Notes on buildings, symmetric spaces, etc.

Fifth part: $B^n_k$-orbits on $p$, Bruhat decompositions, compactifying Schubert cells.

To fix the ideas, suppose $G$ is the complexification of the compact connected group $K$, and put $p = iK$ as usual. Let $\xi \in p$. We propose to determine the $B^n_k$-orbit structure of $p$. For this purpose we consider the flow $\eta_t \mapsto e^{t\xi} \eta_t$ in $p$.

Suppose $e^{t\xi} \eta_t = \eta_t$ for all $t$, i.e. $e^{t\xi} \in B^n_k$. As $e^{t\xi} \in P$ and $P \cap B_k = P_\xi$ (part 4, p. 18), this happens iff $[\xi, \eta] = 0$. Thus $P_\xi$ is the set of fixed points for the flow $\eta_t \mapsto e^{t\xi} \eta_t$.

Let's compute the tangent vector to the curve $e^{t\xi} \eta_t$ at $t=0$. Recall the $G$-action on $K\eta$ is determined by the isomorphism $K/K_\eta \cong G/B_k$. Thus we take the image of $\xi$ in $g/B_k \cong k/k_k$, and apply it to $\eta$ to get what we are after. Let $\alpha$ be a maximal abelian subspace of $p$ containing $\xi$ and

$$g = g_\alpha + \sum_{\alpha \neq \xi} g_\alpha$$

the corresponding root decomposition. Note $\Theta g_\alpha = g_{-\alpha}$.

If $\xi = \xi_0 + \sum \xi_\alpha$ is the corresponding decomposition of $\xi$, then $\Theta \xi_0 = -\xi_0$, $\Theta \xi_\alpha = -\xi_{-\alpha}$.

Hence
\[ y = \sum_{\alpha(\gamma) > 0} (-\xi_{\alpha} + \xi_{-\alpha}) \]

is in \( \mathcal{P} \) and \( \xi - y \in \mathcal{P}_{\alpha} + \sum_{\alpha(\gamma) > 0} q_{\alpha} = \mathcal{P} \). Consequently, the tangent vector to \( e^{t\xi} \times y \) at \( t = 0 \) is

\[ [\xi, y] = \sum_{\alpha(\gamma) > 0} (-\alpha(\gamma) \xi_{\alpha} - \alpha(\gamma) \xi_{-\alpha}) \]

Thus

\[ \frac{d}{dt} \bigg|_{t=0} (e^{t\xi} \times y, \xi) = 1 - \sum_{\alpha(\gamma) > 0} \alpha(\gamma) (|\xi_{\alpha}|^2 + |\xi_{-\alpha}|^2) \leq 0 \]

with equality if \( [\xi, y] = 0 \). Since

\[ |e^{t\xi} \times y - \xi| = |y|^2 + |\xi|^2 - 2(e^{t\xi} \times y, \xi) \]

this means that the distance from \( e^{t\xi} \times y \) to \( \xi \) is strictly increasing in \( t \) - provided \( [\xi, y] \neq 0 \):

**Prop. 1:** (i.e. \( \eta \) is not stationary for the flow \( e^{t\xi} \times \cdot \)), then the distance from \( e^{t\xi} \times y \) to \( \xi \) is strictly increasing in \( t \).

We apply this result as follows. Let \( y' \in \mathcal{P} \) be such that \( y' \notin \mathcal{P}_{\xi} \), and let \( S \) be the set of limit points of the curve \( e^{t\xi} \times y' \) as \( t \to -\infty \). \( S \) is ma-
empty because \( D = G \times \eta' \) is compact. Let \( f \) be the function on \( D \) giving the distance to \( \eta \).

Because \( t \mapsto f(e^{t\xi} \ast \eta') \) is strictly increasing and bounded, it has a limit \( L \) at \( t \to -\infty \), hence \( f(S) = L \).

But \( S \) is stable under the flow \( e^{t\xi} \ast (\cdot) \), so as \( f \) is constant on \( S \) we can conclude \( S \subset f \) from the proposition. Thus we have proved that there exists a point \( \eta \) in \( f \) which is a limit point of \( e^{t\xi} \ast \eta' \) as \( t \to -\infty \).

I next want to describe a nbd. of \( \eta \) in \( G \eta \).

As \( \eta = b^{-1} \eta' \), the function theorem tells me that the map \( B_u^{-1} \to G/B_\eta \cong G \eta \), \( g \mapsto g \cdot \eta \) is a diffeomorphism at the identity. Hence \( B_u^{-1} \) is open in \( G \).

Now if \( g \in B_u^{-1} \cap B_\eta \), then \( e^{-t} \eta \cdot g \eta \) converges at \( t \to +\infty \) and converges to 1 as \( t \to -\infty \). Since \( \eta \) is a matrix whose entries are linear combinations of real exponentials \( e^{-t} \eta \) is constant in \( t \) so \( g = 1 \). Thus we have:

**Prop. 2.** The map \( B_u^{-1} \to G \eta \), \( g \mapsto g \eta \) is a diffeomorphism of \( B_u^{-1} \) onto an open nbd. of \( \eta \) in \( G \eta \).
In virtue of this result and the differentials, exp: $b^u_\eta \rightarrow B^u_\eta$, we therefore can study what's going on near $\eta$ in $G\eta$ using $b^u_\eta$. Furthermore, $G\eta = G/B_\eta$ with $G$ acting by left multiplication.

As $t^\xi$ commutes with $\eta$, the flow $e^{t^\xi}$ normalizes $B^-\eta$, $B_\eta$, and opens up the open set $B^u_\eta$. Thus we get an isomorphism

$$b^u_\eta \sim B^u_\eta \subset G\eta$$

Compatible with the adjoint action of $e^{t^\xi}$ on $b^u_\eta$ and the multiplication action in $G\eta$. However, the action of $Ad(e^{t^\xi})$ on $b^u_\eta$ is simple to analyze in terms of the eigenvalues of $Ad(\xi)$. One sees that only the elements of $b_\eta \cap b^u_\eta$ have a limit point under $Ad(e^{t^\xi})$ as $t \rightarrow -\infty$, and that this limit point is a unique element of $G\eta \cap b^u_\eta$. In particular, the points of $b^u_\eta$ having $0$ as limit point under $Ad(e^{t^\xi})$ as $t \rightarrow -\infty$ are just points of $b_\eta \cap b^u_\eta$.

So we see that if $\eta'$ is a limit point of $e^{t^\xi} \cdot \eta$ as $t \rightarrow -\infty$, then in fact $e^{t^\xi} \cdot \eta'$ converges to $\eta$. Moreover, the set of $\eta'$ with limit $\eta$ is isomorphic to $b_\eta \cap b^u_\eta$. Consider the inclusions:

$$b_\eta \cap b^u_\eta \rightarrow B^u_\eta \cap B^-_\eta \subset B^u_\eta \subset G\eta.$$
If \( g \in B_3^u \), then \( e^{t \xi} (g \times \eta) = (e^{t \xi} g e^{-t \xi}) \times \eta \rightarrow \eta \) as \( t \rightarrow -\infty \). Thus we can conclude that the first two inclusions are isos.

\[ \circ \quad \text{so we obtain:} \]

**Theorem:** Each \( B_3^u \)-orbit on \( \mathfrak{p} \) contains a unique element of \( \mathfrak{p}_3^u \):

\[ \mathfrak{p}_3^u \sim \rightarrow B_3^u \backslash \mathfrak{p} \]

Specifically, given \( \eta' \in \mathfrak{p} \), the limit \( \eta = \lim_{t \rightarrow -\infty} e^{t \xi} \eta' \) exists and is the unique element of \( B_3^u \eta' \cap \mathfrak{p}_3^u \).

If \( \eta \in \mathfrak{p}_3^u \), then one has isos:

\[ B_3^u \cap B_3^u \xrightarrow{\exp} B_3^u \ni \eta \xrightarrow{\gamma} B_3^u \eta \]

**Additions:** The vector field on \( G \eta \) induced by the flow \( e^{t \xi} \) is not the same as the gradient of the distance-squared-to-\( \xi \) function. However, this vector field plays the same role as \( \nabla \) in Morse theory.
Bruhat decomposition: Take \( \xi \) to be a regular element of \( F \), i.e. \( F_{\xi} \) is a maximal abelian subspace of \( F \). I know from parts I and II that \( K_{\eta} \cap F_{\xi} \) is a \( W \)-orbit in \( F_{\xi} \). If \( \eta \in F_{\xi} \), then we have
\[
G_{\eta} = K_{\eta} = \bigcap_{\omega \eta \in W_{\eta}} B_{\xi}^u \omega \eta
\]
Since \( B_{\xi} = G_{\xi} \times B_{\xi}^u \), where \( G_{\xi} \) acts trivially on \( F_{\xi} \), we therefore get
\( B_{\xi} \omega \eta = B_{\xi}^u G_{\xi} \omega \eta = B_{\xi}^u \omega G_{\xi} \eta = B_{\xi}^u \omega \eta \), so
\[
G/B_{\eta} = \bigcap_{\omega \eta \in W_{\eta}} B_{\xi}^u \omega \eta / B_{\eta}
\]
or
\[
G = \bigcap_{\omega \eta \in W_{\eta}} B_{\xi} \omega \eta
\]
Furthermore, I know that \( B_{\xi} \omega \eta = B_{\xi}^u \cap B_{\xi}^u \omega \eta \), hence
\[
B_{\xi} \omega \eta B_{\xi} \equiv (B_{\xi}^u \cap B_{\xi}^u) \times B_{\eta}
\]
The Bruhat decomposition is the special case \( \eta = \xi \): \( B \big/ G/B \cong W \).
Therefore the \( B_{\xi} \)-orbits (= \( B_{\xi}^u \)-orbits) on \( G_{\eta} = B/B_{\eta} \) are indexed by points of \( W_{\eta} \). Each orbit is a cell; the orbit \( B_{\xi} \omega \eta \) is isomorphic to the vector space \( B_{\xi}^u \cap B_{\xi}^u \). Its dimension is the number of hyperplanes \( \xi \cap -\omega \eta \) (counted with multiplicity) crossed in going from \( \xi \) to \( \omega \eta \) in \( F_{\xi} \).
The preceding applies to the following more general situation: G is the group of real points of a real algebraic group arising, as in part III, p. 18, from a compact connected Lie group with involution; \( K \) is the associated maximal compact subgroup and \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) the Cartan decomposition. \( \xi \) is a regular element in \( \mathfrak{p} \), \( \alpha = \mathfrak{p} \xi \), \( C \) = the chamber of \( \xi \) containing \( \mathfrak{g} \), \( B = B_\xi \), \( N \) = normalizer of \( \alpha \) in \( G \), \( Z = Z_\xi \) = centralizer, \( W = N/Z \) the Weyl group.

The theorem on page 5 says

\[ \alpha \mapsto B \backslash \mathfrak{p} \]

(\( \alpha \) is a fundamental domain for the \( B \)-action), and it gives a description of the \( B \)-orbits.

Now I recall that \( C \mapsto G \backslash \mathfrak{p} \), and that the stabilizers \( B_\gamma \) are constant over the strata of \( C \). This \( \alpha \) saw gave rise to a stratification of \( \mathfrak{p} \), preserved by the \( G \)-action. If \( L \) is a stratum of \( \mathfrak{p} \), and we pick \( \gamma \in L \), and \( b \in B \) such that \( b \gamma \in C \), then \( bL \) is a stratum of \( C \), namely the stratum containing \( b\gamma \). This \( bL \) coincides with the stratum of \( \alpha \) containing \( b\gamma \), and multiplication by \( b \) sets up an isomorphism of the strata \( L \) and \( bL \).

**Prop. 3:** The projection \( p : \mathfrak{p} \rightarrow \alpha \) defined by (*) preserves strata. It preserves the
specialization relation between strata.

(For the last statement, suppose \( L \subseteq L' \) and \( bL' \subseteq c \). Then by continuity \( bL \subseteq bL' \subseteq cL \subseteq c \), so \( p(L) \subseteq p(L') \).

Let \( S \subseteq W \) be the set of reflections through the walls of \( C \). If \( s \in S \), let \( C_s \) be the corresponding "panel" of \( C \), and let \( B_s \) be its stabilizer.

Let \( C' \) be a chamber of \( p \) containing \( C_s \), and choose \( g \in G \) such that \( gC = C' \). Then \( g^{-1}C_s = g^{-1}C' = C \), and \( C_s \subseteq C' \) and as \( C \) is a G.fund. domain, \( g \in B_s \). Thus

\[ B_s / B \cong \text{chambers of } p \text{ containing } C_s. \]

Next with \( C' \) as above consider \( pc' \). It is a chamber of \( p \) containing \( C_s \). As \( C_s \) is a panel of \( C \), this means either \( pc' = C \) or \( pc' = sC \). Thus there are 2 \( B \)-orbits on \( B_s / B \) and we get:

\[ B_s = B \cup BsB. \]

Fix a generic element \( \gamma \) of \( C_s \) so that \( \gamma = \beta \). We know that any element of \( K \gamma \) can be moved into or by an element of \( K \gamma \).

If \( C' \) is a chamber of \( p \) containing \( C_s \), and if \( \beta' \) is the point of \( C' \) \( G \)-conjugate to \( \beta \), then...
\[ \xi' \text{ is } K_\eta \text{ conjugate. This shows that } \]
\[ B_s / B \cong K_\eta \xi' \cong K_\eta / K_\xi. \]

But now form inside \( p_\xi \), the sphere \( S_\xi \) obtained by rotating \( \xi \) around the wall \( H_s \), around which \( S_\xi \) reflects.

![Sphere diagram]

\[ H_s \]

\( K_\eta \) preserves \( S_\xi \), so \( S_\xi \) is a union of \( K_\eta \)-orbits. Working in the group \( G_\eta \), we know that each \( K_\eta \)-orbit on \( p_\eta \) intersects or in an orbit for the Weyl group, which in the case of \( G_\eta \) is reflection thru \( H_s \). Thus \( K_\eta \) is transitive on this sphere. So it's clear now that we get:

**Prop. 1:** \( B_s / B \) is a sphere. It is the 1-point compactification of

\[ B_s B / B \cong B^n s B^n s^{-1} = B^n s B^n s^{-1} \]

(where \( B_- = B_{-\eta} \)).

Let us fix an element \( \eta \) of \( \alpha \). We know the orbit \( B_\eta \) is a cell in \( G_\eta \) for it is isomorphic to \( B^n s B^n s^{-1} \)

What we want to do now is to show the...
Embedding $B_Y \subset G_Y$ extends to a map of a closed disk with interior $= B^Y$. This will show that the $B$-orbits of $G_Y$ are the cells for a CW complex structure. At the same time we will determine the closure of $B_Y$ in $G_Y$.

To this end, let us choose a gallery in or

$$C, s_1 C, \ldots, s_i \ldots s_n C, \quad s_i \in S,$$

such that $\eta \in s_i \ldots s_n C$. We denote by $\Gamma(s_1, \ldots, s_n)$ the set of galleries $C_0, C_1, \ldots, C_n$ in $P$ such that $C_0 = C$ and such that $C_{i-1}$ and $C_i$ have their $s_i$-th panel in common. If we choose $g_i \in G$, $i = 1, \ldots, n$, such that $C_i = g_1 \cdots g_i C$, then this condition means that $C_0 = g_1^{-1} \cdots g_i^{-1} C_{i-1}$ and $g_1^{-1} \cdots g_i^{-1} C_i = g_i C$ have the $s_i$-th panel in common. Thus, one gets an isomorphism

$$B_{s_1} \times \cdots \times B_{s_n} / B \cong \Gamma(s_1, \ldots, s_n)$$

$$(g_1, \ldots, g_n) \mapsto (C_0, g_1 C, \ldots, g_n C).$$

Let $\eta_0$ be the point in $C$ such that $\eta = s_1 \cdots s_n \eta_0$, and let

$$\pi : \Gamma(s_1, \ldots, s_n) \rightarrow G_Y$$

be the map sending $(g_1, \ldots, g_n)$ to $g_1 \cdots g_n \eta_0 = \text{the point of } C = g_1 \cdots g_n C \rightarrow \text{conjugate to } \eta$. 

If \( G = (C_0, \ldots, C_n) \in \Gamma(s_1, \ldots, s_n) \), let \( pG = (pC_0, \ldots, pC_n) \) be the image of \( G \) under the projection \( p: F \to \sigma \).

Then \( pG \) is a gallery in \( \sigma \), with \( pC_0 = C \), and \( pC \) have the \( s_i-th \) panel in common. Thus \( pC_i = s_i \ldots s_i' C \), where \( s_i' = s_i \) or \( 1 \). Hence \( p\Phi(G) = \prod s' \ldots s_n \eta_0 \).

Note that \( B \) acts on \( \Gamma(s_1, \ldots, s_n) \) by left multiplication, and \( \Phi \) is compatible with this action, so \( \text{Im } \Phi \) is a union of \( B \)-orbits. Thus we have:

**Prop. 5:** The image of \( \Phi: \Gamma(s_1, \ldots, s_n) \to G \prod s' \ldots s_n \eta_0 \)

is the union of the \( B \)-orbits \( BS(s' \ldots s_n \eta_0) \), where \( (s_1', \ldots, s_n') \) runs over all sequences with \( s_i' = s_i \text{ or } 1 \).

In other terms,

\[ \text{Im } \Phi = \bigcup_{1<s'<s_n} BS(s' \ldots s_n \eta_0) \bigcup_{0<s'<s_n} BS(s' \ldots s_n \eta_0) \bigcup_{1<s'<s_n} BS(s' \ldots s_n \eta_0) \]

Before going on, I need something which should have been developed earlier before starting galleries.

Let \( \eta \in \sigma_i \). We define the length of \( \eta \) with respect to \( C \) to be the number of hyperplanes separating \( \eta \) from \( C_i \). Denote it \( \ell_C(\eta) \). The length of an arbitrary point of \( p \) (with respect to \( C \)) is defined to be the length of its image \( p \) in \( \sigma \). Clearly the length function is constant on each stratum of \( p \), and on each \( B \)-orbit.
Let $\eta \in \mathfrak{o}$, and let $i$ be an interior point of $\mathfrak{c}$. The element $\eta + \varepsilon (i - \eta)$ is regular for $\varepsilon$ small and $\eta > 0$; moreover any hyperplane separating $\eta$ and $\mathfrak{c}$ separates $\eta + \varepsilon (i - \eta)$ for $\varepsilon$ small, and conversely. Let $wC$ be the chamber containing $\eta + \varepsilon (i - \eta)$; then $\eta \in wC$ and $l_C(\eta) = l_C(wC)$.

Note that $wC$ is the unique chamber of $\mathfrak{o}$ containing $\eta$ of the same length as $\eta$. Since every hyperplane separating $\mathfrak{c}$ and $\eta$ automatically separates $\mathfrak{c}$ and $wC$, it follows that the root hyperplanes separating $\mathfrak{c}$ and $wC$ are the same as those separating $\mathfrak{c}$ and $\eta$. Thus $wC = wC$ for $w \neq wC$, then there would exist a root hyp. separating these two and $\eta$ would be on one side of it. Therefore $wC$ is the unique chamber of $\mathfrak{o}$ containing $\eta$ of the same length as $\eta$. From part I we know that $n = l_C(\eta) = l_C(wC)$ is the length of $w$ a reduced decomposition of $w$. Thus we can also describe $l_C(\eta)$ as the least $n$ such that $\exists s_1, \ldots, s_n \in S$ with $\eta \in s_1 \cdots s_n \mathfrak{c}$.

Next, let us compare the B-orbits of $wC$ and $\eta$. According to the above theorem, there is an evident map $B \cdot wC \to B \cdot \eta$ which associates to the chamber $B \cdot wC$ the unique point of this chamber.
which is conjugate to \( \eta \), namely \( \eta \). According to the previous theorem, \( B \eta \) is acted on simply-transitively by the group \( B \cap B^{-\eta} \), which is isomorphic to \( B^{-\eta} \). Also \( B^{-\eta} = \sum y^{n} \) where \( \alpha \) ranges over roots with \( \alpha(\zeta) > 0 \), \( \alpha(\eta) < 0 \). It follows from the nature of \( \omega \) that \( B^{-\eta} \cap B^{-\eta} = B^{-1} B^{-\omega \zeta} \), hence \( B^{-\eta} \cap B^{-\eta} = B^{-1} B^{-\omega \zeta} \), hence \( B \omega C \sim B \eta \).

So we have shown that any point \( \eta \) of \( \omega \) is contained in a unique chamber \( \omega C \) of \( \omega \) with \( \ell(\omega C) = \ell_{\omega}(\eta) \), and that \( B \omega C \sim B \eta \). If \( \eta \) is any element of \( B \eta \), say \( \eta = b \eta \), then \( b \omega C \) is a chamber of \( \omega \) containing \( \eta \) of the same length as \( \eta \). Let \( C' \) be another chamber such that \( \eta \in C' \), \( \ell(\eta) = \ell(\zeta) \). Let \( C' = b_{1} p C' \). Then \( p C' \) is a chamber of \( \omega \) containing \( \eta \) of the same length as \( \eta \). Thus we have \( p C' = \omega C \). But also we have \( b_{1} \eta \in C' \) is conjugate to \( b \eta = \zeta \), hence \( b_{1} \eta = b \eta \). So from \( B \omega C = B \eta \), we conclude that \( C' = b_{1} \omega C = b_{1} \omega C \), which shows \( b_{1} \omega C \) is the unique chamber containing \( \eta \) of the same length.

Thus we have proved:

\textbf{Prop. 6}: Any point \( \eta \) of \( \omega \) is contained in a unique chamber \( C \) with the same length with
Respect to \( C \) as \( I \). Furthermore \( B.C_i \sim B.J \).

Now let us return to the map \( \Phi: \Gamma(s_1, \ldots, s_n) \to \Gamma(\eta) \), and let us assume that \( n = l_c(\eta) \), that is, that \( C, s_i C, \ldots, s_{n-1} C \) is a minimum gallery with \( \eta \in s_i \cdots s_n C \). Denote by \( \Gamma(s_1, \ldots, s_n)^* \) the subset of \( \Gamma(s_1, \ldots, s_n) \) consisting of galleries \( (C_0, \ldots, C_n) \) without (immediate) repetitions i.e. \( C_{i+1} \neq C_i \). Let \((C_0, \ldots, C_n) \in \Gamma(s_1, \ldots, s_n)^* \) and let \((C, s'_1 C, \ldots, s'_i \cdots s'_n C)\) be its projection in \( \eta \).

We show \( s'_i = s_i \) by induction on \( i \). Assuming true for \( j < i \), suppose \( s'_i = 1 \). Then \( C_{i-1}, C_i \) project to \( s_1 \cdots s_{i-1} C \), hence they have the same length.

Moreover \( C_{i-1}, C_i \) have the same \( s_i \)-th panel \((= g_{i-1} \cdots g_1 C_i)\) if \( C_{i-1} = g_{i-1} \cdots g_1 C_i \), which projects to \( s_1 \cdots s_{i-1} C_i \). But because of the minimality of the gallery \((C, \ldots, s_i \cdots s_n C)\), it is clear that \( l_c(s_1 \cdots s_i C) = l_c(s_1 \cdots s_{i-1} C_i) = i-1 \). Thus \( C_{i-1}, C_i \) are two chambers containing a common panel and of the same length as this panel. By Prop. 6 \( C_{i-1} = C_i \), which contradicts \((C_0, \ldots, C_n) \in \Gamma(s_1, \ldots, s_n)^* \). Thus \( s'_i = s_i \), and so we have shown that \((C_0, \ldots, C_n)\) projects to \((C, \ldots, s_i \cdots s_n C)\). In particular, \( \Phi(C_0, \ldots, C_n) \in B.J \).
Because $B$ acts on $\Gamma(s_1, \ldots, s_n)^*$, it is clear that $\Phi$ maps $\Gamma(s_1, \ldots, s_n)^*$ into $B\eta$. In fact the map is 1-1 for if $\Phi(c_0, \ldots, c_n) = b\eta$, then $C_n$ is the chambre containing $b\eta$ of the same length, and $C_{n-1}$ is the chambre containing the $s_n$-th face of $C_n$ and of the same length as this face, etc. So we've proved:

**Prop. 7:** Let $c \in C$ have length $n$ w.r.t. $C$ and let $\gamma \in \Gamma(s_1, \ldots, s_n)$. Then the map $\Phi$ induces a bijection:

$$\Gamma(s_1, \ldots, s_n)^* \xrightarrow{\sim} B\eta$$

Let us now consider $\Gamma(s_1, \ldots, s_n) = B_{s_1} \times B_{s_2} \times \cdots \times B_{s_n} / B$ as a space. According to [Prop. 4], $B_{s_i} / B$ is a sphere with 2 $B$-orbits: a point and $B_1 B / B$. Thus $\Gamma(s_1, \ldots, s_n)$ is a sphere bundle over $\Gamma(s_1, \ldots, s_{n-1})$ having a distinguished section. The image of this section is the set of galleries $(c_0, c_1, \ldots, c_n)$ such that $C_{n-1} = C_n$; let's denote it $Z_n$.

Let $Z_i$ be the subspace of $\Gamma(s_1, \ldots, s_n)$ consisting of galleries with $C_{i-1} = C_i$, $i = 1, \ldots, n$. We have
a tower

\[ \Gamma(s_1, \ldots, s_n) \rightarrow \Gamma(s_1, \ldots, s_{n-1}) \rightarrow \cdots \rightarrow \Gamma(s_1) \rightarrow \text{pt} \]

such that each map is a fibre bundle projection with fibre a sphere and having a distinguished section. \( Z_i \) is the inverse image in \( \Gamma(s_1, \ldots, s_n) \) of the distinguished section of \( \Gamma(s_1, \ldots, s_i) \) over \( \Gamma(s_1, \ldots, s_{i-1}) \). \( Z_i \) is a submanifold of \( \Gamma(s_1, \ldots, s_n) \) of codimension equal to \( d(s_i) = \dim \left( B_{s_i} B / B \right) \). The \( Z_i \) intersect transversally and

\[ \Gamma(s_1, \ldots, s_n) \times \Gamma(s_1, \ldots, s_n) \rightarrow U \bigcup \Gamma(s_1, \ldots, s_n) - Z_i \]

Thus \( \Gamma(s_1, \ldots, s_n) \) is an open dense subset of \( \bigcap \Gamma(s_1, \ldots, s_n) - Z_i \).

In the "complex case" i.e. \( G = \text{complexification} \) of \( K \), I know that \( B_{s_i} B / B \) is \( \cong \mathbb{C}P^1 \) (compare Part I, p. 21), hence \( U Z_i \) is a divisor with normal crossings.

Put \( d(s_i) = \dim \left( B_{s_i} B / B \right) \). This is the dimension of \( \sum \chi_{\alpha} \) where \( \chi \) ranges over the roots with \( \chi(s_i) > 0 \) and \( \chi | H_{s_i} = 0 \). (Recall such roots are proportional, there are at most two such roots \( \chi, 2\chi \).

The dimension of \( \Gamma(s_1, \ldots, s_n) \) is

\[ d = \sum_{i=1}^{n} d(s_i) . \]

**Prop. 8:** There exists a continuous map \( \psi : D^d \rightarrow \Gamma(s_1, \ldots, s_n) \), where \( D^d \) is a disk of dimension \( d \) such that \( \psi(\partial D^d) \subset U Z_i \) and such that \( \psi \) induces a homeomorphism of \( \text{Int}(D^d) \) with \( \Gamma(s_1, \ldots, s_n) \).
(In other words, $(\Gamma(s_1, \ldots, s_n), U_{\mathbb{Z}})$ is a relative cell of dimension $d$.)

This will be proved by induction on $n$. To handle the induction step, consider the following situation. Let $E$ be a fibre bundle over a space $X$ with fibre $S^a$, and let $Z \subset E$ be the image of a section of $E$ over $X$.

**Lemma.** Let $E$ be a fibre bundle over $X$ with fibre $S^a$ and let $Z \subset E$ be the image of a section of $E$. Let $Y$ be a closed subspace of $X$ such that $(X, Y)$ is a relative cell of dimension $b$. (If $\varphi : D^b \to X$ with $\varphi(D^b) \subset Y$ and $\text{Int}(D^b) \cap (X-Y)$.) Then $(E, Z \cup E_Y)$ is a relative cell of dimension $a+b$.

The prop. 8 is proved by induction on $n$ using the lemma in the induction step with $E = \Gamma(s_1, \ldots, s_n)$, $Z = Z_n$, $X = \Gamma(s_1, \ldots, s_{n-1})$, $Y = \Gamma(s_1, \ldots, s_{n-1}^*) - \Gamma(s_1, \ldots, s_{n-1})^*$.

To prove the lemma, note that $E$ is associated to a principal bundle for the group of homeom. of $S^a$ having a basepoint $o$ fixed. Because $D^b$ is contractible, this principal bundle becomes trivial when pulled back to $D^b$ via $\varphi$, hence we get a homeomorphism

$\theta : \varphi^* (E, Z) \leftarrow D^b \times (S^a \cup o)$
over $D^b$. Let $\lambda : D^a \to S^a$ be the standard map such that $\lambda(\partial D^a) = \infty$, $\lambda : \text{Int}(D^a) \to S^a - \infty$.

Let $\psi$ be the map

$$D^{a+b} \cong D^b \times D^a \xrightarrow{\text{id} \times \lambda} D^b \times S^a \xrightarrow{\psi} \varphi^*E \xrightarrow{p_2} E.$$ 

Then

$$\psi(\partial D^{a+b}) = \psi(\partial D^b \times D^a) \cup \psi(D^b \cup \partial D^a) \subset E_y \cup Z$$

and $\psi$ restricted to $\text{Int}(D^{a+b})$ is the homeomorphism

$$\text{Int}(D^a) \times \text{Int}(D^b) \cong \text{Int}(D^b) \times (S^a - \infty) \cong \varphi^*(E - Z)_{\text{Int}(D^b)}$$

$$\cong (E - Z)_{x = y} = E - (Z \cup E_y),$$

which proves the lemma.

Now let's put together the previous five or so propositions. We have seen that the orbit $G\eta_0 = G/B \eta_0 / \eta_0$ is a union of orbits

$$G\eta_0 = \bigsqcup_{\omega \in W_{\eta_0}} B \omega \eta_0,$$

and that each orbit is a cell. Here I suppose $\eta_0 \in C$. Look at the orbit $B\eta$ where $\eta = w_0 \eta_0$.

We can suppose $w$ chosen so that the length $l_c(wC)$ equals $l_c(\eta)$ (Note: $l_c(wC) = l_c(\omega)$ in the standard notation). Let $w = s_1 \cdots s_n$, $n = l_c(\eta)$ be a reduced decomposition, and form the map

$$E : \Gamma(s_1, \cdots, s_n) \to G\eta_0.$$
By propositions 7 and 8 we get a map
\[ D^d \xrightarrow{\varphi} \Gamma(s_1, \ldots, s_n) \xrightarrow{\tilde{\psi}} G\eta_0 \]
which induces a bijection:

\[ \text{Int}(D^d) \sim \Gamma(s_1, \ldots, s_n)^* \sim B\eta. \]

As \( \tilde{\psi}(D^d) \) is compact, it is the quotient space of \( D^d \) by the equivalence relation defined by the map \( \tilde{\psi} \). Since the equivalence classes are points on \( \text{Int}(D^d) \), we see that \( \Gamma(s_1, \ldots, s_n)^* \) in the nbd. of points of \( \text{Int}(D^d) \), \( \tilde{\psi} \) is a homeomorphism. Thus \( \sim \) is a homeomorphism. This proves that the cell decomposition of \( G\eta_0 \), given by the \( B \)-orbits, is a CW decomposition. Furthermore, because \( \Gamma(s_1, \ldots, s_n)^* \) is dense in \( \Gamma(s_1, \ldots, s_n) \), Prop. 5 gives the closure of \( B\eta \). So we get:

**Theorem:** The orbit \( G\eta_0 \sim G/B\eta_0 \) is a CW complex with cells \( B\eta_0 W \), \( \eta_0 \in W/W_\eta_0 \). If \( w = s_1 \cdots s_n \) with \( n = l_0(w\eta_0) \), then

\[ \overline{B\eta_0 w} = \bigcup B s_{i_1} \cdots s_{i_p} \eta_0 \]

1 \( \leq i_1 < \cdots < i_p \leq s_n \)

0 \( \leq p \leq n \).