Fourth part: G-action on \( p \), Iwasawa decomposition.

Let \( K \) be a compact Lie group, \( G \) its complexification, and let \( k, g \) be their Lie algebras, so that \( g = k + i k \). We write \( p \) for the subspace of \( g \) on which \( \theta = -1 \). \( K \) acts on \( p \) by the adjoint action, and we now propose to extend this to a (non-linear) action of \( G \).

Let \( \xi \) be an element of \( p \), and let \( e^{t\xi} \) be the corresponding 1-parameter subgroup of \( G \). If we have an embedding \( K < U_m \), then the image of \( \xi \) in \( gl_m \) is a hermitian matrix. Hence \( e^{t\xi} \) is a matrix whose entries are \( C \)-linear combinations of exponential functions \( e^{\lambda t} \) with \( \lambda \in \mathbb{R} \). Let \( E \) denote the ring of \( C \)-linear combinations of these exponential functions; \( E \) is isomorphic to the group ring \( C[G] \).

Let

\[
B_3 = \{ g \in G | e^{-t\xi} g e^{t\xi} \text{ converges in } G \} \quad \text{as } t \to +\infty
\]

The function \( e^{-t\xi} g e^{t\xi} \) viewed in \( gl_m \) has entries in \( E \), and for \( g \) to be in \( B_3 \) means that no entry involves \( e^{\lambda t} \) with \( \lambda > 0 \), and also that the limit matrix as \( t \to +\infty \) is invertible. \( B_3 \) is a subgp. of \( G \).
If \( g \in B_\xi \) let

\[
(2) \quad l(g) = \lim_{t \to +\infty} e^{-t\xi} ge^{t\xi}.
\]

Then

\[
\lim_{t \to +\infty} \exp(-\frac{\xi}{2} t) l(g) \exp(\frac{\xi}{2} t) = \lim_{t \to +\infty} \exp(-\frac{\xi}{2} (t + t)) ge^{(t + t)\xi} = l(g)
\]

which shows \( l(g) \in G_\xi = \{ x \in G | \text{Ad}(x)\xi = \xi^2 \} \).

Thus we have a homomorphism \( l : B_\xi \to G_\xi \). Its kernel we denote

\[
(3) \quad B_\xi^u = \{ g \in G | e^{-t\xi} ge^{t\xi} \to 1 \quad \text{as} \quad t \to +\infty \}.
\]

If \( x \in G_\xi \), then \( e^{-t\xi} ge^{t\xi} = x \), no \( x \in B_\xi \)
and \( l(x) = x \). Thus we have

**Prop. 1:** \( B_\xi = G_\xi \ltimes B_\xi^u \).

**Example:** Let \( \xi = (\xi_1, \ldots, \lambda_m) \) with \( G = GL_m \).

If \( g = (g_{ij}) \), then

\[
e^{-t\xi} ge^{t\xi} = (e^{-t(\lambda_i - \lambda_j)} g_{ij}).
\]

Suppose \( \lambda_1 < \lambda_2 < \lambda_{a_1 + 1} < \ldots = \lambda_{a_i + a_2} < \ldots = \lambda_{a_i + \ldots + a_r} = \lambda_m \).

\( B_\xi \ni g \iff g_{ij} = 0 \) for \( \lambda_i < \lambda_j \). Thus

\[
(4) \quad B_\xi = \begin{pmatrix}
\ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast \\
& 0 & \ast & \ast \\
& & 0 & \ast \\
& & & 0
\end{pmatrix}
\]
If $G \subseteq \text{GL}_m$, then $B_3(G) = G^0 B_3(\text{GL}_m)$, showing $B_3(G)$ is an algebraic subgroup of $G$.

Suppose next that $k \in K \cap B_3$. Then

$$e^{-ti} e^{ti} = e^{ti} e^{-ti} e^{ti} k e^{-ti}$$

converges as $t \to +\infty$. Suppose more generally that $e^{-ti} e^{\eta}$ converges in $G$ as $t \to +\infty$ where $t, \eta \in F$. Applying Cartan involution $\Theta$:

$$\Theta(e^{-ti} e^{\eta}) = e^{ti} e^{-\eta}$$

we see $e^{-ti} e^{\eta}$ converges in $G$ as $t \to -\infty$ also.

But $e^{-ti} e^{\eta}$ viewed in $\text{GL}_m$ is a matrix with entries in $F$, hence no entry involves $e^{it}$ with $\lambda \neq 0$. Thus $e^{-ti} e^{\eta}$ is constant, hence $= 1$, and therefore $t = \eta$. Thus we have proved:

**Prop. 1:** If $t, \eta \in F$ are such that $e^{-ti} e^{\eta}$ converges in $G$ as $t \to +\infty$, then $t = \eta$. Consequently if $k \in K \cap B_3$, then $k \cdot t = t$, i.e.:

$$K \cap B_3 = K_t.$$
Recall the Cartan decomposition for $G$:

(5) $G = K \times P$, $\exp: \mathfrak{g} \rightarrow \mathfrak{p}$

Taking firsts for the 1-parameter group $e^{it \xi}$ of autors of $G$, we get the Cartan decomposition for $G_\xi$

(6) $G_\xi = K_\xi \times P_\xi$, $\exp: \mathfrak{g}_\xi \rightarrow \mathfrak{p}_\xi$.

Thus the subset $KB_\xi$ of $G$ can be written:

\[
KB_\xi = K \times K_\xi B_\xi B_\xi \quad \quad = K \times K_\xi (G_\xi \times B_\xi^u) \quad \quad = K \times K_\xi (K_\xi \times P_\xi \times B_\xi^u) \quad \quad = K \times P_\xi \times B_\xi^u.
\]

We propose now to show that $G = KB_\xi = K \times P_\xi \times B_\xi^u$. Suppose this has been established for $G = GL_n$, $K = U_n$. Consider a sequence

\[
K \rightarrow K' \rightarrow K''
\]

with $K' = U_m$, $K'' = U_n$ as in Prop. 7 of the 3rd part. It is immediate that for $\xi$ in $\mathfrak{p}$, one has
an exact sequence

\[ B_5 \to B_5' \to B_5'' \]

where \( \tilde{\xi}' = \pi'(\xi) = \pi'(\tilde{\xi}) = \tilde{\pi}(\xi) = \tilde{\pi}(\tilde{\xi}). \) The same will hold for \( \tilde{p}_i, \tilde{P}_i \). Thus \( G' = K \times B_{\tilde{P}_i} \times B_{\tilde{p}_i} \) and similarly with double primes, we can deduce by diagram chasing that the result holds for \( G \).

Suppose \( G = G_{n, n} \) and let \( \tilde{\xi} \) be a Hermitian matrix. To show \( G = K B_{\tilde{\xi}} \), one can conjugate \( \tilde{\xi} \) by any element of \( K \), hence I can suppose \( \tilde{\xi} \) is a diagonal matrix with entries \( \lambda_1, \ldots, \lambda_n \). Then from the formula established at the bottom of page 2, we see \( B_{\tilde{\xi}} \) contains the Borel of upper triangular matrices. But then \( G = K B_{\tilde{\xi}} \) results from the classical Gram-Schmidt orthogonalization process (given any basis \( \tilde{v}_1, \ldots, \tilde{v}_n \), there is an orthonormal basis of the form \( \tilde{v}_j' = a_j \tilde{v}_1 + \cdots + a_j \tilde{v}_j \) with \( a_j > 0 \).) So we have proved:

**Prop. 3**: (Iwasawa decomposition).

\[ G = K \times K_{\tilde{\xi}} B_{\tilde{\xi}} = K \times Q_{\tilde{P}_i} \times B_{\tilde{p}_i} \]

Strictly speaking the Iwasawa decomposition is the following special case: Take \( \tilde{\xi} \) to be a regular element of \( \tilde{P}_i \), i.e. such that \( \tilde{p}_i \) is abelian. Then with standard notation one has: \( \tilde{p}_i = 0 \)
$P_{\xi} = A$, $B_{\xi} = N$ and $G = KAN$.

Comments: As

(7) $K/K_{\xi} \sim G/B_{\xi}$

the algebraic variety $G/B_{\xi}$ is compact. Thus $B_{\xi}$ is parabolic subgroup of $G$.

We now use Prop. 3 to define an action of $G$ on $P$. Given $\xi \in P$ and $g \in G$ we know from $G = KB_{\xi}$ that there exists $k \in K$ such that $k^{-1}g \in B_{\xi}$, i.e.

$$e^{-t^3}k^{-1}g e^{t^3} \text{ converges in } G \text{ as } t \to +\infty$$

(8)

$$k e^{-t}k^{-1} \text{ gets}$$

We define $g_{\xi}$ to be $k_{\xi}$; this is independent of the choice of $k$ as $k$ is unique up to $K_{\xi} = K_{\xi}$. Thus we extend the $K$ action on $K/K_{\xi} \sim G$ to a $G$-action via the isomorphism (7). In virtue of (8) $g_{\xi}$ is the unique element of $P$ such that

(9) $e^{-t}(g_{\xi}) e^{t}$ converges as $t \to +\infty$.

Uniqueness results from Prop. 2. Therefore

Prop. 4: The formula (9) defines an action of $G$ on $P$ extending the adjoint action of $K$. $K$ acts transitively on each
I want to give an intrinsic description of \( \mathfrak{p} \) with its \( G \)-action which is independent of the choice of maximal compact subgroup \( K \). I start with the subset \( \mathfrak{I}' \) of \( \mathfrak{p} \) consisting of elements conjugate to elements of \( \mathfrak{p} \). Because maximal compact subgroups of \( G \) are all conjugate, it follows that \( \mathfrak{I}' \) is independent of the choice of \( K \). If \( G = GL_n \), then \( \mathfrak{I}' \) consists of matrices with real eigenvalues which are semi-simple.

To each element \( X \) of \( \mathfrak{I}' \) we associate the 1-parameter subgroup \( e^{tx} \) in \( G \). Call two elements \( X, Y \) of \( \mathfrak{I}' \) equivalent if \( e^{-tX} e^{tY} \) converges in \( G \) as \( t \to +\infty \). This is an equivalence relation. Let \( \mathfrak{I} \) be the quotient of \( \mathfrak{I}' \) by this equivalence relation. There is an evident map \( \mathfrak{p} \to \mathfrak{I} \) which we now show is bijective. First of all, it is injective by Prop. 2. Next given \( X \in \mathfrak{I}' \) we know that there exists \( g \in G \) such that \( \text{Ad}(g^{1/2}) X = \zeta \in \mathfrak{p} \).

If \( \eta = g \cdot \zeta \), then
\[
e^{-tg \cdot \zeta} e^{tx} = e^{-tg \cdot \zeta} e^{tg \cdot \zeta} e^{-tg \cdot \zeta}^{-1}
\]
converges as \( t \to +\infty \) by the definition of \( g \cdot \zeta \); therefore \( \eta \) is equivalent to \( X \).

Next note that \( G \) acts on \( \mathfrak{I}' \) via the adjoint
action \( \lambda \) on \( g \), and this action preserves equivalence, so one gets a \( G \)-action on \( I \).

Because \( e^{-t g^{-1}} g e^{t \lambda} \rightarrow g^{-1} \) converges as \( t \rightarrow +\infty \), it follows that the action defined as \( \phi \) agrees with this adjoint action of \( G \) on \( I \).

Suppose \( X, Y \in I \) are equivalent:

\[
e^{-t Y} e^{t X} \rightarrow g \quad \text{as } t \rightarrow +\infty
\]

Then \( e^{-s Y} e^{s X} = g \) for all \( s \), so \( \text{Ad}(g)X = Y \), and

\[
ge^{-t X} e^{t X} \rightarrow g
\]

so \( g^{-1} \in B^u_x \), hence \( g \in B^u_x \). Thus if \( X, Y \) are equivalent one has \( Y = \text{Ad}(g)X \) where \( g \in B^u_x \); the converse is evident.

Let's apply this to \( G = \text{GL}_n \). \( X \) is semi-simple with real eigenvalues. If these are arranged in order: \( \lambda_1 > \cdots > \lambda_p \) and if the corresponding eigenspaces are \( W_1, \cdots, W_p \) then \( B^u_x \) is the subgroup of \( \text{GL}_n \) stabilizing the flag

\[
(*) \quad 0 < W_1 < W_1 \oplus W_2 < \ldots < W_1 + \cdots + W_p = \mathbb{C}^n
\]

(Note: \( x_i \in W_i \Rightarrow e^{t X} g e^{-t \lambda_i} x_i = \sum g(x_j) e^{-t(\lambda_j - \lambda_i)} \) if \( g \in B^u_x \) then \( g(x_j) > 0 \Rightarrow \lambda_j > \lambda_i \Rightarrow i > j \). Thus \( g(W_i) \subset W_i + \cdots + W_i \))
If $Y$ is equivalent to $X$, then $Y = g X g^{-1}$ with $g \in B_X$, so $g$ stabilizes the flag and $g$ acts trivially on the quotients. Thus $Y$ is any matrix stabilizing $(\ast)$ and having the same eigenvalue $\lambda_i$ as $X$ does on $W_i \oplus \cdots \oplus W_{i-1}$. Summary:

**Prop. 5:** Let $I'$ be the set of elements of $G$ conjugate to elements of $I$ (call these real semi-simple elements of $G$), and let $I$ be the quotient of $I'$ by the equivalence relation $X \sim Y \iff e^{-t} Y e^{tX}$ converges as $t \to +\infty$. Then $\varphi \sim I'$ and this isomorphism commutes with the action on $\varphi$ and with the adjoint action of $G$ on $I$.

One has $X \sim Y \iff Y = \text{Ad}(g) X$ with $g \in B_X = \{ g \mid e^{-tX} \to g \text{ as } t \to +\infty \}$. In $\text{gl}_n$, two real semi-simple matrices are equivalent iff the associated flags and eigenvalues are the same.

Let's discuss next continuity of the action of $G$ on $\varphi$. What I want to prove is that if $g_n \to g$ is a sequence in $G$ and $\varphi_n \to \varphi$ is a convergent sequence in $\varphi$, then $g_n \varphi_n \to g \varphi$. It is evidently enough to do this for $G = \text{gl}_n$. Let $\varphi$ be the composite map $I' \to I \to \varphi$, whence $g \varphi = p(g\varphi g^{-1})$. Since $g_n \varphi_n \to g \varphi g^{-1}$, it is enough to prove that $p$ is continuous, i.e. that $X_n \to X$ implies $p(X_n) \to p(X)$.
Structure of $B_\frac{\pi}{2}$: Let $g \in B_\frac{\pi}{2}$, where
t $\rightarrow e^{-t/2}get^{t/2}$ is a path in $G$ ending at 1 precisely it is a continuous map of $R \cup \{\infty\}$ into $G$ sending infinity to 1. Because $\exp: g \rightarrow G$ is a local isomorphism, we can find a path $x_t$ starting at 1 ending at 0 such that $x_t = \exp(x_t) e^{-t/2}get^{t/2}$ for $t \geq 0$.

For $t$ sufficiently large $x_t = \log(e^{-t/2}get^{t/2})$ is defined and satisfies

$$x_{t+\varepsilon} = \log(e^{-t/2}e^{-\varepsilon/2}get^{t/2}e^{\varepsilon/2}) = \text{Ad}(e^{-\varepsilon/2})x_t$$

for $\varepsilon$ small. This forces

$$e^{-t/2}get^{t/2} = \exp(\text{Ad}(e^{-(t-\varepsilon)}\psi x_0))$$

for all $t$ as both sides are analytic and agree near 0. Thus if $x_t = ne^{-t-\varepsilon}x_0$ for all $t$ we have

$$x_t = \frac{\text{Ad}(e^{-t/2})x_0}{e^{(t-\varepsilon)}x_0}$$

and

$$e^{-t/2}get^{t/2} = \exp(\text{Ad}(e^{-t/2}x_0))$$

Moreover as $e^{-t/2}get^{t/2} \rightarrow 1$, $x_t = \frac{\text{Ad}(e^{-t/2})x_0}{e^{(t-\varepsilon)}x_0} \rightarrow 0$.

Thus if we put $B_\frac{\pi}{2} = \{x \in G | \text{Ad}(e^{-t/2})x_0 \rightarrow 0 \}$ as $t \rightarrow +\infty$ we know $\exp: B_\frac{\pi}{2} \rightarrow B_\frac{\pi}{2}$ is onto. It also has to be 1-1, because it is 1-1 near 0 and any pair of points can be pulled into a nbhd of
zero using $\text{Ad}(e^{-t^3})$. So

\[ \text{Prop. 6: Let } b_\frac{3}{3}^u = \{ X \in g \mid \text{Ad}(e^{-t^3})X \to 0 \text{ as } t \to +\infty \}. \]

Then $b_\frac{3}{3}^u = \text{Lie}(B_3^u)$ and $\exp : B_3^u \to B_\frac{3}{3}^u$ is a diffeomorphism.

We can also prove this first for $\text{GL}_n$ and then taking subsets where $s = s'$.

Recall $B_3 = G_3 \times B_\frac{3}{3}$, where $G_3$ is the centralizer of $\xi$. We know $\text{Lie}(G_3) = \{ X \mid [\xi, X] = 0 \}$ is the zero eigenspace for $\text{Ad} \tilde{\xi}$; denote it $g_\xi$. So,

\[ \text{Prop. 6': Let } g_\xi, B_3, b_3^u \text{ denote the largest subspaces of } g \text{ invariant under } \text{Ad} \tilde{\xi}, \text{ resp. Then } g_\xi, B_3, \text{ and } b_3^u \text{ are respectively the Lie algebras of } G_3, B_3, \text{ and } B_\frac{3}{3}^u. \]

If $\xi \in 0 \Rightarrow \exists \xi = \text{ an abelian subspace of } \xi$ and $\eta = \eta_0 + \sum_{\alpha \in \Delta} \eta_\alpha$ is the root space decomposition of $\eta$ with respect to $\alpha$, then

\[ g_\xi = \eta_0 + \sum_{\alpha \in \Delta} \eta_\alpha \]

\[ b_\frac{3}{3}^u = \sum_{\alpha \in \Delta, \alpha(\xi) > 0} \eta_\alpha \]

\[ b_\frac{3}{3}^u \Rightarrow g_\xi = g_\frac{3}{3} \oplus b_\frac{3}{3}^u \]

\[ b_\frac{3}{3}^u \Rightarrow g_\xi = g_\frac{3}{3} \oplus b_\frac{3}{3}^u \]
Let's consider the orbit structure of $I$ for the $G$-action. Suppose $G$ is connected. As $G$-orbits coincide with $K$ orbits, we know from the first part of these notes that each $G$-orbit contains a unique point of $C$ where $C$ is a chamber in a maximal abelian subgroup of $G$.

$$C \rightarrow G/I$$

Let $\xi \in C$ and suppose $G$ is connected. We know $G_\xi$ is connected (it has same homotopy type as $K_\xi$), hence $B_\xi = G_\xi \times B^\xi$ is the connected subgroup of $G$ with Lie algebra $g_\xi$. Thus the stabilizer $B_\xi$ depends only on the roots of $G$ with respect to $C$ which vanish at $\xi$.

Let $\alpha_1, \ldots, \alpha_k$ be the simple positive roots. We know $C : \{ x \in \mathfrak{c} | \alpha_i(x) > 0 \quad i = 1, \ldots, k \}$ and that $\alpha_1, \ldots, \alpha_k$ are independent. Moreover any $x \in \mathfrak{h}^+ \mathfrak{c}$ is a linear combination $x = h_1 \alpha_1 + \cdots + h_k \alpha_k$ with $h_i > 0$. Hence $x(\xi) = 0 \iff (h_i > 0 \Rightarrow x_i(\xi) = 0)$. Thus if we stratify $C$ according to the subset of simple roots vanishing at a point, the stabilizers remain constant on the strata. So we get:
Assume $G$ is connected.

**Prop. 7.** Let $\Sigma = \{x_1, \ldots, x_d\}$ be the simple roots of $g$ with respect to the chamber $C$. For each subset $\sigma$ of $\Sigma$, let $C_\sigma$ be the subset of $C$ consisting of points where the $x_i$ in $\sigma$ vanish and the $x_i$ not in $\sigma$ are positive. Then $B_\sigma = B_\tau$ iff $\sigma, \tau$ are in the same stratum of $C$.

**Formula:** Put $B_\sigma = B_\tau$ for $\tau \in C_\sigma$. Then $
abla_{\sigma} \cdot g = \sum_{\alpha} g_{\alpha}$

where $\alpha$ ranges over those positive roots of the form $\alpha = \sum_{i \in \sigma} l_i x_i$ with $l_i > 0$; (call these positive roots with support $\sigma$).

Note that $\sigma \subset \tau$, then $\overline{C_\sigma} \supset \overline{C_\tau}$ and $B_\sigma \subset B_\tau$ so there is a map $G/B_\sigma \to G/B_\tau$. Thus we can form a space by taking $\bigcup_{\sigma} G/B_\sigma \times \overline{C_\sigma}$ and identifying $(x, x, y) \sim (x, x, y)$ for each inclusion $\sigma \subset \tau$. One has a continuous map

$$\bigcup_{\sigma} G/B_\sigma \times \overline{C_\sigma} \to I$$
because $G / B_0 \cong K / K_0$, $K_0 = B_0 \cdot K$, and the $K$ action is continuous. As this map is compatible with the equivalence relation one gets a map

\[(*) \quad \varprojlim \frac{G / B_0 \times C_0}{\text{rel}_n} \rightarrow I\]

which one sees from the fact that both spaces sit over $C$ (recall $K \cap I = C$) and the fibres are the same (note each $i \in C$ is contained in a smallest $C_0$ and $K_0 \subseteq G / B_0$. Because $I$ is Hausdorff, the former space is Hausdorff, and so since both spaces are proper over $C$, it follows (*) is a homeomorphism.

I claim $G$ acts continuously on $Y = \varprojlim \frac{G / B_0 \times C_0}{\text{rel}_n}$. We know it acts continuously on $X = \varprojlim \frac{G / B_0 \times C_0}{\text{rel}_n}$, and the map $X \rightarrow Y$ is proper + surjective. But a proper surjective map is a quotient map, hence $G \times X \rightarrow G \times Y$ is a quotient map, and so the map $G \times X \rightarrow G \times Y$ induces $G \times Y \rightarrow Y$. So we have proved:

**Prop. 8:** $G$ acts continuously on $I$.

Actually the proof assumes $G$ connected, but it suffices to do the proof for $G_n$. 

\[\text{We shall now give} \quad \text{a direct demonstration.} \]
Generalization to the real case:

Let $K$ be a compact group with involution $\sigma$, $G$ the complexification of $K$, and let $\sigma$ be extended to $G$ in anti-holomorphic fashion. We have seen that the decomposition

$$G = K \times P \quad \exp : P \rightarrow P$$

yields on taking $\sigma$-fixpts

$$G^\sigma = K^\sigma \times P^\sigma \quad \exp : P^\sigma \rightarrow P^\sigma.$$  

Recall: $p = i \xi$, so $p^\sigma = i \xi^\tau = \{ x \in g^\sigma \mid \theta x = -x \}$. Now in the preceding, we can take $\sigma$-fixpts to get the following:

$$B^\sigma_3 = \{ g \in G^\sigma \mid e^{-t} \xi e^{t} \text{ converges in } G^\sigma \} \quad \text{as } t \rightarrow +\infty$$

$$B^\sigma_3 = G^\sigma_3 \times B^u_3^\sigma$$

$$G^\sigma_3 = K^\sigma_3 \times \mathbb{R} P^\sigma_3$$

$$G^\sigma = K^\sigma \times P^\sigma \times B^u_3^\sigma$$

$$\exp : B^u_3^\sigma \rightarrow B^u_3^\sigma \quad \text{where}$$

$$B^u_3^\sigma = \{ x \in g^\sigma \mid \text{Re}(e^{-t}\xi^\tau x) \rightarrow 0 \text{ as } t \rightarrow +\infty \}$$

If we take $\xi$ to be a regular element of $p^\sigma$, that means $p^\sigma_3$ is abelian, in fact a maximal abelian...
Subspace of $\mathfrak{p}^\circ$, then
\[ G^\circ = K^\circ \times P_2^\circ \times B_3^u \]
is the Iwasawa decomposition of $G^\circ$.

It is pretty clear that the above notation is awkward. The following notation is more standard. Replace $G^\circ, K^\circ$ by $G, K$, so that now $G$ is a reductive algebraic group over $\mathbb{R}$, and $K$ is a maximal compact subgroup. Similarly we drop $\circ$ from the rest of the notation. If we have occasion the new notation for $(G, K)$ is $(G_c, U)$ so we have the picture:

\[
\begin{align*}
G & \subset G_c \\
U & \subset U \\
K & \subset U
\end{align*}
\]

Again $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\Theta = -1 \text{ on } \mathfrak{p}$, $+1 \text{ on } \mathfrak{k}$.

Suppose now $U$ is connected (this means $G$ as an algebraic group is connected, not necessarily that $G$ as a Lie group is connected, e.g. $G = \mathbb{R}^*$, $G_c = C^*$).
Then I know that the $K$-orbits in $\mathfrak{p}$ are connected. The $K$-orbit $\triangle$ of $\mathfrak{i}$ is $K/K_3 \simeq G/B_3$. I want
to understand the natural stratification on $I = \rho$. I know $K \mathfrak{I} = W \mathfrak{a} = \mathfrak{c}$ where
$\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, and
where $\mathfrak{c}$ is a chamber in $\mathfrak{a}$. Moreover
$\mathfrak{c}$ is described by $\alpha_i(x) \geq 0 \quad i = 1, \ldots, l$ where
$\alpha_i = 0$ are the walls of $\mathfrak{c}$, and $\alpha_1, \ldots, \alpha_l$ are independent. Question: Does $B_\mathfrak{c}$ remain constant
as $\mathfrak{c}$ varies over a stratum of $\mathfrak{c}$? Yes,
because we know this is the case in $G_\mathfrak{c}$ and $B_\mathfrak{c}$ is the $T$-invariant subgroup of the corresponding
stabilizer in $G_\mathfrak{c}$.

In order to understand this point, let's go back
to the $(K, G, K^0, G^0)$ notation. Let $\mathfrak{c}$ be a maximal
abelian subspace of $\mathfrak{p}$, and let the root decomposes $g$

$$g = g_{\mathfrak{c}} + \sum_{\alpha \in \Phi} \alpha \mathfrak{g}^\alpha$$

where $\Phi$ is the set of roots of $G$ with respect to
$\mathfrak{c}$. ($\Phi$ consists of real linear functions on $\mathfrak{c}$). Let $\mathfrak{c}$
be a point of a chamber $\mathfrak{c}$ of $\mathfrak{a}$. Its
centralizer $G_\mathfrak{c}$ is connected with

$$\text{Lie}(G_\mathfrak{c}) = \mathfrak{g}_\mathfrak{c} = \mathfrak{g}_{\mathfrak{c}} + \sum_{\alpha \in \Phi} \alpha \mathfrak{g}^\alpha_{\alpha(\mathfrak{c}) = 0}$$

Because $\sigma = \text{id}$ on $\mathfrak{c}$, $\mathfrak{g}_\mathfrak{c}^\alpha$ is stable under $\sigma$, so

$$\text{Lie}(G_{\sigma}^\mathfrak{c}) = \mathfrak{g}_{\mathfrak{c}}^\sigma + \sum_{\alpha \in \Phi} \alpha \mathfrak{g}^\alpha_{\alpha(\mathfrak{c}) = 0}$$
and 
\[ \text{Lie} (\mathbb{B}_3^\sigma) = \sigma^\sigma \partial + \sum_{\alpha(\xi) > 0} \sigma^\sigma \partial \alpha. \]

It's clear from this that from the stabilizer \( \mathbb{B}_3^\sigma \) we can recover the roots \( \alpha \in \Phi^+ \) which vanish on \( \xi \). Thus we see prop. 7 (p. 13) holds also in the real case.

Suppose \( \xi, \xi' \in \Phi^\sigma \) are such that \( \mathbb{B}_3^\xi = \mathbb{B}_3^\xi' \). Complexifying Lie algebras, we get \( \mathbb{B}_\xi = \mathbb{B}_{\xi'} \); intersecting with \( K \) we get \( K_\xi = K_{\xi'} \); it follows that \( e^{it\xi} \) commutes with \( \xi' \), hence \( [\xi, \xi'] = 0 \). This means that \( \xi, \xi' \) are contained in a maximal abelian subspace \( \alpha \) of \( \Phi^\sigma \). Further, the sign of any root of \( \alpha \) with respect to \( \alpha \) is the same for \( \xi \) and \( \xi' \), hence \( \xi, \xi' \) lie in the same stratum of \( \alpha \).

Lemma: \( P \cap \mathbb{B}_\xi = P_\xi \). More generally, if \( g \in \mathbb{B}_\xi \) is such that \( \Theta g = g^{-1} \), then \( g \in G_\xi \).

Proof: \( \Theta(e^{-t\xi} g e^{t\xi}) = e^{t\xi} g^{-1} e^{-t\xi} = (e^{t\xi} g e^{-t\xi})^{-1} \) converges as \( t \to \infty \). Thus \( e^{t\xi} g e^{-t\xi} \) converges as \( t \to \pm \infty \), so as it has entries which are linear combinations of real exponentials, it is constant, so \( g \in G_\xi \).

(This lemma can be used so: \( \mathbb{B}_3^\sigma = \mathbb{B}_3^\xi \Rightarrow e^{t\xi} e^{P_\sigma B_3^\sigma} = e^{t\xi} e^{P_3^\sigma B_3^\sigma} \), hence \( \xi, \xi' \) commute.)
Summary: \( \mathcal{E}, \mathcal{E}', \mathcal{E}' \) are in the same stratum \( \iff \mathcal{E}_g = \mathcal{E}_{g'} \).  

Consequence: Consider the orbit \( G \cdot \mathcal{E} \subset G / B_\mathcal{E} \). We know that this meets the chamber \( \mathcal{C} \) in exactly one point, namely \( \mathcal{E} \), if we start with \( \mathcal{E}, \mathcal{E}' \).  The stratum \( \mathcal{E}_g \) consists of all points of \( \mathcal{C} \).  

Stratification of \( \mathcal{F}_0 \): Again suppose \( G \) connected, let \( \mathcal{C} \) be a chamber in a maximal abelian subspace \( \Xi \) or of \( \mathcal{F}_0 \), and let \( \Sigma \) be the set of simple roots. For each subset \( \mathcal{T} \) of \( \Sigma \) let \( \mathcal{C}_\mathcal{T} \) be the set of points where the \( \alpha_i \) in \( \mathcal{T} \) vanish and those not in \( \mathcal{T} \) are \( > 0 \). (Thus \( \mathcal{T} = \emptyset \Rightarrow \mathcal{C}_\emptyset = \{0\} \), \( \mathcal{T} = \Sigma \Rightarrow \mathcal{C}_\Sigma = \text{Int} \mathcal{C} \).) We know the stabilizer \( B_{\mathcal{T}}^{\mathcal{E}_g} \) is constant as \( \mathcal{T} \) ranges over \( \mathcal{C}_\mathcal{T} \); denote this stabilizer by \( B_{\mathcal{T}}^{\mathcal{E}_g} \). Then we have a stratification:

\[
\bigsqcup_{\mathcal{T}} \frac{G / B_{\mathcal{T}}^{\mathcal{E}_g}}{\mathcal{C}_\mathcal{T}} \sim \mathcal{F}_0
\]

(set-theoretic isomorphism) because \( \mathcal{C} \) is a fundamental domain for the \( \mathcal{F}_0 \)-action on \( \mathcal{F}_0 \).  

Thus \( \mathcal{F}_0 \) is broken up into strata \( g \mathcal{C}_\mathcal{T} \), \( g \in G \), \( \mathcal{T} \subset \Sigma \). According to the discussion on p. 18 we have:

Assertion: \( \mathcal{E} \) and \( \mathcal{E}' \) are in the same stratum.
\[ B_i = \mathcal{B}_i \quad \text{(in fact it suffices that Lie}(B_i) = \text{Lie}(B'_i)). \]

**Consequence:** Let \( \gamma \in C \) and consider the orbit \( G\gamma = G/B_{\gamma} \). We know this orbit meets \( C \) in exactly one point, namely \( \gamma \). Since the stratum of \( \gamma \) in \( p\gamma \) is the stratum of \( \gamma \) in \( C \), it follows that all the points of the orbit \( G\gamma \) except \( \gamma \) belong to different strata, hence their stabilizers differ from \( B_{\gamma} \). But the stabilizer of \( g\gamma \) is \( gB_{\gamma}g^{-1} \). Thus \( g \notin B_{\gamma} \implies gB_{\gamma}g^{-1} \neq B_{\gamma} \) and we get:

**Cor:** \( B_{\gamma} \) is its own normalizer.

**Rank 1 case:** This means \( \dim \alpha = 1 \). (One could generalize a bit and only require \( \text{card } \Sigma = 1 \).) In this case the orbits for \( K \) in \( p\alpha \) are the spheres around \( 0 \) because \( C \) is a ray containing zero. Thus what we have \( 0 \) is a sphere \( G/K = K/M \) of dimension \( \dim(p\alpha) - 1 \) and the \( G \)-space \( p\alpha \) may be viewed as the open disk associated to this action of \( G \). The action of \( G \) on \( p\alpha \) is evidently continuous, but probably not differentiable, because otherwise it would be linear, as it is homogeneous.