July 11, 1975.

Wiener factorization: Let $f(z)$ be a holomorphic matrix function in a nbd of S^1 ~~and~~ such that $f(z)$ is positive definite for z in S^1. Then there exists a ~~holomorphic~~ matrix function $g(z)$ holomorphic inside S^1 or in a nbd of S^1 such that

$$f(z) = g(z)^* g(z). \qquad z \in S^1$$

Proof: Using f as a clutching function one gets a holomorphic vector bundle on $\mathbb{P}^1$ which is topologically trivial, hence trivial via Grothendieck. This means f has a factorization

$$f = g_- \, g_+$$

where g_+ is holomorphic inside and in a nbd of S^1 and where g_- is holomorphic outside S^1 in a nbd of S^1 and at $z = +\infty$. Moreover such a factorization is unique up to $g_+ \mapsto c g_+$, $g_- \mapsto c g_-$ where c is an invertible matrix.

~~Since $f(\frac{1}{\bar z})^* = f(z)$ by the reflection principle one gets~~

By the reflection principle

~~something here reminiscent about~~ [illegible]

$$f(\bar z^{-1})^* = f(z)$$

so

$$g_+(\bar z^{-1})^* g_-(\bar z^{-1})^* = g_-(z) g_+(z)$$

is another factorization of f. By uniqueness there is a constant matrix c in GL_n such that

$$c\, g_+(z) = g_-(\bar z^{-1})^*$$

$$g_-(z) c^{-1} = g_+(\bar z^{-1})^*$$

I will suppose g_+ chosen so that $g_+(1) = 1$. Then $c = g_-(1)^*$, [illegible] so $c = f(1)^* = f(1)$ being positive definite [illegible] has a positive definite square root. Then if

$$g = c^{1/2} g_+$$

we have $g(\bar z^{-1})^* = g_+(\bar z^{-1})^* c^{1/2} = g_-(z) c^{-1/2}$ so

$$g(\bar z^{-1})^* g(z) = g_-(z) g_+(z) = f(z)$$

as was to be proved.

Garland – Raghunathan proof uses a density theorem of the following sort:

Claim: $\mathcal{K}'$ is dense in ΩK when K is simply-connected.

It suffices to show $\overline{\mathcal{K}'}$ contains a nbd. of 1 since ΩK is connected. Can find enough maps $SU_2 \longrightarrow K$ so that

$$SU_2 \times \cdots \times SU_2 \longrightarrow K$$

is a submersion at 1; hence any nbd of 1 in ΩK lifts into $\Omega SU_2 \times \cdots \times \Omega SU_2$; this reduces us to the case of SU_2. By using

$$S^1 \times S^1 \times S^1 \longrightarrow SU_2$$

we can suppose we have a small loop in the maximal torus of SU_2 which we want to approximate by a Laurent loop.

Next note that if $\gamma(t)$ is a smooth path in K starting at 1, then $\gamma(\frac{t}{n})^n \longrightarrow \exp(t\,\gamma'(0))$, hence the closure of $\mathcal{K}'$ will contain exponentials of tangent vectors to paths in $\mathcal{K}'$ starting at 1.

Now I consider the path in $\mathcal{K}'$, $K = SU_2$:

$$\left(zP_{L_t} + I - P_{L_t}\right)\left(z^{-1}P_{L_0} + I - P_{L_0}\right)$$

where $L_t = \mathbb{C}(1,t)$ and P_L is the orthogonal projection on L_t. Calculation gives

$$P_{L_t} = \begin{pmatrix} \frac{1}{1+|t|^2} & \frac{t}{1+|t|^2} \\ \frac{\bar{t}}{1+|t|^2} & \frac{|t|^2}{1+|t|^2} \end{pmatrix} \sim \begin{pmatrix} 1 & t \\ \bar{t} & 0 \end{pmatrix} \quad \text{mod } |t|^2$$

hence the element in $\mathcal{K}'$ is

$$\left(z\begin{pmatrix} 1 & t \\ \bar{t} & 0 \end{pmatrix} + I - \begin{pmatrix} 1 & t \\ \bar{t} & 0 \end{pmatrix} \right) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z & (z-1)t \\ (z-1)\bar{t} & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (z-1)t \\ -(z^{-1}-1)\bar{t} & 1 \end{pmatrix}$$

Similarly replacing z by z^n and adding we can get the tangent vectors

$$\begin{pmatrix} 1 & a(z)t \\ -\bar{a}(z^{-1})\bar{t} & 1 \end{pmatrix} \qquad t^2 = 0$$

where $a(z)$ is any Laurent poly vanishing at 1.

Observe that the matrices of the form

$$\begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} \qquad a \in \mathbb{C}$$

form the space k_α for SU_2 and the diagonal maximal torus. Hence we have a fairly systematic way of ~~making loops~~ approximating loops of the form

$$\exp\begin{pmatrix} 0 & a(z) \\ -\bar{a}(z) & 0 \end{pmatrix}$$

by Laurent polynomial loops.

Direct proof that $\mathcal{U}'_n \longrightarrow \Omega U_n$ is a (weak) homotopy equivalence along Atiyah–Bott lines:

Start with a family $X \xrightarrow{\alpha} \Omega U_n$ with X compact. By Fejér approximation one can homotop it into a family α of loops given by Laurent polynomial matrices without singularities on S^1. Shifting by mult. by z^N we can assume the family is given by polynomials matrices α, non-singular on S^1. Then one forms the associated outgoing space αH^n which is of finite codim in H^n. We know that α is canonically homotopic to the family of scattering matrices associated to αH^n.

So what one wants to show is that the space $\mathcal{S}$ of outgoing subspaces D in H^n of a given codimension is homotopy equivalent to the subspace $\mathcal{S}_0$ consisting of D such that $D \supset z^N H^n$. But by triangulation $\mathcal{S}_0$ is a strong defm. retract of a nbd. U of $\mathcal{S}_0$ in $\mathcal{S}$. And the family of autos. $z \longmapsto az$ pulls $\mathcal{S}$ down into U. . . .

July 24, 1975. Closures of Schubert cells.

Suppose I consider an orbit in the building say $K\eta_0 \xleftarrow{\sim} G/P_{\eta_0}$ for some point η_0 in the fundamental chambre C_0. I have seen that the B^u-orbits (B = Borel corresponding to C_0) on $K\eta_0$ are indexed by $W\eta_0 = K\eta_0 \cap E$.

Let $\eta = w\eta_0 \in W\eta_0$. I have constructed a compactification of $B^u\eta$ using the space $\Gamma(s_1, \dots, s_n)$ of galleries

$$C_0, \; g_1C_0, \; \dots, \; g_1\cdots g_n C_0$$

of type $s_1, \dots, s_n$, where $s_i = s_{\alpha_i}$ is a minimal sequence of fundamental reflections such that $\eta = s_{\alpha_1}\cdots\cdots s_{\alpha_n}\eta_0$. As

$$\Gamma(s_1, \dots, s_n) = (B \cup Bs_1B) \times^B \cdots \times^B (B \cup Bs_nB)/B$$

I saw that the closure of $B^u\eta$ consists of the B^u-orbits of the point

$$s_{i_1} \cdots s_{i_p} \eta_0$$

where $1 \le i_1 < \dots < i_p \le n$ is any subset of $\{1, \dots, n\}$.

~~Next note that~~ What is the difference between $s_1 \cdots s_n$ and $s_1 \cdots \hat{s}_i \cdots s_n$ on the other: If

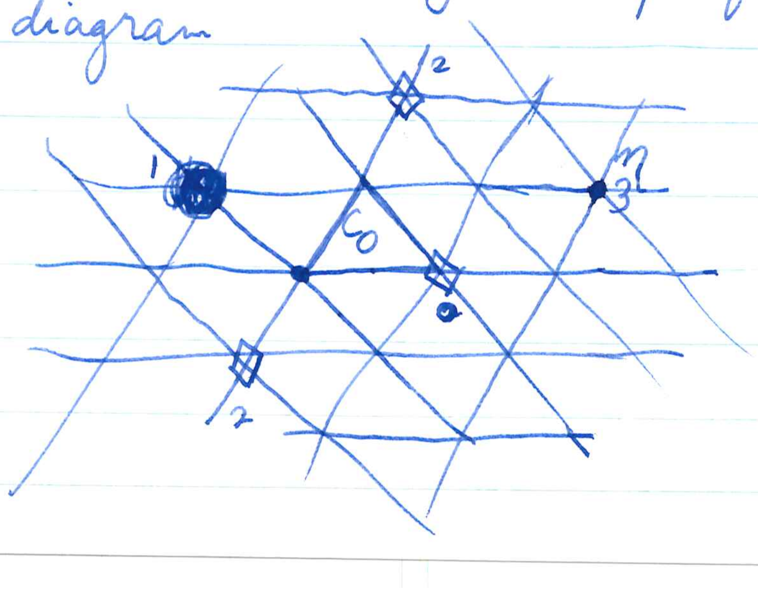
$$u\, s_1 \cdots\cdots s_n = s_1 \cdots \hat{s}_i \cdots s_n$$

then

$$u = s_1 \cdots s_{i-1}\, s_i^{-1}\, s_{i-1}^{-1} \cdots s_1^{-1}$$

is conjugate to s_i. Thus u is a reflection. In fact if we recall that $s_i = s_{\alpha_i}$ and that for $w = s_{\alpha_1} \cdots\cdots s_{\alpha_n}$ a reduced decomposition we know that the roots $s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ are exactly the ones changing sign, then we see that u is reflection in a hyperplane crossed in going from C_0 to η.

So we have the following procedure for determining the cells in the closure of $B^u\eta$, namely reflect η thru a hyperplane crossed in coming to η from C_0, and repeat the process. Thus for the diagram



the closure of $B^u\eta$ is the cells indicated by $\diamond$ and $\bullet$.

Next I want a procedure for writing down the chain complex of the orbit $K\eta_0$. I have a cell for each element $\eta = w\eta_0$ of $W\eta_0$. If I choose a minimal representation

(1) $$\eta = s_{\alpha_1} \cdots\cdots s_{\alpha_n} \eta_0$$

then in the chain complex the boundary of the generator belonging to η will involve the codimension 1 orbits in $\overline{B^u \eta}$. These I find by looking at those elements

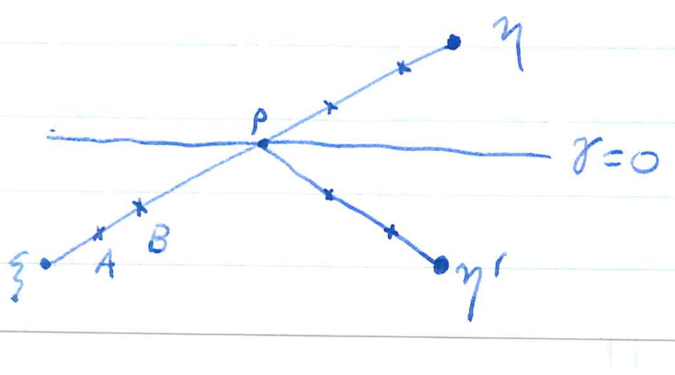
(2) $$\eta' = s_{\alpha_1} \cdots\cdots \hat{s}_{\alpha_i} \cdots s_{\alpha_n} \eta_0$$

which are of codimension 1, which means

i) α_i is a root of multiplicity 1

ii) the representation (2) is minimal.

Geometrically this means that if a straight line from $\xi \in \mathrm{Int}(C_0)$ to η is drawn and then reflected as it crosses a root γ of multiplicity 1, then no root hyperplanes are crossed backwards along the reflected branch, i.e.



I want the hyperplanes thru A, B not to cross $P\eta'$.

For each simple root α_i $i = 1, \dots, l$ we choose an orientation of U_{α_i}. Then we get an orientation on the cell

$$B^u \eta \xleftarrow{\sim} U_1 s_1 \times \cdots \cdots \times U_n s_n$$

which, it seems, depends on the representation (1).

Suppose orientations chosen for all the cells. I want next to calculate the coefficients in $\partial[\eta]$ of $[\eta']$ where η, η' are as above and $[\eta]$ denotes the chain given by η with its orientation.

We know

$$B^u \eta \simeq (Bs_1 B) \times^B \cdots \times^B (Bs_i B) \times^B \cdots \times^B (Bs_n B)/B$$

$$B^u \eta' \simeq (Bs_1 B) \times^B \cdots (\quad)^B \cdots (Bs_n B)/B$$

Put $P_i = Bs_i B \cup B$. Then $B^u \eta \cup B^u \eta'$ is

$$(U_1 s_1 \times \cdots \times U_{i-1} s_{i-1}) \times P_i \times^B \left(Bs_{i+1} \cdots s_n B / B \right)$$

Because α_i is a root of multiplicity 1, $P_i / B \cong \mathbb{R}P^1$.

Moreover $Bs_{i+1}\cdots s_n B/B$ is an affine space isomorphic to the subspace spanned by the positive roots reversed by $s_{i+1}\cdots s_n$. These roots are $\alpha_{i+1}, s_{i+1}(\alpha_{i+2}), \dots, s_{i+1}\cdots s_{n-1}(\alpha_n)$.

It seems clear to me that $P_i \times^B (Bs_{i+1}\cdots s_n B/B)$ is the vector bundle over $\mathbb{R}P^1$ associated to the representation of B with characters given by the above roots. To see what line bundles over the circle $\mathbb{R}P^1$ we get I take the character χ apply it to H_{α_i} and reduce the integer mod 2. So it seems I get:

<u>Assertion:</u> $B^u\eta \cup B^u\eta'$ is a vector bundle over the circle S^1 which is orientable or not depending on whether

$$m_{i+1}\gamma_{i+1}(H_{\gamma_i}) + \dots + m_n\gamma_n(H_{\gamma_i})$$

is even or not. Here $\gamma_j = s_1\cdots s_{j-1}(\alpha_j)$ are the roots encountered as we build up η, and m_j is the multiplicity of γ_j.

The coefficient of $[\eta']$ in $\partial[\eta]$ should be 0 if this bundle is orientable and ± 2 if it is not. The sign depends on how the choice of orientation for $[\eta']$ differs from that provided by (2).