Let $V$ be a vector space over a field $K$ of countable infinite dimension. Form a poset consisting of subspaces $W$ of $V$ such that $\dim W = \dim V/W = \infty$, with $W_1 < W_2$ iff $W_1 \subset W_2$ and $\dim(W_2/W_1) = \infty$. Is $X$ contractible?

Again let $F$ be a finite subset of $X$. Pick a maximal subset $\{x_1, \ldots, x_k\}$ of $F$ such that $y = x_1 \oplus \cdots \oplus x_k$ has infinite dimension. Then for all other $x$ in $F$, $x \oplus y$ is finite-dimensional and so we can shrink $y$ successively still keeping its infinite dimensional until we get $x_1 \oplus \cdots \oplus x_k$ such that for all $x \in F$ either $x \oplus x = 0$ or $x \subset x$. Now let $z'$ be a subspace of $z$ of infinite dimension in $z$.

I consider sending $x$ in $F$ to $x + z'$. If $x \oplus z = 0$, then $x + z'/x + z' \simeq z/z'$ is inf. dim, so $x + z' \in X$. Also $x < x + z'$ in $z'$. On the other hand if $z \subset x$, then $x + z' = x \in F$ and $x = x + z' > z'$ because $z/z' \subset x/z'$. Finally if $x \subset x_2$, we want to show that $x_1 + z' < x_2 + z'$. This is clear if either $z \subset x_1$ or if $x \oplus x_2 = 0$. If $z \subset x_2$ and $x_2 \oplus x_1 = 0$, then $x_1 + z' < x_2 = x_2 + z'$ because $x_2/x_1 + z' \supset x_1, x_2/z' \sim z/z'$ is infinite dimensional. Therefore $X$ is contractible.
Given a $p$-simplex $x_0 < x_1 < \ldots < x_p$ in $X$, there exists a decomposition $V = V_0 \oplus \ldots \oplus V_{p+1}$ such that $x_i = V_0 \oplus \ldots \oplus V_{i}$, where $V_i$ is infinite-dimensional hence isomorphic to $V$. Thus $G = \text{Aut}(V)$ acts transitively on the $p$-simplices, the stabilizers of a $p$-simplex being a group:

\[
\begin{array}{c}
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\end{array}
\]

with $p+1$ blocks in the diagonal positions, and where the entries come from the ring $\text{End}(V)$. So now make the usual assumptions that guarantee the nilpotent radical doesn't contribute to homology:

a) homology with coefficients in $\mathbb{Q}$

or b) homology with coefficients in $\mathbb{F}_p$ where $p^{-1} \in K$.

Then the spectral sequence becomes

\[E_1^{pq} = H_q(G^{p+1}) \Rightarrow 0\]

and as before this implies $\tilde{H}_*(G) = 0$. 
Fix a vector space $M$ over $K$, and consider the groupoids $E^M$ consisting of exact sequences of $K$-vector spaces:

$$0 \rightarrow V \rightarrow E \rightarrow M \rightarrow 0$$

with $\dim(V)$ countable infinite; morphisms are isomorphisms over $M$. On $E^M$ we have a product functor

$$(E_1, E_2) \mapsto E_1 \times_M E_2$$

which induced a product on $H_\ast(E^M)$ which is commutative and associative. Note that $E^M$ is equivalent to the group

$$\text{Aut}(M \oplus M/M) = \begin{bmatrix} id_M & 0 \\ \text{Hom}(M, V) & \text{AdV} \end{bmatrix}$$

In addition we have an infinite sum functor $\Sigma$ defined as follows. Given $E \rightarrow M$, consider inside $E \times E \times E \times M \times \cdots$ the subspace formed of sequences $(e_1, e_2, e_3, \cdots)$ such that $\{e_1, e_2, e_3, \cdots\}$ is finite. Call this subspace $(E/M)^{\infty}$, whence we have an exact sequence

$$0 \rightarrow V^{\infty} \rightarrow (E/M)^{\infty} \rightarrow M \rightarrow 0$$
which we define to be \( \Sigma(E/M) \). So the K-theory of \( E/M \) is trivial. This means that the embedding

\[
\begin{bmatrix}
    \text{id}_M & 0 & 0 \\
    \text{Hom}(M, V) & \text{Aut}(V) & 0 \\
    0 & 0 & \text{id}_V
\end{bmatrix}
\subset
\begin{bmatrix}
    \text{id}_M \\
    \text{Hom}(M, V) & \text{Aut}(V \oplus V) \\
    \text{Hom}(M, V)
\end{bmatrix}
\]

should induce the zero map on \( H_* \).

It seems necessary to review stability for a field. Let \( V_0 \) be a subspace of \( V \) and let \( P(V/V_0) \) be the poset of subspaces \( W \subset V \) such that \( W + V_0 = V \). According to Lusztig, this complex has the homotopy type of a bouquet of spheres of its dimension (which is \( \dim(V_0) - 1 \)). It follows that we get a Lusztig sequence.

\[
\ldots \rightarrow \bigoplus J(W, V/W_0) \rightarrow \bigoplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \ldots
\]

\[
\begin{align*}
    & W + V_0 = V \\
    & \dim(W + V_0) = 0
\end{align*}
\]

So let me consider the analogous infinite situation. I want to understand the map on homology.
induced by the inclusion

\[ \text{Aut}(V_0) = \left[ \begin{array}{cc} \text{id}_M & \circ \\ \circ & \text{Aut} V_0 \end{array} \right] \subseteq \left[ \begin{array}{cc} \text{id}_M \\ \circ & \text{Aut} V_0 \end{array} \right] = \text{Aut}(M \oplus V_0) \]

so I will want to make the group \( \left[ \begin{array}{cc} \text{id}_M \\ \circ & \text{Aut} V_0 \end{array} \right] \) act on a poset whose minimal elements are the subspaces \( W \) of \( V \) such that \( W \oplus V_0 = V \). (I have changed notation from \( 0 \rightarrow V \rightarrow E \rightarrow M \rightarrow 0 \) to \( 0 \rightarrow V_0 \rightarrow V \rightarrow M \rightarrow 0 \).)

Thus in the infinite situation let \( V \) be the poset consisting of subspaces \( W \) of \( V \) such that \( W + V_0 = V \), such that \( W \cap V_0 \) is of finite codimension in \( V_0 \), and such that \( \dim(W \cap V_0) \leq 0 \) or \( \infty \)

Define \( W_1 < W_2 \) if \( W_1 \cap W_2 \) and \( \dim(W_2/W_1) = \infty \).

Put \( G = \text{GL}(V/M) \) for the group of automorphisms of \( V \) inducing the identity on \( M \). Does \( G \) act transitively on the simplices of \( V \)?
November 5, 1975,

Let $I$ be a groupoid with product functor $\times: I \times I \to I$. Assume $I$ has two iso. classes: $0, 1 \in N$ such that $0 \perp N = N$, $1 \perp N = N$. I want to consider the poset $\mathcal{X}$ consisting of iso.

\[ \alpha: N \perp N \cong N \]

with $\alpha < \alpha'$ iff \[ \exists \gamma, \gamma' \in \mathcal{X} \]

\[ \begin{array}{ccc}
N \perp N & \xrightarrow{\alpha} & N \\
| & \gamma \perp \text{id} & |
\end{array} \]

\[ (N \perp N) \perp N \]

\[ N \perp (N \perp N) \]

\[ N \perp N \]

commutes. Assume $\gamma, \gamma'$ uniquely determined essentially because $I$ is faithfull.

Better description. Consider the simplicial groupoid, which in degree $p$ is $S^{p+2}$

\[ S \times S \times S \cong S \times S \times S \cong S \times S \]

There is an augmentation to $S$ so I can form the
fibre simplicial set over an object \( N \). Thus a \( p \)-simplex in \( X \) is a partitioning of \( N \) into \((p+1)\)-pieces:
\[
M_0 + \cdots + M_{p+1} \sim N
\]

Assume \( X \) is contractible. Does it follow that I get a stability theorem:
\[
e^*: H_* (\text{Aut } N) \xrightarrow{\sim} H_* (\text{Aut } (N))
\]

By letting \( G = \text{Aut } N \) act on \( X \) I get a spectral sequence
\[
E^{ij}_1 = H_q (G^{p+1}) \Rightarrow 0
\]

Can I produce a relative spectral sequence?

\[
\dots \rightarrow C_1 (x) \rightarrow C_0 (X) \rightarrow \mathbb{Z} \rightarrow 0
\]

Let \( G' \) be the image of \( G \rightarrow G \) obtained from \( N + N \xrightarrow{\sim} N \) trivial action on the first factor.
Let \( X' \) be the corresponding subcomplex of \( X \); it consists of partitions of the second factor \( N + N \rightarrow N \).
Then I get a map \((G', X') \rightarrow (G, X)\).

Next I have to compute the relative terms.
\[ X_0 = \frac{G}{G'' \times G'} \quad X_0' = \frac{G'}{(G')}'' \times (G')' \]

Redo: Start with \( N' < N \), specifically given by \( N' \perp N'' = N \). Let \( G' = \text{Aut}(N') \) be viewed as a subgroup of \( G = \text{Aut}(N) \). We can also view \( X(N') \) as a complex of \( X(N) \).

In effect given

\[ N_0 \perp \ldots \perp N_{p+1} = N' \]

we send it to the partition

\[ N_0 \perp \ldots \perp N_{p+1} \perp (N'_{p+1} \perp N'') = N. \]

(This corresponds to the obvious map of the building of \( V' \) into the building of \( V \) when \( V' \subset V \).)

Fix a \( p \)-simplex in \( X(N') \),

\[ N_0 \perp \ldots \perp N'_{p+1} = N'. \]

and let \( G'_{(p+2)} \) denote its stabilizer. Then

\[ G'/G'_{(p+2)} \rightarrow X(N')_p \]

The image of this simplex in \( X(N) \) is \( N_0 \perp \ldots \perp N_{p+1} = N \)

where \( N'_0 = N_0, \ldots, N'_p = N_p, \quad N'_{p+1} = N'_{p+1} \perp N'' \). Denote the stabilizer in \( G \) of this by \( G_{(p+2)} \) so that
\[
G / G_{(p+1)} \rightarrow X(N)_{p}
\]
Then the inclusions \[ X(N)_{p} \subseteq X(N)_{p} \] correspond to the map \( G' / G'_{(p+1)} \rightarrow G / G_{(p+1)} \) induced by the inclusions
\[ G' \subseteq G \]
\[ G'_{(p+1)} \subseteq G_{(p+2)} \].
Now
\[
G'_{(p+1)} = \text{Aut}(N_{0}) \times \cdots \times \text{Aut}(N'_{p+1})
\]
\[ \cong \]
\[
G_{(p+2)} = \text{Aut}(N_{0}) \times \cdots \times \text{Aut}(N'_{p+1} \cap N'')
\]
and so we see that
\[
(G'_{(p+2)}, G_{(p+2)}) \simeq G_{p+1} \times (G, G').
\]
Therefore the relative spectral sequence should have the \( E' \)-term
\[
E'_{pq} = H_{q}(G^{p} \times (G, G')) \Rightarrow 0
\]
If I know that \( H_{q}(G, G') = 0 \) for \( q < r \), then
\[
E'_{pr} = H_{r}(G^{p} \times (G, G')) = \bigoplus_{i+j=r} H_{i}(G^{p}) \otimes H_{j}(G, G')
\]
\[ = H_{0}(G^{p}) \otimes H_{r}(G, G') = H_{r}(G, G') \]
So you have to compute \( d_i : E'_1 \rightarrow E'_0 \) and show that it's zero.

\[
\begin{array}{c}
G \times G \times G \\
\downarrow \text{id} \times \text{id} \times (\text{id} \circ \text{id}) \\
G \times G \times G \\
\end{array} 
\begin{array}{c}
\downarrow \text{id} \times \text{id} \times (\text{id} \circ \text{id}) \\
G \times G \times G \\
\end{array}
\begin{array}{c}
\downarrow \text{id} \times \text{id} \\
G \times G \\
\end{array} 
\begin{array}{c}
\downarrow \text{id} \\
G \\
\end{array}
\]

This induces

\[
G \times G \times (G, G') \xrightarrow{\text{id} \times \text{id}} G \times (G, G') \xrightarrow{\text{id}} (G, G')
\]

and so what happens for \( H_r \) is that this is the same as for the subgadget

\[
H_r(G, G') \xrightarrow{\text{id}} H_r(G, G') \xrightarrow{N \perp} H_r(G, G')
\]

\((N \perp)\) is an idempotent operator, and we know it's surjective, hence it must be the identity. But then from:

\[
\begin{array}{c}
H_n(G') \xrightarrow{N \perp} H_n(G) \rightarrow H_n(G, G') \rightarrow H_{n-1}(G') \rightarrow H_n(G) \\
\downarrow \text{same} \downarrow \text{same} \downarrow \text{same} \downarrow \text{same} \\
H_n(G') \rightarrow H_n(G) \rightarrow H_n(G, G') \rightarrow H_{n-1}(G) \rightarrow H_n(G, G')
\end{array}
\]

we see \((N \perp)^2 = 0\) which concludes the proof.
I will now attempt to make the above stability proof geometric. Let $\mathcal{P}$ be an associative monoid of the homotopy type $BG$ and let $M' = pt \ast M$ be the associated monoid with 1. We can form the simplicial space

\[
\ldots \xrightarrow{\times} M \times (M')^2 \times M \xrightarrow{\times} M \times M' \times M \xrightarrow{i} M \times M
\]

which is obtained in the usual way by letting $M'$ act on the left and right on $M$. By hypothesis on the contractibility of $X$, the augmentation $M \times M \to M$ given by 1 gives a homotopy equivalence of $(\ast)$ with $M$.

Let $a$ be the basepoint of $M$. Right multiplication by $a \in M$ furnishes an embedding of $(\ast)$ into itself, so we can form the quotient

\[
\ldots \xrightarrow{\times} M \times (M')^2 \times M/\langle a \rangle \xrightarrow{\times} M \times M' \times M/\langle a \rangle \xrightarrow{i} M \times M/\langle a \rangle
\]

which will be hom to $M/\langle a \rangle$ via the evident augmentation. So assume $H^\ast (M/\langle a \rangle)$ begins in dimension zero. From the homology spectral sequence $\mathcal{P}$ get an exact sequence

\[
H_r \left( M \times M' \times M/\langle a \rangle \right) \xrightarrow{i} H_r \left( M \times M/\langle a \rangle \right) \xrightarrow{i} H_r \left( M/\langle a \rangle \right) \to 0
\]

\[
H_r \left( M/\langle a \rangle \right) \xrightarrow{i} H_r \left( M/\langle a \rangle \right) \xrightarrow{i} H_r \left( M/\langle a \rangle \right) \to 0
\]
so I can argue as before.

November
October 7, 1975:

Notation: $N = \text{the groupoid of countable}$
$\text{infinite sets and bijections with } + \text{ operation, } N'$
$= \{ \emptyset \} \cup N$. $M = \text{a groupoid with product operation}$
$\text{and which is associative and commutative, } M' = \{ pt \} \cup M$
I suppose I am given a functor

$\theta : N' \rightarrow M'$

compatible with products such that $\theta(N) < M'$.

Example: Take $M$ to be countable infinite sets
with morphisms defined to be $\sim$ mod finite
sets.

I want to establish stability for $M$.

Let $L$ be an object of $M$. What is the
unimodular complex in this situation?

I construct the following poset $X(L)$. An element
is a "mono" $\theta(N) \rightarrow L$. More precisely
it is an isomorphism $\theta(N) \times L'' \cong L$ modulo
isos. of $L''$ (I continue to assume $L$ faithful).

$x: \theta(N) \to L$ is $\beta: \theta(N) \to L$ iff

$\exists \alpha: N \to N$ such that $\alpha \theta(N) = \beta$.

I guess I have to assume $\theta$ faithful.

Put $G = \text{Aut}(L)$, $\Sigma = \text{Aut}(N)$. Then if

I fix a vertex $\theta(N) \times L'' \cong L$ and let $G'' = \text{Aut}(L'')$ viewed as a subgroup of $L$, $G$ acts

transitively on $(p-1)$-simplices, so

$$\{ (p-1)\text{-simplices} \} \cong G / \Sigma^p \times G''$$

$$\{ \theta(N_0, \ldots, N_{p-1}) \to L \}.$$ 

But this is the fibre of $N^p \times M \to M$

over $L$.

Hence contractibility of $X(L)$ tells us that the

simplicial object with augmentation

$$N \times (N')^2 \times M \equiv N \times N' \times M \Rightarrow N \times M \Rightarrow M$$

is contractible.

However using the fact that $N$ is acyclic we can compute the $E_1$ term of the spectral sequence

$$E_{pq}^1 = H_q(M) \quad \text{for all } p,$$
Recall $d_i : \eta(n) \eta^p \rightarrow \eta(n') \eta^p \rightarrow M$

$d_i : (n_0, n_1, \ldots, n_p, 1) \mapsto (n_0, \ldots, n_i = n_{i+1}, \ldots, 1)$

so $d_i : E^1_p \rightarrow E^1_{p-1}$ is identity if $0 \leq i \leq p-1$

is mult. by $e$ if $i = p$

Thus the $E^1$-term is

$\rightarrow H_*(M) \xrightarrow{e} H_*(M) \xrightarrow{id-e} H_*(M) \xrightarrow{e} H_*(M) \xrightarrow{\cdots} H_*(M)$

$E^1_{2*} \xrightarrow{E^1_{1*}} E^1_{0*}$

since $e$ is idempotent it follows that the sequence is exact, so

$E^2_{p*} = \begin{cases} 0 & p > 0 \\ eH_*(M) & p = 0 \end{cases}$

so the spectral sequence collapses and we get

$H_*(M) \xrightarrow{id-e} H_*(M) \xrightarrow{e} H_*(M) \rightarrow 0$

is exact. Algebraically this implies $e$ is the identity.

Notice the preceding calculation of the $E^2$-term is completely independent of what $M$ is. $M$
can be any space on which $\mathbb{N}$ acts. This leads to

**Question:** Do the simplicial space

\[(\star) \quad n \times n \times m \Rightarrow n \times m\]

of the same homology as $(\mathbb{N})^m$ for a geometric reason?

**Possible proof.**

\[
\begin{align*}
n \times (n')^2 \times m & \Rightarrow n \times n' \times m \Rightarrow n \times m \\
\downarrow & \\
(n')^2 \times m & \Rightarrow n' \times m \Rightarrow m
\end{align*}
\]

But we know, by virtue of commutativity, that on $\langle n', m \rangle$ the operation $M \rightarrow M \oplus \Theta(N)$ is homotopic to the identity.
Next don't assume $N$ is acyclic, but instead suppose $\overline{R} = H_\ast(N)$ is a ring with identity whence
\[ R = H_\ast(N') = k \times \overline{R} \]

Now the $E^1$ term for

\[ N \times (N')^2 \times M \to N \times M \to \eta \times M \to \eta \times M \]

is
\[ \overline{R} \otimes \overline{R} \otimes H_\ast(m) \to \overline{R} \otimes \overline{R} \otimes H_\ast(m) \to \overline{R} \otimes H_\ast(m) \]

and I recognize this as a standard construction for computing $\text{Tor}$: So
\[ E_{p\ast}^2 = \text{Tor}_p^R(\overline{R}, H_\ast(m)) \]

But $\overline{R} = eR$ is projective as an $R$-module, so
\[ E_{p\ast}^2 = \begin{cases} 0 & p \neq 0 \\ eH_\ast(m) & p = 0 \end{cases} \]

So homologically at least I see that $(\ast)$ has the homology type of $N^{-1}M$.

Question again is whether the above argument can be made geometric. Possibility: Show that
the two maps from \((x)\) to itself given by \(x \mapsto e \cdot x\) on \(M\) and \(y \mapsto e \cdot y\) on \(N\) are homotopic. Then use the fact you have stability for \(N\).

Question: Can you relate the fact that

\[
\Longrightarrow m \times (m') \xrightarrow{\phi} m \times m \longrightarrow m
\]

is a seq with the space

(2) \(\Longrightarrow m' \times m \Rightarrow m\)

(h-orbit of \(m'\) acting on \(M\)). Is the latter contractible?

(2) is essentially the category \(<M', M>\) which consists of objects of \(M\) with \(\Rightarrow\) for morphisms. It is a monoid \(\exists X \times X \leftarrow X\), hence I know at least that its homology is zero. In the case of countable infinite sets I know the category is contractible.

Can I use (2) to prove stability. Again I get a spectral sequence

\[
E^{1}_{pq} = H_{q}(m' \times (m, ma)) \Rightarrow 0
\]
Consider the category consisting of all finitely generated injective $N$-modules which is injective with respect to $N$-modules. This is the category $\mathcal{Mod}_N$. The category consists of all finitely generated injective $N$-modules.

Let $\mathcal{M}$ be the category of all finitely generated injective $N$-modules. We have a functor $\mathcal{M} \to \text{Set}$ defined by $\mathcal{M} \to \text{Nat}$. The diagram commutes.
November 8, 1975

Let $N$ be the groupoid of countable infinite sets and all isos. between them; let $N' = pt + N$. Equip $N'$ with the operation of disjoint union.

Form $\langle N', n \rangle$. The objects are those of $N$, a morphism $N_1 \to N_2$ is given by an isom. $N \sqcup N_1 \to N_2$ modulo isos of $N \in N'$. Thus a morphism in $\langle N', n \rangle$ is simply an injection whose complement is empty or infinite.

Claim $\langle N', n \rangle$ is contractible: Use the cone construction:

$$N \to N \sqcup N_0 \leftarrow N_0$$

are maps in $\langle N', n \rangle$ and $N \to N \sqcup N_0$ is a functor.

Next consider the simplicial groupoid which in degree $p$ consists of $p$-simplices $N_0 \to N_1 \to \cdots \to N_p$ in $\langle N', n \rangle$ and their isomorphisms. The obvious augmentation to $\langle N', n \rangle$ is a homotopy equivalence; this
would be true for any category. Picture of this simplicial category

\[
N \times (N')^2 \xrightarrow{\begin{aligned} 1 \times \text{id} & \quad \text{and} \\ \rho_{123} & \end{aligned}} N \times N' \xrightarrow{\begin{aligned} 1 & \quad \text{and} \\ \rho_1 & \end{aligned}} N
\]

\[
(N_0, X_1, X_2) \quad (N_0 \perp X_1) \quad \xmapsto{\epsilon} \quad N_0 \perp X_1
\]

\[
N_0 \rightarrow N_0 \perp X_1 \rightarrow N_0 \perp X_1 \perp X_2
\]

We get a functor from this simplicial cat to itself by adding on the left a fixed object \( E \) of \( N \). It sends \( N_0 \rightarrow \cdots \rightarrow N_p \) into \( E \perp N_0 \rightarrow E \perp N_1 \rightarrow \cdots \rightarrow E \perp N_p \). This gives us a map

\[
N \times (N')^2 \xrightarrow{\begin{aligned} 1 \times \text{id} & \quad \text{and} \\ \rho_{E1} & \end{aligned}} N \times N' \xrightarrow{\begin{aligned} 1 & \quad \text{and} \\ \rho_1 & \end{aligned}} N
\]

and hence a relative spectral sequence

\[
E_1^{pq} = \bigwedge^q \text{Norm} \left( N \times N', E \perp N \right) \xrightarrow{\begin{aligned} 1 \times \text{id} & \quad \text{and} \\ \rho_1 & \end{aligned}} N \times N' \xrightarrow{\begin{aligned} 1 & \quad \text{and} \\ \rho_1 & \end{aligned}} N
\]

\[
E_1^{pq} = \bigwedge^q \text{Norm} \left( (N, E \perp N) \times N', E \right) = \bigwedge^q \text{Norm} \left( (N, E \perp N) \times N' \right) = H_q \left( (N, E \perp N) \times N' \right)
\]

So now suppose \( H_q(N, E \perp N) = 0 \) for \( q < k \).
whence \( E'_{p \theta} = 0 \) for \( \theta < r \) and we have

\[
E'_{p \theta} = H_n(N, E \perp \eta)
\]

Now \( d_i : N \times (N^p)^p \rightarrow N \times (N^p)^p \)

\[
(N, x_1, \ldots, x_p) \mapsto \begin{cases} 
N + x_1, x_2, \ldots, x_p & i = 0 \\
N, x_1, x_2, \ldots, x_{i+1}, x_p & 1 \leq i \leq p \\
N, x_1, \ldots, x_{p-1} & i = p
\end{cases}
\]

Because the \( E \perp \eta \)

\[
H_n((N, E \perp \eta) \times N^p) = H_n(N, E \perp \eta)
\]

induced by a basepoint \( pt \rightarrow N^p, \cdot \rightarrow E \perp \eta E \)

it follows that \( d_i \) on \( E'_{p \theta} \) is the identity for \( 1 \leq i \leq p \) and for \( i = 0 \) the map induced by \( N \rightarrow N \perp E \). Denote this by \( \cdot e \).

So the \( E'_{p \theta} \)-term looks:

\[
H_n(N, E \perp \eta) \rightarrow H_n(N, E \perp \eta) \rightarrow H_r(N, E \perp \eta)
\]

\[
p = 2 \quad p = 1 \quad p = 0
\]

So \( \cdot e \) is a projection on \( H_n(N, E \perp \eta) \) and \( \cdot e - \text{id} \)

is onto so \( \cdot e = 0 \). So we get nothing at all this way.
\[ \pi_1(M/\mathcal{M}a) = 0 \] geometrically. Hence we want to show that any covering of \( M \) which is trivial over \( \mathcal{M}a \) is trivial. So let \( F \) be a covering, i.e. a functor from \( M \) to sets. I am assuming that if \( N = N' \sqcup N'' \) with \( N', N'' \) infinite, then \( \text{Aut}(N') \) acts trivially on \( F(N) \). It follows that \( \text{Aut}(N'') \) acts trivially also.

Now also I know that
\[ \ldots M \times M' \times M \Rightarrow M \times M \]
is hom to \( M \). This means that to give a covering of \( M \) is the same as giving a covering of \( M \times M \) equipped with descent data. Specifically this means that if I give a functor \( F' \) on pairs \( N', N'' \) together with isos,
\[ F'(N' \sqcup X, N'') = F'(N', X \sqcup N'') \]
satisfying some sort of transitivity, then \( F'(N', N'') \) depends only on \( N' \sqcup N'' \). Therefore it should follow that because \( F(N) \) is acted on trivially by \( \text{Aut}(N') \times \text{Aut}(N'') \), it is a trivial \( \text{Aut}(N) \)-set.

Specifically the argument goes as follows: To show \( g \in \text{Aut}(N) \) acts trivially on \( F(N) \): This depends only on the splitting \( gN' \sqcup gN'' = N \) which is a vertex.
in the contractible complex $X(N)$. So we choose a path $x_0, x_1, \ldots, x_n = x$ in the complex and put $x_i = g_i - g_i x_0$. So it is enough to worry about a $g$ which gives rise to a one-simplex in $X(N)$.

\[ N = N_1 \sqcup N_2 \sqcup N_3 \]

But I can choose this isom. to be the identity on an infinite subset of $N$. 
November 10, 1975

Fix $A$ in $M$. Assuming $M$ is associative one has that the maps $\lambda_A(M) = A \cdot M$, $\rho_A(M) = M \cdot A$ commute, hence $\lambda_A$ induces a map

$$\overline{\lambda}_A : \text{e} M / M \cdot A \rightarrow \text{e} M / M \cdot A$$

where $\text{e} M / M \cdot A$ denotes the cone of $\rho_A$. I want to show that $\overline{\lambda}_A$ is null-homotopic using the commutativity isomorphism.

Recall properties of the cone. Given $f : x \rightarrow y$, one puts $\text{Cone}(f) = \text{Cyl}(f) / x \times 0$ where $\text{Cyl}(f) = x \times [0,1] \supset y \times 1$.

Given

$$x \xrightarrow{g'} x'$$

$$\begin{array}{ccc}
  f & \downarrow h & f' \\
  \downarrow h & & \downarrow h \\
  y & \xrightarrow{h \circ g} & y'
\end{array}$$

$h$ a homotopy $g' \leftarrow f \rightarrow f'$. One constructs a map $\text{Cyl}(f) \rightarrow \text{Cyl}(f')$ as follows. First one forms the comm. square

$$\begin{array}{ccc}
  x & \xrightarrow{g'} & x' \\
  \downarrow h \circ g & & \downarrow h \\
  \text{Cyl}(f) & \xrightarrow{h \circ g} & y'
\end{array}$$

$X \xrightarrow{\text{ht}_G} Y$.

$$\begin{array}{ccc}
  y & \mapsto & \text{ht}_G (y) \\
  (x,t) & \mapsto & h_t(x)
\end{array}$$

$h_0(x) = f' \circ g'$

$h_1(x) = g \circ f$
Then one takes cylinders
\[ \text{cyl}(i_0) \rightarrow \text{cyl}(f') \]
and identifies \( \text{cyl}(i_0) \) with \( \text{cyl}(f) \).

If we have
\[
\begin{array}{ccccccc}
X & \xrightarrow{g_1} & X' & \xrightarrow{g_2} & X'' \\
\downarrow h & \downarrow f' & \downarrow h_2 & \downarrow f'' \\
Y & \xrightarrow{g_1} & Y' & \xrightarrow{g_2} & Y''
\end{array}
\]
then the map \( \text{cyl}(f) \rightarrow \text{cyl}(f') \rightarrow \text{cyl}(f'') \) is homotopic to the map \( \text{cyl}(f) \rightarrow \text{cyl}(f'') \) associated to \( g \circ h_1 + h_2 \circ g \).

I want to apply this to the following situation:
\[
\begin{array}{cccc}
M & \xrightarrow{\lambda_A} & M & \xrightarrow{id} & M \\
S_A & \downarrow & \downarrow h_1 & \downarrow \text{id} & \downarrow S_A \\
M & \xrightarrow{id} & M & \xrightarrow{\lambda_A} & M
\end{array}
\]
Here \( h_1 \) is the homotopy from \( \lambda_A \) to \( S_A \) furnished by the
Commutativity axiom. \( \theta : A \perp X \to X \perp A \). 

\( h_2 \) is furnished by the inverse of \( \theta \). What is the composite homotopy from \( \lambda_A \circ \text{id} \circ \rho_A \) to \( \lambda_A \circ \text{id} \circ \rho_A \)?

\[
(p_A \circ \text{id} \circ \lambda_A)(x) = (A \perp X) \perp A \\
\sim (\lambda_A \circ \text{id} \circ \lambda_A)(x) = A \perp (A \perp X) \\
\sim (\lambda_A \circ \text{id} \circ p_A)(x) = A \perp (X \perp A)
\]  

(by \( h_2 \))

(by \( \lambda_A \))

However, the coherence axiom for commutativity says that

\[ (A \perp X) \perp B \sim A \perp (X \perp B) \]

\[ A \perp X \perp B \sim X \perp A \perp B \]

\[ X \perp B \perp A \]

commutes. The isomorphism I'm after is the special case when \( B = A \) of

\[ A \perp X \perp B \sim B \perp (A \perp X) \sim B \perp (X \perp A) \]
Unfortunately this automorphism of \( A \times X \times A \) is not the identity:

\[
(a, x, a') \mapsto (a', x, a)
\]

This argument does show that \( \mathcal{I}(\overline{T}_A)^2 \) is null-homotopic, because let \( \mathcal{V} \) be the endomorphism of \( \text{Cone} f_A \) that we have defined above using the comm. isom. Then it’s clear that \( \mathcal{Y}^2 \) is the same as \( \mathcal{I}(\overline{T}_A)^2 \), because the problem with the commutativity isomorphism is of order 2. On the other hand, \( \mathcal{Y} \) is null-homotopic because \( \mathcal{Y} \) factors through \( \text{Cone}(\text{Id}_m) \).

Since \( A^2 = A \), one sees that \( \overline{T}_A \) is idempotent, \( \overline{T}_A^2 = \overline{T}_A \), hence null-homotopic.

Program: I want to give a geometric proof that \( M/\Sigma(A) \) is contractible; the point is not to use spectral sequences but instead to see what spaces actually occur.

Return to the space

\[
\exists \quad M \times M \times M \Rightarrow M \times M \rightarrow M
\]
and form the cone on the map $p_A$:

$$(m \times (m/ma)^2) \wedge (m/ma) \Rightarrow (m \times m) \wedge (m/ma) \Rightarrow m \wedge (m/ma)$$

This space has an augmentation to $m/ma$ which is a reg. Introduce the skeleta of the realization. Let $Y$ be the realization of the above simplicial space and let $F_p Y$ be its skeleta. Then

$$F Y / F_{p-1} Y = Y / Y_{\text{deg}} \wedge \Delta(p) / \partial \Delta(p)$$

$$= (m^{p+1} \cup pt) \wedge (m/ma) \wedge S^0$$

Assume I know that $H_*(m/ma)$ begins in degree $p$ and $H_*(F Y / F_{p-1} Y)$ begins in degree $p+2$. Hence

$$H_* (F_0 Y) \Rightarrow H_* (F_1 Y) \Rightarrow H_* (F_2 Y) \Rightarrow \cdots \Rightarrow H_* (Y)$$

But $F_0 Y = (m \cup pt) \wedge (m/ma)$ and $M$ is connected, so $H_* (m/ma) = H_* (F_0 Y)$. So

$$H_* (m/ma) \Rightarrow H_* (Y) = H_* (m/ma)$$

But on the other hand I know this map is zero. \: $H_* (M/ma)$ begins in degree $r+1$. etc.
I have a similarity in the preceding with Čech cohomology in sheaf theory. I should go over the latter.

So let \( X \) be a space. Given a presheaf \( F \) and a covering \( U \) of \( X \) I get a complex

\[
C^*(U, F)
\]

whose homology groups one denotes \( H^*(U, F) \). Then

\[
\check{H}^*(X, F) = \lim_{\longrightarrow} H^*(U, F).
\]

is the Čech cohomology.

What is the nature of \( C^*(U, F) \)? A presheaf is a functor on the category of open sets. Associate to \( U \) the sieve \( S(U) \) consisting of \( \{ U_i \} \) open sets contained in members of \( U \). I claim that

\[
H^*(U, F) = \operatorname{R} \lim_{\longrightarrow} F
\]

I have to check the effaceability. To put it another way, I can show that if \( \{ U_i \} \) is a family of object covering \( \mathcal{R} \) such that all fibre
products exist: $U_{i_1} \times \ldots \times U_{i_p}$, then the nerve:

$$
\exists \prod_{(i, j) \in I} U_{i} \times U_{j} \Rightarrow \prod_{i \in I} U_{i}
$$
is acyclic. This is clear.


So I see that $H^*(U, F)$ is just the cohomology of the presheaf $F$ pulled back to the subcat $R(U)$ of $\text{Open}(X)$.

$$
R(U) \leftarrow \text{Open}(X)
$$

$$
\begin{array}{ccc}
\text{Open}(X) & \overset{\delta_i}{\leftarrow} & R(U) \\
\downarrow & & \downarrow \\
\text{Open}(X)^\wedge & \overset{i^*_!}{\rightarrow} & R(U)^\wedge
\end{array}
$$

$$(i^*_!(F))(V) = \lim_{U \in R(U), V \subseteq U} F(U)$$

$$=
\begin{cases}
0 & V \in R(U) \\
F(V) & V \in R(U)
\end{cases}$$

Since $i^*_!$ is exact, it follows that $i^*$ preserves injectives.

**Lemma:** If $F$ is an injective presheaf, then $F$ restricted to any cribble $R$ is also injective.
Alternative approach to stability. Let me consider the analogue of the unimodular complex. Fix $N$ in $\mathcal{N}$ and consider the set of all embeddings $u: N \to \mathcal{N}$ with infinite complement. Make these into a simplicial complex by calling $(u_0, \ldots, u_p)$ a simplex if $u_0 N \supseteq \cdots \supseteq u_p N$ embeds in $\mathcal{N}$ with infinite complement.

If the unimodular complex is contractible, then I get an acyclic complex

$$\cdots \to \bigoplus_{(u_0, u_1)} \mathbb{Z} \to \bigoplus_{u_0} \mathbb{Z} \to \mathbb{Z} \to 0$$

(this is not the complex of chains on the simplicial complex, but it should still be acyclic). This complex should furnish a spectral sequence

$$E^1_{pq} = H_q(G)$$

with each $d_1$ multiplication by $a$. Thus

$$d_1 = \begin{cases} \text{mult by } a & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$$

Then I can use induction: If $H_q(G) = 0$, $1 \leq q \leq r$

then $E^2_{0,n} = \text{Coker } \{ H_n(G) \to H_n(G) \} = 0$, so $a$ is onto.
Also \( E_{1,\alpha}^2 = \ker \left[ H_\alpha(G) \to H_\alpha(G) \right] \) is zero, so \( \alpha_\cdot \) is injective.

Suppose we try to prove contractibility of the unimodular complex by a variant of Kervaire–Lusztig.

Let \( F \) be a finite subcomplex. I want to regard a simplex in the unimodular complex as giving me a partition of \( N \) into infinite pieces. Take the partitions of \( N \) associated to the simplices of \( F \) and form their infimum, i.e., the finite non-empty intersection. Now throw away the finite sets in this partition. Then one gets \( N = A_1 \cup \ldots \cup A_m \) (finite set) such that for any simplex \( \sigma \) each \( A_i \) is a subset of one of the blocks of \( \sigma \).

Let \( F_i \) be the subcomplex of \( X(N) \) consisting of simplices \( \alpha_i : N^p \to N \) such that \( A_i \) is in the complement of this embedding. Choose \( V_i : N \to A_i \) with inf. complement for each \( i = 1, \ldots, m \).

Then the vertex \( V_i \) can be joined to \( F_i \).

Furthermore the simplex \( \{ V_{i_1}, \ldots, V_{i_p} \} \) can be joined to
$F_{i_1} \cap \ldots \cap F_{i_p}$ for $1 \leq i_1 < \ldots < i_p \leq m$. It follows (I think) that $\bigcup_{i=1}^{m} F_i$ can be contracted to a point in $\prod_{i=1}^{m} X(N)$.

Furthermore, I claim $F \subseteq \bigcup_{i=1}^{m} F_i$. In effect, given $u : N^p \to N$, we know each $A_i$ is contained in $\text{Im} \ u$ or in the complement of $u$, and not all $A_i$ are contained in $\text{Im} \ u$ as $N - UA_i$ is finite. Hence $u \in F_i$ for some $i$.

Thus it is clear that the unimodular complex is contractible, and so we again get a stability result.

Let $S$ be a set. Form the complex

$$\to \bigoplus \mathbb{Z} \to \prod_{(s_0, \ldots, s_p)} \mathbb{Z} \to \prod_{(s_0, \ldots, s_p)} \mathbb{Z} \to 0$$

where the sum is taken over the sequences $(s_0, \ldots, s_p)$ of distinct elements of $S$. What is the homology of this complex? Let's use the Kervaire–Lusztig argument.

Let $F_s$ be the subcomplex formed using the set $S - \{s_i\}$. We try to show that any cycle $Z_s$ with support in $F_{s_1} \cup \ldots \cup F_{s_p}$ is homologous to one in $F_{s_1} \cup \ldots \cup F_{s_{p-1}}$.

For example, let $Z$ have support in $F_{s_1}$. Here $S = \{s_0, s_1\}$. 


Then let \( T_1 \) be the cone operator
\[
T_1(s_0, \ldots, s_m) = (1, s_0, \ldots, s_m)
\]
so that \( dT_1 + T_1 d = id \). More precisely,
\( T_1 : F_1 \to C \) and
\[
dT_1(s_0, \ldots, s_m) = (s_0, \ldots, s_m) - \sum (-1)^i (1, s_0, \ldots, \hat{s_i}, \ldots, s_m)
\]
so \( dT_1 + T_1 d = \text{inclusion} \ F_1 \to C \).

So if \( z \) is a cycle in \( F_1 \), then \( dT_1 z = z \), so \( z \) is homologous to zero.

Let \( z \) be a cycle in \( F_1 + \cdots + F_p \) and write
\[
z = u \circ \nu
\]
with \( u \in F_1 + \cdots + F_{p-1} \) and \( \nu \) in \( F_p \).

Then \( du = dv \in (F_1 + \cdots + F_{p-1}) \cap F_p \). Consider
\[
d(T_p \nu) + T_p (dv) = \nu
\]
Then
\[
z = u - \nu = u - T_p (dv) - d(T_p \nu)
\]
so all that remains is to show \( T_p (dv) \in F_1 + \cdots + F_{p-1} \), i.e. that \( T_p ((F_1 + \cdots + F_{p-1}) \cap F_p) \subset F_1 + \cdots + F_{p-1} \). To let \( (s_0, \ldots, s_k) \in F_i \cap F_p \). Then \( T_p (s_0, \ldots, s_k) = (p, s_0, \ldots, s_k) \) still belongs to \( F_i \). Thus everything works and we have proved:
Prop: The complex

\[ \rightarrow \bigoplus \mathbb{Z} \rightarrow \cdots \rightarrow \bigoplus \mathbb{Z} \rightarrow \bigoplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \]

\( (s_1, \ldots, s_8) \) runs over sequences of distinct elements of \( S \) is acyclic except in dimension \( n = \text{card}(S) \).

Consider the following poset \( J \) attached to a set \( S \). The elements of \( J \) are finite sequences \( (s_1, \ldots, s_8) \) of distinct elements of \( S \). In other words, embeddings \( u: \{1, \ldots, 8\} \rightarrow S \).

We have \( (s_1, \ldots, s_8) \leq (t_1, \ldots, t_p) \) iff \( \exists \) \( 1 \leq i_1 < i_2 < \cdots < i_8 \leq p \) such that \( s_i = t_{i_j}, 1 \leq j \leq 8 \).

In other words, \( s: \{1, \ldots, 8\} \rightarrow S \) is \( \leq t: \{1, \ldots, p\} \rightarrow S \) iff there exists a monotone map \( i: \{1, \ldots, 8\} \rightarrow \{1, \ldots, p\} \) such that

\[ \{1, \ldots, 8\} \xrightarrow{i} \{1, \ldots, p\} \]

\[ s 

commutes. Note that \( i \) is unique if it exists.

Next note that for any element \( \tau \) of \( J \) the poset \( \{ \tau | \tau \leq \sigma \} \) is isomorphic to the set of proper subsets of \( \{1, \ldots, 8\} \) if \( q = \text{size of } \tau \). Thus
it should be that the complex in the proposition is the Lustig sequence associated to the poset \( \mathbb{T} \), hence \( \mathbb{T} \) should be spherical.

(But now recall what you were doing about stability for the symmetric groups. I formed a category \( C \) of pairs \( (S_1, S_2) \) of finite sets of same card such that a map \( (S_1, S_2) \to (S_1', S_2') \) consists of a pair of injections \( i_1, i_2 \) isomorphic between the complements. Now you wanted to see what the relative terms were if you filtered by size.

Thus fix \( (S, S) \) and you want to calculate \( C/(S, S) \), which can be identified with the poset consisting of a pair of splittings

\[
\begin{align*}
S &= S_1' \sqcup S_1'' \\
S &= S_2' \sqcup S_2''
\end{align*}
\]

together with an isomorphism \( S_1'' \sim S_2'' \). Also we want \( S_2'' \sqcup S_1' \neq \emptyset \). Thus this poset is not the same as \( \mathbb{T} \) above.)
Let $V$ be a vector space over $k$ of dim. $V_0$. Let $X(V)$ be the poset consisting of subspaces of inf. dim and codim, in which $W_1 < W_2$ if $W_1 \subseteq W_2$ and $W_2/W_1$ has inf. dim. I have seen that $X(V)$ is contractible. $X(V)$ gives a spectral sequence abutting to $0$ with $E_1$-term:

\[ \cdots \rightarrow H^*(\bigtriangleup) \rightarrow H^*(\square) \rightarrow H^*(\bigcirc) \]

Let $V_0$ be an elt of $X(V)$ and let $Y(V, V_0)$ denote the subposet consisting of subspaces $W$ such that $W + V_0 = V$ and $W \cap V_0 \in X(V_0)$. (Thus)

\[ \begin{array}{c}
\downarrow \\
V \\
\downarrow \\
W \\
\downarrow \\
W \cap V_0 \\
\downarrow \\
0,
\end{array} \]

I let $\mathcal{O} \cdot \text{Aut}(V \cup V_0) = \left( \frac{\text{id}_{V/V_0}}{x} \bigg| \text{Aut}(V_0) \right)$ act on $Y(V, V_0)$ and I get a spectral abutting to zero with $E_1$-term.
\[
H_x\begin{pmatrix}
1 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix} \rightarrow H_x\begin{pmatrix}
0 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix} \rightarrow H_x\begin{pmatrix}
0 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix}
\]

Suppose I try to understand \( H_1 \).

\[
H_1\begin{pmatrix}
1 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix} \leftrightarrow H_1\begin{pmatrix}
\star \\
1 \\
\star \\
\star \\
\star \\
\end{pmatrix} \oplus H_1\begin{pmatrix}
0 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix}
\]

\[
H_1\begin{pmatrix}
1 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix} \leftrightarrow H_1\begin{pmatrix}
\star \\
\star \\
\star \\
\star \\
0 \\
\end{pmatrix} \oplus H_1\begin{pmatrix}
1 \\
\star \\
\star \\
\star \\
0 \\
\end{pmatrix}
\]

Now, stably, I should know that

\[
H_8\begin{pmatrix}
1 \\
0 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix} \leftarrow H_8\begin{pmatrix}
1 \\
0 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix}
\]

is the zero map, because I have \( \infty \) sums. This I find that

\[
H_1\begin{pmatrix}
1 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix} \leftarrow H_1\begin{pmatrix}
1 \\
0 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix} \leftarrow H_1\begin{pmatrix}
1 \\
0 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix}
\]

is onto giving me \( H_1\begin{pmatrix}
1 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix} = H_x\begin{pmatrix}
1 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix} \)

as well as

\[
H_8\begin{pmatrix}
\star \\
0 \\
\star \\
\star \\
\star \\
\end{pmatrix} \rightarrow H_1\begin{pmatrix}
\star \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix}
\]

\[
H_1\begin{pmatrix}
0 \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix} \rightarrow H_1\begin{pmatrix}
\star \\
\star \\
\star \\
\star \\
\star \\
\end{pmatrix}
\]

implying \( H_1\begin{pmatrix}
1 \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
\star \star \star \\
0 \\
\end{pmatrix} = 0 \).
Therefore it seems that I get at least over a field.

Go back to stability for $\Sigma_n$. I have seen that the map

$$H_*(\Sigma_n, \Sigma_{n-1}, \alpha) \xrightarrow{a.} H_*(\Sigma_{n+1}, \Sigma_{n+1}, \alpha)$$

when iterated has square zero.

**Hypothesis:** The image of the preceding map is killed by 2. Thus, if we invert 2, this map is zero. Let's see if I can establish linear stability with this hyp. No

So I will use the complex

$$0 \to J_n \to \bigoplus \mathbb{Z}_{(s_i, s_i^*)} \to \bigoplus \mathbb{Z} \to \mathbb{Z} \to 0$$

in which $\Sigma_n$ acts, and the corresponding one for $\Sigma_{n+1}$.

$junk \to junk \to H_*(\Sigma_n, \Sigma_0) \to \ldots \to H_*(\Sigma_{n+1}, \Sigma_{n+1})$

To get $E^2_{0,h} = H_*(\Sigma_{n+1}, \Sigma_n^\alpha) = 0$ we need to know $H_{n-1}(\Sigma_{n-1}, \Sigma_{n-2}) = 0$. Doesn't work.
Let $A$ be a ring, let $M$ be an $A$-module. The poset of frames of $M$, denoted $\text{F}(M)$, consists of unimodular sequences $(u_1, \ldots, u_p)$ in $M$ with the inclusion ordering. I assume that $A$ is such that

Let's axiomatize the arguments a little.

If $A$ is a ring let $\text{F}(A)$ be the poset consisting of sequences $(a_1, \ldots, a_p)$ in $A$ such that there exists an element $a_{p+1}$ such that

$$A^{p+1} \xrightarrow\sim A, \quad (x_1, \ldots, x_{p+1}) \mapsto \sum x_i a_i$$

Assume $\text{F}(A)$ is contractible. Then it gives me a Lustig sequence:

$$\rightarrow \oplus \mathbb{Z} \quad \rightarrow \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

and hence a spectral sequence abutting to 0 with

$$E'_p = \begin{cases} H_\delta \left( \frac{1 \times \text{G}}{\text{G}} \right) & p > 0 \\ H_\delta \left( \text{G} \right) & p = 0 \end{cases}$$

If $H_x \left( \frac{1 \times \text{G}}{\text{G}} \right) = H_x \left( \text{G} \right)$, then $d_1$ becomes 0 in even degrees, multi. by $a$ in odd degrees, so we get
November 17, 1975.

Let $S$ be a finite set of card $n$. Let $Y_2(S)$ be the simplicial complex whose $p$-simplices are $\{V_0, \ldots, V_p\}$, where $V_i$ is a card 2 subset of $S$ and $V_0, \ldots, V_p$ are disjoint. Then $\dim Y_2(S) = \left[\frac{n}{2}\right]-1$.

I would like to show that $Y_2(S)$ has the homotopy type of a bouquet of spheres of dimension $\left[\frac{n}{2}\right]-1$.

Fix $s_0 \in S$ and let $S' = S - \{s_0\}$. One obtains $Y_2(S')$ from $Y_2(S)$ by removing simplices containing a vertex $v$ with $v = \{s, s_0\}, s \in S'$. Thus

$$Y_2(S) = \bigcup_{s \in S'} Y_2(S') \cup \text{Cone Link } \{s, s_0\} \cup \text{Link } \{s, s_0\}$$

Moreover, $\text{Link } \{s, s_0\} = Y_2(S' - \{s\})$. Now if $n$ is odd, we know that $Y_2(S')$ is a bouquet of $m-1$ spheres, and $Y_2(S' - \{s\})$ is a bouquet of $(m-2)$-spheres. Thus $Y_2(S)$ will be a bouquet of $m_\alpha = \left[\frac{n}{2}\right]-1$ spheres. The critical case then is when $n = \text{card } S$ is even $= 2m$.

Suppose $n = 2m$. We know $Y_2(S')$ is a bouquet of $(m-2)$-spheres and also the same is true for $Y_2(S' - \{s\})$. We
have

\[ \tilde{H}_{m-1}(Y_2(S')) \rightarrow \tilde{H}_{m-1}(Y_2(S)) \rightarrow \bigoplus_{A} \tilde{H}_{m-2}(Y_2(S'-a)) \]
\[ \circlearrowleft \tilde{H}_{m-2}(Y_2(S')) \rightarrow \tilde{H}_{m-2}(Y_2(S)) \rightarrow 0 \]

Thus what I need to know is that \( A \) is surjective. Now the preceding argument showed that

\[ \tilde{H}_{m-2}(Y(S')) \leftarrow \tilde{H}_{m-2}(Y(S'')) \bigoplus \bigoplus_{A \in S''} \tilde{H}_{m-3}(Y(S''-a)) \]
\[ \uparrow_{2m-1} \uparrow_{2m-2} \]

It appears therefore that the correct inductive hypothesis involves surjectivity of

\[ \bigoplus_{A \in S} \tilde{H}_{m-2}(Y_2(S-a)) \rightarrow \tilde{H}_{m-2}(Y_2(S)) \]

if \( |S| = 2m+1 \). Granting this we see that \( \tilde{H}_{m}(Y(S)) \) is concentrated in degree \( m-1 \) if \( n = 2m+2 \).

Now suppose \( n = 2m+1 \), whence we have

\[ 0 \rightarrow \tilde{H}_{m-1}(Y_2(S')) \rightarrow \tilde{H}_{m-1}(Y_2(S)) \rightarrow \bigoplus_{A \in S'} \tilde{H}_{m-2}(Y_2(S'-a)) \rightarrow 0 \]

Applying induction hypothesis to \( Y_2(S'-a) \) we know that \( \tilde{H}_{m-2}(Y_2(S'-a)) \) is generated by \( \tilde{H}_{m-2}(Y_2(S'-a-t)) \). Thus
\[ \oplus \tilde{H}_{m-2} \left( Y_2 \left( S - \{s_0, s, t\} \right) \right) \]

\[ H_{m-1} \left( Y_2 (S) \right) \rightarrow \oplus \tilde{H}_{m-2} \left( Y_2 \left( S - \{s_0, t\} \right) \right) \]

Let's guess that the dotted arrow arises by means of the map

\[ \sum Y_2 \left( S - \{s_0, s, t\} \right) \rightarrow Y(S) \]

one obtains from the contractions using the vertices \{s_0, s\} and \{s, t\}.

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This just doesn't work. If \( \text{card}(S) = 4 \), then the simplicial complex \( Y_2(S) \) is a disjoint union of 1-simplices, hence it is not connected.
Recall Legals funny idea about group completion. He takes a free monoid \( M \) and constructs \( M[a^{-1}] \). Actually it is enough to construct \( M a^{-\infty} M \) for this has the correct homology and probably the correct fundamental group.

First step: analyze \( Ma^{-1}M \). We have cocart.

\[
(Ma \times M) \cup (M \times aM) \subset M \times M
\]

\[
M \subset Ma^{-1}M
\]

Now lets find a category having the homotopy type of \( Ma^{-1}M \).

Recall that if \( G \) is a group with subgroups \( G_1, G_2 \) we know how to interpret the space

\[
B(G_1 \cap G_2) \cup BG_1 \cap BG_2
\]

as a category, namely, as the fibred category over \( G \) with fibre the poset of left cosets for the family \( G_1, G_2, G_2 \) of \( G \).

So it seems that the category I seek consists of finite sets \( E \) with autors and pairs \( (E, F) \) with
autos. A morphism $E \rightarrow (F_1, F_2)$ should consist of a reduction of $(F_1, F_2)$ to $(\text{Max} M \cup (M \times aM))$ (which means we fix an element of $F_1$ or $F_2$ or both) and an isomorphism of $E$ with $F_1 \cup F_2$ minus this element. This won’t work very simply.

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Form the category $C$ consisting of triples $(E, k, F)$ where $E, F$ are (say sets) and $k$ is an integer. A map $(E, k, F) \rightarrow (E', k', F')$ consists of a pair of isos.

$$E \sim E' \oplus A^\mu$$

$$F \sim A' \oplus F'$$

such that $\mu + k' = k$. Think of $(E, k, F)$ as $E \oplus A^{-k} \oplus F$; $E' \oplus A^{-k'} \oplus F' = E' \oplus A^\mu \oplus A^{-k-k'} \oplus A' \oplus F' = E' \oplus A^\mu \oplus F$.

What is the homotopy type of $C$?

$$\{(E, k)\} \times \{(l, F)\} \rightarrow C$$

$$(E, k), (l, F) \mapsto (E, k+l, F)$$

Here $(E, k)$ denotes the fibred category over the ordered set $\mathbb{I}$ associated to the functor.
\( k' \leq k \rightarrow (E \rightarrow E \oplus A^{k-k'}) \)

\( \{ (l, F) \} \) is similarly defined. Call this function \( f \).

\( (E, m, F) \backslash f \) consists of \((E', k'), (C', F')\) plus nicer.

\( E \oplus A^{\nu} \rightarrow E' \)

\( F \oplus A^{\mu} \rightarrow F' \)

Thus I should be able to identify \((E, m, F) \backslash f \) with \( (\nu, \mu, k', l') \) such that \( \nu + m + \mu = k' + l' \).

with \( (\nu_1, \mu_1, k'_1, l'_1) \leq (\nu_2, \mu_2, k'_2, l'_2) \)

meaning \( \nu_2 - \nu_1 = k'_2 - k'_1 \geq 0 \)

\( \mu_2 - \mu_1 = l'_2 - l'_1 \geq 0 \).

Thus doesn't work.

Conjecture: \( C \) seq to \((E, k') \times \{ (l, F) \}\).

Assume this for now. Inside of \( C \) we have the subcategory \( C_{<0} \) consisting of \((E, k, F)\) with \( k < 0 \).

To \((E, k, F)\) in \( C_{<0} \) we can associate the object

\( E \oplus A^{-k} \oplus F \)

of \( M \). Moreover to the arrow

\( (E', k', F') \rightarrow (E, k, F) \) in \( C_{<0} \) given by

\( E' \oplus A^{\mu} \approx E \)

\( A^{\nu} \otimes F' \approx F \)

\( \mu + k' + \nu = k \)
I can associate the isomorphism

\[ E' \oplus A^{-k} \oplus F' = E' \oplus A^\mu \oplus A^\nu \oplus A^{-k} \oplus F' \]
\[ \sim E \oplus A^{-k} \oplus F \]

Thus I have a functor from \( C_{<0} \) to \( M \) which carries arrows into isomorphisms.

**Question:** Fix an object \( M \) of \( M \) and look at \( C_{<0} / M \). What is this?

Objects of \( C_{<0} / M \) can be identified with decompositions of \( M \):

\[ M = E \oplus A^\mu \oplus F \]

A morphism \( (M = E' \oplus A^\nu \oplus F') \rightarrow (M = E \oplus A^\mu \oplus F) \) consists of \( E' \oplus A^\mu = E \), \( A^\nu \oplus F' \oplus F = F \), \( \mu + (-\rho') + \nu = -\rho \) or \( \mu + \rho + \nu = \rho' \). What this gadget is is clear—it is a poset of layers in \( M \) such that the layer has some sort of additional structure, i.e. reduction to an ordered free module.

Note that if \( E, F \in \text{N} \) is not the point category, then \( C \) is \( \text{N} \times \text{N} \) acting on \( \text{Z} \) which is not homotopy equivalent to \( \text{N} \) acting on \( \text{Z} \times \text{N} \) acting on \( \text{Z} \).
so the conjecture is wrong. However we have a functor

\[ C \rightarrow \langle \mathbb{N} \times \mathbb{N}, \mathbb{Z} \rangle \]

\[ (E, \kappa, \phi) \mapsto \kappa \]

with fibres \( M \times M \)
Let $M$ be a free simplicial monoid, and let $G$ be the associated simplicial group. I want to construct a category realizing $G$.

Look at $M(M)^{-1} \subset G$. We have a canonical reduced word description of the elements of $G$. Elements of $M$ are written $m = s_i \ldots s_k$, $s_j \in S$ the generating set for $M$. If

$$m(m)^{-1} = s_k \ldots s_1$$

then this nerve ought to be isomorphic to $M \cdot M^{-1}$

of the category defined by $\Delta M$ acting by right multiplication on $M \times M$, then this nerve ought to be isomorphic to $M \cdot M^{-1}$.

Next look at $M(M)^{-1} \cdot M$. I start with $M \times M^{-1} \times M$ which suffices to describe what's happening with reduced words of the form $s_1 \ldots s_l \ldots s_i \ldots s_k$, with $l \leq m$. Then I have to work in what happens when $l = m$.

We have a functor

$$M \times M \rightarrow M^m \times M^{-1} \times M$$
sending \((m, m') \mapsto (m, e, m')\) and we also have the product functor \(M \times M \to M\). So the conjecture is that the diagram

\[
\begin{array}{ccc}
M \times M & \xrightarrow{\cdot} & M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\cdot} & M \cdot M^{-1} \cdot M
\end{array}
\]

is homotopy-\(\simeq\) cartesian.

Take \(M = N\).

\(N \times N \times N\) modulo \(N \times N\) acting by \((x, y)\) \(\cdot\) \((m, n, p)\) = \((m+x, x+n+y, p+y)\)
Here's a crucial point where algebraic and topological $K$-theory differ. Topologically: $K^{-1}(X)$ is the Grothendieck group formed out of couples $(E, \Theta)$ where $E \in \mathcal{P}_X$ and $\Theta \in \text{Aut}(E)$; one introduces relations coming from exact sequences, homotopies of the auto. $\Theta$, and also $(E, \text{id}_E) \mapsto 0$.

Now suppose we were to try to do the same thing in algebraic $K$-theory. We form the Grothendieck group of couples $(E, \Theta)$. These are the same thing as $A[T, T^{-1}]$-modules which are finite and flat over $A$. If $A$ is a field this $K$-theory is a direct sum: Let $F = F^*$ be alg. closed. Then

$$K_*(\text{mod} \text{flow } F[T, T^{-1}]) = \bigoplus_{\lambda \in F^*} K_*(F) = \mathbb{Z}[F^*] \otimes K_*(F)$$

So far we have considered only the relations coming from exact sequences among the couples $(E, \Theta)$. So how might we handle homotopies?

Try $F^* \otimes K_*(F)$. Ignoring uniquely divisible stuff, this is

$$\text{Tor}_1(F^*, K_*(F)) = \text{Tor}_1(\mu, K_*(F)).$$
In degree $n$, it is $\text{Tor}_1(F^*, K_{n-1}(F)) = K_{n-1}(F)(\pm 1)$
and I want it to be something like $K^{-1}_n(F) = K_{n+1}(F)$

So in some funny way, what I have to do is to formulate some sort of algebraic $K^{-1}_x(A)$
which is to be constructed out of $(E, \Theta)$ and satisfies $K^{-1}_x(A) \cong K_{x+1}(A)$. This should be
what Vilardi + Wagner have done. Then I will want to relate $K^{-1}_x(F)$ to $K_x(F) \otimes (F, \Theta)$
with torsion coefficients present. At this point
we will get some sort of periodicity.

Now what has all this to do with topological periodicity?

Funny thing is that if I, instead of couples $(E, \Theta)$, consider $P \in P_{\Delta T^{-1}}$, then the
exact sequence relations give me the gps

$$K_x(A[T, T^{-1}])$$

so if I further kill the direct summand coming from $T=1$ I get

$$K_n(A[T, T^{-1}])/K_n A = K_{n-1} A$$ if $A$ reg.
Today's pairing:
\[
\text{Rep}(G, \mathbb{P}_A) \times \mathbb{P}_{A[G]} \to \mathbb{P}_A
\]
\[
(V, M) \to V \otimes_{A[G]} M
\]
Perhaps this induces a map
\[
K_i^J(BG; A) \otimes K_j^J(A[G]) \to K_{i+j}^J(A)
\]
Take \( G = \mathbb{Z} \). Then
\[
K_i^J(B \mathbb{Z}; A) = K_i^J(A) \oplus K_{i+1}^J(A)
\]
\[
K_i^J(A[T, T^{-1}]) = K_i^J(A) \oplus K_{i-1}^J(A)
\]
Thus the map is probably the usual cup product.

Notice how things are backward. \( A[T, T^{-1}] \) is the coordinate ring of \( \text{Spec} A \times \mathbb{G}_m \). Topologically
\[
K_q^J(X \times \mathbb{C}^*) = K_q^J(X \times S^1) = K_q^J(X) \oplus K_q^J(SX)
\]
\[
= K_q^J(X) \oplus K_q^J(SX)
\]
\[
= K_q^J(X) \oplus K_q^J(SX)
\]
\[
= K_q^J(X) \oplus K_{q+1}^J(X)
\]
However we have
$K_1(A[1, T^{-1}]) = K_1 A \oplus K_{1-1} A.$

Or in another direction I recall that topological connected $K^{-1}(X)$ can be defined using bundles over $X$ equipped with a decomposition relative to $S^1$, a $\mathbb{C}^*$. Over $\mathbb{C}$ a bundle $E$ decays rel. to $\mathbb{G}_m$ is the same as a bundle plus an automorphism.

**Idea:** A bundle $E$ with an auto $\Theta$ is something like a finite module over $\mathbb{G}_m$, i.e., a sheaf over $X \times \mathbb{G}_m$ proper over $X$. It is like a section of some gadget $\mathfrak{F}$ over $X \times \mathbb{G}_m$ having support proper over $X$. But I have seen in duality theory that the functor $\mathfrak{F}$ (when put into the derived category) is unusual. Let's guess that what I am after is something over $X \times \mathbb{P}^1$ which dies canonically on $X \times 0$ and $X \times \infty$, modulo stuff with support on $X \times 1$.

\[
\begin{align*}
\xymatrix{\ldots & K_0(X \times \mathbb{P}^1) \ar[r] & K_0(X) \times K_0(X) } \\
\xymatrix{\ldots & K_1(X \times \mathbb{P}^1) \ar[r] & K_1(X) \times K_1(X) }
\end{align*}
\]
This shows that

\[ 0 \rightarrow K_1(X) \rightarrow (?) \rightarrow K_0(X) \rightarrow 0. \]

More completely: \[ K_n(X \times \mathbb{P}^1) = K_n(X) \cdot 1 \oplus K_n(X) \cdot (\mathcal{O}(1) - 1), \]
and the latter factor dies on \( X \times 0 \) and \( X \times 1 \).

Now inside \( K_n(X \times \mathbb{P}^1) \) is \( \text{Im} K_n(X \times 1) \) which \( = K_n(X) \cdot (\mathcal{O}(1) - 1) \). Thus it does seem that what we get is \( K_1(X) \).

Perhaps I want to think of \( K_1(X) \) as being \( K_0 \) of \( n \) bundles over \( X \times \mathbb{G}_m \) with proper support over \( X \).