Wiener factorization: Let \( f(z) \) be a holomorphic function in a nbd of \( S^1 \) such that \( f(z) \) is positive definite for \( z \) in \( S^1 \). Then there exists a matrix function \( g(z) \) holomorphic inside \( S^1 \) as in a nbd of \( S^1 \) such that
\[
 f(z) = g(z)^* g(z), \quad z \in S^1
\]

Proof: Using \( f \) as a clutching function as gets a holomorphic vector bundle on \( P^1 \) which is topologically trivial, hence trivial via Grothendieck. This means \( f \) has a factorization
\[
 f = g_- - g_+
\]

where \( g_+ \) is holomorphic inside and in a nbd of \( S^1 \) and where \( g_- \) is holomorphic outside \( S^1 \) in a nbd of \( S^1 \) and at \( z = +\infty \). Moreover such a factorization is unique up to \( g_+ \mapsto c g_+ \)
\[
 g_- \mapsto c g_-. \quad \text{where} \ c \ \text{is an invertible matrix.}
\]

\[
 f(1) \mapsto f(-1)
\]

By the reflection principle
something here, I'm not sure about...?

\[ f(z) = f(z)^* \]

So

\[ g_+(z)^* = g_-(z)^* \]

is another factorization of \( f \). By uniqueness there is a constant matrix \( c \) in \( \text{GL}_n \) such that

\[ c g_+(z) = g_-(z)^* \]

\[ g_-(z) c^{-1} = g_+(z)^* \]

I will suppose \( g_+ \) chosen so that \( g_+(1) = 1 \). Then \( c = g_-(1)^* \) and \( c^2 f(1)^* = f(1) \) being positive definite has a positive definite square root. Then if

\[ g = c^{1/2} g_+ \]

we have

\[ g^*(z)^* = g_+(z)^* c^{1/2} = g(z) c^{-1/2} \]

so

\[ g(z)^* g(z) = g_-(z) g_+(z) = f(z) \]

as was to be proved.
Raghunathan's proof uses a density theorem of the following sort.

Claim: If $K'$ is dense in $\mathbb{R}K$ when $K$ is simply-connected, then it suffices to show $K'$ contains a nbhd. of 1 since $\mathbb{R}K$ is connected. Can find enough maps $SU_2 \to K$ so that

$$SU_2 \times \cdots \times SU_2 \to K$$

is a submersion at 1; hence any nbhd of 1 in $\mathbb{R}K$ lifts into $SU_2 \times \cdots \times SU_2$. This reduces us to the case of $SU_2$. By using

$$S^1 \times S^1 \times S^1 \to SU_2$$

we can suppose we have a small loop in the maximal torus of $SU_2$, which we want to approximate by a Laurent loop. Smoothly.

Next note that if $\gamma(t)$ is a path in $K$ starting at 1, then

$$\gamma(\frac{1}{n}) \to \exp(t\gamma'(0))$$

hence the closure of $K'$ will contain exponentials of tangent vectors to paths in $K'$ starting at 1.

Now I consider the path in $K'$, $K=SU_2$:

$$\left(z^{-1}P_{L_t} - I + P_{L_0}\right)\left(z^{-1}P_{L_t} + I - P_{L_0}\right)$$

where $L_t = \mathfrak{L}(1,t)$ and $P_{L_0}$ is the orthogonal projection on $L_t$. Calculation gives
\[ p_t = \left( \begin{array}{c} \frac{1}{t-1} \\ \frac{1}{t+1} \\ \frac{1}{t^2+1} \\ \frac{1}{t^2-1} \end{array} \right) \sim \left( \begin{array}{c} 1 \\ t \\ 0 \end{array} \right) \mod \mathfrak{h}^2 \]

hence the element in \( \mathfrak{h}' \) is
\[
\left( \frac{z}{t} \right) + \left( \frac{1}{t} \right) \left( \begin{array}{cc} z^{-1} & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & (z-1)t \\ (z-1)t & 1 \end{array} \right) \]

Similarly replacing \( z \) by \( z^n \) and adding we can get the tangent vectors
\[
\begin{pmatrix}
1 & a(z)t \\
-a(z)t & 1
\end{pmatrix}
\]
t² = 0

where \( a(z) \) is any Laurent polynomial vanishing at 1.

Observe that the matrices of the form
\[
\begin{pmatrix}
a & 0 \\
-a & 0
\end{pmatrix}
\quad a \in \mathbb{C}
\]
form the space \( \mathfrak{h}' \) for \( SU_2 \) and the diagonal maximal torus. Hence we have a fairly systematic way of approximating loops of the form
\[
\exp \left( \begin{pmatrix} 0 & a(z) \\ -a(z) & 0 \end{pmatrix} \right)
\]
by Laurent polynomial loops.
Direct proof that $U_n \to \Omega U_n$ is a (weak) homotopy equivalence along Atiyah-Bott lines.

Start with a family $X \to \Omega U_n$ with $X$ compact. By Fejer approximation, one can homotop it into a family of loops given by Laurent polynomial matrices without singularities on $S^1$. Shifting by mult. by $z^n$, we can assume the family is given by polynomials matrices, non-singular on $S^1$. Then one forms the associated outgoing space $\mathcal{X} \times H^n$ which is of finite codim in $H^n$. We know that $\mathcal{X}$ is homotopic to the family of scattering matrices associated to $\mathcal{X}$.

So what one wants to show is that the space of outgoing subspaces $D$ in $H^n$ of a given codimension is homotopy equivalent to the subspace $\mathcal{D}$ consisting of $D$ such that $D \to \mathcal{X} = H^n$. But by triangulation, $\mathcal{D}$ is a strong def. retract of a nbd. $U$ of $\mathcal{D}$ in $S$. And the family of auto. $z \to az$ pull s. down into $U$. 

\[
[\omega_1 \Lambda \Omega \Lambda = (\omega_2) \Lambda] \quad (a + b) \Lambda 
\]

Suppose I consider an orbit in the building $\eta_0 \in G/P$ for some point $\eta$ in the fundamental chamber $C_0$. I have seen that the $B^u$-orbits (corresponding to $C_0$) on $K\eta_0$ are indexed by $W\eta_0 = K\eta_0 E$.

Let $\eta = w\eta_0 \in W\eta_0$. I have constructed a compactification of $B^u\eta$ using the space $\Gamma(s_1, \ldots, s_n)$ of galleries $w_0, w_1, \ldots, w_n$. We have $s_i = a_i$ is a minimal sequence of fundamental reflections such that $\eta = A_{s_1} \cdots A_{s_n} \eta_0$. As

$$\Gamma(s_1, \ldots, s_n) = \bigl( B u B a_i B \bigr)^B \times \cdots \times B (B u B a_n B) / B$$

I saw that the $B^u$ closure of $B^u\eta$ consists of the $B^u$-orbits of the point

$$s_1, \ldots, s_i \eta_0$$

where $1 \leq i_1 < \cdots < i_p \leq n$ is any subset of $\{1, \ldots, n\}$. 
What is the difference between $S_1 \ldots S_n$ and $S_1 \ldots \hat{S}_i \ldots S_n$ on the other? If

$$u S_1 \ldots S_n = S_1 \ldots \hat{S}_i \ldots S_n$$

then

$$u = S_1 \ldots S_{i-1} S_i^{-1} S_{i+1}^{-1} \ldots S_n$$

is conjugate to $S_i$. Thus $u$ is a reflection. In fact, if we recall that $S_i = S_{i-1} S_i S_{i-1}^{-1}$ and that for $v = S_{i-1} \ldots S_{i-n}$ a reduced decomposition we know that the roots $S_{i-1} \ldots S_{i-1} (x_i)$ are exactly the ones changing sign, then we see that $u$ is reflection in a hyperplane crossed in going from $C_0$ to $\gamma$.

So we have the following procedure for determining the cells in the closure of $B^\gamma$, namely reflect $\gamma$ thru a hyperplane crossed in going to $\gamma$ from $C_0$.

The closure of $B^\gamma$ is the cells indicated by $\circ$ and $\bullet$ for the diagram.
Next I want a procedure for writing down the chain complex of the orbit $K_{\eta_0}$. I have a cell for each element $\eta = w\eta_0$ of $W\eta_0$. If I choose a minimal representation

(1) \[ \eta = s_{n_1} \cdots s_{n_k} \eta_0 \]

then in the chain complex the boundary of the generator belonging to $\eta$ will involve the codimension 1 orbits in $B^*\eta$. These I find by looking at those elements

(2) \[ \eta' = s_{n_1} \cdots s_{n_k} \eta_0 \]

which are of codimension 1, which means

i) $s_i$ is a root of multiplicity 1

ii) the representation (2) is minimal.

Geometrically this means that if a straight line from $e \in \text{Int}(C_0)$ to $\eta$ is drawn and then reflected as it crosses a root $\mathfrak{r}$ of multiplicity 1, then no root hyperplanes are crossed backwards along the reflected branch, i.e.

\[ \mathfrak{r} = 0 \]

I want the hyperplanes through $A, B$ not to cross $P_{\eta'}$. 
For each simple root $\alpha_i$, $i = 1, \ldots, l$ we choose an orientation of $U^*_{\alpha_i}$. Then we get an orientation on the cell

$$B^* \eta \leftarrow U_1 s_1 \times \cdots \times U_n s_n$$

which, it seems, depends on the representation (1).

Suppose orientations chosen for all the cells. I want next to calculate the coefficients in $2[\eta]$ of $[\eta']$ where $\eta, \eta'$ are as above and $[\eta]$ denotes the chain given by $\eta$ with its orientation.

We know

$$B^* \eta \leftarrow (B s_1 B')^B \times (B s_i B')^B \times \cdots \times (B s_n B')^B$$

$$B^* \eta' \leftarrow (B s_1 B')^B \times (B s_i B')^B \cdots (B s_n B')^B$$

Put $P_i = B s_i B \circ B$. Then

$$B^* \eta_0 \times B^* \eta'_0$$

$$(U_1 s_1 \times \cdots \times U_{i-1} s_{i-1}) \times P_i \times (B s_i t_1 \cdots s_n B')^B$$

Because $\alpha_i$ is a root of multiplicity 1, $P_i / B \cong \mathbb{RP}^1$. 
Moreover, $B_{s_{i+1}} \cdots s_n B/B$ is an affine space isomorphic to the subspace spanned by the positive roots reversed by $s_{i+1} \cdots s_n$. These roots are $s_{i+1}, s_{i+1}(x_{i+2}), \ldots, s_{i+1} \cdots s_n(x_n)$.

It seems clear to me that $P \times B (B_{s_{i+1}} \cdots s_n B/B)$ is the vector bundle over $\mathbb{R}P^1$ associated to the representation of $B$ with characters given by the above roots. To see what line bundles over the circle $\mathbb{R}P^1$ we get, I take the character $\chi$ and apply it to $H_{x_i}$ and reduce the integer mod 2. So it seems I get:

Assertion: $B^\chi \cup B^\chi'$ is a vector bundle over the circle $S^1$ which is orientable or not depending on whether

$$m_1 \chi_1 (H_{x_1}) + \cdots + m_n \chi_n (H_{x_n})$$

is even or not. Here $\chi_j = s_1 \cdots s_{j-1}(x_j)$ are the roots encountered as we build up $\chi$, and $m_j$ is the multiplicity of $\chi_j$.

The coefficient of $[\chi']$ in $\mathbb{H} \chi$ should be 0 if this bundle is orientable and $\pm 2$ if it is not. The sign depends on how the choice of orientation for $[\chi']$ differs from that provided by (2).