June 17, 1975. Morse theory + Lie groups.

Let $K$ be a compact connected group with Lie algebra $\mathfrak{g}$, let $(\cdot, \cdot)$ be an invariant inner product on $\mathfrak{g}$, let $E$ be a maximal abelian subspace of $\mathfrak{g}$ and $T$ the corresponding maximal torus, and $\Phi \subset \text{Hom}(E, \mathbb{R})$ the set of roots of $\mathfrak{g}$ with respect to $T$. Let $O$ be an orbit of $K$ in $\mathfrak{g}$: $O = K\cdot \mathfrak{g}$. Choose a regular element $\mathfrak{g}$ of $E$ (i.e. $\langle \mathfrak{g}, \mathfrak{g} \rangle \neq 0$ for all $\mathfrak{g} \neq 0$) and consider the function on $O$

$$f(k\cdot \mathfrak{g}) = \| k\cdot \mathfrak{g} - \mathfrak{g} \|^2 = \| \mathfrak{g} \|^2 + \| \mathfrak{g} \|^2 - 2 \langle k\cdot \mathfrak{g}, \mathfrak{g} \rangle$$

Calculation shows $f$ has $0$ on $E$ for its set of critical points, that each critical point $k\cdot \mathfrak{g}$ is non-degenerate and has index equal to the number of hyperplanes crossed in going along the straight line from $\mathfrak{g}$ to $k\cdot \mathfrak{g}$.

Consequences from Morse theory:

1) $K\cdot \mathfrak{g} \cap \mathfrak{g} = \emptyset$. This implies conjugacy of maximal abelian subspaces of $\mathfrak{g}$, and also that if $W$ = group of auts of $E$ induced by elements of $K$, then $K\cdot \mathfrak{g}$ is a $W$-orbit on $E$.

$$W\cdot \mathfrak{g} = K\cdot \mathfrak{g}$$
2) $K_\eta$ has a cell decomposition indexed by points of $W_\eta$ (say $\eta \in E$), the dimension of the cell indexed by $\omega\eta$ being the number of root hyperplanes separating $\omega\eta$ and $\xi$. This implies the homology is free over $\mathbb{Z}_2$, $H_0 \cong 0$ in odd dimensions, and a basis is given by the cells.

As $K_\eta$ is connected, $f$ has a unique minimum value $\Rightarrow W_\eta$ meets cone $C_\xi = \{ x \mid x(\xi) > 0 \}$ for $x(\xi) > 0$ in exactly one pt:

$$C_\xi \rightarrow W \setminus E.$$ 

Further $\eta \in C_\xi \iff |\eta - \xi| < |\omega\eta - \xi|$ for $\omega\eta \neq \eta$.

Since $K_\eta$ has no 1-cells, $\pi_1(K_\eta) = 0$.

Thus stabilizers of $\eta$ in $K$ are connected, and $T$ is self-centralizing. (These facts are usually proved by showing every $T$ is conjugate to an element of $T$, and that a group generated by a torus and a centralizing element is gen. by a single element).
Next I wish to consider the group $K$ acting by conjugation on itself. Bott-Samelson consider geodesics starting perpendicular to an orbit:

They prove the geodesic is perpendicular to all orbits it crosses, that a point $t$ is a conjugate point if $\dim K_t > \dim K_0$, in which case the Jacobi fields along $l$ vanishing at $t$ all arise from $\text{Lie}(K_t)/\text{Lie}(K_0)$. This result is "variational completeness" of the action of $K$ on itself.

Suppose $t$ is a regular element of $T$ (I mean that the centralizer of $t$ in $K$ is $E$). We consider the space $\mathcal{N} = \mathcal{N}(K; t_0, 0)$ of paths joining $t$ to $0$. Critical points for the energy function on $\mathcal{N}$ are geodesics $l$ joining $t$ to $0$ and perpendicular to $0$. $l$ has to be perpendicular to the orbit $Kt$. Conversely $l \subset T \Rightarrow l \perp \mathcal{N}$. Thus the critical points are geodesics in $T$ from
may be identified with lines joining to a point of \( p^{-1}(O \cap T) \), where \( p : E \to T \) is the exp. map, and \( O \) is a given point in \( p^{-1}(e) \). These critical points turn out to be non-degenerate, and the index of \( l \) is \( \delta \) twice the number of hyperplanes of the form \( x(x) = n, x \in E, n \in \mathbb{Z} \) crossed in going along \( l \).

Consequences from Morse theory:

1) \( \partial T = \emptyset \). This means every element is conjugate to an element of \( T \), and implies \( \partial T \) is a \( W \)-orbit of \( T \):

\[
W \backslash T \overset{\sim}{\longrightarrow} \mathbb{K} \backslash K
\]

2) \( \Lambda(K, t, 0) \) has the homotopy type of a CW complex with even-dimensional cells, so these cells have to be a basis for the integral homology.

Take \( O = 1 \) and look at \( H_0(A) = \text{free abelian group gen. by } \pi_0 \Lambda = \pi_1 K \). Thus we find \( |\pi_1 K| \) number of points of \( p^{-1}(1) \) contained in the small chamber \( C'_1 \). Choose \( O \) to be an interior point of fundamental cone close to \( O \), whence

\[
C'_1 = \{ x \in E \mid 0 \leq x(1) \leq 1 \text{ all } x \in \mathbb{R}^+ \}.
\]
Assume now that $K$ is simply-connected. $\Lambda$ has the type of the fibre of the inclusion $\emptyset \hookrightarrow K$ over $t_0$, hence $\pi_1 K = \pi_0 \emptyset = 0 \implies \pi_0 \Lambda = 0$. Consequently, there is a unique point of $p^{-1}(\emptyset \cap T)$ contained in $C'$. So:

$$C' \sim W | T \sim K \setminus K.$$ 

Next note that because $\Lambda$ has no 1-cells, $\Lambda$ is simply-connected $\implies \emptyset$ is simply-connected. Therefore the centralizer of any element of $K$ is connected.

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Next we consider the symmetric space situation: $K$ compact connected Lie group and $\sigma$ is an involution. $X = K/K^\sigma$ is the symmetric space. We can identify $X$ with the component of $\{ x \mid \sigma x = x^{-1}, x \in K \}$ containing the identity, or with the orbit of 1 under the action $k \cdot y = ky(\sigma k)^{-1}$.

For the linear situation, we look at the action of $K^\sigma$ on the tangent space to $X$ at 1 which is $k_\perp = \{ x \in K \mid \sigma x = -x \}$.
Let $E_0$ be a maximal abelian subspace of $K_-$, and $S$ the corresponding torus. Let $\mathfrak{h}$ be the set of roots of $K$ with respect to $S$. Let $\xi \in E_0$ be regular, i.e. $\alpha(\xi) \neq 0$ for $\alpha \in \mathfrak{h}$. Let $K^{\xi}$ be a $K$-orbit in $K_-$, and consider the function

$$f(\xi \eta) = |\xi \eta - \xi|^2 = |\eta|^2 + |\xi|^2 - 2(\xi \eta, \xi)$$

Suppose $\xi \eta$ is a critical point. Then for $X \in K_-$

$$f(\exp(X) \xi \eta) = (1 + |X|^2)|\eta|^2$$

$$- 2(1 + [X, X] + [X, X] [X, X] \xi | \xi \eta, \xi)$$

hence $0 = ([X, \xi] \xi) = ([X, \xi] \xi) = ([X, [\xi, \xi]])$. Since $[\xi, \xi] \in K_-$ this implies $[\xi, \xi] = 0$, so $\xi \eta \in E_0$. Thus $K^{\xi} \cap E_0$ is the set of critical points.

The Hessian at a critical point $\xi \eta$ is

$$-2((\text{ad} X)^2 \xi \eta, \xi) = 2([\xi, X], [\xi, X])$$

I need to understand the representations of $S' = \mathbb{C}/\mathbb{Z} \times S$. Let $V$ be an irreducible complex rep. of $S'$. If $S$ acts trivially, then it is a character of $\mathbb{Z}/\mathbb{Z}$. Otherwise if the character $X: S \to S'$ occurs, so does $X^{-1}$, and $X \neq X^{-1}$ for $X \neq 0$, $V$ is 2-dimensional. $V$ is induced from a non-trivial character of $S$. $V^*$ is induced from $X^{-1}$ so $V^* \cong V$. But $S'$ can't have non-trivial
quaternion characters as only $-1$ is of order $2$ in $S^3$. So $V$ must be the complexification of a real $2$-dimensional representation. In fact one takes $X: S \rightarrow S^1$ and lets $S^1$ act on $C$ thru $X$ and $\sigma$ acts via $-$. This gives us the irreducible real reps. $V$ of $S'$ on which $S$ acts non-trivially.

(Actually $S'$ is a generalized dihedral group so we know its representations are all defined over $\mathbb{R}$)

So therefore we know that as an $S$-module $k$ is the direct sum of its centralizer $m \oplus \mathbb{E}_0$ plus root spaces $k_{\pm \alpha}$ indexed by each pair $\pm \alpha$ in $\mathbb{E}_0$. If I select one of this pair, then $k_{\pm \alpha}$ gets a complex structure such that $\exp(\mathfrak{a})$ is multiplication by $e^{2\pi i \mathfrak{a}}$. Moreover $\sigma$ is a conjugation for this complex structure.

Let $X$ have the component $X_\alpha$ in $k_{\pm \alpha}$, $X_0$ in $m$. Then
\[
[k_{\alpha}, X_\alpha] = 2\pi i \alpha(\mathfrak{k}_{\alpha}) X_\alpha
\]
\[
[k_0, X_\alpha] = 2\pi i \alpha(\mathfrak{k}_0) X_\alpha
\]
and the inner product is $\langle (4\pi)^2 \alpha(\mathfrak{k}_{\alpha}) \alpha(\mathfrak{k}_0) X_\alpha X_\alpha \rangle$. Thus the Hessian is:
\[
-2(\text{ad } X)^2 k_{\alpha} \langle \mathfrak{k}_{\alpha} \rangle = (8\pi^2) \sum_{\alpha \in \mathbb{E}_0^+} \alpha(\mathfrak{k}_{\alpha}) \alpha(\mathfrak{k}_0) |X_\alpha|^2.
\]

For the critical point to be non-degenerate this form must be non-singular on $k_{\alpha} \oplus \mathbb{E}(k_{\alpha}) k_0$, that is, for $X$ such that $X_{\alpha} = 0$ for $\alpha(\mathfrak{k}_{\alpha}) = 0$, and $X_0 = 0$. Clear.
Finally the index of the critical point is the number of $\lambda$ such that $\alpha(\lambda) < 0, \alpha(\lambda) > 0$ for each $\lambda$ counted with multiplicity $\frac{1}{2} \dim k^\lambda$. Note that $\frac{1}{2} \dim k^\lambda = \dim (k^\lambda \ominus k)$ is the excess over $2 \dim \mathfrak{m}$ of the stabilizer in $\mathcal{K}^0 \oplus E_0$ of a general point of $x = 0$ in $E_0$.

So Morse theory tells us that the orbit $K^0 \eta E_0$ has a cell decomposition with cells indexed by points of $K^0 \eta E_0$ with indices given by the number of root hyperplanes crossed counted with multiplicities.

From the fact that $K^0 \eta E_0$ is non-empty we get the $K^0$-conjugacy of maximal abelian subspaces of $k^\mathfrak{m}$, moreover $K^0 \eta E_0$ is a $W_0$-orbit of $E_0$, where $W_0 = \mathcal{K}^0$ subgroup of $\text{Aut}(E_0)$ consisting of $\Theta$ induced by acts of $K^0$.

Now we can not yet at this point argue that the points of index 0 correspond to components of $K^0 \eta$. For this one needs to show that each cell in $K^0 \eta$ is the image of a birational map from a closed manifold. This will show that the cells give a basis for the mod 2 homology of $K^0 \eta$.
Let's admit the fact that the cells in $K^{\eta}$ give a basis for the mod 2 homology. I will replace $K^\circ$ by its connected component $K_0$; this won't affect any of the calculations made so far.

So now $K_0^{\eta}$ is connected, hence there is exactly one zero-cell. This implies that there is exactly one point of the $W_0$-orbit $W_0^{\eta} = K_0^{\eta} \cdot e_0$ (say $\eta \in K_0^{\eta} \cdot e_0$) which is in the chamber $C_0 = \{ x \in E_0 \mid x(3) > 0 \text{ if } x(1) > 0 \}$ containing $\eta$. And this point is where $|k\eta - \delta|$ is minimum:

$$\eta \in C_0 \iff |k\eta - \delta| > |\eta - \delta| \quad \forall k \in K_0^{\eta} \setminus k_0^{\eta}$$

Thus $C_0 = W_0^{\eta} \cdot e_0 = K_0^\circ \setminus k_0^{\eta}$.

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**Digressions:** If $K$ is 1-connected, then the group of lattice points in $E$ (those in $p^{-1}(1)$, $p : E \to T$) is generated by the root vectors $H_a$. In effect, given a lattice point $\lambda$, one can move it by reflections into $C_0$. But zero is the only lattice point in $C_0$. Thus $\lambda$ is in the orbit of the group $W$ gen. by reflections thru $x \in E$, and $W$ preserves the lattice $\Sigma_{\mathbb{Z}H_a}$, so $\lambda \in \Sigma_{\mathbb{Z}H_a}$. [Diagram or further text related to lattice points and group actions could be included here.]
Note that $C'$ is described by $C' = \{ x | 0 \leq \alpha_i(x) \leq 1 \}$ for all $x \in \Phi^+$. Suppose $V$ is simple, i.e. $\mathfrak{g}$ is an irreducible module, hence has a maximal root $\Psi = [X, X^]\Psi = 0$ all $x \in \Phi^+$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the simple roots. If $x \in \Phi^+$, then there is a sequence $1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq n$ such that

$$\alpha_{i_1}, \alpha_{i_1} + \alpha_{i_2}, \ldots, \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_m} = \Psi$$

is a sequence of roots. (This results from the fact that the weight vector $X_\alpha$ has to be of the form

$$X_\alpha = c \; \text{ad} \; (X_{-\alpha_{i_1}}) \ldots \text{ad} \; (X_{-\alpha_{i_{m-1}}}) \; X_{-\alpha_{i_m}}$$

Thus it is clear that $\alpha_{i_1}(x) \geq 0, \ldots, \alpha_{i_1}(x) \geq 0, \Psi(x) \leq 1$ forces $\alpha(x) \leq 1$ for all $x \in \Phi^+$. We conclude therefore that $C'$ is the simplex described by $\alpha_1, \alpha_2 \geq 0, \Psi \leq 1$ in the irreducible case.
Suppose again \( K \) simple and \( I \)-connected, and let \( K_\alpha \) be the centralizer of \( \alpha \in I \). I have seen \( K_\alpha \) is the connected group containing \( I \) with roots \( \Phi(\bar{r}) = \{ \alpha | \alpha(\bar{r}) \in \mathbb{Z} \} \) where \( \bar{r}(\bar{r}) = \alpha \). The question is whether \( K_\alpha \) can be different from the centralizer of a torus, i.e. whether \( \Phi(\bar{r}) = \{ \alpha | \alpha(\bar{r}) = 0 \} \) for some \( \bar{r} \in E \).

Suppose \( \bar{r} \in C \), whence \( 0 \leq \alpha(\bar{r}) \leq 1 \) for \( \alpha \in \Phi^+ \). If \( \psi(\bar{r}) < 1 \), then \( \Phi(\bar{r}) = \{ \alpha | \alpha(\bar{r}) = 0 \} \), and we are ok. Let then \( \psi(\bar{r}) = 1 \) and arrange the simple roots in order so that

\[ \alpha_1(\bar{r}) < \alpha_2(\bar{r}) < \ldots < \alpha_n(\bar{r}) \]

Let \( \alpha \in \Phi(\bar{r})^+ \). If \( \alpha(\bar{r}) = 0 \), then \( \alpha \) is a linear combination of \( \alpha_1, \ldots, \alpha_n \) (in general \( \alpha = \sum n_i \alpha_i \) uniquely with \( n_i \geq 0 \)), and conversely. Call \( \Delta \subset \Phi(\bar{r}) \) the set of roots which are lin. combinations of \( \alpha_1, \ldots, \alpha_n \).

If \( \alpha(\bar{r}) = 1 \), we have \( \alpha = \psi + \alpha_1 + \ldots + \alpha_n \), and it is clear that \( 1 \leq \alpha_i(\bar{r}) < \psi \), \( i = 1, \ldots, r \).

Assume that \( \Phi(\bar{r}) = \{ \alpha | \alpha(\bar{r}) = 0 \} \) for some \( \bar{r} \). Then \( \alpha_i(\bar{r}) = 0 \) \( i = 1, \ldots, r \). If \( \psi = \sum n_i \alpha_i \) is any pos. root not in \( \Delta \), then \( \psi(\bar{r}) = 1 \) and \( \psi(\bar{r}) = 0 \). So there's a problem if when we write \( \psi = \eta_1 \alpha_1 + \ldots + \eta_n \alpha_n \), some \( \eta_i > 1 \).
Example: Suppose \( \Phi = n_1 \alpha_1 + \ldots + n_k \alpha_k \) where say \( n > 1 \). (Then \( l \geq 2 \).) Take \( \xi \) to be the point of \( C' \) where \( \alpha_1(\xi) = \ldots = \alpha_{k-1}(\xi) = 0 \) and \( n_2 \alpha_k(\xi) = 1 \).

Suppose \( \eta \) such that \( \eta(\xi) \in \mathbb{Z} \) \( \iff \alpha(\eta) = 0 \). Then \( \alpha_1(\eta) = \ldots = \alpha_{k-1}(\eta) = \eta(\eta) = 0 \), so \( \eta \) would have to be zero, and we get a contradiction.

- \( G_2 \) has \( \Phi = \alpha_1 + 2 \alpha_2 \)
- \( G_2 \) has \( \Phi = 2 \alpha_1 + 3 \alpha_2 \).

Examples of symmetric spaces:

Start by classifying the symmetric spaces arising from involutions on \( G_2 \) and \( U_n \).

First look in fiber originating from \( \alpha_1(\xi) \neq 0 \) and \( \eta(\eta) \neq 0 \).

First some general considerations. Suppose \( \sigma \) is a given involution on \( K \). Then I might look for involutions \( \tau \) of the form

\[
\tau(x) = y \sigma(y) y^{-1}.
\]

For this to be an involution means \( y \sigma(y) \in \text{center} \). Involution \( \tau \) and \( x \mapsto z \frac{1}{z} (z^{-1} \sigma z) z^{-1} \) have
conjugate field groups, hence isomorphic symmetric spaces. Thus
\[ z \tau (z^{-1} x z) z^{-1} = z y (z^{-1} x z)^{-1} z y^{-1} \tau^{-1} \]
we see that the different kinds of symmetric spaces we get from involutions of the form \( \tau x = y \sigma x y^{-1} \)
are described by:
\[ \{ y \in K^*/ | y'y \in \text{center} \} \text{ action: } z \cdot y = z y z^{-1} \).

Now look at \( U_n \) and first take \( \sigma \) to be complex conjugation (or \( x \rightarrow (x^*)^{-1} \)).
If \( z \) is a scalar matrix then \( z z^{-1} = z^2 \)
and so given \( y \) with \( y y \in \text{Center} \) we can modify it so that \( y y = 1 \).
Let \( V \) be the eigenspace of \( y \) with eigenvalue 1. Then
\[ y v = \lambda v \Rightarrow y (y^{-1} v) = \lambda (y^{-1} v) \Rightarrow v = \lambda (y^{-1} v) \Rightarrow y (y^{-1} v) = \lambda v \]
so \( V \) is stable under \( \tau \). Thus if I select \( z \) so as to have the same eigenspace as \( y \) and also \( \tau z = z^{-1} \), then \( \sigma z = z \), and
so we conclude \( \Box \) is a single point.
Corresp. symm. space is \( U_n / O_n \).
Take $\sigma$ to be the identity in $U(n)$. If $y^2 \in$ center, then multiplying $y$ by a scalar, we can suppose $y^2 = 1$; so we are classifying involutions in $U_n$ up to conjugation, and multiplication by $\pm 1$. So the type of symmetric spaces obtained are the Grassmannians

$$U(n)/U(p) \times U(n-p)$$

$0 \leq p \leq \frac{n}{2}$

Next consider $SU_n$ with $\sigma = \text{id}$. whence $y^2 = 1, y^n = 1$. So $y$ has two eigenvalues, and we get the Grassmannians again for the symmetric spaces.

Consider $SU_n$ with $\sigma = \text{id}$ and let $yy^* = 1, y^n = 1$. If $n$ is odd, then we can change $y$ so that $yy^* = 1$, whence the eigenspaces of $y$ are stable under $\sigma$. Let $W$ be an eigenspace of $y$. By what we know about $U_n$, we can find a $z$ in $W^* \cap N$ and we can extend $z$ to an element of $SU_n$. Thus we can suppose $y = 1$ in $W^*$, $y = 1$ in $W$ so
Consider $SU_n$ with $\sigma x = \overline{x}$ and let $y\overline{y} = 1$ be a scalar matrix. Then $\overline{\overline{y}} = \overline{y}$
and $(\overline{y}, y) = 1$ as $I$ is in the center, so $\overline{\overline{y}} = y$.
whence $y = \pm 1$. Note that $y^n = \det y \det \overline{y} = 1$, so $y = 1$ if $n$ odd.

If $V_1$ is where $y = \lambda$, then $\overline{V_1}$ is where $\overline{y} = \overline{\lambda}$ or where $y = \overline{\overline{\lambda}}^{-1} = \overline{\lambda}$.
Thus $\overline{V_1} = V_{\overline{\lambda}}$.

If $y = -1$, then $V_1 = V_{1}$ and so we can find a line $L$ with $L = L$ stable under $y$. By what we've seen for $U_n$, there exists $z$ in $L^\perp$ such that $z \overline{z}^{-1} = y$ in $L^\perp$; hence extending $z$ to an element of $SU_n$, we can arrange $y$ to be $1$ in $L^\perp$, whence $y = 1$. Thus if $y = 1$, we get only the symmetric space:

$SU_n/SO_n$

If there is a subspace $W$ stable under $y$ and $y$, then we can arrange that $y$ be $1$ on $W^\perp$, whence $y = 1$.

If $y = -1$, we have $V = V_1 + W$, where these are interchanged under $y$. If $n = 2m$, suppose $n$ is even, whence $\chi_m(x) = \chi_m(-x)$. If $n = 2m + 1$, we can arrange $n = 1$. 

The sentence was not clear and may need further clarification or rephrasing.
Consider a $SU_n$ with $\sigma x = \bar{x}$. If $y \bar{y}$ is in the center, say $y \bar{y} = i$, then $y, \bar{y}$ commute and $J = (y \bar{y})^{-1} = \bar{y} y = y \bar{y} = i$, so $J^2 = \pm 1$.

Now recall that if we interpret elements as transfs. of $\mathbb{C}^n$, then $\sigma(x) = \sigma \circ x \circ \sigma^{-1}$, where $\sigma =$ conjugation in $\mathbb{C}^n$. Thus our involution is $x \mapsto y \sigma(x) y^{-1} = (y \sigma) \circ x \circ (y \sigma)^{-1}$.

But $y \sigma$ is an anti-linear transf. of $\mathbb{C}^n$ such that $y \sigma \circ y \sigma = y \bar{y} = 1$. Thus if $J = i$, we get a real structure on $\mathbb{C}^n$, whereas if $J = -1$, we get a symplectic structure. Thus the symmetric spaces in question are $SU_n/SO_n \quad SL_n/Sp[\mathbb{C}^n]$ when $n$ is even.
It is desirable to work out the Grassmannian examples carefully. Take the case where $\sigma$ is conjugation on $U_n$ by:

\[
\sigma = \begin{pmatrix}
-I_p & 0 \\
0 & I_q
\end{pmatrix}
\quad p+q=n
\]

Then $K^\sigma = U_p \times U_q$.

First analyse $\{ y \in U_n \mid \sigma(y) = y^{-1} \} = \tilde{X}$.

Let $V_1$ be an eigenspace of $y \mapsto \sigma y = y^{-1}$. Then

\[
y v = \lambda v \implies \sigma(y)v = \lambda^{-1} v
\]

\[
y^{-1}(\sigma v) = \lambda v
\]

\[
y(\sigma v) = \lambda^{-1} v
\]

So $\sigma V_1 = V_{1^{-1}}$. If $\lambda \neq 1$, we can move $\lambda$ to 1.

\[
\lambda \rightarrow 1
\]

doing the same for $1^{-1}$. This deforms $y$ into those $y$ such that $\sigma y = y$ and $y^2 = 1$.

Note: $\det(y^*) = \det(\sigma y) = \det(y^{-1})$ so $\det(y) = \pm 1$. This is an invariant of the component
of \( X \) to which \( y \) belongs. Therefore \( X \) is not connected.

Consider the group generated by \( \sigma \) and \( y \). It is a dihedral group so its irreducible complex representations are as follows:

- \( \dim = 2 \) with \( y \) acting as \( \lambda, \lambda^{-1} \), \( \lambda \neq \pm 1 \).
- \( \dim = 1 \) with \( y = -1 \), \( \sigma = \pm 1 \).

Letting \( \lambda \) move around the unit circle we can specialize into the sums of the two characters. Thus as \( y \) varies over \( \{ y | Ty = y^{-1} \} \) the difference between the \( \dim \) of the \( \sigma = +1, y = -1 \) eigenspace and the \( \sigma = -1, y = -1 \) eigenspace doesn't change. If this is zero we can deform \( y \) to 1. Thus the components are indexed by:\n
\[
\begin{array}{c}
\sigma: & \frac{1}{2} & \frac{1}{2} \\
y: & \frac{1}{2} & \frac{1}{2} \end{array}
\]

so the invariant is \( p' - q' \).
Let $\sigma$ be an involution on $K$, given by an inner automorphism:
$\sigma x = \sigma_0 x \sigma_0^{-1}$ where $\sigma_0 \in Z$. Then

$\{ y \mid y \cdot \sigma(y) \in Z \} \sim \{ u \mid u^2 \in Z \}$

$u \sigma_0 \sim u \sigma_0$

(for $u \sigma_0 \sigma_0 u \sigma_0^{-1} = u^2 \sigma_0^2 \in Z \Rightarrow u^2 \in Z$). And moreover if $\sigma_0^2 = 1$, then

$\{ y \mid y \cdot \sigma(y) = 1 \} \sim \{ u \mid u^2 = 1 \}$

$u \sigma_0 \sim u \sigma_0$

Furthermore conjugation action on the right corresponds to twisted conjugation on the left:

$z (u \sigma_0) \sigma(z^{-1}) = z u \sigma_0 \sigma_0 z^{-1} \sigma_0^{-1} = z u \sigma_0^{-1} \sigma_0$

Therefore when $\sigma$ is the inner auto produced by $\sigma_0$ of order 2, the set of $y$ such that $\sigma(y) = y^{-1}$ is nothing but the different elements of order 2 in $K$.
Given an involution \( \sigma \) on \( K \), the symmetric spaces associated to involutions of the form \( \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} \) (related symmetric spaces to \( K/K^\sigma \)) are the \( K \)-orbits under \( \sigma \)-twisted conjugation on the set \( \{ y \in K/\mathbb{Z} \mid y \sigma(y) \in \mathbb{Z} \} \).

So if I want the symmetric spaces associated to \( U_n \) and the involutions related to \( \sigma = \text{id} \), I should look at the conjugacy classes of involutions, and these are the Grassmannians.

If \( \sigma = - \) on the other hand, and \( y \sigma = \bar{y} \), I have seen that \( f = \pm 1 \).

If \( f = +1 \), \( y \sigma \) is a conjugation, and if \( f = -1 \), \( y \sigma \) is a symplectic structure. One gets one \( U_n \)-orbit, when \( n \) is odd, and 2 if \( n \) is even.
Time now to understand the Grassmannians as symmetric spaces in detail. To start with, an involution \( \sigma \) in \( U_n \) with \( p \) eigenvalues \(-1\). If \( T \) is a maximal torus normalized by \( \sigma \), then \( T \) corresponds to a decomposition of \( \mathbb{C}^n \) into lines which are permuted by \( \sigma \). It is clear that \( T \) contains a maximal 1-reversed torus \( S \) iff there \( p \) orbits of order 2. Here I suppose \( 2p \leq n \). So I can divide \( \mathbb{C}^n \) into

\[
\mathbb{C}^n = L_1 \oplus L_2 \oplus \cdots \oplus L_p \oplus \Gamma
\]

where each \( L_i \) is a line and \( \sigma L_i = L_i \), whence if \( S \) acts on \( L_i \) thru \( X_i \), then it acts on \( L_i \) thru \( X_i^2 \). \( \Gamma \) is a trivial representation of \( S \) and \( \sigma \).

The good way to do the above is to choose a max. \( \sigma \)-reversed torus \( S \), then to consider the "dihedral" group \( \mathbb{Z}/2 \ltimes S \) acting on \( \mathbb{C}^n \) and to decompose it into irreducible representations

\[
(L_1 \oplus L_1) \oplus \cdots \oplus (L_p \oplus L_p) \oplus \Gamma
\]

where \( \Gamma \) is a trivial representation (as \( 2p \leq n \)). The group \( \mathbb{M} \) is the centralizer of this dihedral.
group, hence consist of scalars in each $L_i$. And any auto of $\Gamma$:

$$M = (S^1) \times \ldots \times (S^1) \times \text{Aut}(\Gamma).$$

June 20, 1975. The Grassmannians as symmetric spaces.

Start with $\mathbb{P}^1(C) = S^2 = \frac{U(2)}{U(1) \times U(1)}$, which is the conjugation class of the involution

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $U(2)$. $S^2 = \{e_1 + e_2\}$, a line will contain a vector $x_1 e_1 + x_2 e_2$ unique up to a non-zero scalar, and we associate to it the point $z = \frac{x_2}{x_1}$ of the sphere. Hence

$$\begin{align*}
e_1 &\leftrightarrow 0 \\
e_2 &\leftrightarrow \infty.
\end{align*}$$

Maximal flat submanifolds of $S^2$ are the great circles. So our reversed torus $S$ will be a double covering of $\mathbb{P}^1(R) \subset \mathbb{P}^1(C)$. Thus $S$ is the torus consisting of the rotations
\[ e^\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad 0 \leq \theta \leq 2\pi \]

and \( e^\theta \) maps \( e^0 = (1, 0) \) to \( (\cos \theta, \sin \theta) \) which corresponds to \( \tan \theta \in P^1(\mathbb{R}) \).

Note that if we identify \( S \subset U(2) \) properly with a flat submanifold of \( K \), then we map \( e^\theta \) to \( e^{\theta/2} \), \( \theta \in [0, \pi] \), \( \theta = 0 \) being a basket of \( S \). Thus \( e^{\theta/2} \) corresponds to \( \tan \theta \).

The group \( W_0 = \mathbb{Z}/2\mathbb{Z} \) and is generated by \( \sigma \).

The effect of \( \sigma \) on \( S^2 \) is given by \( \sigma(x) = -x \).

\( K^\sigma \) orbits on \( S^2 \) are latitude lines and

\[ \frac{\tan(\theta)}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2} \]

is a fundamental domain.

Next, generalize to \( X = \frac{U(2n)}{U(n) \times U(n)} \).

\[ \sigma = \begin{pmatrix} -I_n & I_n \\ I_n & I_n \end{pmatrix} \]

\( K^\sigma = U(n) \times U(n) \)
The Grassmannian we think of as \( n \)-planes is \( \mathbb{C}^n \oplus \mathbb{C}^n \), and the orbit of \( S \) is \( X \) containing those \( n \)-planes which decompose \( A = L_1 \oplus \ldots \oplus L_n \) 
where \( L_i \in \mathbb{C}e_i \oplus \mathbb{C}e_{n+1} \).

\[
M = \left( \begin{array}{ccc} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \ldots & \frac{1}{\lambda_n} \\
\end{array} \right) = \Delta T \in U(n) \times U(n)
\]

The orbit of \( S \) in \( X \) will be denoted \( \overline{S} \); it consists of planes stabilized by \( M \). Of course \( \overline{S} = (\mathbb{P}^1)^n \). The Weyl group \( W_0 \) is a semi-direct product \( \bigwedge^2 \cong (\mathbb{Z}/2)^n \).

Next I want to compute the diagram of this symmetric space. It is first necessary to transform \( S \) into the diagonal matrices. In this case \( S \) will appear as matrices of the form...
and \( T \) interchanges the \( i \)-th and \((i+1)\)-th entries. The group \( W_0 \) permutes the angles \( \theta_i \) and changes their signs; it is therefore a subgroup of \( \Sigma_{2n} \). The centralizing \( S \) of \( T \) consists of matrices

\[
\begin{pmatrix}
  e^{i\theta_1} & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & e^{i\theta_n}
\end{pmatrix}
\]

and it might be convenient to just \( \theta_{n+i} = \theta_i' \) for \( i = 1, 2, \ldots, n \). Then \( S \) consists of those elements of \( T \) such that \( \theta_i' = -\theta_i \).

I should maybe think of the angle \( \theta_i \) as being a linear function on the Lie algebra \( \mathfrak{t} \); \( \theta_i(x) \) is the \( i \)-th entry of the element \( x \).
Positive region of \( E_0 \) will be described by 
\[ \theta_1 \geq \ldots \theta_n \geq \theta'_n \geq \theta'_{n-1} \geq \ldots \geq \theta'_1 \]

Suppose now I see what happens to a pos. root of \( T \) when it's restricted to \( S \).

**Case 1:** \( \alpha = \theta_i - \theta_j \), \( i \neq j \). This restricts to the root \( \theta_i - \theta_j \) on \( E_0 \). Note that \( \alpha \sigma = \theta'_i - \theta'_j = -\alpha \).

**Case 2:** \( \alpha = \theta_i' - \theta_j' \), \( i \neq j \). This restricts to the root \( \theta_i' - \theta_j' \) on \( E_0 \). Note that \( \alpha \sigma = \theta_j' - \theta_i' = -\alpha \).

**Case 3:** \( \alpha = \theta_i - \theta_j' \). This restricts to the root \( \theta_i + \theta_j' \) on \( E_0 \). Note that \( \alpha \sigma = \theta_i' - \theta_j' = -\alpha \) when \( i \neq j \), but \( = -\alpha \) when \( i = j \).
Thus the positive roots in $E_6$ are:

$\theta_i - \theta_j \quad i < j \quad$ multiplicity 2

$\theta_i + \theta_j \quad i < j \quad$ mult. 2

$2\theta_i \quad$ mult. 1.

So the infinitesimal diagram for $\mu = 2$ is:

The Weyl group reflects around the lines; it is the dihedral group of order 8. Incidentally, this diagram is the same as for $SO(4)$ except that the multiplicities in the latter are all 2.

$$\dim(\mathbb{K}/\mathbb{M}) = 2 + 1 + 2 + 1 = 6 \quad \Rightarrow \quad \dim \frac{u(2) \times u(2)}{T} = 4 + 4 - 2$$

Yes.
The big diagram will have in addition the planes where the roots have integral values (I replace $\theta_i$ by $(2\pi)\theta_i$ say). For $n=2$

For $n=1, \quad 2\theta_1$ is only root and it has mult. 1.

Diagram is

\[
\begin{align*}
\theta = 0 & \quad \theta = \frac{\pi}{2} & \quad \theta = \pi
\end{align*}
\]

I get a fundamental domain using

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

with $0 \leq \theta \leq \frac{1}{2} \cdot 2\pi = \pi$
It is necessary to work out the relation between my building approach to $\Omega X$ and the Bott-Samelson approach.

In my approach, I have a building $X^\circ$, an action of $G^0$ on $X^\circ$ with quotient $C^\circ$, and the choice of $C^\circ$ gives me a parahorique $B_0$ such that

$$B_0 \backslash X^\circ = E_0.$$ 

Thus, quite canonically, points of $E_0$ are to be interpreted as cells in $X^\circ$. The basic CW complexes in my theory are the $G^0$-orbits in $X^\circ$; there is one for each $\eta \in C_0$. The CW complex $K^\circ_\eta \cong G^0_\eta$ has the homotopy type of the space of paths starting at any point of $G^0_\eta \subset B_0$ and ending on the orbit $K^\circ_\eta$.

In the path interpretation, we have given to elements of $X^\circ$, $K^\circ_\eta$ is the space of paths $h$ in $K$ such that $h(t) = h(-t)$ and $h(t) \in K^\circ_\eta$.

In the Bott-Samelson approach, one uses broken geodesics in $X$ perpendicular to $K^\circ$-orbits from $t_0$ to $K^\circ_\eta$.

Conclusions: Do not think of $K^\circ_\eta$ as paths in the symmetric spaces, but rather as an orbit in the building which intersects $E_0$ in a $G^0_\eta$-orbit.
and which has a cell decomposition with cells having their centers in $\tilde{\infty}$. (The point is that $X^\circ$ is just a contractible fibre space over $X$, hence not necessarily canonically identifiable with a space of paths.)

So the spaces that make sense algebraically are of the homotopy type of the spaces considered by Bott-Samelson, namely, paths from a point $x_0$ to a $K^\circ$-orbit. (Recall that $P(x_{10}, K^\circ \eta)$ has the homotopy type of the fibre of the inclusion $K^\circ \eta \subset X$.)

So the next point is that if we want the loop space of the symmetric space, then we want to choose the orbit $K^\circ \eta$ to be a point. So I want all roots $\alpha$ to have integral values at the points $\eta$.

June 21, 1975:

Recall that the building approach gives you the spaces $K^\circ \eta$, $\eta \in C'$, which have the homotopy of the fibre of the inclusion $K^\circ \eta \subset X$. Thus if $\eta$ is fixed under $K^\circ$, the space $K^\circ \eta$ has
the homotopy type of $\Sigma X$.

I am interested in the inclusion

\[ K^0 \eta \subset K^0 \eta. \]

The former space is isomorphic to $K^0 \eta \subset B\mathbb{K}$, and has a cell decomposition indexed by $W_0 \eta$, whereas $K^0 \eta$ has a cell decomposition indexed by $\overline{W}_0 \eta$. It seems clear that since $\eta \in \mathcal{C}_0$, the multiplicities are the same so in fact $K^0 \eta$ should be a subcomplex of $K^0 \eta$.

I wanted to do the example of $X = G_n(C^\infty)$.

\[
\sigma = \begin{pmatrix} -I \\ I \end{pmatrix} \quad K^0 = \begin{pmatrix} U(n) & 0 \\ 0 & U(n) \end{pmatrix}
\]

and

\[
\sigma \cdot \eta = \begin{pmatrix} 0 & \mathbb{C} \\ -\mathbb{C}^* & 0 \end{pmatrix}
\]

Take $\eta \eta = \begin{pmatrix} I \\ -I \end{pmatrix}$. Then

\[
e^{2\pi t \eta} = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}
\]

So if $t = \frac{1}{2}$ we get

\[
e^{\pi \eta} = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}
\]
which is centralized by \( K^\sigma \):

\[
\begin{pmatrix} A & B \\ B & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & B^{-1} \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & AB^{-1} \\ -BA^{-1} & 0 \end{pmatrix}
\]

So the orbit \( K^\sigma \eta \) is isomorphic to:

\[
\frac{U(n) \times U(n)}{\Delta U(n)} = U(n) \quad (A, B) \mapsto AB^T
\]

Take the inner product \( \text{tr} \ A^*A = \sum_{i,j} |a_{ij}|^2 \) on \( \mathbb{R}_{2n} \). On \( K \) it becomes:

\[
\langle \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix}, \begin{pmatrix} C_1^{*} \\ C_2^{*} \\ C_3^{*} \\ C_4^{*} \\ C_5^{*} \\ C_6^{*} \end{pmatrix} \rangle = 2 \times \text{tr} \ C^*C
\]

whence on polarizing:

\[
\langle \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}, \begin{pmatrix} C_1^{*} \\ C_2^{*} \\ C_3^{*} \end{pmatrix} \rangle = \text{tr} \left( C_1^{*}C_2 + C_2^{*}C_1 \right)
\]

The Morse function on \( K^\sigma \eta \) we use is \( \eta \mapsto -\langle \phi \eta, \xi \rangle \) where \( \xi \) is a generic pt of \( C_0 \). Take:

\[
\xi = \begin{pmatrix} \lambda_1 \\ \lambda_1 \end{pmatrix} \quad A = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix}
\]

where the \( \lambda_i \) are real and decreasing \( \lambda_1 > \cdots > \lambda_n > 0 \).
Then
\[- \begin{pmatrix} 0 \\ -\lambda I \end{pmatrix} \begin{pmatrix} A \\ \lambda \end{pmatrix} = - \text{tr} (A^* \lambda + \lambda A)\]
is the Morse function to consider; call it \( f(A) \).

\[ f(A(I + \epsilon B)) = - \text{tr} \left( (I - \epsilon B)A^* \lambda + \lambda A (I + \epsilon B) \right) \]
\[ = - \text{tr} (A^* \lambda + \lambda A) + \epsilon \text{tr} (BA^* \lambda - \lambda AB) \]

do a critical point is when

\[ tr (BA^* \lambda - \lambda AB) = 0 \]
for all \( B \).

\[ tr B \cdot (A^* \lambda - \lambda A) = 0 \]

so \( A \) critical \( \iff \) \( A^* \lambda = \lambda A \) \( \iff \) \( \lambda A \) is hermitian.

Now if \( \lambda A = H \) hermitian, then

\[ (\lambda A)(\lambda A)^* = \lambda AA^* \lambda = \lambda^2 = H^2 \]

so therefore \( H \) will be a square root of \( \lambda \)

ie:

\[ H = \begin{pmatrix} \pm \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pm \lambda_n \end{pmatrix} \]

It follows that \( A \) is a diagonal matrix with \( \pm 1 \) entries.
June 22, 1975 (35 yrs old)

I still don't understand the Bruhat decomposition in the symmetric space case.

\[ K \text{ compact connected with involution } \sigma, \]
\[ S \text{ a maximally-reversed torus, } T \text{ a max. torus containing } S, \]
\[ E = \text{roots of } K \text{ with respect to } T, \]
\[ E_0 = \text{roots of } K \text{ with respect to } S. \]

Basic point was to look at \( K \) as a representation of \( S' = \{ \sigma^{-1} \} \times S. \) Then it splits as a direct sum according to the cliff reps of \( S': \)

\[ K = (m \oplus E_0) \oplus \sum_{\beta \in E_0^+} k_\beta \]
\[ \sigma = 1 \quad -1 \quad \beta \in E_0^+ \]
\[ S \text{ trivial} \]

where \( k_\beta \) is isomorphic to a direct sum of the representation \( \chi_\beta \) of \( S' \) on \( C \) where

\[ \sigma(z) = \bar{z} \]
\[ \exp(z) \cdot z = e^{2\pi i \beta(\eta)} z \]
I am going to try to prove the Bruhat decomposition. The method goes like this: suppose given the group $G$ with subgroups $B$ and $N$ with $BnN < N$ and $W = N/BnN$ a Coxeter group gen. by $s_i$. Then I have to prove

a) $BsbwB = BawB$ if $l(sw) = l(ws) + 1$

b) $\overset{\circ}{B} \overset{\circ}{B} = B \cup BsbB$

The general case goes like this:

$l(sw) = l(ws) - 1$, then $l(sw) = l(sw) + 1$ so

$$BsbwB = BsbBsbBsbwB$$

$$= (B \cup BsbB)BawB$$

$$= BawB \cup BawB.$$  

Example: Recall that I can identify the spherical building of $G^r$ with the sphere in $\mathbb{R} (or \mathbb{F}^r \subset G^r)$, I want to describe the action of $G^r$ on $S(p)$.

The idea is as follows. To an element $g$ of $p$ I associate the geodesic $e^{t\xi} \cdot 0$ in the symmetric spaces $\Omega \subset G^r$, the geodesic
get \( e^{t \xi} \bigstar \) should be asymptotic to a unique geodesic of the \( e^{t \eta} \). Then \( g \star e^{t \xi} = \eta \).

Recall the group \( K^* \) is the fixed point group for the Cartan involution which I will denote by \( g \mapsto (g^*)^{-1} \). Then I can embed the symmetric space inside \( G^* \) via \( g \circ \delta \mapsto gg^* \), (and we get the positive component of these \( heG \to h^* = h \)).

So now \( \eta \) is defined by
\[
ge^{2t \xi} g^* \sim e^{2t \eta} \quad \text{as } t \to \infty.
\]

Let \( P_\xi = \{ geG^* | e^{t \xi} \text{ is bounded as } t \to \infty \} \).

The associated finite factor group \( G = GL(n, \mathbb{C}) \) with \( \sigma = \text{id} \), \( \bigstar = \text{conjugate transpose} \). If
\[
\xi = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{pmatrix}
\]
with \( i < j \Rightarrow \lambda_i > \lambda_j \)

Then
\[
(ge^{t \xi} \bigstar)^* \bigstar = \sum \lambda_i e^{2t \lambda_i} \bigstar 
\]
declares that
\[
\sum \lambda_i ^2 = 0 \Rightarrow \gamma_{ij} = 0.
\]
Thus if we arrange the $x_i$ in decreasing order

then \((e^{-t_j}g e^{t_i})_{ij} = e^{(\lambda_j - \lambda_i)}g_{ij}\). So

\[
\lambda_i < \lambda_j \Rightarrow g_{ij} = 0
\]

Thus $P_\xi$ is the parabolic

Note that the limit

\[
\lim_{t \to \infty} e^{-t_j}g e^{t_i} = g^0
\]

is just the seductive part of $g$ and it centralizes $\xi$.

So now I can understand the asymptotic behavior of $g e^{t_i} \xi^*$ when $g \in P_\xi$. I pull this back via $e^{-t_j}$. Better I have $g e^{t_i} \xi^0$ which I pull back to $e^{-t_j}g e^{t_i} \xi^0$ which converges to $g^0 \xi^0$. 
So at least when \( g \in P_g \) I see that the asymptotic behavior is

\[
ge^{-i.0} \sim e^{-i.0} \]

where \( \sim \) means the distance between these points approaches zero.

**General case:** Because \( P_g \) is parabolic I know \( \frac{K}{K_g} = G/P_g \), hence \( \exists k \in K \) such that \( \tilde{g}^{-1}k \in P_g \). Then

\[
k^{-1}g^{-1}e^{+i(\kappa - \eta)} \sim e^{+i(\kappa - \eta)}
\]

or

\[
g^{-1}e^{+i.0} \sim e^{+i(\kappa - \eta)} \kappa(k^{-1}g) \eta
\]

Thus I see that the action of \( \tilde{g} \) on \( P \) is described by the asymptotic behavior of geodesics.

It should be possible to compactify \( G/K \) by adding limit points corresponding to \( S(g) \).
Existence of asymptotics:

Assertion: For any $g \in G$ and $\xi \in \mathfrak{p}$, there is a unique $\eta = g * \xi$ in $\mathfrak{p}$ such that $e^{-t\eta} g e^{t\xi}$ converges as $t \to \infty$.

First we want to establish the uniqueness. Thus if $e^{-t\eta_1} g e^{t\xi}$ and $e^{-t\eta_2} g e^{t\xi}$ are convergent so is $e^{-t\eta_1} e^{t\eta_2}$. Hence I want to prove:

Lemma: If $\eta, \xi \in \mathfrak{p}$ and $e^{-t\eta} e^{t\xi}$ is convergent, then $\xi = \eta$.

Proof: If I embed K inside $U(n)$, then I can reduce to the case where $G = GL(n)$ and $\eta, \xi$ are hermitian matrices.

If $e^{-t\eta} e^{t\xi} \to g$ as $t \to \infty$

then $e^{-s\eta} g e^{s\xi} = g$ for any $s$

or $g e^{s\xi} g^{-1} = e^{s\eta}$ or $g \cdot \xi = \eta$. 
So we can rewrite:

$$g e^{-t\xi} g^{-1} e^{t\xi} \rightarrow g$$

or

$$e^{-t\xi} g e^{t\xi} \rightarrow 1.$$

Since I am trying to show $g = 1$, I can suppose by conjugating with a unitary matrix, then $\xi$ is diagonal

$$\xi = \begin{pmatrix} \lambda_1 \\ & \ddots \\ & & \lambda_n \end{pmatrix}$$

with $\lambda_1 \geq \ldots \geq \lambda_n$. Then

$$(e^{-t\xi} g e^{t\xi})_{ij} = e^{t(\lambda_j - \lambda_i)} g_{ij}$$

so $\lambda_i \leq \lambda_j \Rightarrow g_{ij} = 0$. Thus if $\mu_1, \ldots, \mu_n$ are the distinct eigenvalues, $g$ is of the form

$$g = \begin{pmatrix} I & \ast & \ast \\ 0 & I & \ast \\ 0 & 0 & I \end{pmatrix}$$

But consider $g^* g^{-1} \xi = \eta \Rightarrow g^* g^* g^* \xi = \eta \Rightarrow g^* g^* \xi^* = \eta$.

$$\Rightarrow g^* g \xi = \xi.$$
If \( g = \begin{pmatrix} I & A \\ \hline I & \ast \\ \hline 0 & \ast \end{pmatrix} \)

then

\[
g^* g = \begin{pmatrix} I & 0 \\ \hline A^* & I \\ \hline 0 & I \end{pmatrix} \begin{pmatrix} I & A \\ \hline 0 & I \\ \hline \ast & \ast \\ \hline \ast & \ast \end{pmatrix}
= \begin{pmatrix} I & A \\ \hline A^* & A^* + I \end{pmatrix} \begin{pmatrix} I & A_2 & \cdots & A_n \\ \hline A_2^* & \ast & \ast & \ast \\ \hline \vdots & \ast & \ast & \ast \\ \hline A_n^* & \ast & \ast & \ast \end{pmatrix}
\]

and this can centralize \( \xi \) only if \( A = 0 \).

Thus we see \( g = 1 \), proving the lemmas.
So next we want to establish the existence of an \( \eta \) such that

\[
e^{-\eta g} e^t \frac{g}{e^t}
\]

converges. Enough to do for a set of \( g \) which generate \( G \) (with all \( \frac{g}{e^t} \) for each \( g \)),

because

\[
e^{-t g t(g^* t)} g_{1,2} e = e^{-t g t(g^* t)} g_{1,2} e g_{1,2} e^{-t g t(g^* t)} e.
\]

So I can suppose \( g \) is in the neighborhood of \( 1 \).

Next we can use Lie algebra theory to show we that any \( g \) near \( 1 \) can be represented \( g = k \cdot u \) where \( k \in K \) and \( u \in P \) where \( u \in P = \{ u \mid e^{-t} u e^{t} \text{ converges} \} \).

Then I have

\[
ge^t = e^{t k} e^{-t} u e^t.
\]

and we win.

---

**Note:** That the signature

**Lemma:** \( \eta \) is a \( e^{\eta t} e^{\frac{t}{2}} \) bounded \( t \to \infty \)

\[\Rightarrow \eta = \frac{1}{2}.
\]

**Proof:** Enough to do for \( G_{n} \). The
The point is that the matrices $\eta, \zeta$ have real eigenvalues, hence the entries of $e^{\eta t} e^{\zeta t}$ will be linear combinations of exponentials: $e^{at}$ for $a \in \mathbb{R}$. Such a function if bounded as $t \to +\infty$ will be convergent, so we can use the preceding proof.

Consider the asymptotic behavior of $e^{t J}$ where $J$ is any element of $\mathfrak{g}$. First take $G = \text{GL}_n \mathbb{C}$. If $n = 1$, then

$$e^{-t \Re J} e^{it \Im J} = e^{i \Im J}$$

is bounded. Write $J = S + I_n$ Jordan decomposition,

$$e^{t J} = e^{t S} e^{t I_n}$$

polynomial in $t$.

Basic asymptotic behavior result is:

Prop: For any $J \in \mathfrak{g}$, there is a unique $\eta \in \mathfrak{g}$ such that $e^{-t \eta} e^{t J}$ has polynomial growth as $t \to +\infty$.

Uniqueness: if $e^{-t \eta_1} e^{t J} = e^{-t \eta_2} e^{t J}$ is of polynomial
the because we know its entries are linear combinations of real exponentials, we have that polynomial growth \( \Rightarrow \) convergence, whence preceding arguments show that \( y_1 = y_2 \).

\[ S = S_0 + S_n \]

we have

\[ e^{tS} = e^{tS_0} e^{tS_n} \]

polynomial function of \( t \)

hence we can suppose that \( S \) is semi-simple.

I claim I can find \( g \) such that if 

\[ \text{Ad}(g) \cdot S \]

is split according to \( g^o = k^o \cdot q \),

then the two components commute. Assume this for the moment and let \( \text{Ad}(g) \cdot S = S' + S'' \) be this decomposition. Then

\[ g^{-1} e^{tS} g = e^{tS'} \]

bounded \( (e \in K) \)

\[ e^{tS} = g e^{tq} e^{tS'} g^{-1} \]

bounded

so we win. As for the claim, it should be
known that any semi-simple $I$ is contained in a Cartan subalgebra fixed under the Cartan involution $\tilde{\tau}$.

$$e^{t\tilde{\tau}} = e^{t\tilde{\tau}_0} e^{t\tilde{\tau}_n}$$

$$= g e^{t\eta} e^{t\eta'} g^{-1} e^{t\tilde{\tau}_n}$$

$$= e^{t g \eta} e^{-t g \eta} g e^{t\eta} e^{t\tilde{\tau}_n}$$

fast convergence oscillatory polynomial growth

A way of describing the building without using a specific $K$ is to consider all $I$ inside of which are semi-simple and have only real eigenvalues. Put an equivalence relation on $I$ by saying $I_1 \sim I_2$ iff $e^{-t_1} e^{t_2}$ is convergent.

(* conjugate to an element of $E_0$, or generating a split form in $G$.*)
Review: If $K$ is compact connected, then $G$, $\sigma$, $\xi$, $\text{cartan involution of } G$ with respect to $K$. One defines $\sigma$ on $G$ so that $\sigma \cdot \xi = \xi \cdot \sigma$, which is the $C$-linear extension of $\sigma$ to $G$. This extended $\sigma$ is anti-linear.

Basic facts about $G, K$. A. Pat X = $G/K$, and identify it with the identity component of $G$ via $gK \mapsto g\xi g^*$. We know from looking at Lie algebras that for $g$ small we have $g = e^{2\pi i} \xi$ with $\xi K$. Hence, $gg^* = e^{2\pi i}$. This implies that $\exp: \xi K \mapsto e^{2\pi i}K = e^{2\pi i}X$.

One has the map

$$\eta \mapsto e^{2\pi i} \xi \rightarrow e^{2\pi i}K \rightarrow e^{2\pi i}X$$

and one can compute its differential. I can do this in $G(n, C)$ and I find the differential is non-singular because $\xi$ is hermitian, hence its eigenvalues are real, so are not identified by $e^{2\pi i}$. Thus the map $\eta \mapsto e^{2\pi i}X$ is etale.

Next, for $g$ small the above theorem applies...
I want to show that the map
\[ ik \longrightarrow X \]
\[ \eta \longrightarrow \mathbb{C}^n \]
is a diffeomorphism. Possible proof:
Verify for $K = U(n)$, $G = GL_n$ and descend via a $K \longrightarrow U(n) \llarrow U(m)$ presentation.
This will prove bijectivity, and the rest is clear because the map is étale. Incidentally this shows $\pi_1 G$ is connected if $K$ is.

Next one has obviously $G^\sigma/K^\sigma \rightarrow (G/K)^\sigma$, and $(ik)^\sigma = \pi_0$, whence
\[ \pi_0 \longrightarrow \pi_0 \]
so in particular $\pi_0 K^\sigma = \pi_0 G^\sigma$.

Next I need to show the $G^\sigma$-orbits on $\pi_0$ are the same as the $K^\sigma$-orbits. Go back over the action of $G^\sigma$ on $\pi_0$, which is defined as follows: Given $g \in G^\sigma$ and $x \in \pi_0$, $g \ast x$ is the
unique element of \( \mathcal{P} \) such that
\[
e^{-t(g*\delta)} \mathcal{P} e^{t\delta}
\]
converges. I shall recall the proof that \( g*\delta \) exists.

First, suppose \( g \) is near the identity, in which case \( g = e^\delta \) and we can work in the Lie algebra. So I look at the action of \( \text{Ad}(e^{t\delta}) \) on \( g^\mathbb{T} = \mathbb{R}^\mathbb{T} \oplus \mathcal{P} \).

Suppose first try to understand
\[
g = \mathbb{T} \oplus i\mathbb{R}.
\]
Let \( \mathfrak{h} \) be the Lie algebra of \( \mathbb{T} \mathbb{K} \), \( \mathfrak{a} = i\mathbb{R} \). Choose usually basis
\[
g = h + \sum_{\alpha \in I} C X_\alpha \quad \quad X_\alpha^* = X_{-\alpha}
\]
\[
h = \mathbb{T} \oplus i\mathbb{R} \quad \quad \mathfrak{a} \text{ contains the } H_\alpha.
\]
\( \alpha(\delta) \) is real if \( \delta \in \mathfrak{a} \). Thus
\[
\text{Ad}(e^{t\delta}) = 0 \text{ on } \mathfrak{h}
\]
\[
= e^{t\alpha(\delta)} \text{ on } \mathbb{C} X_\alpha,
\]
and so the roots divide up into those such that \( \alpha(\delta) > 0 \), \( = 0 \), \( < 0 \).
I let $P_\xi$ be the Lie subgroup of $G$ with the Lie algebra

$$\text{Lie}(P_\xi) = \mathfrak{h} + \sum_{\alpha(i) > 0} \mathfrak{c}_\alpha$$

$$= \{ \eta \in \mathfrak{g} \mid \text{Ad}(e^{t\xi})\eta \text{ converges as } t \to \infty \}$$

Since $\mathfrak{g}_\xi = \mathfrak{k} + \text{Lie}(P_\xi)$ any element of $\mathfrak{g}$ near the identity is a product $g = ke^u$ where $k \in K$ and $u \in P_\xi$. If $u = \exp(\eta)$ then

$$e^{-t\xi}ue^{t\xi} = \exp \left( \text{Ad}(e^{t\xi})\eta \right)$$

converges so

$$ge^{t\xi} = ke^{t\xi}e^{-t\xi}ue^{t\xi}$$

$$= e^{t\text{Ad}(k)\xi}ke^{-t\xi}ue^{t\xi}$$

showing that $g*\xi = \text{Ad}(k)\xi$.

Next I want to show $G = \bigoplus P_\xi$. First remark it that $K P_\xi$, contains an open neighborhood of 0 which can be chosen $K$-invariant. This follows from the above Lie algebra study.
plus the implicit function theorem. Now suppose \( g_2 \in KP_3 \) and \( g_1 \in U \). Then
\[
g_2 = k_2 p_2, \quad k_2 \in K, \quad p_2 \in P_3
\]
\[
g_2 \times k = k_2 \cdot k
\]

Because \( U \) is \( K \)-invariant, \( U \subseteq KP_3 \cdot k \) for any \( k \)

and \( k \cdot P_3 \cdot k^{-1} = P_{k \cdot 3} \), so choosing \( k = k_2 \)

we have \( g_1 = k_1 \cdot p_1 \) where \( p_1 \in P_{k_2 \cdot 3} = k_2 \cdot P_3 \cdot k_2^{-1} \)

\[
k_2^{-1} p_1 \cdot k_2 \in P_3
\]

Thus \( g_1 g_2 = k_1 p_1 \cdot k_2 p_2 = k_1 k_2 \cdot k_2^{-1} p_1 \cdot k_2 p_2 \)

\( \in K \cdot p_3 \)

It follows that \( KP_3 \) is stable under multiplication by \( U \), which generates \( G \).

(Precisely: \( \bigwedge_{k \in K} KP_3 \cdot k^{-1} = \bigwedge_{k \in K} KP_{k \cdot 3} = U \)

contains an open neighborhood of \( K \) hence generates \( G \).)
Suppose now that $g \cdot \xi = \xi$, and let $g = kp$ with $k \in K$ and $p \in P_\xi$. Then 

$$g \cdot \xi = k \cdot \xi = \xi$$

where $k$ centralizes $\xi$. But the centralizer of $\xi$ in $K$ is connected, hence $K_\xi \leq P_\xi$. Thus $g \in P_\xi$. So we have proved 

$$K/K_\xi = G/P_\xi.$$

In particular, if I take $\xi$ to be a regular element of $G$, then I get 

$$K/\Gamma = G/B$$

as the Iwasawa decomposition 

$$G = K \times _{\max} B.$$
$CP^n$ as a symmetric space.
Continue with the Grassmannians $U(p+q)/U(p) U(q)$
when $p>q$.

$$K^\circ = \begin{pmatrix} U(p) & 0 \\ 0 & U(q) \end{pmatrix}, \quad \tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The maximal reductive torus can be taken to be

$$S = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The space $\mathfrak{so}$ is generated thereby

$$\mathfrak{so} = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
To see what the roots are and their multiplicities, it is more convenient to use a basis consisting of the eigenvectors of $S$.

$$S: \begin{pmatrix} e^{i\theta} & e^{i\theta} \\ e^{-i\theta} & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \mathbf{1} \end{pmatrix}$$

The Weyl group permutes the angles $\theta_i$ and changes sign.

$Lie(T)$ consists of:

$$\begin{pmatrix} id_1 \\ id_2 \\ \vdots \\ id_n \end{pmatrix}$$

where $\lambda'_i = \lambda_{p+i}$.

$\sigma$ acts on $Lie(T)$ by interchanging $\lambda_i$ and $\lambda'_i$.

Thus

$$\lambda_{i \cdot \sigma} = \lambda'_i$$

as functions on $Lie(T)$. 
Let us now compute the restriction of the roots of \( R \) with \( T \) to \( \text{Lie}(S) \).

\[
\lambda_2 \rightarrow 0 \\
\lambda_{p+1} \rightarrow -\Theta_i \\
\lambda_i \rightarrow 0 \\
\lambda_i \rightarrow \Theta_i, \quad i = 1, \ldots, p
\]

\( \lambda_i \rightarrow 0 \), \( i > 2p \).

Fundamental chamber in \( \text{Lie}(S) \) is given by

\( \Theta_1 \geq \Theta_2 \geq \cdots \geq \Theta_p \geq 0 \). This is contained in the chamber

\[
\lambda_1 \geq \cdots \geq \lambda_p \geq \lambda_{p+1} \geq \cdots \geq \lambda_{2p+1} \geq \cdots \geq \lambda_n.
\]

Now take \( \alpha = \lambda_i - \lambda_j \) with \( i < j \).

Case 1: \( 1 \leq i < j \leq p \) \( \left( \alpha \big| \text{Lie}(S) \right) = \Theta_i - \Theta_j \)

\[
\rightarrow \Theta_j - \Theta_i
\]

Case 2: \( 1 \leq i \leq p, \ p+1 \leq j \leq 2p \) \( \left( \alpha \big| \text{Lie}(S) \right) = \Theta_i + \Theta_j \)

\[
\rightarrow -\Theta_i - \Theta_j
\]

Case 3: \( 1 \leq i \leq p, \ 2p < j \) \( \left( \alpha \big| \text{Lie}(S) \right) = -\Theta_i - \Theta_j \)

\[
\rightarrow \Theta_i
\]

Case 4: \( p+1 \leq i < j \leq 2p \) \( \left( \alpha \big| \text{Lie}(S) \right) = \Theta_{2p} - \Theta_i - \Theta_{i-p} \)

\[
\rightarrow \Theta_{2p} - \Theta_{i-p}
\]

Case 5: \( p < k \leq 2p < j \) \( \left( \alpha \big| \text{Lie}(S) \right) = -\Theta_i - \Theta_{i-p} \)

\[
\rightarrow -\Theta_i - \Theta_{i-p}
\]
Thus the roots of the Grassmannian are the following:

\[
\begin{align*}
\Theta - \Theta & \quad 1 \leq i < j \leq p \quad \text{multiplicity} = 2 \\
\Theta + \Theta & \quad 1 \leq i < j \leq p \quad \text{multiplicity} = 1 \\
2\Theta & \quad 1 \leq i \leq p \\
\Theta^2 & \quad 1 \leq i \leq p
\end{align*}
\]

Suppose \( p = 1 \). Then this becomes:

\[
\begin{align*}
2\Theta & \quad \text{mult. 1} \\
\Theta & \quad \text{mult. 2(n-2)}
\end{align*}
\]

Picture:

\[
\begin{array}{ccccccc}
& 1 & 2n-3 & 1 & 2n-3 & 1 & 2n-3 \\
\Theta = 2\pi & 0 & \Theta = \pi & \Theta = 2\pi & \Theta = 3\pi & \Theta = 4\pi
\end{array}
\]

Here the symmetric space is \( CP^{n-1} \). The reversed torus \( S \) corresponds to the circle

\[ RP^1 \subset CP^1 \subset CP^{n-1} \]

The stabilizer of this circle is \( M = \mathbb{R} \times \mathbb{S}^1 \times U(n-2) \subset U(2) \times U(n-2) \)

The special points are \( 0, \infty \in RP^1 \). The stabilizer in \( K^0 \) of \( 0 \) is \( K^0 \); the stabilizer in \( K^0 \) of \( \infty \) is \( S \times S^1 \times U(n-2) \). Thus

\[
\frac{K^0}{M} = \frac{U(2) \times U(n-1)}{AS^1 \times U(n-2)} = S^{2n-3}
\]
When we compute the loop space, we meet the new diagram in the Gysin sequence. We can head from a conjugate point. Which are perpendicular to the $K^n$ orbit.

$$\Omega_0 : K^n / M = S^{2n-3}$$

$$\Omega_\infty : S^n \times S^{n-2} / M = S^1$$

So the Kac-Moody series is

$$\frac{1 + t}{1 - t^{2n-2}}$$

which corresponds to

$$\Omega S^{2n-2} \rightarrow \Omega \mathbb{C}P^{n-1} \rightarrow S^1$$

Features of this example:

i) both $\theta, 2\theta$ roots

ii) high multiplicities for a root.

Consider next the sphere $S^n$, $n \geq 2$. Here I can calculate the roots and the multiplicities without going to the group. Maximal flat submanifolds are the great circles. So
\[ K^n / M = S^{n-1} \]

and so the diagram has to be:

\[
\begin{array}{ccc}
\mathbb{R}^m & \rightarrow & \mathbb{R}^m \\
\mathbb{R}^n & \rightarrow & \mathbb{R}^n \\
\mathbb{R}^n & \rightarrow & \mathbb{R}^n \\
\mathbb{R}^n & \rightarrow & \mathbb{R}^n \\
\end{array}
\]

\[ e = \pi \quad e = 0 \quad e = \pi \quad e = 2\pi \]

Roots for \( O(2n) \): The Lie algebra \( \mathfrak{so}(2n) \) consists of skew-symmetric matrices.

A maximal torus is:

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

To see what the roots are, I can suppose \( m = 2 \), and \[ M_2(\mathbb{R}) \], acted on by multiplying:

\[ M_2(\mathbb{R}) = \text{Hom}_\mathbb{R} (\mathbb{R}, \mathbb{R}) \]

with \( \begin{pmatrix} r_0 & r_0 \\ r_0 & -r_0 \end{pmatrix} \) acting as \[ A \mapsto e^{i\theta} A e^{-i\theta} \].

\[ M_2(\mathbb{R}) \text{ breaks up into } \mathbb{C} \oplus \mathbb{C} \text{, e.g. } \sigma = \text{ complex conjugation, so we get the roots} \]

\[ \theta \pm i \theta \].
Thus the roots for $O(2m)$ are the function $\theta_i \pm \theta_j$ for $1 \leq i < j \leq m$. The Weyl group permutes the $\theta_i$ and changes their signs. Only an even number of signs can be changed.

$$W = \sum_m \times \{ x \in \mathbb{Z}^n \mid x_i = 0 \}$$

To get the sphere we consider the involution $\theta_i \to -\theta_i$, $\theta_j \to \theta_j$. The only root is $\theta_j$; its multiplicity is $2m-2 = (2m-1) - 1$.

Calculate Dynkin diagram of $O(2m)$.

Choose generic point in Lie(T) say $i \in (e_i)$, and suppose it lies in the open region $i > \cdots > i_m > 0$. Then the chamber containing $i$ is described by $\theta_i + \theta_j$ has same sign as $i_j + i_j$ or zero.

Positive roots are therefore

$$\theta_i + \theta_j \quad 1 \leq i < j \leq m$$

$$\theta_i + \theta_j \quad 1 \leq i < j \leq m$$

and the fundamental chamber is described by

$$\theta_i > \theta_j \quad i < j$$

$$\theta_i + \theta_j > 0$$

But $\theta_i + \theta_j$ is minimum for $\theta_i + \theta_j$. So
the fundamental character is

\[ \theta_1 > \theta_2 > \cdots > \theta_m > -\theta_{m-1} \]

and the fundamental roots are

\[ \theta_1 - \theta_2, \quad \theta_{m-1} - \theta_m \]

so we get the diagram

\[ \begin{array}{c}
1 & 1 & 1 & 1 \\
\theta_1 - \theta_2 & \cdot & \cdot & \cdot \\
& \theta_{m-1} - \theta_m & \cdot & \cdot \\
& & & \theta_{m-1} + \theta_m \\
\end{array} \]

(one line between vertices means the angle is \( \frac{2\pi}{3} \))

over vertex goes something proportional to length

\[ n(2m+1) \]: same forms and the same roots

\[ \theta_i + \theta_j, \quad 1 \leq i < j \leq m \], but we have some more roots.

Critical case \( m = 1 \):

\[ \begin{pmatrix} \cos \theta & -m \cos \theta \\ \sin \theta & \cos \theta \end{pmatrix} \times \\
\begin{pmatrix} \cos \theta & \cos \theta \\ \sin \theta & \cos \theta \end{pmatrix} \times \\
\begin{pmatrix} 1 \end{pmatrix} \]

Extra root is just \( \theta \). So the roots are

\[ \pm (\theta_i - \theta_j), \pm (\theta_i + \theta_j) \quad 1 \leq i < j \leq m \]

and \[ \pm \theta_i \quad 1 \leq i \leq m \]
The Weyl group is clearly 
\[ \Sigma_m = \prod_{i=1}^{m-1} (\mathbb{Z}/m) \]
so the fundamental chamber is clearly 
\[ 0 > e_2 > \cdots > e_m > 0 \]
and the simple roots are 
\[ e_1 - e_2, \quad \ldots, \quad e_{m-1} - e_m, \quad e_m. \]

Thus the Dynkin diagram is

![Dynkin Diagram](attachment:image.png)

\( \text{Sp}(2m) \), a subgroup of \( \text{U}(2m) \) commuting with \( J \).

First take \( m = 1 \). Let \( J \) denote the auto-

**if** \( C \in \mathbb{C}_1 \) such that \( J(x+iy) = 2y - x \). Then

I am often than unitary matrices commuting

with \( J_1 \). Let \( J \) be the linear operator with

\( J(1) = 1 \), \( J(x) = -x \). Then \( J = x \cdot J \) where

also \( x(1,1) = (2,2) \).

In general, I work in \( C^{2m} \) with

basis \( e_1, \ldots, e_{2m} \) and \( J e_i = e_{i+m} \) \( J^2 = -1 \).

Thus

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
and I want those unitary matrices $A$ such that

$$\sigma^T A \sigma = A \sigma^T$$

i.e., such that $J A J^{-1} = A$.

It follows that

$$\begin{pmatrix} 1 & \alpha^* \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ -\alpha^* & 1 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$$

This means we want all unitary matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Thus $Sp_1 = SU_2$.

The Lie algebra consists of matrices of the above form which are skew-hermitian $\Rightarrow \times$ skew-hermitian, $\beta$ symmetric. Maximal torus

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$
In Roots include $\Theta_i - \Theta_j$ for $i < j$, $2\Theta_i$, and $\Theta_i + \Theta_j$ for $i < j$.

\[
e^{i\Theta_i}, \quad e^{i\Theta_j}
\]

\[
e^{i(\Theta_i + \Theta_j)}
\]

We say group is clearly $\Sigma_2 \times (\mathbb{Z}_2)^n$, hence a fundamental chamber is given by $\Theta_1 > \ldots > \Theta_m \geq 0$, so the simple roots are

$\Theta_1 - \Theta_2, \quad \Theta_m - \Theta_m - \Theta_m, \quad \ldots$

and the Dynkin diagram is

\[
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\]

Basic proof technique in the case of a group $G$ is to consider for any root $\alpha$, the map $\Sigma L_2 \to G$ associated to the triple $X_{\alpha}, Y_{\alpha}, H_{\alpha}$. This map is unique up to picking out $X_{\alpha}$. Also if $G$ is simply-connected it is an embedding, because one can suppose $\alpha$ simple, and the simple $H_{\alpha}^\mathbb{Z}$ form a $\mathbb{Z}$-lattice for the lattice points, when $\mathbb{Z}$ is 1-connected.
What this technique means is that the centralizer of the hyperplane \( z = 0 \) in \( h \) is \( h + q^x + q^{-x} \). By effect if \( \xi \) is generic such that \( x(\xi) = 0 \) then \( \beta = 0 \Rightarrow \beta \) proportional to \( x \) whence \( \beta = \pm x \). These centralizers of hyperplanes are exactly the sort of things one considers in the Morse theory.

I want to look at the analogous things in the symmetric space situation. Thus given a root \( x \in \Phi^+ \) I consider the torus killed by \( x \) and its centralizer which will be:

\[(\star) \quad m + (\Delta E + k + k_{2x})\]

and whose complexification is:

\[q^{2x} + q^{-2x} + (m + m) + q^x + q^{-x}\]

It is clear that \( E_0 \) is still a maximal abelian subspace of \((\star)\) because the \( \Delta \)-minus space is contained in \( E + k + k_{2x} \).

Next take \( \chi(\xi) = 0 \) for any \( \xi \) in the center of \((\star)\) which has to be in \( m + E_0 \) in fact it is contained in \( m + \) the part of \( E_0 \) killed by \( x \) so therefore...
it is clear that the semi-simple part of (x) is going to be associated to a rank 1 symmetric space.

Rank 1 symmetric spaces are spheres, projective spaces, quaternionic projective spaces, and the complex rank 1 symm. spaces are projective spaces. It is the octonion projective plane.

(Therefore ranks 1 symm. spaces are projective spaces.)

Just to be complete I should work out the diagrams and multiplicities of the quaternionic projective spaces.

We had the max. cones in Sp(n):

\[ e^{i0}, \quad e^{i\Theta}, \quad e^{-i0}, \quad e^{-i\Theta} \]

One takes the reflection in the \(\Theta\) angle (for involutivity), so the restriction to \(E_0\) sends \(\Theta\) to \(-\Theta\) and \(\Theta_m\) to 0. Then we get

\[ \Theta \quad \text{mult.} \quad 2(\Theta - 1) \]

\[ 2\Theta \quad \text{mult.} \quad 1 \]
For calculation we take interchange of $\theta_1, \theta_2$ i.e. conjugation by

$$
\sigma: \begin{pmatrix}
0 & 1 \\
+1 & 0
\end{pmatrix}
$$

$\theta_1 \rightarrow \theta, \quad \theta_2 \rightarrow -\theta, \quad \theta_3, \ldots, \theta_m \rightarrow 0$

Roots are:

| $2\theta$ | mult. | 3 | (comes from $\theta_1 - \theta_2$, $\theta_3 - \theta_4$) \\
|----------|-------|---|----------------------------------|
| $\theta$  | mult. | $m-2$ | $(\theta_1 - \theta_j, j > 2)$ \\
|          |       | $m-2$ | $(-\theta_2 + \theta_j, j > 2)$ \\
|          |       | $m-2$ | $(\theta_1 + \theta_j, j > 2)$ \\
|          |       | $m-2$ | $(-\theta_2 - \theta_j, j > 2)$ \\
|          |       | $+ \frac{m-2}{4(m-2)}$ |                                   |

So the diagram is:
so the Poincaré series of the loop space is 
\[1 + t^3 + t^{4m-2} + t^{3+4m-2} + \ldots\]
\[= \frac{(1+t^3)}{(1-t^{4m-2})}\]
which agrees with the fibering
\[\mathbb{S}^3 \rightarrow \mathbb{S}^{4m-1} \rightarrow \mathbb{H}P^{m-1}\]
\[\Omega \mathbb{S}^{4m-1} \rightarrow \Omega \mathbb{H}P^{m-1} \rightarrow \mathbb{S}^3\]

For a rank 1 symmetric space, the spherical building is a sphere, namely \(S(p)\), and the apartment is a pair of anti-podal points. It is a non-dimensional complex.

We can interpret as think of a rank 1 symmetric space as a projective space, in which closed geodesics live in projective lines. Observe that projective lines are spheres.

\[\mathbb{P}_1^3 = S^1\]
\[\mathbb{P}_1^2 = S^2\]
\[\mathbb{P}_1^1 = S^1 \rightarrow S^1 \rightarrow S^1\]
\[\mathbb{P}_1^0 = S^1 \rightarrow S^2 \rightarrow S^2\]
From Bott-Jamerson we know that
the space of paths in $X$ from a point $t_0 = p(5)$ to the orbit $K_{\infty}$ is described by
geodesics joining $t_0$ to $p_\infty(K_{\infty} \cap S) = p(W_\infty)$. Now described means we get a CW complex whose cells
form a basis for the mod 2 homology.In particular, if the space of paths in
the number of points of $p_\infty(W_\infty)$ in the same chamber as $t_0$ in diagram. Suppose $0 < x(5_0) < 1$
for any positive root $x(t_0) < 1$ for all $x \in \mathbb{Z}_0$.

**Theorem.** Suppose $K$ simply-connected.

Taking $n = 1$, we find $\pi_1 X = \pi_0 X$ is in one
one correspondence with $C' \cap p_\infty(1)$. But as $S \cap T$
and $C' \subseteq C'$ and $C'$ contains a unique lattice $y$,
it follows $\pi_1 X = 0$. Hence $K_{\infty}$
is connected. Thus the orbits $K_{\infty}$ are
connected and go from the fibration
\[ \text{paths}(t_0, \delta_0) \rightarrow K_{\infty} \rightarrow X \]
we see that $\pi_0 \text{ paths}(t_0, K_{\infty}) = 1$, hence $\delta_0$.\]
$C_0$ contains exactly one point of $p_0^*(\mathcal{W}_0)$. Thus

$$C_0 \longrightarrow W_0 \backslash S \longrightarrow K_0 \backslash X.$$ 

Go back to Lie alg. case. Suppose $K$ compact connected, let $K_0 = \text{identity component of } K$. Apply Morse theory to the orbit $K_0\eta$ and the function $H_0' - \frac{1}{2}$ where $J$ is regular in $E_0$.

Consequence: Because this function has a minimum $K_0\eta \cap E_0 \neq \emptyset$, whence $K_0$-conjugacy for maximal subspaces of $K_0$, and $K_0\eta \cap E_0 = W_0\eta$ (for $\eta \in E_0$).

Next you have to show critical points of index 1 are realized by circles, so that $F_1$ critical points of index 0 in each component. This shows that $C_0$ contains exactly one point of each $W_0$-orbit.

This means $W_0$ is a reflection group. First—only the identity carries a chamber into itself. Next given hyperplane $x = 0$, pick two chambers $C_1, C_2$ have this hyperplane as common wall. Then
there is some element $w$ such that $wC_1 = C_2$. Thus $w$ has to be the reflection through $x=0$. Next let's try to understand the $K^0$-orbit.

I claim that any element of $K^0/K_0$ is represented by a $k \in K^0$ which normalized $C_0$; this is clear. The induced transfer of $C_0$ is of finite order, so has a fixed point in the interior. The centralizer of $\tilde{g}$ in $K$ is connected, and has Lie algebra $\mathfrak{m}$ and $\mathfrak{e}_0$, so the centralizer of $\tilde{g}$ centralizes $\mathfrak{e}_0$. Thus any element of $K^0$ normalizing $C_0$ centralizes it, hence it is in $M$. \[ K^0/M \text{ is connected, so all } K^0\text{-orbits are connected.} \]

June 27, 1975:

Let $\theta$ be an automorphism of a compact comm. Lie group $U$. Let $\phi$ be the automorphism $K = U \times U$ given by

$$\phi(x, y) = (\theta^{-1}y, \theta x)$$

Then $\phi^2(x, y) = (\theta^{-1}(\theta x), \theta^{-1}(\theta^{-1}y)) = (x, y)$, so $\phi$ is an involution. Clearly $K^0 = \{ (x, \theta x) \mid x \in U \} = \overline{I}$. 
\[ \frac{K}{K^0} \to U \quad (x, y) \mapsto x(\theta^{-1} y)^{-1} \quad \text{and} \]

the \( K \)-action on \( U \) becomes

\[(x, y) \cdot u = xu \theta^{-1} y^{-1}.\]

I was hoping that the twisted conjugation

\[ (x, \theta x) \cdot u = u \iff \theta x = x \]

would include the homogeneous space \( U^0 \).

Thus I want to find a \( u \in U \) such that

But \( (x, \theta x) \cdot u = xu x^{-1} \), so this means \( U^0 \)

the centralizer of \( x \). Thus nothing new arises.

Let \( \theta \) be an automorphism of the compact connected

Let \( K \) act on itself by twisted

Conjugation \( x * y = x y \theta x^{-1} \), whence the orbit

Consider the local situation first, that is,

in a tubular nbhd of \( K * 1 \). This is

completely described by \( K^0 \) acting on \( K/K^{0} \).

Now this action has to preserve the different eigenspaces

of \( \theta \) on \( K/K^{0} \).
Let $K$ be a connected compact Lie group acting linearly on a Euclidean space $V$. Let \( \xi \) be a generic point of $V$. A tubular neighborhood $K \xi$ is a disk bundle in $K \times K \xi W$ over $K \xi$. Because \( \xi \) is generic, it must act trivially on $W$. Thus the orbit $K \xi$ meets the fibre\footnote{This means by a principal orbit type.} set $V K \xi$ transversally at $\xi$. (This is what \( Z \) means by a principal orbit type.)

Put $H = K \xi$ and $E = V K \xi$.

Recall that there are only finitely many orbit types of $H$.

**Question:** Given any point $\eta$ of $E$, is the orbit $K \eta$ perpendicular to $E$ at $\eta$? More precisely, is $X \cdot \eta$ perpendicular to $E$ for any $X \in \mathfrak{h}$?

Idea might be that this is true if $K \eta = H$, and that the set of such $\eta$ in $E$ is dense. Consider the map

$$K \mathfrak{h} \times E \rightarrow V$$

$$X + \xi \rightarrow X \cdot \eta$$

which is linear. The set of $\eta$ in $E$ for which $K \mathfrak{h} \rightarrow V$ is injective is Zariski.
open and non-empty hence dense. This it seems that given any \( y \in E \), \( x, y \) will be the limit of \( x, y \) where \( y \) is such that \( x, y \in E \), hence \( x, y_0 \in E \).

Now fix \( z \in E \) with centralizer exactly \( H \), and consider the Morse function on an orbit \( Kz \) given by

\[
|Kz - z|^2 = |z|^2 + |z| - 2(z, Kz)
\]

This has inf. behavior

\[
|z|^2 + |z|^2 - 2(z, Kz, z) = \text{constant} - 2(z, Kz, z) - 2(z, Kz, z)
\]

so it has critical points where

\[
(z, Kz, z) = -(Kz, z, z) = 0
\]

for all \( x \in H \), i.e., where \( Kz \in E \). Thus every orbit meets \( E \), which shows among other things that there is a unique principal orbit type.

Consider the Hessian on the tangent space to the orbit at the critical point \( z = Kz \in E \). The tangent space is \( \mathfrak{h}/\mathfrak{h}^* \), where \( \mathfrak{h} = \{ x \mid x, y = 0 \} \) is a subalgebra containing \( z \).
The Hessian in the bilinear form \((x, y) \mapsto \langle x, y, \frac{i}{2} \rangle\) which is symmetric, so \(\langle x, y, \frac{i}{2} \rangle = 0\) as we are at a critical point.

\[-\langle xy, \frac{i}{2} \rangle = \langle y, x \rangle\]

But we have seen that the vector \(y\) for any \(x \in \mathbb{R}^k\) is perpendicular to \(E\), hence of the form \(x\eta\) for some \(x\). This shows the Hessian is non-degenerate.

Different version of p.72: Take \(x \in V\) and write \(V = E \oplus \mathbb{R} \cdot \mathbf{e}\) where \(E\) is the orthogonal complement of \(x\), hence stable under the \(\mathbb{R}\)-action.

Given \(x \in V\), we know that for \(v\) near \(x\) in \(E\), \(\dim E \leq \dim \mathbb{R} v\). Thus if we choose \(E\) so that \(\dim \mathbb{R} v\) is minimal, then \(\dim E = \dim \mathbb{R} v\) for all \(v\) in some nbhd of \(x\). Let \(E\) be the orthogonal complement of \(x\).

Given \(x \in V\), we know that \(x\) has minimal \(E\)-projection \(x_E\). Let \(E\) be the largest element of \(E\) such that \(x_E \neq 0\). Let \(x\) be in \(E\).

First note that \(x_E \in E\): \(x \in E\), \(y \in \mathbb{R} x\), \(c \in E\)

\[
\langle x_E, y_c \rangle = -\langle y, x_E \rangle = -\langle x, y \rangle + \langle z, y \rangle = 0
\]
Now, let's calculate the stabilizer of $\xi + \epsilon$
when $\epsilon$ is very small. Given $X \in \mathbb{K}$ write
it $X = X' \oplus X'' \in \mathbb{K}_1 \oplus \mathbb{K}_2$. Then

$X(\xi + \epsilon) = (X'' + X''') \oplus X' \epsilon$

Thus if $p_1$ is the projection of $V$ onto $\mathbb{K}_1$
we have

$p_1(X(\xi + \epsilon)) = p_1(X'' + X' \epsilon)$

Now because $X'' \in \mathbb{K}_2$ we have an estimate

$|X''| > \epsilon |X''|$

and as we have an estimate

$|p_1(X' \epsilon)| \leq C|X'| |\epsilon|

if $\epsilon$ is suff. small we will have

$X(\xi + \epsilon) = 0 \implies X'' = 0 \implies X' = 0.$

By the choice of $\xi$, this means that $\xi_{1+\epsilon} = \xi_1$.
As this holds for all small $\epsilon$ we see that

$\xi_1 \cdot E = 0.$

If $we E$, then $k_1 \in E$, so

$E_1 \oplus E_2$, which is true for $E_1$.
You made an error: \((G/H)^H = N/H\) not \(H/H\). Consequently, we must see what can be salvaged.

Again let \(i\) be generic in \(V\) whence a tubular subalgebra of \(K\) is of the form \(K_x K_i W\) where \(K_x\) acts trivially on \(W\).

Put \(H = K_i\), and let \(N = N_K(H)\). Then \(N/H\) acts on \(V^H\).

The problem is as follows: Choose \(i\) generic, then at \(i\) we know that the orbit \(K_i\) is perpendicular to \(E\) by definition of \(E\).

But we want \(K_i \eta\) to be perpendicular to \(E\) for all \(\eta\) in \(E\), and at least for \(\eta\) in \(E\) near to \(i\).

\[
\begin{array}{c|c}
K_i & \eta \\
\hline
E
\end{array}
\]

But this doesn't always happen. For example assume \(K\) acts freely on \(SV\), then whenever all \(i\) are generic. If the perpendicular to one orbit is perpendicular to nearby orbits, we get a flat connection.
So let's return to the auto-\(\Theta\) of \(F\)
and to the action of \(F^0\) on \(F_1/F^0\). The
action preserves the eigenspaces of \(\Theta\), so we
shall let \(V\) be one of these eigenspaces.

Why do we get a good situation when \(\Theta^2 = I\).
Let \(F\) be generic and \(E\) be the orthogonal space to \(F^0\).

\[-(\eta, x) = (X, \eta, x) = (X, [\eta, t])\]

so that \(\eta \in E \iff F^0 \subseteq [\eta, t].\)

But when \(\Theta^2 = I\),
\([\eta, t] \subseteq F^0\) so \([\eta, t] = 0 \iff \eta \in E.
Actually one proceeds as follows. One takes \(E\) to be a
maximal abelian subspace of \(F\), then one constructs
the roots of \(F\) with respect to \(E\), and from this
root decomposition, one can see that \(F^0\).\(E\) is the
orthogonal complement to \(E\).

So if \(K\) acts on a manifold \(X\) and
\(x \in X\) is of the principal type (this means that
\(K_x\) acts trivial in the normal space to the orbit
\(K_x \cdot x\) at \(x\) then the really good situation
is where the a tubular neighborhood around
the orbit is not just topologically a
product of the orbit and the isotropy subgroup, but
also preserving the metric.

Return to loop space of a symmetric space and work out the theory.

$K$ compact and connected with

involutions $\sigma$. I used the Bott-Samelson theory to understand the fibre of the inclusion $K^\sigma x \to X$ where $x = K/K^\sigma$. Choosing a generic element $t_0$ of $X$, one gets $S, E_0, W_0$ as before. If $x = s \in S$, then the fibre $P(x; K^\sigma t_0)$ has the homotopy type of a cw complex with cells indexed by points of $p^{-1}(W_0s)$. Suppose $t_0 = p(q_0)$ with $q_0 \in C_0' = \{ q \in E_0 | q \sigma_0 = 0, s(q) = 1 \}$. The theory shows among other things that $\pi_0(P(x; K^\sigma t_0)) = p^{-1}(W_0s) \cap C_0'$. Holgaasen's method, which is based on the following isomorphism,

$K^\sigma/M \times S_n \to X_n$

plus the fact that $\pi_1(X_0) = \pi_1(X)$ because the singular set has codimension $\geq 2$, also shows $p^{-1}(1) \cap C_0'$

$= \upsilon^{-1}(X)$.
So by these style arguments I know that $C'$ is a fundamental domain for $(S, W_0)$ on $(X, K^0)$, when $K$ is simply-connected. Thus, $C'$ will also be a fundamental domain for $K^0$ acting on $X$.

I want to define an action of $G$ on $X$, following the procedure used for the $G$ action on $M$. Here $K$ is assumed to be connected, compact, and with given maximal torus $T$. Let me take an element $x$ in the apartment of $X$: then $x \in E = \frac{1}{2\pi i} \log(t)$ and

$$x(t) = e^{2\pi it x}$$

I recall that $G$ is generated by the 1-parameter subgroups

$$x_a(t) = \exp(t x_a)$$

and also $H(E)$.

$$x^{-1} x_a(t) x =$$

(My idea is to let $F$ be meromorphic near $\infty$, $f = \sum a_n z^n$.)
Convention: We will work near \( z = 0 \) to start with. Thus \( f \) will consist of series \( \sum_{n > -N} a_n z^n \) which are convergent near zero. I can identify elements of \( G \) with certain holomorphic maps from punctured disks to \( \mathbb{C} \). I should think of my objects as holomorphic for \( \text{Im } t > \text{const.} \) with values in \( G \).

Suppose given an element \( x \) of \( E \) and an element \( g \) of \( G \). Then

\[
\tilde{x}^{-1} g \tilde{x} = e^{-2\pi i t x} g(e^{2\pi i t} x) e^{2\pi i t x}
\]

is a function on a strip \( \text{Im } t > \text{const.} \) with values in \( G \). I want to consider those \( g \) such that this converges as \( \text{Im } t \to +\infty \).

Let \( g(z) = \exp (f(z) X_a) = X_a(t) \).

Then

\[
\tilde{x}^{-1} g \tilde{x} = \exp \left( \text{Ad}(e^{-2\pi i t x}) f(z) X_a \right)
\]

\[
= \exp \left( f(z) e^{-2\pi i t x} X_a \right)
\]

\( f \) holm. and \( t \to 0 \) at \( 0 \), the above function will
converge as $\text{Im } t \to +\infty$ iff

$$e^{2\pi i t (m - a(x))} \text{ converges}$$

which is the case if $m - \pi(x) \geq 0$.

Let $P_x$ denote the set of $g \in G$ such that $g^{-1} x g$ has a limit as $\text{Im } t \to +\infty$.

Then clearly $P_x$ is a subgroup of $G$.

We see that it contains

$$\begin{bmatrix}
\eta_x(t) \\
H(R)
\end{bmatrix} \quad \text{for } a(x) \leq \text{ord } t$$

---

I want to show that given $\xi \in X$ and $g \in G$, there exists a unique $\eta \in X$ such that $g^{-1} \eta \xi$ has a value at $z = +\infty$, i.e.

converges as $\text{Im } t \to +\infty$.

Uniqueness. If $\eta^{-1} \xi$, $\eta_1^{-1} \xi$ converge at $\text{Im } t \to +\infty$, then so does $\eta^{-1} \eta_1$. Thus we have to show that $\eta^{-1} \xi$ converges in $G$ as $\text{Im } t \to +\infty$.

Now $\xi$ being an element of $X$ we have

$$\xi(t+1) = \xi(t) \cdot \xi(1)$$
and similarly for \( \eta \), so

\[(\eta^{-1} \xi)(t+1) = \eta(t)^{-1} \cdot (\eta^{-1} \xi)(t) \cdot \xi(t)\]

so \( \eta(t) T = T \xi(t) \) when \( T = (\eta^{-1} \xi)(t \to \infty) \).

Choose \( X \) with \( e^X = \xi(t) \), whence

\[\xi(t) = f(e^{-tx}) e^{tx}\]

with \( f \in \mathbb{K}' \). Then \( e^{tx} \xi^{-1} = \eta(t) \), so

\[\eta(t) = e^{tx} \xi^{-1} f^{-1}(z) e^{tx}\]

with \( f \in \mathbb{K}' \). Hence

\[(\eta^{-1} \xi)(t) = e^{-t(Xt^{-1})} \cdot (f^{-1} f)(z) e^{tx}\]

\[= T e^{-tx} \cdot T^{-1}(f^{-1} f)(z) e^{tx}\]

converges, as \( t \to \infty \). To show this is constant, I can suppose \( X \) diagonal with diagonal entries \( \lambda_j \), \( j \in \mathbb{R} \).

\[T^{-1}(f^{-1} f)(z) = \sum_{n \in \mathbb{N}} a_n z^n\]

Hence so we have now-

\[e^{-tx} \cdot \psi(z) e^{tx}\]

converging as \( \text{Im} t \to -\infty \)

where \( \psi(z) = T^{-1}(f^{-1} f)(z) \)
Lemma: If \( \eta \) are self-adjoint matrices such that \( e^{-t\eta}e^{t\xi} \) converges in \( GL_n \) as \( t \to +\infty \). Then \( \xi = \eta \).

Proof: By converging in \( GL_n \) I mean that \( T = \lim_{t \to +\infty} e^{-t\eta}e^{t\xi} \) is an invertible matrix. It follows that \( T^{-1} = \lim_{t \to +\infty} e^{-t\xi}e^{t\eta} \). But because \( \xi \) and \( \eta \) are self-adjoint,

\[
T^* = \lim_{t \to +\infty} e^{t\xi}e^{-t\eta} = \lim_{t \to +\infty} e^{-t\eta}e^{t\xi}
\]

Now the matrix \( e^{-t\xi}e^{t\eta} \) has entries which are linear combinations of exponentials \( e^{at} \) with \( a \in \mathbb{R} \). If such a function converges, both as \( t \to +\infty \) and as \( t \to -\infty \) it is necessarily constant. Therefore \( e^{-t\xi}e^{t\eta} \) is constant, so \( \xi = \eta \).

Analogous lemma will be this:

Let \( \xi, \eta \in GL_n \) be such that \( \eta^{-1} \xi \) converges as \( \text{Im } t \to +\infty \). Then \( \eta = \xi \).

Special case \( \eta = 1 \). The matrix \( \xi(t) = \text{fie}^{2\pi i t} \)

(\( x \) self-adjoint) has entries which are linear combinations of exponentials \( e^{2\pi itx} \) with \( x \in \mathbb{R} \).
Thus if it is bounded as both \( \text{Im} t \to +\infty \) and \( \text{Im} t \to -\infty \), then it is constant.

But \( \xi(t) = (\xi(t)^*)^{-1} \) if \( t \) is real, so

\[
\xi(t) = (\xi(t)^*)^{-1}
\]

for all \( t \in \mathbb{C} \). But also \( \text{Im} t \to -\infty \) \( \text{Im}(\xi) \to +\infty \) and the right side converges, hence \( \xi(t) \) converges in both directions, hence it is constant.

The same argument works with the function \( (\eta^{-1}(\xi))(t) \) which is a linear combination of exponentials \( e^{2\pi i a} \) with \( a \in \mathbb{R} \).

I am still trying to show that for any \( \xi \in X \) and for any \( g \in V \), there exists a unique \( \eta \) in \( X \) such that

\[
\eta^{-1} g \xi
\]

converges as \( \text{Im}(t) \to -\infty \). We have already seen

the uniqueness holds, so it remains to prove the existence of \( \eta \). Note

that if \( g \in \mathbb{R} \), then \( g \xi g(1)^{-1} \in \mathbb{R} \), so

\[
\eta = g \xi g(1)^{-1}
\]
Thus \( g \cdot \bar{g} = g \cdot g(1) = 1 \) if \( g \in G \).

Let \( I = \{ g \in G \mid \forall x \in X, \exists k \in X \implies k \cdot g \in P \} \).

1. \( I \) is a subgroup. Given \( g_1, g_2 \in I \) and \( g \in X \), choose \( h_1 \) so that
   \[ h_1^{-1} g_1 \in P \]
   and choose \( h_2 \) so that
   \[ h_2^{-1} g_2 \in P \]
   Then \( h_2^{-1} h_1^{-1} g_1 g_2 \in P \), so
   \[(h_1 h_2)^{-1} g_1 g_2 = h_2^{-1} h_1^{-1} g_1 h_2^{-1} g_2 \in P.
\]

2. \( I = \{ g \in G \mid \forall \bar{g} \in \text{fund. domain for } K \implies \exists k \in V \implies k \cdot g \in P \} \).

Given \( g \in G \) and \( \overline{g} \in \text{fund. domain} \), choose \( k \) such that \( k \cdot g = \overline{g} \).

I want to return to the action of $G$ on $p = ik$ and to see if I can define this by descent from $G^k$. Let $\xi \in P$ and recall that we have defined

$$P^u_\xi = \{ \gamma \in G \mid e^{-i \xi} \gamma e^{i \xi} \text{ converges} \}$$

$$P^u_\xi = \{ \gamma \in G \mid e^{-i \xi} \gamma e^{i \xi} \rightarrow 1 \}$$

Let $G^\xi_\xi$ be the centralizer of $\xi$ in $G$. If $\gamma \in P^u_\xi$, it is clear that if

$$g_0 = \lim_{t \to \infty} e^{-t \xi} \gamma e^{t \xi}$$

then $e^{-i \xi} (g_0) e^{i \xi} = (g_0)$, hence $(g_0) \in G^\xi_\xi$. Thus

$$P^u_\xi = G^\xi_\xi \times P^u_\xi.$$

Let $G^\xi_\xi = K \cap G^\xi_\xi$; it's clear that because $\xi^* = \xi$, $G^\xi_\xi$ is stable under the Cartan involution, hence $K_\xi$ is a maximal compact subgroup.

Let $X = \exp P_\xi$, $X_\xi = X \cap G^\xi_\xi$. Then

$$G^\xi_\xi = K_\xi \times X_\xi, \quad X_\xi = \exp (P_\xi).$$

(Take Lie points under the group of inner automorphisms which preserves the involution.)
Now the claim is that
\[ G = K \times K \times P_3 \]
\[ = K \times K \times (G_3 \times P_3) \]
\[ = K \times X_3 \times P_3 \]
and this can be proved by descent from \( G_{10} \). In fact, the only point really is that \( G = K \cdot P_3 \), for \( K \cap P_3 = K_3 \):
\[ e^{t \cdot s' \cdot k} \cdot e^{t \cdot s' \cdot k} = e^{t \cdot s' \cdot k} \]
which converges \( \iff k_3 = i \). For \( G_{10}(a) \), this results by Gram-Schmidt.

So at this point we can define the \( G \)-action on \( p \) and we know that \( K \) acts transitively on each orbit. We can identify \( S(y) \) with the spherical building.

Next, you want to generalize things to \( G \times \mathbb{R} \). Let \( \gamma \in \mathbb{R} \).
\[ P_i = \{ g \in G \mid \exists g_1 \} \text{ converges as } \text{Int} \to -\infty \]
\[ P_i = \{ g \in G \mid \exists g_1 \} \to 1 \text{ as } \text{Int} \to -\infty \]
Because \( g(t+1) = g(t) \cdot g(1) \) we have

\[
g^{-1}(t+1) \cdot g^{-1}(t) = g^{-1}(t) \cdot g(1)
\]

and so taking the limit as \( t \to -\infty \) we get

\[
g_0 = g(1) \cdot g_0 \cdot g(1)
\]

where \( g_0 \) is this limit. Thus we get a homomorphism \( F_g \to G \) with kernel \( P_g \).

Let \( g \in G \). I want to manufacture an element of \( P_g \) mapping to \( g \). Suppose

\[
\gamma(t) = e^{-2\pi it \cdot X}
\]

Then

\[
(\gamma \cdot \gamma^{-1})(t) = e^{2\pi it \cdot X} \cdot e^{-2\pi it \cdot X} = e^0 = 1
\]

will be an element of \( P_g \), such that

\[
\gamma^{-1} \cdot (\gamma \cdot \gamma^{-1}) = 1
\]

is constant, hence \( \gamma \cdot \gamma^{-1} e P_g \) maps to \( g \).

In general

\[
\gamma(t) = f(t) \cdot e^{2\pi it \cdot X}
\]

so

\[
(\gamma \cdot \gamma^{-1})(t) = f(t) \cdot e^{2\pi it \cdot X} \cdot e^{-2\pi it \cdot X} \cdot f(t)^{-1}
\]

\[
= f(t) \cdot e^{2\pi it \cdot X} \cdot e^{-2\pi it \cdot X} \cdot f(t)^{-1}
\]
is an element of \( G \) such that \( \bar{g} \cdot g = g \) is constant.

Put \( \bar{g} = \frac{1}{\bar{g}} \), where \( \bar{g} \cdot g = \alpha \), is constant,

where \( \alpha \) is constant.

Also note that
\[
K \cap P_i = \{ k \in K \mid k \cdot i = i \}.
\]

Because if \( k \in K \cap P_i \), then
\[
\bar{k}^{-1} k \cdot i = \bar{k}^{-1}(k \cdot i) = k(i) \implies k(i) \text{ converges as } i \to \infty.
\]

implies \( k^{-1}(k \cdot i) \) converges \( \iff \) \( k \cdot i = i \). Thus
\[
K \cap P_i = \{ k \mid k \cdot i = i \},
\]

and
\[
K \cap P_i = \bar{K} = K \cap P_i
\]

implies \( k^{-1}(k \cdot i) \text{ converges } \iff \) \( k \cdot i = i \). Thus
\[
K \cap P_i = \{ k \mid k \cdot i = i \},
\]

and
\[
K \cap P_i \to K(i),
\]

so now I want to prove the basic formula:

\[
\bar{g} = \bar{k} \times X_i \times P_i
\]

\[
= K \times X_i \times P_i
\]
Note that the isomorphism
\[ G_9 \xrightarrow{\sim} G_{3;1} \]
\[ \xi \mapsto \eta^{-1} \mapsto \xi \]
commutes with $\ast$ because $\xi$ is unitary, so that $\xi \ast \eta$ will be the image of $X_{i;1}$. This formula obviously can be proved by descent from $G_{i;1}$.

Cells: Go back to the spherical building and let $\xi \in \mathfrak{p}$. Consider the 1-parameter group of motions $\eta \mapsto e^{t \xi} \ast \eta$ on $\mathfrak{p}$. Fix $\eta$: 
\[ e^{t \xi} \ast \eta = \eta \iff e^{t \xi} \in \mathfrak{p} \cap X = \mathfrak{p} \]
\[ e^{t \xi} \text{ centralizes } \eta. \]

What I want to do is to retract the building for $G$ onto the building for $G_9$ with the fibres of the retraction being $P_{3}^\ast$-orbits. Thus you wish to show that each $P_{3}^\ast$-orbit has a unique fixedpoint for the group $e^{t \xi}$.

Uniqueness: Consider $\eta \in \mathfrak{p}_3$ and $e^{t \xi} \ast \eta = \eta$ and $u \in \mathfrak{p}_3$. Then
\[ e^{-t\mathfrak{g}} \cdot (u \cdot \eta) = e^{-t \mathfrak{g}} \cdot \eta \rightarrow \eta \]

as \( t \rightarrow +\infty \).

**Existence:** The idea might be to show that given any point \( \eta \) in the building the limit \( e^{-t\mathfrak{g}} \cdot \eta \) exists, say \( \eta = \eta_0 \), and that \( \eta \in P^u_i \eta_0 \). But this won't be any good for descent purposes until we pin down the actually elements of \( P^u_i \). 

But actually, elements of \( P^u_i \) are good for descent purposes until we pin down the actually elements of \( P^u_i \) as \( g \cdot \eta_0 = \eta \).

Note that \( P^u_i = \exp(\text{Lie } P^u_i) \) because you can see this in \( \text{GL}_n \). So we can certainly describe

\[ P^u_i \eta_0 = P^u_i / P^u_i \eta_0 \]

in Lie algebra terms. Then I am interested in these elements of the Lie algebra such that

To really get a feeling for what's going on, I should think of \( \mathfrak{g} \) as a flag with real eigenvalues prescribed for each quotient, and the eigenvalues are in order. So therefore given two points \( \xi, \eta \) what are important are the two flags. Since these two flags are "refined" by a tower, this means I can find elements \( p \in P^u_i \), \( q \in P^u_i \) such that
\[ p \cdot p^{-1} \text{ and } q^{-1} \cdot q \text{ commute.} \]

First suppose \( i \) commutes with \( q^{-1} \cdot q \).

Then
\[
e^{-tq} e^{i\gamma} e^{tq} = \left( e^{-tq} e^{tq} \right) e^{i\gamma} (qe^{-tq} e^{tq})
\]
\[
\rightarrow q^{-1} e^{i\gamma} q \quad \text{as } t \rightarrow +\infty.
\]

showing that \( e^{i\gamma} \cdot \eta = \eta \).

In general we know \( p \cdot p^{-1} \) commutes with \( q^{-1} \cdot q \) so we find
\[
e^{-tq} p e^{i\gamma} p^{-1} e^{tq} \rightarrow q^{-1} p e^{i\gamma} p^{-1} q.
\]

i.e.
\[
p e^{i\gamma} p^{-1} \cdot \eta = \eta.
\]

\[
e^{i\gamma} \cdot (p^{-1} \cdot \eta) = \left( e^{i\gamma} \right) (p^{-1} \cdot \eta).
\]

Therefore we see that in the \( G_{\alpha} \) case that

in the orbit of \( P_{\alpha} \eta \) there is a

fixed point, which is unique and given

by

\[
\lim_{s \rightarrow -\infty} e^{s \cdot i\gamma} \cdot \eta.
\]
But one can be a bit more specific. The idea is that $q$ is semi-simple and $g \times q$ is $g \times g^{-1}$ written as $u(g \times q)u^{-1}$ with $u \in P_g$. In addition to having $G = K \times X_\xi \times P_g$, I want to know that any admissible set of $P_g$ is of the form $u \times u$, $x \in P_g$, so $G_f$

Assertion: $\forall g \in G, \xi \in \mathbb{F}$ we have

$g \cdot \xi \cdot g^{-1} = u(g \times \xi)u^{-1}$

with $u \in P_g$.

Proof: $G = K \times X_\xi \times P_g$. Let $g = k \times u$ be the decomposition of $g$ corresponding to this product. Then $g = t \times k$ where $t = k \times u (k \times u)^{-1}$

Now $u \in P_g \times P_g \times X_\xi \Rightarrow k \times u \times k^{-1} \in P_g$

$\Rightarrow k \times u \times k^{-1} \times k \times P_g \times k^{-1} = P_g \times k \times k^{-1}$.

Also $k \times k^{-1} = g \times g^{-1}$, since $g \times g^{-1}$ since $u \in P_g$. Thus

$g \times g^{-1} = t \times k \times k^{-1} \times t^{-1} = t (g \times g^{-1})t^{-1} \times t \in P_g$.
Note that if (\(\ast\)) holds, then

\[
\lim_{t \to -\infty} e^{-t(\alpha \cdot x)} g e^{t \gamma} g^{-1} = \lim_{t \to -\infty} e^{-t(\gamma \cdot x)} u e^{t(\gamma \cdot x)} u^{-1} = u^{-1}
\]

showing that \(u\) is unique.

July 4, 1975:

I am presently trying to establish the Bruhat decomposition for the spherical building. \(g\) is a fixed element of \(P\) and I want to describe the \(P_\gamma\)-orbits on \(P\).

Assertion: Every \(P_\gamma\)-orbit contains a unique point centralized by \(g\):

\[
P_\gamma \backslash P \rightarrow P_\gamma = \{ \eta \mid \exists \gamma \in P, \eta = \gamma \}
\]

Furthermore, if \(\eta \in P_\gamma\), then

\[
P_\gamma \cap P_\gamma \gamma \sim P_\gamma \gamma
\]

\[
u \mapsto u \gamma
\]

Also \(P_\gamma \gamma = \{ \eta \mid e^{-\alpha \gamma} \eta \rightarrow \eta \text{ as } s \to \infty \}.\)
It should be enough to prove this for $G_n$. Suppose we have an embedding $G \to G'$ and the situation

$$G \to G' \to G''$$

and the theorem is true for $G'$ and $G''$.

Given $\eta \in \mathfrak{g}$ we know that the limit

$$\lim_{t \to +\infty} e^{-t \xi} \star \eta$$

exists, call it $\eta_0$; it is the unique $\xi$-fixed point in $\mathfrak{p}'^u \cap \mathfrak{p}'^{-u}$. Moreover there is a unique $u \in \mathfrak{p}'^u \cap \mathfrak{p}'^{-u}$ such that $\eta = u \star \eta_0$. Clearly $\eta_0 \in \mathfrak{p}'^u$, and we must have by uniqueness $\alpha(u) = p(u)$ so $u \in \mathfrak{p}'^u \cap \mathfrak{p}'^{-u}$.

Suppose I want to prove

Lemma: \[ [\xi, \eta] = 0. \] Then any element $u \in \mathfrak{p}'^u$ can be uniquely written $u = u^- u^+$ where $u^- \in \mathfrak{p}'^- \cap \mathfrak{p}'^{-u}$ and $u^+ \in \mathfrak{p}'^+ \cap \mathfrak{p}'^u$. 
Consider $P_\eta^u \ast \eta$. This is stable under $e^{s\eta}$ because $e^{s\eta} P_\eta^u e^{-s\eta} = P_\eta^u$. What I want to show is that $P_\eta^u \ast \eta \subset P_{-\eta}^u \ast \eta$, or equivalently that

$$(e^{s\eta} u) \ast \eta \rightarrow \eta \quad s \rightarrow +\infty$$

for any $u \in P_\eta^u$. Now we know that

$$\lim_{s \rightarrow \infty} (e^{s\eta} u) \ast \eta = f(u \ast \eta)$$

exists for every element $u \in P_\eta^u$. Also

$$f(e^{s\theta} (u \ast \eta)) = e^{s\theta} f(u \ast \eta)$$

because $\theta$ and $\eta$ commute. Unfortunately, I can't argue that $f$ is continuous.

* Possible argument:

Argue that $P_\eta^u \ast \eta$ is an open subset of $\eta$ in $\mathbb{G}_u \eta$. Thus for $u$ large

$$e^{-s\eta}(u \ast \eta) \in P_{-\eta}^u \ast \eta$$

But latter is invariant under $e^{\theta}$ as $u \ast \eta e^{\theta} \in P_{-\eta}^u \ast \eta$

So assume we know $P_{-\eta}^u \subset P_{-\eta}^u \ast \eta$ which can be proved by descent because $\eta \ast P_{-\eta}^u \subset P_{-\eta}^u \ast \eta$. 
i.e., \( P^u \cap P^- = \emptyset \) (because if \( x \in P^u \cap P^- \), then \( e^{-t|X|}e^{|X|} \) converges at both \( t = \pm \infty \) hence is constant \( \pm \infty \) is 1, so \( x \in \emptyset \)).

Then I can write \( u \cdot y = u^- \cdot y \) \( \mathbb{C} \), i.e. \( u = u^- u^+ \) with \( u^- \in P^u \), \( u^+ \in P^- \). Next argue that this splitting has to commute with the \( \gamma \) action, since \( e^{t \gamma (u^- \cdot y)} \to 0 \) and the isomorphism \( P^u \to P^u \) is topological, we get \( e^{-t^2} e^{t^2} \to 1 \), hence \( u^- \in P^u \) and also for \( u^+ \).

**Problem:** I know that \( P^u \setminus \mathbb{K} \gamma \) can be identified with \( \mathbb{K} \gamma \setminus \mathbb{P}^u \). When \( \gamma \) is regular, this is a \( \mathbb{W} \)-orbit! What is its structure if \( \gamma \) is not regular?

Take the case of \( \mathbb{G}_k \). The orbit \( \mathbb{K} \gamma \) can be identified with orthogonal splittings

\[ V = W_1 \oplus \cdots \oplus W_m \]

where \( \dim(W_i) = d_i > 0 \) \( d_1 + \cdots + d_m = n \), are fixed. We want the \( \gamma \)-fixed points where \( \gamma \) gives a decomposition \( V = u_1 \oplus \cdots \oplus u_k \). This means I want
the set of decompositions

\[ V = \bigoplus_{1 \leq i \leq k} \bigoplus_{1 \leq j \leq m} T_{ij} \]

such that

\[ W = \bigoplus_i T_i, \quad U_i = \bigoplus_j T_{ij} \]

The obvious invariant of such a decomposition are the integers \( t_{ij} = \dim T_{ij} \). Supposing these are fixed, then what I am looking at is a sequence of flags:

\[ U_1 = T_{11} \oplus \cdots \oplus T_{1m} \]

\[ U_2 = T_{21} \oplus \cdots \oplus T_{2m} \]

such that \( \dim (T_{ij}) = t_{ij} \). So in this case we see that each component of \( (K/\overline{\mathbb{F}_q})^3 \) is a product of flag manifolds, and the different components are indexed by families \( (t_{ij}) \) such that \( \sum t_{ij} = \dim W_j \).

\[ \sum t_{ij} = \dim U_2. \]

In fact note that each component is an orbit for the group \( \text{Aut}(U_1) \times \cdots \times \text{Aut}(U_k) = \mathbb{K}_1 \). So the conjecture will be that each component of \( (K/\overline{\mathbb{F}_q})^3 \) is a \( \mathbb{K}_1 \)-orbit.
Because \( P_3 = G_3 \times P_3 \), it follows that
\[
P_3 \backslash K_\eta = G_3 \backslash (P_3 \backslash K_\eta) \approx G_3 \backslash (K_\eta)^2.
\]

Now if I choose a Borel \( B \subseteq \subseteq P_3 \) say \( B = P_3 \) where \( \xi \) is a perturbation of \( \xi \), then I know
\[
P_3 \backslash K_\eta = W_\eta \approx W/W_\eta
\]
(here I assume \( \eta, \xi \in E \)). Because \( P_3 = P_3 W_\eta P_3 \), it's more or less clear that
\[
P_3 \backslash K_\eta \approx W_\eta \backslash W/W_\eta.
\]

Concerning the action of \( K_\xi^\circ \) on \( P_3 \): \( \xi \) is \( \bullet \) of principal orbit type iff \( K_\xi^\circ \) acts trivially on the normal space to the orbit \( K^\circ \xi \) at \( \xi \). This normal space may be identified with the space \( P_3 = \{ \eta \in P | [\xi, \eta] = 0 \} \) because of the identity
\[
([X, \xi], \eta) = (X, [\xi, \eta])
\]
and the fact that \( \xi, \eta \in P \Rightarrow [\xi, \eta] \in K_\xi^\circ \). Thus \( \xi \) is of principal orbit type \( \iff \) \( K^\circ \xi \) acts trivially on \( P_3 \). But the really interesting point is that \( \xi \)
is abelian for if \( \eta_1, \eta_2 \in \mathfrak{g}_\xi \), then \([\eta_1, \eta_2] \in \mathfrak{g}_\xi\).

hence

\[
([\eta_1, \eta_2], [\eta_1, \eta_2]) = ([\eta_1, [\eta_2, \eta_2]], [\eta_1, \eta_2])
\]

(geometrically the significant point is that at any point \( \eta_1 \in \mathfrak{g}_\xi \), the orbit \( \mathcal{K}_{\eta_1} \) is perpendicular to \( \mathfrak{g}_\xi \), i.e. \([\eta_1, \mathfrak{g}_\xi] = 0\).