mathcal{O}^2$, a canonical isomorphism. I want to know if the isomorphism γ_0 can be lifted to a family

$$\gamma : K \simeq \mathbb{C}[z]^2$$

over $Q_d^{(2)}(\mathbb{G}_m)$, this family can be topological.

Case $d=1$: Here E is a 1-dimensional quotient of $\mathbb{C}^2$ so that $Q_d^{(2)}(\mathbb{A}^1) = \mathbb{P}^1 \times \mathbb{A}^1$. If $E' = \operatorname{Ker}\{\mathbb{C}^2 \to E\}$, then one has ~~an~~ a canon exact sequence

$$0 \to \mathbb{C}[z]\otimes E' \longrightarrow K \longrightarrow \mathbb{C}[z]\otimes E \longrightarrow 0$$

whence over $Q_d^{(2)}(\mathbb{A}^1)$ we have an exact sequence

$$0 \longrightarrow \mathbb{C}[z]\otimes\mathcal{O}(-1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{C}[z]\otimes\mathcal{O}(1) \longrightarrow 0$$

Topologically this sequence splits, and also $\mathcal{O}(-1)\oplus\mathcal{O}(1)$

$\simeq \mathcal{O}^2$. Thus $\mathcal{K} \simeq \mathbb{C}[z]^2 \otimes \mathcal{O}$ at least topologically.

In fact suppose that one has a family $X \longrightarrow Q^{(2)}_{1}(\mathbb{A}^1)$ of quotients parameterized by an affine scheme X. Then

$$0 \longrightarrow \mathbb{C}[z] \otimes \mathcal{O}(-1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{C}[z] \otimes \mathcal{O}(1) \longrightarrow 0$$

is an exact sequence of bundles over the affine scheme $X \times \mathbb{A}^1$, hence it splits, so over X we have

$$\mathcal{K} \simeq \mathbb{C}[z]^2 \otimes \left(\mathcal{O}(-1) \oplus \mathcal{O}(1)\right)$$

as desired.

An observation. Consider $Q^{(2)}_1(\mathbb{P}^1)$. Then

$$K = \mathrm{Ker}\,\{\mathcal{O}^2 \twoheadrightarrow E\}$$

is always isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}$, hence it has a canonical filtration

$$0 \longrightarrow \mathcal{O} \longrightarrow K \longrightarrow \mathcal{O}(-1) \longrightarrow 0$$

which is essentially what we just used.

Now try $d=2$. Then $K = \mathrm{Ker}\,\{\mathcal{O}^2_{\mathbb{P}^1} \to E\}$ can be $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O}(-2) \oplus \mathcal{O}$. Classify what occurs. For support $a_1 + a_2$ $a_1 \neq a_2$ one gets $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ when the two quotients E_{a_1}, E_{a_2} of $\mathbb{C}^2$

are different, and $\mathcal{O}(-2) \oplus \mathcal{O}$ when they are the same.
It is clear that $K = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ exactly when $\mathbb{C}^2 \xrightarrow{\sim} E$ and $K = \mathcal{O}(-2) \oplus \mathcal{O}$ when this map is not injective.

Here perhaps is how to trivialize K over a family X. X divides up into the open set U where $\mathbb{C}^2 \xrightarrow{\sim} E$ and the closed set $X-U$ when this isn't an isomorphism. On $X-U$ the bundle K should admit sections necessarily non-vanishing. Such a section should give a section of $K \otimes \mathcal{O}(1)$ vanishing only at ∞. But $\Gamma(\mathbb{P}^1, K \otimes \mathcal{O}(1))$ is a rank 2 vector bundle on X because $H^1(\mathbb{P}^1, K \otimes \mathcal{O}(1)) = 0$. So we have a section of the bundle $\Gamma(\mathbb{P}^1, K \otimes \mathcal{O}(1))$ over $X-U$. We want to extend this to a section over X which is no-where zero on U. Can extend to all of X but not so as to be $\neq 0$.

January 11, 1976

~~Summary $Q_d^{(2)}(\mathbb{A}^1)$ is the space of quotients E of $\mathbb{C}[z]^2$ of length d, $\mathcal{K}$ is the family of $\mathbb{C}[z]$-modules over $Q_d^{(2)}(\mathbb{A}^1)$ with fibre $K = \mathrm{Ker}\{\mathbb{C}[z]^2 \to E\}$ at a quotient E. It might be easier~~

$Q_d^{(2)}(\mathbb{P}^1)$ is the scheme of quotients $\mathcal{O}^2 \to M$ on $\mathbb{P}^1$ of length d. Same as the scheme of all exact sequences

$$0 \longrightarrow K \longrightarrow \mathcal{O}^2 \longrightarrow M \longrightarrow 0$$

with $\mathrm{length}(M) = -\deg(K) = d$. Thus it's the same as pairs (E, u), where $E = K^\wedge$ is a bundle of rank 2 and degree d, and where $u : \mathcal{O}^2 \hookrightarrow E$ is a pair of generically independent sections.

Take $d = 2$. Then K is a bundle of degree -2 on $\mathbb{P}^1$, hence $K \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a + b = -2$, $a \geq b$, and since $K \subset \mathcal{O}^2$, one has $a, b \leq 0$. Thus $K \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $K \simeq \mathcal{O} \oplus \mathcal{O}(-2)$. I know K becomes trivial over $\mathbb{A}^1 = \mathbb{P}^1 - \infty$. This means I can find a map $\mathcal{O}^2 \longrightarrow K(m\infty)$ m sufficiently large which is an isomorphism off ∞.

Normalize: $t = z$ is the canonical rational function on $\mathbb{P}^1$. Let us consider quotients M of $\mathcal{O}^2$ with support not meeting $t = 1$, whence $K\langle 1 \rangle \simeq \mathcal{O}^2\langle 1 \rangle$ and we can normalize our map $\mathcal{O}^2 \longrightarrow K(m\infty)$ to be this canonical isom. over $t = 1$.

Let K be a vector bundle on $\mathbb{P}^1$. We have an inductive system

$$K \subset K(1) \subset K(2) \subset \cdots$$

defined by the section $t^{-1} : \mathcal{O} \longrightarrow \mathcal{O}(1)$. The union of this inductive system is $\Gamma(\mathbb{A}^1, K)$ which is a vector bundle over $\mathbb{A}^1$. A map $\gamma : \mathcal{O}^2 \longrightarrow K(m)$ which is an isomorphism off ∞ is the same thing as a unimodular subspace in $\Gamma(\mathbb{A}^1, K)$ which is contained within $\Gamma(\mathbb{P}^1, K(m)) \subset \Gamma(\mathbb{A}^1, K)$.

So let me consider the problem of the canonical family on $Q_2^{(2)}(\mathbb{A}^1)$.

$$0 \longrightarrow K \longrightarrow \mathcal{O}^2 \longrightarrow M \longrightarrow 0$$

We have $K \simeq \mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. To distinguish between these look at $\Gamma(K)$:

$$0 \longrightarrow \Gamma(K) \longrightarrow \mathbb{C}^2 \longrightarrow \Gamma(M)$$

$$\Gamma(K) = 0 \iff \mathbb{C}^2 \longrightarrow \Gamma(M) \text{ is an isom.}$$

~~Suppose I fix the support of M, say~~
~~The support of M is the divisor (a+b) where~~

If $K = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then $\Gamma(K(1))$ contains a unique unimodular subspace. If $K = \mathcal{O} \oplus \mathcal{O}(-2)$ then $\Gamma(K(1))$ contains no unimodular subspace. $\Gamma(K(2))$ is 4 dimensional. Canonical sequence

$$0 \to \mathcal{O} \to K \to \mathcal{O}(-2) \to 0$$

$$0 \to \Gamma(\mathcal{O}(2)) \to \Gamma(K(2)) \to \mathbb{C} \to 0$$

Any unimodular subspace of $\Gamma(K(2))$ has to project onto $\mathbb{C} = \Gamma(\mathcal{O})$ and the kernel must be the unique unimodular line in $\Gamma(\mathcal{O}(2))$. Thus unimodular subspaces V are subspace of $\Gamma(K(2))$ ∋

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\cdot t^2"] & V \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{O}(2)) \arrow[r] & \Gamma(K(2)) \arrow[r] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

These V form an affine space of dimension 2; in fact one wants all lines in $\Gamma(K(2))/\mathbb{C}t^2$ complementary to the hyperplane $\Gamma(\mathcal{O}(2))/\mathbb{C}t^2$.

Question: Given an affine family of such K is it always possible to find a unimodular subspace within $\Gamma(K(2))$?

If $K = \mathcal{O}(-1) + \mathcal{O}(-1)$, then $\Gamma(K(2)) = \Gamma(\mathcal{O}(1))^2$ is 4 dimensional. How many subspaces are unimodular. A subspace V can be obtained from a 2×2 matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are degree ≤ 1 polys. and the determinant $\alpha\delta - \beta\gamma \in \mathbb{C}^{\times}$. I can arrange that the matrix of constant terms be the identity. Thus

$$A = I + t\nu$$

where because A is invertible, we know ν is nilpotent, hence $\nu^2 = 0$. It appears therefore that the unimodular subspaces of degree $\leqslant 1$ in $\mathbb{C}[t]^2$ can be identified with all nilpotent 2×2 matrices. The cone of nilpotent matrices has dim 2.

More generally a unimodular subspace in $\mathbb{C}[t]^n$ of dimension n can be identified with a matrix of polys

$$I + A_1 t + \cdots + A_m t^m$$

where the family A_i of matrices is nilpotent i.e. $A^{\alpha} = 0$ $|\alpha| \geqslant$ constant.

Summary: I have shown that within $\Gamma(K(2))$ the unimodular subspaces form a contractible 2 dimens.

[redacted] variety. But more is true maybe: Suppose I have a family of K's and a corresponding family of 2 dimensional subspaces V in $\Gamma(K(2))$. If V is unimodular at some point x of the [redacted] parameter scheme X then it is unimodular in some neighborhood ~~of~~ x. This is [redacted] false! For we may have a section [redacted] of $\mathcal{O}(1)$ vanishing at ∞ at x and at points $\neq \infty$ at all points $\neq x$.

[redacted] Summary: If K is a vector bundle on $\mathbb{P}^1$ of degree -2 [redacted] embeddable in $\mathcal{O}^2$, then $K \simeq \mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. We have calculated the variety of unimodular subspaces in $\Gamma(K(2))$. Now suppose we have a family of such bundles. Is it possible to [redacted] find a family of unimodular subspaces if the parameter scheme is affine. [redacted]

~~Simpler question: Is it possible to find a~~

Over $Q_2^{(2)}(\mathbb{P}^1)$ we have the canonical vector bundle $K \mapsto \Gamma(K(2))$ which I denote E. It is of rank 4. Inside of $\mathrm{Grass}_2(E)$ we have the set Z of unimodular subspaces.

Conjecture: There is a subscheme Z of $\mathrm{Grass}_2(E)$ such that maps $X \longrightarrow Z$ are the same as families $X \longrightarrow Q_2^{(2)}(\mathbb{P}^1)$ $\quad x \longmapsto K(x)$ together with a choice of unimodular subspace:

$$\mathcal{O} \otimes V(x) \hookrightarrow K(x)(2)$$

January 13, 1975

Summary: $X = Q_2^{(2)}(\mathbb{P}^1)$ is the scheme of quotients M of $\mathcal{O}^2_{\mathbb{P}^1}$ of length d. On $X \times \mathbb{P}^1$ we have a canonical exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{M}$ is a 2-dimensional bundle over X. The fibres K of $\mathcal{K}$ over points of X are isom. to $\mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$, and I know I can find unimodular subspaces for K inside $\Gamma(\mathbb{P}^1, K(2))$. Let $\mathcal{E} = p_*(\mathcal{K}(2))$, $p : X \times \mathbb{P}^1 \longrightarrow X$. Then $\mathcal{E}$ is a rank 4 bundle over X. I would like to show that the unimodular subspaces of $\mathrm{Grass}_2(\mathcal{E})$ form a subscheme Y of $\mathrm{Grass}_2(\mathcal{E})$ representing the obvious functor.

Suppose I just look at the open set of X where some point, say $t = 1$, is not in the support of M. Then I can look at all maps $\mathcal{O}(-2)^2 \longrightarrow \mathcal{O}^2$ reducing to the identity at $t = 1$. Then we put down also the condition that this map have rank ≤ 2 when tensored with $\mathcal{O}_\infty / m_\infty^2$, and with $\mathcal{O}_0 / m_0^2$.

In fact suppose I consider inside $\Gamma(\mathcal{O}(2))^2$ which is six dimensional, all those 2 dimensional subspaces V such that the map

$$\mathcal{O} \otimes V \longrightarrow \mathcal{O}(2)^2$$

has at ∞ cokernel of dimension 2

$t^{-2}v_1, t^{-2}v_2$

This means that $v_1, v_2, t^{-1}v_1, t^{-1}v_2$ generate a 4 dimensional subspace of $(\mathcal{O}(2)/m_\infty^3)^2$, where $V = \mathbb{C}v_1 + \mathbb{C}v_2$. This condition defines a locally closed subvariety of the Grassmannian whose points consist of subbundles $K \subset \mathcal{O}^2$ with $K \simeq \mathcal{O}(-2)^2$ such that $\mathcal{O}^2/K$ has dim 2 at ∞ and dim 2 off ∞. Is it not true that this variety is the same as a point of $Q_2^{(2)}(A^1)$ plus a choice of unimodular subspace?

Question: Let E be a bundle on $\mathbb{P}^1$, say $\mathcal{O} \oplus \mathcal{O}(2)$. Classify all unimodular subspaces of $\Gamma(E(m))$ for m large. Are these varieties contractible?

I've seen that if $E = \mathcal{O}^2$ then the unimodular space in $\Gamma(\mathcal{O}(m))^2$ can be identified with polyn. matrices

$$I + tA_1 + \cdots + t^m A_m$$

which are invertible, ~~which is equivalent to the matrices $A_1, \dots, A_m$ being strictly upper triangular for some flag. (?)~~ Canically contractible.

Now given $E = \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n)$ $\quad p_1 \leq \cdots \leq p_n$ it embeds in $\mathcal{O}(p_n)^n$ with cokernel at ∞, so a similar description of unimodular subspaces is possible except there are degree conditions on the entries of the matrix. Again conically contractible.

Proof: For $B = I + tA_1 + \dots + t^m A_m$ to be contractible it is necessary & sufficient that $\det(B) = 1$. But then $B_\lambda = I + t\lambda A_1 + \dots + t^m \lambda^m A_m = B(\lambda t)$ is also contractible for all $\lambda \in \mathbb{A}^1$.

So we consider the product P of $Q_d^{(2)}(\mathbb{A}^1)$ with the space of lattices containing $\mathcal{O}^2_\infty$ of index-d. There is a canonical vector bundle of degree 0 on $P \times \mathbb{P}^1$. The subspace of P where the bundle E over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}^2$, can be described as the place where $H^1(E(-1)) = 0$, hence it is open in P. Call this open set $\mathcal{U}$. Then $\mathcal{U}$ is flat over $Q_d^{(2)}(\mathbb{A}^1)$. A point of $\mathcal{U}$ is a bundle $K \subset \mathcal{O}^2$ in $Q_d^{(2)}(\mathbb{A}^1)$ together with an extension of K to a "regular" bundle of degree 0 on $\mathbb{P}^1$. The question is whether given a map $X \longrightarrow Q_d^{(2)}(\mathbb{A}^1)$ with X affine, does $\exists$ a lifting to $\mathcal{U}$?

Let K be a bundle on $\mathbb{P}^1$. To give a rational map $\mathcal{O}^2 \longrightarrow K$ which is an isomorphism off ∞ means that we first find a bundle K' agreeing with K off ∞ and such that K' is isom. to $\mathcal{O}^2$ and then we choose an isomorphism of K' and $\mathcal{O}^2$. Thus if we divide out by the action of GL_2, we see that a unimodular subspace of $\Gamma(\mathbb{A}^1, K)$ is the same as a lattice for O_∞ commensurable with K_∞ such that the resulting bundle K' is of degree 0 and regular. [redacted] In my problems K appears as a subbundle of O^2 with no support at ∞, hence [redacted] perhaps it is natural that $K' \supset K$.

So what's happening is this. Given a bundle E over $\mathbb{P}^1$ we [redacted] twist it to make it nice, then we consider all subbundles [redacted] all $E' \subset E$... such that (i) E/E' supported at ∞ (ii) $E' \simeq O^2$, i.e. $\deg(E')=0$ and [redacted] $H^1(\mathbb{P}^1, E(-1))=0$. [redacted] This set of E' is an open subscheme of the scheme of lattices of given index in E at [redacted] ∞.

Recall the following yoga. Given a filtered ring A with increasing filtration

$$F_0A \subset F_1A \subset F_2A \subset \cdots\cdots$$

such that $F_pA \cdot F_qA \subset F_{p+q}A$ and $gr(A)$ is a poly ring we were able to prove a homotopy [redacted] property. Can I use this idea here. Such a filtered ring is simply an affine space bundle over A, when [redacted] it is commutative.