Lattices and Scattering Matrices

Let \( \Delta = \mathbb{C}[z, z^{-1}] \) be the ring of Laurent polynomials with only finitely many \( a_m \neq 0 \). Let \( S^1 \) denote the unit circle in \( \mathbb{C} \). \( \Delta \) may be viewed as a subring of continuous complex-valued functions on \( S^1 \). Conjugation on functions induces the involution:

1) \[ p = \sum a_m z^m \quad \mapsto \quad \overline{p} = \sum a_m z^{-m} \]

in \( \Delta \). Let \( \Delta^0 \) denote the subring of elements such that \( p = \overline{p} \). Then

2) \[ \mathbb{R}[x, y]/(x^2 + y^2 - 1) \xrightarrow{\sim} \Delta^0 \]

where \( x \mapsto \frac{1}{2}(z + z^{-1}) = \cos \Theta \), \( y \mapsto \frac{1}{2i}(z - z^{-1}) = \sin \Theta \) (if \( z = e^{i\Theta} \)).

Let \( g = (g_{ij}) \), \( g_{ij} \in \Delta \), \( 1 \leq i, j \leq n \) be an \( (n \times n) \)-matrix in \( \Delta \). Then \( z \mapsto g(z) \) is a map from \( S^1 \) to \( n \times n \) matrices. For \( g(z) \) to be a unitary matrix for each \( z \in S^1 \), means that

\[
(g(z) \ast g(z))_{ij} = \sum_k \overline{g_{ki}(z)} g_{kj}(z) = \delta_{ij}
\]

for each \( z \), or equivalently that in \( \Delta \) we have

3) \[ \sum_k \overline{g_{ki}} g_{kj} = \delta_{ij} \]
In other words \( g \) makes \( S' \) into \( U_n \) iff \( g \) is a unitary matrix over the ring with involution \( \Delta \).

Let \( U_n \) (or simply \( U \)) denote the group of such matrices.

We can give another interpretation of \( U \) as follows. We equip \( \Delta \) with the \( \mathbb{C} \)-hermitian inner product which is the restriction of the \( L^2 \)-inner product for functions on \( S' \):

1) \[ \langle p_1, p_2 \rangle = \int_{S'} p_1(z) \overline{p_2(z)} \frac{dz}{2\pi i}. \]

This inner product is the one such that \( z^m, m \in \mathbb{Z} \), is an orthonormal basis for \( \Delta \).

Let \( \Delta^n \) be the space of column vectors with entries in \( \Delta \), let \( e_i, 1 \leq i \leq n \), be the standard basis for \( \Delta^n \). We interpret matrices \( g = (g_{ij}) \) over \( \Delta \) as entries of \( \Delta^n \) in the usual way:

2) \[ g e_k = \sum_{k=1}^n g_{ki} e_k. \]

Extend (2) to \( \Delta^n \) in the obvious way so that \( z^m e_i \) is an orthonormal basis of \( \Delta^n \). For the matrix \( g \) to preserve the inner product (1) means:

\[ s_{lm} s_{ij} = \langle g(z^l e_i), g(z^m e_j) \rangle = \left( \sum_k z^l g_{ki} e_k \right) \left( \sum_k z^m g_{kj} e_k \right) \]
\[ = \sum_{k} (z^{l-m}, f_{k}^{i} f_{k}^{j}) = (z^{l-m}, \sum_{k} f_{k}^{i} f_{k}^{j}) \]

i.e. that 3) holds. Therefore we see that \( U \)

is the group of autos. of \( \Delta^{n} \) preserving the

\( \Delta \)-module structure and the inner product.

In fact we see from this calculation just

made that there is a 1-1 correspondence between

elements \( g \in U \) and sequences \( v_{1}, \ldots, v_{n} \)

of elements of \( \Delta^{n} \) such that \( \{z^{m} v_{i}, m \in \mathbb{Z}, 1 \leq m \} \)

is an orthonormal subsets of \( \Delta^{n} \); the correspondence

sends \( g \) to the sequence \( g_{e, i} \), and to a

sequence \( v_{i} \) the matrix \( (g_{ij}) \) such that \( v_{i} = \sum g_{ij} e_{j} \).

(Remark: The above corresponds to the fact (proved

in Scattering Theory) that \( \Delta^{n} \) in \( \mathcal{L}^{2}(S^{1})^{\otimes n} \)

commuting with multiplication by \( z \) are in a one-one

correspondence with measurable maps form \( S^{1} \) to \( U_{n} \).

modulo null-set equivalence.)

By a lattice for \( \mathcal{L}^{2}(S^{1}) \) in \( \Delta^{n} \) we

mean a \( \mathcal{L}^{2}(S^{1}) \)-submodule \( L \) which is free of

rank \( n \). Since \( \mathcal{L}^{2}(S^{1}) \) is a PID, such an \( L \) is

the same thing as a \( \mathcal{C} \)-subspace of \( \Delta^{n} \) such that

\( z^{-1} L \subset L \), and such that

\( 6) \quad z^{-N} \mathcal{L}^{2}(S^{1})^{\otimes n} \subset L \subset z^{N} \mathcal{L}^{2}(S^{1})^{\otimes n} \)

for some \( N \). Let \( \mathcal{L} \) denote the set of \( \mathcal{L}^{2}(S^{1}) \)-lattices.
Clearly we have
\[ \frac{\text{GL}_n(\mathbb{A})}{\text{GL}_n(\mathbb{C}[x^{-1}])} \rightarrow \mathbb{L}. \]

Put \( \Lambda_0 \) for the lattice \( \mathbb{C}[x^{-1}]^n \), and let \( L \) be any lattice (for \( \mathbb{C}[x^{-1}] \) is understood) such that \( z^{-N} \Lambda_0 \subset L \subset z^N \Lambda_0 \). Denote by \( E \mathbb{A} \Delta^n \) the \( \mathbb{A} \)-subspace of \( \Delta^n \) with basis \( \{ z^m e_i \} \) with \( p \leq m \leq q \). Let \( W \) denote the subspace of elements of \( L \subset E \mathbb{A} \Delta^n \).

Put \( \Lambda_0 \) for the lattice (for \( \mathbb{C}[x^{-1}] \) is to be understood) \( \mathbb{C}[x^{-1}]^n \). Note that \( z^8 \Lambda_0 \cap (z^p \Lambda_0)^\perp \) has basis \( \{ z^m e_i \} \) for \( p < m < q \), hence for \( 0 < p < q \)

\[ z^8 \Lambda_0 = z^8 \Lambda_0 \cap (z^p \Lambda_0)^\perp + z^p \Lambda_0 \]

so if \( L \) is a lattice with \( z^p \Lambda_0 \subset L \subset z^8 \Lambda_0 \), we have

\[ L = L \cap (z^p \Lambda_0)^\perp + z^p \Lambda_0. \]

If also \( z^p \Lambda_0 \subset z^L \), we have

\[ z^L = z^L \cap (z^p \Lambda_0)^\perp + z^p \Lambda_0. \]

Let \( N = \) be the orthogonal complement of \( z^L \cap (z^p \Lambda_0)^\perp \) inside \( L \cap (z^p \Lambda_0)^\perp \). Then we have

\[ L \cap (z^p \Lambda_0)^\perp = N \oplus z^L \cap (z^p \Lambda_0)^\perp. \]
so \[ L = N + z^L N. \] On the other hand \( N \) is perpendicular to \( z^L N (z^L N)^{\perp} \) and \( z^L N \), hence \( N \) is perpendicular to \( z^L L \). Thus we have

\[ L = N \oplus z^L L \]

where \( N = \{ x \in L \mid x \text{ perpendicular to } z^L L \} \).

The dimension of \( N \) is \( n \) as \( L \) is free of rank \( n \) over \( \mathbb{C}[z^{-1}] \). Let \( v_1, \ldots, v_n \) be an orth. basis for \( N \). Since the spaces \( z^m N \) are mutually perpendicular, \( \{ z^m v_i \} \) is an orth. set, so \( \exists! g \in U \) such that \( g(v_i) = z^m v_i \). It follows that \( g(L_0) = L \). Therefore \( U \) acts transitively on \( L \).

If \( g(L_0) = L_0 \), with \( g \in U \), then \( g \) preserves the orth. complement of \( z^L N \) in \( L_0 \), which is \( \mathfrak{g} e_1 + \cdots + \mathfrak{g} e_n \) where \( \mathfrak{g} e_i \) is regarded as the subgroup of constant matrices. Thus \( g \in U_n \).

\[ U / U_n \rightarrow L. \]

Note that the homomorphism \( U \rightarrow U_n \) sending \( g \) to \( g(1) \) is the identity on \( U_n \). If \( U' \) be the kernel of this map, then we have

\[ U = U_n \times U' \]

and \( U \) says that \( U' \) acts simply-transitively on \( L \).
So we see that for each \( L \) in \( L \) there is a unique \( g \in U \) such that \( g\Lambda_0 = L \). We call \( g \) the scattering matrix associated to \( L \). (The terminology comes from scattering theory. The closure of \( L \) in \( L^2(S^1)^m \) is an "incoming space" for the unitary operator of multiplying by \( z \), and \( g \) is its scattering operator.) In general incoming spaces form a homogeneous space isomorphic to \( \text{Meas}(S^1, U_n)/U_n \) for the group of unitary autors of \( L^2(S^1)^m \) commuting with \( z \) which is essentially \( \text{Meas}(S^1, U_n) \).

The topology on \( U \): Let \( F_{p,q} U \) denote the subset of \( U \) consisting of \( g \) such that \( g_{ij} \in \mathbb{C}^m \sum C \in m \). Then \( F_{p,q} U \) is a closed subset of \( C^N \) for \( N = n^2 (q - p + 1) \). Also it is a bounded subset, because \( g(z) \in U_n \Rightarrow |g_{ij}(z)| \leq 1 \), and one gets bounds on the coefficients of \( g_{ij}(z) \) using the formulas:

\[
a_n = \frac{1}{2\pi i} \oint \frac{p(z)}{z^{n+1}} \, dz
\]

for \( p(z) = \sum a_i z^n \in \Lambda \).
Therefore $F_{p,q} U$ is compact in the topology obtained by considering coefficients.

Another way of seeing $F_{p,q} U$ is compact is to note that $F_{p,q} U / U_n = F_{p,q} U'$ corresponds under the isomorphism $U \sim L$ to the set of lattices $L$ such that $z^p L_0 \subseteq L \subseteq z^q L_0$. (In effect $z^{-1}L_0 \subseteq z^{-1}L \Rightarrow (z^{-1}L_0^\perp) = (z^{-1}L)^\perp \Rightarrow L \cap (z^{-1}L^\perp) \subseteq \sum_{m \geq 0} z^m C^n \Rightarrow g \in F_{p,q} U'$.

Conversely, if $g \in F_{p,q} U'$, then $L \cap (z^{-1}L^\perp)$ is in $z^q L_0$, so $L \subseteq z^q L_0$; also $z^{-1}L^\perp \subseteq z^{-1}z^p L_0^\perp$ for the same reason, so $L \subseteq z^p L_0$.) On the other hand, the set of $L$ such that $z^p L_0 \subseteq L \subseteq z^q L_0$ is isomorphic to a closed subset of the union of the Grassmannians of subspaces in $(z^q L_0 / z^p L_0)$, hence this set of $L$ is compact because the Grassmannians are.

We can now define a topology on $U$ by requiring a set to be closed if its intersection with any $F_{p,q} U$ is compact. (Thus $U = \bigcup_{n \geq 0} F_{p,q} U$ as $g \to \infty$, $p \to \infty$ is given the inductive limit topology.) Clearly $U$ is a compactly generated space.
Note that multiplication \( F_{p_1}U \times F_{p_2}U \rightarrow F_{p+p_1+p_2}U \) is continuous and hence gives a continuous map \( U \times U \rightarrow U \) provided to product is taken in the compactly generated spaces. In this manner \( U \) becomes a topological group in the compactly generated category.

Suppose \( R \) is a ring containing \( \mathbb{C}[[x]] \) and flat over \( \mathbb{C}[[x]] \) where \( x \) denote \( x \). Put \( \pi = x^{-1} \). Let \( R \) be a ring flat over \( \mathbb{C}[[x]] \) such that \( \mathbb{C} \xrightarrow{} R / \pi R \), and let easy \( F = R[[x^{-1}]] \cong \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} R \). It is easy to see that

\[
\mathbb{C}[[x]] / \pi^m \mathbb{C}[[x]] \xrightarrow{} R / \pi^m R
\]

for all \( m \). Hence for all \( p,q \)

\[
\pi^p \mathbb{C}[x] / \pi^p \mathbb{C}[x] \cong R / \pi^p R
\]

and so there is a 1-1 correspondence between \( \mathbb{C}[x] \)-lattices \( \mathbb{C}[z, z^{-1}] \) and \( \mathbb{R} \)-lattices in \( F^n \) given by \( L \mapsto R \otimes_{\mathbb{C}[x]} L \). Here, by
$R$-lattice, I mean a free $R$-submodule of $F^n$. $GL_n(F)$ acts transitively on these $R$-lattices, so we have:

11) $L = GL_n(C[\pi, \pi^{-1}])/GL_n(C[\pi]) \rightarrow GL_n(F)/GL_n(R)$.

Combining 7), 9), and 11) we get

12) $U_n/U_n \sim GL_n(F)/GL_n(R)$

or equivalently

13) $GL_n(F) = U_n \times U_n GL_n(R)$
Special paths in $\text{GL}_n$

As usual identify $\text{gl}_n = \text{Lie}(\text{GL}_n(C))$ with $M_n(C)$ (matrices over $C$), and $\exp : \text{gl}_n \rightarrow \text{GL}_n$ with $A \mapsto e^A = \sum_{m=0}^{\infty} \frac{1}{m!} A^m$. Denote by $\text{GL}_n(\Delta)'$ the subgroup of $g \in \text{GL}_n(\Delta)$ such that $g(1) = 1$.

Lemma: Let $A, B \in \text{gl}_n$ be such that $e^A = e^B$. Then there is a unique $g \in \text{GL}_n(\Delta)'$ such that

$$e^{\omega A} - e^{-\omega B} = g(e^{2\pi i\omega})$$

for $0 \leq \omega \leq 1$.

The uniqueness is clear as a holomorphic function is determined by its values on the unit circle. To prove the existence of $g$, we can conjugate $A, B$ by the same matrix. We can suppose if we want, by splitting $C^n$ according to the eigenvalues of $e^A = e^B$, that $e^A$ has a single eigenvalue.

Write $A = A_s + A_n$ where $A_s$ is semi-simple, $A_n$ is nilpotent, and $[A_s, A_n] = 0$. Do similarly for $B$. Then

$$e^A = e^{A_s} e^{A_n} = e^{B_s} e^{B_n}.$$

By uniqueness of the decomposition of $\text{GL}_n$ as the product of a semi-simple + nilpotent elements, we have $e^{A_s} = e^{B_s}, e^{A_n} = e^{B_n}$. But the exponential map
is a bijective between nilpotent and unipotent matrices, so $A_n = B_n$. Hence

$$e^{\omega A} e^{\omega B} = e^{\omega A_n} e^{\omega B_n} e^{-\omega B_n} e^{-\omega B_n} = e^{\omega A_n} e^{-\omega B_n}$$

So we have reduced to the case where $A, B$ are semi-simple.

In this case $e^A = e^B = \lambda I$ so writing $\lambda = e^{i\mu}$ and replacing $A, B$ by $A - \mu I, B - \mu I$ we reduce to showing that $e^A = I \implies e^{\omega A} = g(e^{2\pi i \omega})$ with $g \in GL_n(D)$.

Decomposing over the eigenvalues of $A$, we can suppose $A = \lambda I$, whence $e^A = 1$, so $\lambda = 2\pi i n$ with $n \in \mathbb{Z}$. Then

$$e^{\omega A} = e^{(2\pi i n)\omega} I = g(e^{2\pi i \omega})$$

where $g(z) = z^n I$. QED.

**Corollary**: If $A, B \in \text{Lie}(U_n) = \text{skew-adjoint (n×n)}$-matrices, and $e^A = e^B$, then there is a unique $g \in U_n$ such that for $0 \leq \omega \leq 1$, we have

$$e^{\omega A} e^{-\omega B} = g(e^{2\pi i \omega}).$$

In effect, $g$ exists by the lemma, and it takes $s'$ into $U_n$, hence it is in $U_n$. 
By a special path in $GL_n$ I mean a map $h: [0, 1] \rightarrow GL_n$ which is of the form

$$h(w) = g(e^{2\pi i w}) e^{wX}$$

for some $X \in gl_n$ and $g \in GL_n(\Delta)'$, such a map $h$ extends uniquely to a holomorphic map of $C$ into $GL_n$ such that

$$h(w+1) = h(w) e^X$$

Let $P_n$ be the set of special paths in $\mathbb{A} GL_n$. We have an action of $GL_n(\Delta)'$ on $P_n$ given by $(gh)(w) = g(e^{2\pi i w}) h(w)$; this is a free action. We have a map $P_n \rightarrow GL_n$ given by $h \mapsto h(1)$, which is constant on $GL_n(\Delta)'$ - orbits. Suppose $A$ is an element of $GL_n$. Because exponential is onto for $GL_n$, there exists an $X \in gl_n$ such that $e^X = A$; hence $e^{wX} \in P_n$ lies over $A$. If $h_0$ is any element of $P_n$ with $h(1) = A$, say $h(w) = g(e^{2\pi i w}) e^{wY}$, then $h(w) = g'(e^{2\pi i w}) e^{wX}$ where $g' = g e^{wY} e^{-wX}$ is in $GL_n(\Delta)'$ by the lemma. Thus $GL_n(\Delta)'$ acts transitively on the fibres of $P_n$ over $GL_n$ and we have a principal bundle (at least on the level of sets).

2) $GL_n(\Delta)' \rightarrow P_n \rightarrow GL_n$

A special path $h(w)$ extends uniquely to a holomorphic map $w \mapsto h(w)$ from $C$ to $GL_n$ satisfying
3)  
\[ h(w+1) = h(w) e^x. \]

Suppose given a linear first order DE in \( \mathbb{C}^n \):

4)  
\[ \frac{dy}{dz} = P(z)y, \]

where \( y \) is a column vector of length \( n \) and \( P(z) \) is a \((n \times n)\) matrix of analytic functions in \( \mathbb{C}^n \).
Using \( e^w = z \) this can be transformed into:

5)  
\[ \frac{dy}{dw} = Q(w)y, \]

where \( Q \) is holomorphic in \( \mathbb{C} \) and \( Q(w+1) = Q(w) \).

(In fact, \( \frac{dy}{dw} = \frac{dy}{dz} \frac{dz}{dw} = zP(z)y \) so \( Q(w) = e^w P(e^w) \).) The solution of 5) starting at \( v \) when \( w = 0 \) is

\[ y = h(w)v, \]

where \( h \) is the matrix function holomorphic in \( \mathbb{C} \) such that

6)  
\[ h(0) = 1. \]

Using \( Q(w+1) = Q(w) \), one sees a solution matrix (i.e. satisfies 6)) so one sees that:

7)  
\[ h(w+1) = h(w)h(1). \]

Conversely, given a holomorphic map \( h: \mathbb{C} \rightarrow GL_n \)
satisfying 7) one sees it is the solution matrix of the DE 5) with \( Q = h' h^{-1} \).

Therefore a holomorphic map \( h: \mathbb{C} \rightarrow \text{GL}_n \) satisfying 7) is the same thing as a linear holomorphic first order DE 7) in \( \mathbb{C}^* \). If we choose \( X \) so that \( e^X = h(1) \), then \( h(w) e^{-wX} = f(e^{\omega}) \) where \( f \) is holomorphic in \( \mathbb{C}^* \). By definition one says that the DE 7) has regular singular points at \( 0, \infty \) if \( f \) is meromorphic, i.e. if \( f \in \text{GL}_n(\Delta) \). Thus we see that elements of \( T_n \) are the same thing as the solution matrices of DE's with regular singular points at \( 0, \infty \).

Continuation (June 1975 after 2 weeks interruption).

Let us consider the problem of putting a topology on \( \text{GL}_n(\Delta)' \) and \( P_n \) so that \( P_n \) becomes a principal \( \text{GL}_n(\Delta)' \) bundle over \( \text{GL}_n \). First observe that \( \exp: \text{gl}_n \rightarrow \text{GL}_n \) is a covering in the following sense. I recall that \( \exp \) is etale at those matrices \( A \) such that no two eigenvalues of \( A \) differ by \( \pm 2\pi i n \) with \( n \) a non-zero integer. In other words if \( \lambda_1, \lambda_2 \) are eigenvalues of \( A \) such that
Given $B \in \text{GL}_n$ recall how one finds $A \in \text{gl}_n$ with $e^A = B$. One factors $B = B_s B_u$, puts $B_u = \exp(\log B_u)$, and lifts $B_s$ to a semi-simple matrix $A_s$ so that if $\lambda_1, \lambda_2$ are two eigenvalues of $A_s$ with $e^{\lambda_1} = e^{\lambda_2}$, then $\lambda_1 = \lambda_2$. Thus any matrix commuting with $B_s$ commutes also with $A_s$; in particular $A_u = \log(B_u)$ commutes with $A_s$. Now put $A = A_s + A_u$, and note that $\exp$ is etale at $A$. Therefore we see that $\exp$ maps the etale points of $\text{gl}_n$ onto $\text{GL}_n$, which is what I mean by $\exp$ being a covering.

Addition: 1) Given $B$ there is a unique solution of $e^A = B$ such that the eigenvalues $\lambda$ of $A$ satisfy $0 \leq \text{Im}(\lambda) < 2\pi$. The exponential map is etale at such a point $A$.

(Proof goes as follows:\n\n2) Derivation of the formula for the differential of $\exp$ at a point $A$.

We identify the tangent space to $\text{gl}_n$ at $A$ with $\text{gl}_n$ by associating to $X$ the vector $A + eX$ ($e^2 = 0$ as usual). Under $\exp$ this vector goes to $e^{A + eX}$ which is a tangent vector to $\text{GL}_n$ at $e^A$. We identify the tangent space to $\text{GL}_n$ at $e^A$ with...
by associating to \( Y \) the vector \( e^A(I+\varepsilon Y) \).

In terms of these identifications the differential of \( \exp \) is \( X \mapsto \text{coeff. of } \varepsilon \text{ in } e^{-A}e^{A+\varepsilon X} \).

Recall
\[
e^{-tA}e^{tA} = e^{-t \text{ad} A} X ( = \sum_{n=1}^{\infty} (-t \text{ad} A)^n X)
\]

for both sides satisfy
\[
\phi'(t) = -(\text{ad} A) \phi(t) \quad \phi(0) = X.
\]

Hence
\[
\frac{d}{dt} e^{-tA}e^{t(A+\varepsilon X)} = e^{-tA}(-A + A + \varepsilon X)e^{t(A+\varepsilon X)}
\]
\[
= e^{-tA} \varepsilon X e^{tA} \quad \text{as } \varepsilon \to 0
\]

Integrating we get
\[
e^{-tA}e^{t(A+\varepsilon X)} = I + \varepsilon \sum_{n=0}^{\infty} \frac{t^n}{n!} (\text{ad} A)^n X
\]

Thus
\[
\text{coeff of } \varepsilon \text{ in } e^{-A}e^{A+\varepsilon X} = \sum_{n=0}^{\infty} \frac{(-\text{ad} A)^n X}{(n+1)!}
\]

Thus
\[
\text{dexp}_A (X) = \left( 1 - e^{-\text{ad} A} \right) \left( \frac{1}{\text{ad} A} \right) (X)
\]

3) Since \( \text{ad}(A) = \text{ad}(A_1) + \text{ad}(A_2) \) is a Jordan decomposition of \( \text{ad}(A) \), one sees that the eigenvalues of \( \text{ad}(A) \) are \( \lambda_1 - \lambda_2 \) where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \). Now \( \frac{1 - e^{-x}}{x} \) vanishes
exactly when \( x \) is of the form \( 2\pi n \), \( n \) an integer \( \neq 0 \). Thus \( \exp \) is etale at \( A \) exactly when no two eigenvalues of \( A \) differ by \( 2\pi n \), \( n \) an integer \( \neq 0 \).

Let us now return to putting a topology on \( P_n \). The pull-back of \( P_n \) with respect to \( \exp : \text{OGL}_n \to \text{GL}_n \) is canonically a product \( \text{GL}_n(\Delta)^{\prime} \times \text{OGL}_n \). Let \( P_n = \text{OGL}_n \times \text{OGL}_n \); if \( (X, Y) \in P \), let \( f_{X,Y} \) be the element of \( \text{GL}_n(\Delta)^{\prime} \) such that \( f_{X,Y}(e^{2\pi it}) = e^{tx}e^{-ty} \). Then \( (X, Y) \to \text{GL}_n(\Delta)^{\prime} \) is a cocycle:

\[ f_{X,Y} f_{Y,Z} = f_{X,Z} \]

which describes the twisting of \( P_n \) over \( \text{GL}_n \). (Specifically, \( P_n \) is the cokernel of the pair of arrows

\[ \text{GL}_n(\Delta)^{\prime} \times P \to \text{GL}_n(\Delta)^{\prime} \times \text{OGL}_n \to P_n \]

\[ (x, (X, Y)) \to (x, X) \to (x f_{X,Y}, Y) \to (x e^{tx}, e^{-ty}) \]

Now to topologise \( P_n \) so that it becomes a principal \( \text{GL}_n(\Delta)^{\prime} \) bundle over \( \text{GL}_n \) we have to put a topology on \( \text{GL}_n(\Delta)^{\prime} \) such that...
The function $f_{xy} = e^{tx} e^{-ty}$ is continuous.

Review the nature of $f_{xy} = e^{tx} e^{-ty}$ and calculate its degree. $f_{xy}$ depends only on the semi-simple parts of $X$ and $Y$; let $\lambda_j$ and $\mu_j$ be the eigenvalues of $X, Y$ respectively. We look at what happens in the eigenspace of $e^x = e^y$ with eigenvalue $\lambda$. We choose $\varepsilon$ such that $e^{\varepsilon} = \lambda$, whence $f_{xy} = e^{t(X-\varepsilon)} e^{t(Y-\varepsilon)}$.

The eigenvalues of $X-\varepsilon$ are $\lambda_j - \varepsilon = 2\pi i n_j$ where $n_j \in \mathbb{Z}$, so $e^{t(X-\varepsilon)}$ has degree $\max |n_j|$. Assuming $\varepsilon$ chosen with imaginary part in $[0, 2\pi)$ we see

$$|n_j| = \left| \frac{1}{2\pi} (\text{Im}(\lambda_j) - \varepsilon) \right| < \frac{1}{2\pi} |\text{Im} \lambda_j| + 1$$

Thus

$$\text{deg} f_{xy} \leq \max \left( \frac{1}{2\pi} |\text{Im} \lambda_j| \right) + \max \left( \frac{1}{2\pi} |\text{Im} \mu_j| \right) + 2$$

and we obtain:

**Lemma:** The degree of $f_{xy}$ is bounded if $(x, y)$ range over a bounded subset of $(G_n)^2$.

Filter $G_n(\mathbb{A})'$ by degree:

$$F^k G_n(\mathbb{A})' = \{ \sum a_i z_i \text{ in } G_n(\mathbb{A})' \} \ | \text{each } F^k G_n(\mathbb{A})' \text{ is an affine variety over } \mathbb{C} \},$$

hence it has a natural topology.
Let $GL_n(\Delta)'$ be endowed with the inductive limit topology (this is clearly the finest topology we might want to consider on $GL_n(\Delta)'$).

Recall that for $L$ locally compact we have

\[ \lim_{\alpha} (L \times X_\alpha) = L \times \lim_{\alpha} X_\alpha. \]

Thus the continuous maps

\[ F^N_N GL_n(\Delta)' \times F^N_N GL_n(\Delta)' \to F^N_{N+1} GL_n(\Delta)' \]

induce a continuous map in the limit as $N \to \infty$:

\[ GL_n(\Delta)' \times F^\infty GL_n(\Delta)' \to GL_n(\Delta)' \]

Because of the lemma it follows therefore that

\[ (x, (x,y)) \mapsto xy \]

is continuous for $(x,y)$ in a neighborhood of each point of $F$, hence this map is continuous.

___

Proof of *) : This is a consequence of the fact that for a locally compact space $L$, there is a mapping space functor adjoint to the product $X \times L$. Hence

\[ \text{Hom}(L \times \lim_{\alpha} X_\alpha, Y) = \text{Hom}(\lim_{\alpha} X_\alpha, Y) = \lim_{\alpha} \text{Hom}(X_\alpha, Y) \]

\[ = \lim_{\alpha} \text{Hom}(\lim_{\alpha} \lim_{\beta} X_\alpha, Y) = \text{Hom}(\lim_{\alpha} \lim_{\beta} X_\alpha, Y) \]
so we have seen that $\mathbb{P}_n$ becomes a principal $\text{GL}_n(A)$-bundle over $\text{GL}_n$. An intriguing point which might be useful goes as follows.

Suppose we consider $\mathbb{P}_n$ only those matrices $X$ whose eigenvalues $\lambda_j$ are such that $0 \leq \frac{1}{2\pi} \text{Im}(\lambda_j) \leq 1$. It should be the case that $\mathbb{P}_n$ is obtainable by taking the inverse of this set of $X$. Thus, if $I$ assume

For any $a \in \mathbb{R}$, let $U_a \subset \text{GL}_n$ be the subset consisting of $X$ with whose eigenvalues $\lambda_j$ satisfy

$$a < \frac{1}{2\pi} \text{Im}(\lambda_j) < a + 1.$$

Then $\exp : U_a \to V_a$ where $V_a$ is the open set in $\text{GL}_n$ consisting of matrices having no eigenvalue on the ray: $\text{arg} = 2\pi a$. The $U_a$ cover $\text{GL}_n$ (in fact any $n+1$ of them do).

Suppose $a < b < a + 1$ and $X \in U_a$, $Y \in U_b$ are such that $e^X = e^Y$. We can decompose $\mathbb{C}^n$ into $V' \oplus V''$, where $V'$ is the sum of the generalized eigenspaces corresponding to the eigenvalues of $X$ in the interval $(a, b)$ (resp. $(b, a + 1)$).
Let $X \in U_a$, $Y \in U_b$ be such that $e^X = e^Y$. Let $\lambda$ be an eigenvalue of $X$ and let $W_{\lambda}$ be the corresponding generalized eigenspace. Then $W_{\lambda}$ is the generalized eigenspace of $e^X$ with eigenvalue $e^\lambda$, because distinct eigenvalues of $X$ map to distinct eigenvalues of $e^X$. Similarly, $W_{\lambda}$ is the generalized eigenspace of $Y$ with eigenvalue $e^\mu$, $\mu$ being the unique eigenvalue of $Y$ with $e^\mu = e^\lambda$.

On $W_{\lambda}$, $X = \lambda I + N$, $N$ nilpotent, and $Y = \mu I + N = X + 2\pi i n I$, where $n \in \mathbb{Z}$.

Suppose now that $a < b < a + 1$, and let $V'$ (resp. $V''$) be the sum of the $W_{\lambda}$ such that $a < \frac{1}{2\pi} \text{Im} \lambda < b$ (resp. $b < \frac{1}{2\pi} \text{Im} \lambda < a + 1$). Then on $V''$ we have $e^Y = e^X$ and on $V'$ we have $e^Y = e^X + 2\pi i$. Therefore,

$$f_{XY} = e^X e^{-Y} = e^{z^{-1}I_Y} z^{-1} I_Y \otimes I_Y.$$
Suppose next that $a < b < c < a + 1$ and we have $X \in U_a, Y \in U_b, Z \in U_c$ such that $e^X = e^Y = e^Z$. Let $C^n = V_1 \oplus V_2 \oplus V_3$ where $V_1$ (resp. $V_2, V_3$) is the sum of the general eigenspaces of $X$ corresponding to $1$ with $a < \frac{1}{2\pi} \text{Im}(X) < b$ (resp. in $(b, c)$ in $(c, a + 1)$). Then

\[ f_{xy} = z^{-1} I_{V_1} \oplus I_{V_2} \oplus I_{V_3} \]

\[ f_y = z^{-1} I_{V_2} \oplus I_{V_3} \oplus I_V \]

\[ f_{xz} = z^{-1} I_{V_1} \oplus z^{-1} I_{V_2} \oplus I_{V_3} \]

What's intriguing about this is that we see the cocycle $\delta$ on the family $\{U_a | 0 < a < 1\}$ will take values in the partial monoid of projectors in $C^n$. (The operation is $(E_1, E_2) \mapsto E_1 + E_2$ and is defined on pairs $E_1, E_2$ such that $E_1 E_2 = E_2 E_1 = 0$.)
Special Paths in $U_n$ and $SU_n$

$U_n = \text{group of maps } S^1 \to U_n \text{ given by Laurent polynomials topologized with the inductive limit topology.}$

$U' = \text{subgroup consisting of } f \text{ such that } f(1) = 1.$

Then $U_n = U' \times U_n$ where $U_n$ is identified with the subgroup of constant maps.

Call a path $h: [0,1] \to U_n$ special if it is of the form

$$h(t) = f(e^{2\pi it}) \exp(tX)$$

where $f \in U'$ and $X \in \text{Lie}(U_n) = \text{skew-hermitian matrices.}$ Let $X_n$ be the set of special paths. As for $G_n$ we get a principal bundle

$$U' \to X_n \to \phi \to U_n$$

where $\phi(h) = h(1)$.

Suppose $R, F$ as on page 8. Let $X_n$ be the simplicial complex whose vertices are $R$-lattices $L$ in $F^n$ and whose simplices are chains $L_0 < L_1 < \ldots < L_8$

such that $T_{L_8} < L_0$. Our aim is to construct a
bijection \(|X_n| \to X_n\), that is, to triangulate \(X_n\) via \(X_n\).

We represent elements of \(\text{Lie}(U_n)\) in the form \(2\pi i A\) where \(A\) is a hermitian matrix. Any unitary matrix \(\Theta\) can be uniquely represented \(\Theta = e^{2\pi i A}\) where \(A\) is hermitian and its eigenvalues are in \([0,1]\); notation: \(0 \leq \sigma \leq 1\).

We begin by triangulating the set \(D\) of hermitian matrices \(A\) with \(0 \leq \sigma \leq 1\).

Given \(A\) in \(D\) let \(\lambda_1, \ldots, \lambda_g\) be the eigenvalues of \(A\), not 0 or \(1\), arranged in decreasing order. Then we have an decomposition

\[
\mathbb{C}^n = W_0 \oplus W_1 \oplus \ldots \oplus W_{g+1}
\]

where \(A = \lambda_i\) on \(W_i\), where \(\lambda_1 = \lambda_0 > \lambda_1 > \ldots > \lambda_{g+1} = 0\), and where \(W_1, \ldots, W_g\) are \(= 0\) but \(W_0, W_{g+1}\) may be zero. Let \(Y\) be the simplicial complex whose simplices are the chains of subspaces of \(\mathbb{C}^n\), and we associate to \(A\) the point of \(1Y\)

\[
(\lambda_0 - \lambda_1) V_0 + (\lambda_1 - \lambda_2) V_1 + \cdots + (\lambda_g - \lambda_{g+1}) V_g
\]

where \(V_0 = W_0\), \(V_1 = W_0 + W_1\), \(\ldots\), \(V_g = W_0 + \ldots + W_g\).

(Think of \(\lambda_i - \lambda_{i+1} \to 1\) to get \(V_i\).)
Conversely given a point \( \sum_{i=0}^{b} t_i V_i \) of \( \mathcal{Y} \) with \( V_0 < V_1 < \cdots < V_b \) and \( \sum t_i = 0 \), \( t_i > 0 \) we associate to this point to operator \( A = \sum t_i \text{proj}_{V_i} \) which has eigenvalue \( \lambda_i = t_i + t_{i+1} + \cdots + t_b \) on \( V_i \oplus V_{i-1} = V_i \).

These two constructions are inverse to each other and so give a bijection \( \mathcal{Y} \rightarrow \mathcal{D} \). Actually if we puts the usual topology on simplices of \( \mathcal{Y} \), this becomes a homeomorphism by compactness.

Note that the condition \( A < 1 \) means that \( V_0 = V_b = 0 \), and that \( 0 < A \) means that \( V_0 \oplus C^n \). Thus if from \( \mathcal{Y} \) we delete chains with \( V_0 = 0 \), \( V_b = C^n \), the resulting simplicial complex which is the suspension of the Tits complex made out of proper subspaces has realization the boundary of \( \mathcal{D} \) which \( \sim S^{n^2-1} \) (\( \mathcal{D} \) is a closed convex body with non-empty interior).

We identify \( R \)-lattices \( L \) such that \( R^n \subset L \subset \pi^{-1}R^n \) with subspaces \( V \) of \( C^n \) via the formula: \( L = R^n + \pi^{-1}V \), \( V = L \cap R^n \). In this way \( \mathcal{Y} \) becomes identified with the subcomplex of \( \mathcal{X}_n \) made up of lattices between \( R^n \) and \( \pi^{-1}R^n \).

On the other hand \( \mathcal{D} \) maps to \( \mathcal{X}_n \) by sending \( A \) to \( e^{2\pi itA} \). So we have maps
The vertical maps are as above. For $X_n$ this is clear, where the vertical maps are defined using the $U_n$ action; note $U_n \subset GL_n F$. Now given a simplex $\sigma : l_0 \prec \cdots \prec l_d$ of $X_n$, there is a unique element $f$ of $U_n$ such that $f l_0 = R_n$. Let $|Y|^*$ be the open star of the vertex $O_j$; then $|Y|^* \sim D^* = \{ A \in O \mid A < 1 \}$. We see then that any $f \in |X_n|$ is congruent under $U_n$ to a unique point of $|Y|^*$; since the analogous thing is so for $X_n$, $D^*$ we get the desired bijection

$$|X_n| \sim U_n' \times |Y|^* \sim U_n' \times D^* \sim \mathbb{R}_n.$$

Formulas for $SU_n$. In this case special paths may be represented

$$f(e^{2\pi i t}) e^{tX},$$

where $X$ is skew hermitian of trace 0 and $f \in SU_n'$, i.e. $\det(f) = 1$. 

Start with the simplest example:

\[ \text{SL}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{C}) \]
\[ 0 \subset 0 \]
\[ \text{SO}_2 \subset \text{SU}_2 \]

The non-compact symmetric space is \( \text{SL}_2(\mathbb{R})/\text{SO}_2 = \mathbb{H} \), upper half plane, the compact form is \( \text{SU}_2/\text{SO}_2 \) in \( \mathbb{H} \).

Maximal flat submanifolds of \( S^2 \) are the great circles; the stabilizers of any one in \( \text{SO}_2 \) is cyclic of order 4, because \( \text{SU}_2 \) acts on \( S^2 \) thru \( \text{SO}_3 \).

Consider broken geodesics starting from the north pole in \( S^2 \). The possible directions are described by points of the equator which is \( \mathbb{R}(\mathbb{R}) \). It is clear that broken geodesics ending at the north or south pole which do not cancel are the same as geodesics in the building of \( \text{SL}_2 \)

Consider next the situation:
$$\text{SL}_n(\mathbb{R}) \subset \text{SL}_n(\mathbb{C})$$

$$U \subset U$$

$$\text{SO}_n \subset \text{SU}_n$$

and the building of $\text{SL}_n$ over $\mathbb{R}[[[x]]][[x^{-1}]] = F$. The vertices of the same type as the 0-point may be identified with lattices of index 0 over $\mathbb{R}^n$. Such lattices can be complexified and they give rise to algebraic loops $f: S^1 \rightarrow \text{SL}_n$ such that if $f(z) = \sum a_m z^m$

then $a_m$ is a real matrix. Hence $\overline{f(z)} = f(\overline{z})$.

If $f(z)$ is projected into the symmetric space $\text{SU}_n/\text{SO}_n$, then $z \mapsto f(z) : \text{SO}_n$ for $z = e^{2\pi it^1}$, $0 \leq t \leq \frac{1}{2}$ is a loop in the symmetric space. Thus we get a map from vertices of the building into loops in $\text{SU}_n/\text{SO}_n$.

Notice that if $U/K$ is a compact symmetric space, $K =$ fixed points of an involution on $U$, then a map $f: S^1 \rightarrow U$ such that $f(\overline{z}) = f(z)$ is the same thing as a path $[0,1] \xrightarrow{f} U$ such that $f(0) = 1$, $f(1) \in K$. The space of these is the homotopy fibre of $K \hookrightarrow U$, which is also $\Omega(U/K)$.
Here's the idea: We have the situation

\[ G_0 \to G \]
\[ K \to K \]

where $G_0$ is a real semi-simple, adj., group, with maximal compact subgroup $K_0$ and complexification $G$. $K$ is a maximal compact of $G$. The building for $G$ over $F = R[[\pi]] [[\pi^{-1}]]$ should be the fixes under conjugation for the building of $G_0$ over $F = C[[\pi]] [[\pi^{-1}]]$. Vertices of the latter have been identified with alg. maps $s^i \to K$, so vertices of the former building are alg. maps $s^i \to K$ compatible with conjugation. We hope this would be a hey.

Go back to $SO_n \subset SL_n$ example. We have found inside of $SL_n(R[[\pi^{-1}]]$ the group $I_2$ of Laurent polynomial matrices

\[ f(z) = \sum a_m z^m \]

such that $f(s^i) \subset SL_n$ and $f(z) = \overline{f(z)}$. $I_2$ acts on the building $X_0$ of $SL_n$ over $F$, and $I_2'$ acts simply-transitively on the vertices of index 0. We want to understand the orbit structure of $I_2$ on $X_0$. $I_2/I_2' = SO_n$. 
Is it possible to exhibit the building $\mathbf{SL}_n$ over $F_0$ as a bundle over the symmetric space $\mathbf{SU}_n/\mathbf{SO}_n$? So start with a point in the building given by a simplex $L_0 < \cdots < L_\ell$ and $t_i > 0$, $\sum_{i=0}^\ell t_i = 1$.

I have to recall the formulas relating $\mathbf{SL}_n$ building over $F$ to special paths.

If $\Theta \in \mathbf{SU}_n$ its eigenvalues may be represented uniquely in the form $e^{2\pi it_1}, \ldots, e^{2\pi it_n}$

where $t_1 > \cdots > t_n > t_1 - 1$, $\sum t_i = 0$.

Thus I can lift $\Theta$ to a self-adjoint $X$ of trace 0 whose eigenvalues have spread ≤ 1.

Given such an $X$ let

$t_1 > t_2 > \cdots > t_\ell > t_{\ell+1} = t_1 - 1$

be its eigenvalues and let $V = W_1 \oplus \cdots \oplus W_\ell \oplus W_{\ell+1}$ be the corresponding eigenspaces. Here $W_1 \sim W_\ell \neq 0$

but $W_{\ell+1}$ might be 0. The simplicial records of $X$ are then $t_1 - t_2, \ldots, t_\ell - t_{\ell+1}$ and the
vertices are the subspaces

\[ zW_1 \oplus W_2 \oplus \cdots \oplus W_{g+1} \]

\[ zW_1 \oplus zW_2 \oplus W_3 \oplus \cdots \oplus W_{g+1} \]

\[ zW_1 \oplus \cdots \oplus zW_g \oplus W_{g+1} \]

which correspond to lattices

\[ \mathbb{R}^n < L_1 < \cdots < L_g < z\mathbb{R}^n. \]

\textbf{Note:} $\text{SU}_n$ acts on the $X$ which are self-adjoint of trace 0 and of spread $\leq 1$. The orbit space is an $(n-1)$-simplex. If we require spread $(X) = 1$, then we get the realization of the Tits complex. This is the same as the rays in the space of self-adjoint matrices of trace 0, so the realization of the Tits complex is the unit sphere in the space of self-adjoint matrices.

Special paths: The tangent space to $K/K_0$ at origin can be identified with the space $k^* = k^{-1}$ eigenspace of $K$ under the involution. Note that if $X, Y \in k^*$ and $\exp(X) = \exp(Y)$.
then \( \exp(tX) \exp(tY) = f(e^{2\pi it}) \quad f \in K' \)

is an algebraic loop such that

\[
\frac{f(e^{2\pi it})}{f(e^{-2\pi it})} = \exp(-tX) \exp(tY) = f(e^{-2\pi it})
\]

So denote by \( K' \) the group of algebraic loops in \( K \)

\( f(z) = f(\bar{z}) \), \( K' \) the ones preserving basepoint,

and define a special path to be one in \( K \)

of the form

\[
h(t) = f(e^{2\pi it}) \exp(tX)
\]

with \( f \in K' \) and \( X \in \mathbb{K} \).

Endpoint map. Given the special path \( (*) \)

we associate to it the right coset

\( K_0 \cdot h(\frac{1}{2}) \).

Because \( f(z) = f(\bar{z}) \), it follows

that \( f(t) = f(-t) \) so \( f(t) \in K_0 \). Thus

\[
K_0 \cdot h(\frac{1}{2}) = K_0 \exp(\frac{1}{2}X)
\]

Let \( Y \) be another element of \( \mathbb{K} \) such that

\[
K_0 \exp(\frac{1}{2}X) = K_0 \exp(\frac{1}{2}Y)
\]

i.e.

\[
\exp(\frac{1}{2}X) \exp(-\frac{1}{2}Y) = K_0.
\]

Applying the

involution we get

\[
\exp(-\frac{1}{2}X) \exp(\frac{1}{2}Y) = \exp(\frac{1}{2}X) \exp(\frac{1}{2}Y)
\]

or

\[
\exp(Y) = \exp(X).
\]
which means we can write $h(t)$ in the form

$$h(t) = f(e^{2\pi i t}) \exp(tX) \exp(-tY) \exp(tY).$$

$\in \mathfrak{X}'$

Therefore set-theoretically at least we get the principal bundle

$$\mathfrak{X}' \to \mathfrak{X} \to K/K_0.$$

where $\mathfrak{X}$ is the set of special paths and

$$\phi(h) = h(\frac{1}{2})^{-1} K_0.$$

Action of $\mathfrak{X}'$ on $\mathfrak{X}$:

$$(f, h)(t) = f(e^{2\pi i t}) h(t) f(1)^{-1}$$

$$= f(e^{2\pi i t}) f(1)^{-1} \exp(t \text{Ad} f(1)(X))$$

(note: $f(1) \in K_0$).

So from this we will get the orbits of $\mathfrak{X}'$ on $\mathfrak{X}$ are the same as the orbits of $K_0$ on $K/K_0$.

Check: If $h(t) = f(e^{2\pi i t}) \exp(tX)$

then $(g \cdot h)(t) = g(e^{2\pi i t}) h(t) g(1)^{-1}$.
\[ \phi(g \cdot h) = \left[ g(1) f(-1) \exp(\frac{1}{2}X) g(1) \right]^{-1} K_0 \]
\[ = g(1) \exp(-\frac{1}{2}X) K_0 = g(1) \phi(h) \]

**Example:** Suppose we consider a compact group \( U \) considered as a symmetric space
\[ U \xrightarrow{\Delta} U \times U \quad (x, y) = (y, x) \]

Then an element of \( K^* \) is an alg. map \( z \mapsto (f(z), g(z)) \) such that \( (f(z^\ast), g(z^\ast)) = (g(z), f(z)) \) i.e. \( g(z) = f(z^\ast) \).

Thus \( U \mapsto K^* \), \( f \mapsto (f(z), f(z^\ast)) \). \( K^* \) consists of paths
\[ (f(e^{2\pi it}), f(e^{-2\pi it})) e^{t(x, -X)} \]
\[ = \begin{bmatrix} h(t) & h(-t) \end{bmatrix} \]

where \( h(t) = f(e^{2\pi it}) e^{tx} \) is in \( X_0(U) \). So \( X = X_0(U) \). The endpoint map is:
\[ (f(e^{2\pi it}) e^{tx}, f(e^{-2\pi it}) e^{-tx}) \mapsto (f(-1)e^{tx}, f(-1)e^{-tx})^{-1} \Delta U \]

so if we identify \( U \times U / \Delta U \mapsto U \) via \( (x, y) \mapsto xy^{-1} \)
This goes to $e^{\frac{1}{2}x}f(-1)^{-1}f(-1)e^{-\frac{1}{2}x} = e^{-x}$. Thus we get the same fibration over $U$.

**Question:** Suppose $h$ is a special path in $K$, $\overline{h(t)} = f(e^{2\pi it})e^{tx}$. If $h$ is in $X$, then $\overline{h(t)} = h(-t)$. Conversely does an $h$ in $X$ such that $\overline{h(t)} = h(t)$ belong to $oX$?

Consider such an $h$. Then $\overline{h(1)} = h(-1) = h(1)^{-1}$.

If we can write $h(1) = e^y$ with $\overline{y} = -y$, then

$$\overline{h(t)} = g(e^{2\pi it})e^{ty}$$

and

$$\overline{h(t)} = g(e^{2\pi it})e^{-ty} = g(e^{-2\pi it})e^{-ty}$$

implies $g \in oK$ so $h \in oX$. Thus we reach

**Question:** Given $x \in K$ such that $\overline{x} = x^{-1}$, is $x = e^x$ where $\overline{x} = -x$?

Obvious counterexample. Suppose the involution is trivial and $x$ is an element of $K_0$ of order 2. In general if this question has an affirmative answer then every element of order 2 in $K_0$ must be of form $e^x$ with $\overline{x} = -x$. 
Case of trivial involution in $K$. Let $f: S \to K$ be algebraic such that $f(\bar{z}) = f(z)$. Then we get a map of $C$-variables $f: G_m \to G$ such that $f(z) = f(\frac{1}{z})$. Quotient of $G_m$ by the action of $\mathbb{Z}_2: z \mapsto z^{-1}$ is $\mathbb{G}_a$, the map being $x' = \frac{1}{2}(z + z^{-1})$. (Clearly $x$ generates the algebra of invariant functions).

So we get $f(z) = g(z + z^{-1})$ where $g: \mathbb{G}_a \to G$ is algebraic, and $g(x) \in K$ for $x$ in $\mathbb{R}$. If $u \in A(K)$, then $ug(x)$ is a polynomial in $x$ which remains bounded for $x \in \mathbb{R}$. This is possible only if $ug(x)$ is constant. Thus $g$, hence $f$ has to be constant.

So we see that in the case of the trivial involution, there are no non-trivial special paths in the symmetric space: $\mathcal{X} = \mathbb{G}_a \times K$.

Now go back to the first question on page 9. Suppose $h(t) = f(e^{2\pi it})e^{tx}$ is a special path in $K$ such that $h(t) = h(-t)$. This means
\[ f(e^{2\pi it})e^{tx} = f(e^{-2\pi it})e^{-tx} \]
or
\[ e^{tx} + tX = \left[ f(e^{2\pi it}) \right]^{-1} f(e^{2\pi it}) e^{tX} \]
so nothing more than $e^{tX} = t$. 
Also in the trivial involution case, suppose given a special path \( h(t) = f(e^{2\pi i t}) e^{tx} \) in \( K \) such that \( h(t) = h(-t) \). Then
\[
f(e^{2\pi i t}) e^{tx} = f(e^{-2\pi i t}) e^{-tx}
\]
So setting \( t = \frac{1}{2} \) we get
\[
f(-1) e^{\frac{1}{2}x} = f(-1) e^{-\frac{1}{2}x}
\]
or \( e^x = 1 \), whence \( h \in K \) has to be constant by the preceding.

**Proposition:** Let \( h(t) = f(e^{2\pi i t}) e^{tx} \) be in \( K \) and satisfy \( h(t) = h(-t) \). Then \( h \) is in \( K \).

**Proof:** \( f(-1) e^{\frac{1}{2}x} = f(h(\frac{1}{2})) = h(\frac{1}{2}) = f(-1) e^{-\frac{1}{2}x} \)
hence if \( y = h(\frac{1}{2}) = f(-1) e^{\frac{1}{2}x} \) then we have
\[
y = y e^{-x} \quad \text{or} \quad e^x = \overline{y}^{-1}y.
\]
Now I know that every element of \( K/\mathbb{K}_0 \) is of the form \( \overline{e^y} \mathbb{K}_0 \) where \( \overline{y} = -y \). Hence
\[
\overline{y}^{-1} \mathbb{K}_0 = \overline{e^y} \mathbb{K}_0
\]
or \( \overline{y}^{-1} = \overline{e^y} \mathbb{K}_0 \) so
\[
e^x = \overline{e^y} \mathbb{K}_0 \mathbb{K}_0^{-1} \overline{e^y} = e^y
\]
This means \( h \) can be put in the form \( g(e^{2\pi i t}) e^y \).
whence \( g(\overline{z}) = g(z) \). Q.E.D.

The general picture. We have

Identify \( K \) with the unit disk.

Consider the inclusion map \( \mathcal{K}' \subset \mathcal{K} \)
which gives us a map \( \mathcal{O}(K / K_0) \rightarrow \mathcal{O}(K) \).

It takes a path \( \lambda : [0, 1] \rightarrow K \) starting at 1
ending in \( K_0 \) and associate to it the loop in \( K \)
given by

\[
\lambda(e^{2\pi it}) = \begin{cases} 
\lambda(2t) & 0 \leq t \leq \frac{1}{2} \\
\overline{\lambda(-2t)} & -\frac{1}{2} \leq t < 0
\end{cases}
\]

What is the composition with the map \( \mathcal{O}(K) \rightarrow \mathcal{O}(K / K_0) \)?

We get the loop

\[
\lambda(2t)K_0 \quad 0 \leq t \leq \frac{1}{2}
\]
\[
\overline{\lambda(-2t)K_0} \quad -\frac{1}{2} \leq t < 0
\]

in \( K / K_0 \). This is the \underline{difference} of the loops \( \lambda(t)K_0 \)
and \( \overline{\lambda(t)K_0} \).
General picture: We have identified $X$ with the building associated to $G$ over the local field $\mathbb{F} = \mathbb{Q}[[z]][[z^{-1}]].$ Now the involution on $K$ extends to a $\mathbb{C}$-anti-linear involution on $G,$ which defines a real semi-simple group $G_0$ whose maximal compact subgroup is $K_0$:

$$G_0 \subset G$$

$$K_0 \subset K$$

It should be the case that the involution $h \mapsto h(-t)$ on $X$ corresponds to the natural involution on $G(F)$ with fixed set $G_0(F_0),$ $F_0 = \mathbb{R}[z^{-1}][z].$ Thus it should be possible to identify $X$ with the building of $G_0$ over $F_0.$

It is necessary to understand root theory for symmetric spaces. Start with standard situation:

$$G_0 \subset G$$

$$U \subset U$$

$$K_0 \subset K$$
where \( K \) is a simply-connected compact group, \( G \) its complexification, \( \mathcal{G}_0 \) is a semi-simple real algebraic group with complex, \( G \) and \( K \), its maximal compact. One has Cartan involutions \( \theta, \theta_0 \) of \( \mathcal{G}_0 \) with respect to \( \mathcal{G} \) and \( K \), and the conjugation involution \( \tau \) of \( (G, K) \) with fixed points \( (\mathcal{G}_0, K_0) \).

**Lie algebra decomposition**

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus i \mathfrak{g}_0
\]

\[
\mathfrak{g} = \mathfrak{k} \oplus i \mathfrak{k}
\]

\[
\mathfrak{k}_0 = \mathfrak{k}_0 \oplus i \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{g}_0 \oplus i \mathfrak{k}
\]

\[
\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0
\]

\[
\mathfrak{k} = \mathfrak{k}_0 \oplus i \mathfrak{p}_0
\]

Next one lets \( \mathfrak{o}_0 \) be a maximal abelian subspace of \( \mathfrak{p}_0 \). It corresponds to a maximal split torus \( S_0 \) of \( \mathcal{G}_0 \), having identity component \( A_0 \), or \( \mathfrak{o}_0 \) extends to a Cartan subalgebra \( \mathfrak{h}_0 \) of \( \mathcal{G}_0 \). Diagram of \( \mathfrak{g}_0 \) and \( \mathfrak{p}_0 \).
\[ T_0 \cap H_0 = H \]

\[ T = T_0 T^- \]

\[ \text{Lie}(T^-) = \text{i} \omega. \quad \text{Thus } T^- \text{ is a maximal torus of } K \text{ on which } T \text{ acts as } -1. \quad \text{Now we have root decompositions} \]

\[ \mathfrak{g} = h + \sum_{\alpha \in \Phi} \mathbb{C} \mathfrak{h} \]

\[ \mathfrak{k} = i \mathbb{E} \oplus \sum_{\alpha \in \Phi_+} \mathbb{R} \{ \mathfrak{h}_\alpha - \mathfrak{h}_{-\alpha} \} i \mathfrak{h}_\alpha + i \mathfrak{h}_{-\alpha} \]

where \( \mathbb{E} \) is spanned by the \( H_\alpha = [\mathfrak{h}_\alpha, \mathfrak{h}_{-\alpha}] \).

Better: One starts with \( \mathfrak{g}_0 \) and shows there exists a compact form \( \mathfrak{k} \) in \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{C} \), invariant under \( \Theta_0 \).

Involutions are as follows

\[ \sigma_0 = \sigma_0^+ \circ \sigma_0^- \quad \text{involution (anti-linear)} \]

\[ \sigma_1 = \mathbb{k} + i \mathbb{k} \quad \Theta \]

\[ \sigma_2 = \mathbb{k} \mathfrak{h} + \mathfrak{h} \mathfrak{k} \quad \text{involution (linear)} \quad \Theta_2 = \Theta \circ \sigma_2 \]
\[
oalign{\bigskip} o_{\theta_0} = k_0 + f_0 \quad \text{involution} \quad \Theta = \Theta_0 \quad \text{on} \quad o_{\theta_0}.
\]
\[
oalign{\bigskip} k = k_0 + ip_0 \quad \text{involution} \quad \sigma = \Theta_0 \quad \text{on} \quad k.
\]

Next let \( o_{\theta_0} \) be a maximal abelian subspace of \( f_0 \) and \( h_0 \) any maximal abelian subalgebra of \( o_{\theta_0} \) containing \( o_{\theta_0} \). Then for \( X \in h_0, Y \in o_{\theta_0} \),
\[
[x - \Theta X, y] = [x, y] - \Theta [x, \Theta y] = 0
\]
and as \( x - \Theta x \in f_0 \), maximality of \( o_{\theta_0} \) implies \( X - \Theta X \in o_{\theta_0} \); thus \( \Theta h_0 \subset h_0 \), so
\[
h_0 = t_0 + o_{\theta_0}
\]
Similarly \( k = t_0 + i o_{\theta_0} \).

Next I want to look at the roots of \( o_{\theta} \) with respect to \( h \).
\[
o_{\theta} = h + \sum_{x \in \mathbb{C}} e_x
\]
Now \( \Theta_0 \) which is \( \mathbb{C} \)-linear moves this around. It preserves \( h \) \( = h_0 \otimes \mathbb{C} \), hence we get
\[
h = t_0 \otimes \mathbb{C} \bigoplus o_{\theta_0} \otimes \mathbb{C}
\]
\[
\Theta_0 = 1 \quad \Theta_0 = -1
\]
So actually it is better to ask about the $h_0$-action on $g_f$.

Improvement: Suppose we start with the involution $\sigma$ in $K$ simply-connected and compact. Choose a torus $T$ in $K$ which is maximal such that $\sigma = -1$ on $T$, and extend to a maximal torus $T$ of $K$. Claim $T$ is stable under $\sigma$. In effect if $X \in \text{Lie}(T)$, then for $Y$ in $\text{Lie}(T)$ we have

$$[X - \overline{X}, Y] = [X, Y] - \overline{[X, Y]} = [X, Y] + \overline{[X, Y]} = 0$$

as $T$ is abelian. Thus $X - \overline{X}$ generates a 1-parabolic subgroup centralizing $T$, and reversed by $\sigma$. i.e. $X - \overline{X} \in \text{Lie}(T)$, so $\sigma \text{Lie}(T) \subseteq \text{Lie}(T)$.

Possible notation: $\text{Lie}(T) = 2\pi i \alpha$, i.e. $\alpha$ plays the role of $E$ before.

$\alpha$ is the Lie algebra of a maximal split torus $S_0$ of $G_0$. We know that $S_0$ acting on $G_0$ splits into a sum of characters. Let $E_0 \subseteq \alpha^*$ be the set of these characters; these are called the roots of $G_0$ with respect to $S_0$. We have
a surjection $\Phi \rightarrow \Phi_0 \cup \{0\}$.

$\Phi_0$ is the centralizer of $S_0$ in $G_0$. It is the reductive group containing $H_0$.

It has the roots $\alpha_s$ where $\alpha$ vanishes on $\Phi_0$.

$\Phi_0$ consists of the roots $\alpha$ such that $\alpha(\alpha_0) \neq 0$.

$\Theta_0$ is an involution (linear) of $G$ preserving $h$, hence

$$[H, X_\alpha] = \alpha(H) X_\alpha$$

$$[\Theta H, \Theta X_\alpha] = \alpha(H) \Theta X_\alpha$$

$$[H, \Theta X_\alpha] = \alpha(\Theta H) \Theta X_\alpha$$

Thus $\Theta g^\alpha = g^{\alpha \Theta_0}$. There are two cases depending on whether $\alpha \Theta_0 = \alpha$ or $\alpha \Theta_0 \neq \alpha$.

If $\alpha \Theta_0 = \alpha$, then for $H \in \Phi_0$ we have

$$\alpha(H) = \alpha(\Theta_0 H) = -\alpha(H)$$

so $\alpha(H) = 0$. Thus $\alpha \Theta_0 = \alpha \Rightarrow \alpha(H) = 0$.

And so $\alpha$ appears in $Z_0$.

Recall that $\Theta X_\alpha = -X_\alpha$.

Hence $\Theta g^\alpha = g^{-\alpha}$. Thus $g^\alpha + g^{-\alpha}$ is fixed.
If \( \alpha | \alpha = 0 \), then for any \( H \in h \) we have \( H - \theta H \in \alpha \) so \( \alpha(H) = \alpha(\theta H) \).

Conversely, if \( \alpha \theta_0 = \alpha \), then \( H \in \alpha \) implies

\[
\alpha(H) = \alpha(\theta_0 H) = -\alpha(H)
\]

so \( \alpha(H) = 0 \), hence \( \alpha | \alpha = 0 \). In this case

\[
\theta_0 \cdot \gamma \alpha = \gamma \alpha \theta_0 = \gamma \alpha \quad \theta_0 X_\alpha = \pm X_\alpha, \quad \theta_0 X_\alpha = -X_\alpha \quad \text{then} \quad X_\alpha \in \mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}, \quad \text{and if} \quad H \in \alpha
\]

\[
[H, X_\alpha] = \alpha(H)X_\alpha = 0
\]

which contradicts \( \alpha \) being a maximal abelian subspace of \( \mathfrak{p} \). Thus \( \theta_0 X_\alpha = X_\alpha \) so \( X_\alpha \in \mathfrak{k}_0 \otimes \mathbb{C} \).

Recall \( Z < G \) is the centralizer of the maximal split torus \( S \). I want to select a Borel subgroup \( B \) in \( G \) such that \( ZB \) is a subgroup. Recall \( E \) is the real vector space generated by the lattice \( \text{Hom}(\mathbb{G}_m, H) \), and \( \text{Hom}(\mathbb{G}_m, S) \) generates the subspace \( \alpha_0 \). Now I choose \( B \) so that \( \alpha_0 \cap C \) has a non-empty interior point of \( \alpha_0 \), where \( C \) is the chamber in \( E \).
determined by $B$. In other words I take the roots of $G$ with $H$ and restrict them to $\alpha_0$, thereby dividing $\alpha_0$ into cones. I then take an open cone in $\alpha_0$ and an open cone C in $\Phi$ of which the former cone is a face. Then I get a set of positive roots $\Phi_+^+ \subset \Phi_0$, and $\Phi_+^+ \subset \Phi_0$, such that if $\alpha \in \Phi_+^+$ is such that $\alpha/\alpha_0 \subset \Phi_0^+$, then $\alpha \in \Phi_+^+$.

Now note that if $\alpha \neq \alpha_0$, i.e., $\alpha/\alpha_0 \neq 0$, then $\alpha, \alpha_0$ has opposite signs on $\alpha_0$, so $\alpha \in \Phi_+^+ \implies \alpha_0 \in \Phi_0^-$. The parabolic group $ZB$ we get has the roots $\alpha \in \Phi_+^+$ such that $\alpha/\alpha_0$ is either $0$ or in $\Phi_+^+$.

Note that the unipotent radical of $ZB$ has Lie algebra $n^+ = \oplus_{\alpha \in \Phi_+^+}$, that both $\Theta_0$ and $\Theta_0^{-1}$ carry this into $n^-$, $\alpha/\alpha_0 \in \Phi_0^+$, hence $n^+$ is stable under $\Theta = \Theta_0 \Theta$. Thus $n^+$ gives us a nilpotent group $N^+_0$ in $G_0$ with Lie algebra $n^+_0 = n^+ \cap g_0$. Moreover, the roots of $n^+_0$ with respect to $g_0^+$ are the restrictions $\alpha/\alpha_0$, which are $\Theta_0$ with $\alpha \in \Phi_+^+$. So the weight space decomposition of $g_0$ looks like:

$$g_0 = g_0^+ + \sum_{\beta \in \Phi_0^+} g_0^\beta$$

$$n^+_0 = \sum_{\beta \in \Phi_0^+} g_0^\beta$$