The building for $\text{SL}_n$ over a field with discrete valuation.

Let $R$ be a discrete valuation ring with quotient field $F$ and residue field $R/\pi R$. Let $V$ be a vector space over $F$ of dimension $n$. By an $R$-lattice in $V$ we shall mean an $R$-submodule which is free of rank $n$.

Using lattices we shall define two simplicial complexes $X, X'$ associated to $V$. $X'$ is the simplicial complex whose vertices are the $R$-lattices in $V$ and in which a simplex is a chain of lattices $L_0 < L_1 < \ldots < L_g$ such that $\pi L_g \subseteq L_0$. Note that if a lattice $L$ is fixed, then the lattices $L'$ such that $\pi L < L' < L$ are in 1-1 correspondence with subspaces of the $n$-dimensional vector space $L/\pi L$ over the residue field $R/\pi R$. Consequently $L_0 < L_1 < \ldots < L_g$ is a simplex of maximum dimension in $X'$ iff $L_0 = \pi L_8$, $L_i/L_{i-1}$ is one-dimensional over $R/\pi R$, and $g = n$. Thus $\dim(X) = n$ and every simplex is contained in a $n$-simplex.

By the homothety class of a lattice $L$ we mean the chain $\mathcal{C}(L) = \{\pi^j L, j \in \mathbb{Z}\}$. $X$ is the simplicial complex whose vertices are the homothety classes of lattices, and in which a set of vertices is a simplex iff the union of these homothety classes is a chain. Thus a
simplex of \( X \) may be viewed as a chain of lattices closed under multiplying by \( \pi, \pi^{-1} \). If the chain is enumerated: \( \sigma = \{L_j, j \in \mathbb{Z}\} \), starting with some lattice \( L_0 \) in \( \sigma \), and putting \( L_j = \text{least member of } \sigma \text{ containing } L_{j-1} \), then \( \pi L_j = L_{j-k} \) for some \( k \), and \( \sigma \) is a \((k,1)\)-simplex with the vertices \( \text{cl}(L_0), \ldots, \text{cl}(L_{k-1}) \). A simplex of maximal dimension is one such that \( \dim(L_i/L_{i-1}) = 1 \) and the dimension of such a simplex is \( n-1 \), hence \( \dim(X) = n-1 \). The \((n-1)\)-simplices of \( X \) will be called chambers. Every simplex is the face of some chamber.

There is an evident simplicial map from \( X' \) to \( X \). We can lift \( X \) back to a subcomplex of \( X' \) as follows. Given \( 0 \neq w \in V \), it is clear that each homothety class of lattices contains a unique lattice \( L \) such that \( L \cap Fw = Rw \). The subcomplex of \( X' \) made up of such lattices \( L \) maps isomorphically onto \( X \). Another method is to fix a lattice \( Rw \) in \( \Lambda^n V \) and define the volume of any lattice \( L \) to be the integer \( v(L) \) such that \( \Lambda^n L = \pi^* v(L) Rw \). Any homothety class of lattices contains a unique \( L \) with \( 0 < v(L) \leq n \), and the subcomplex of \( X' \) made up
of these lattices \( \Lambda \) maps isomorphically onto \( X \).

The complex \( X \) is the building for \( \text{SL}_n \) over \( F \) in the sense of Bruhat and Tits. \( X' \) is in some sense the building for \( \text{GL}_n \) over \( F \) (compare...)

\[ \text{Apartments:} \]

Suppose now that \( V = F^n \), the vector space of column vectors of length \( n \) over \( F \), and let \( e_1, \ldots, e_n \) be the standard basis. Call a lattice special if it is of the form \( L = \sum_{i=1}^{n} R_i \alpha_i e_i \) with \( \alpha_i \in \mathbb{Z} \). The subcomplexes of \( X' \) and \( X \) made up of special lattices will be denoted \( A' \) and \( A \) and called the fundamental apartments of \( X' \) and \( X \) respectively.

We propose to determine the realizations of \( A' \) and \( A \). \( A' \) may be identified with the complex whose vertices are elements of \( \mathbb{Z}^n \) and whose simplices are chosen for the geometrical situation in \( \mathbb{Z}^n \) such that we identify a vertex of \( A' \) with a sequence of integers \( (m_1, \ldots, m_n) \) and \( \alpha_i \) associated.
We propose to determine the realizations of \( A' \) and \( A \). Recall that an ordered simplicial complex is one equipped with a partial ordering on its vertex such that each simplex is linearly ordered. The product of two ordered simplicial complexes is naturally an ordered simplicial complex such that \( |K_1 \times K_2| = |K_1| \times |K_2| \) for the compactly generated topology.

Let \( D \) be the ordered simplicial complex of dimension one having integers with the usual ordering for vertices and \( \mathbb{R} \) consecutive integers for 1-simplices. Then \( |D| = \mathbb{R} \).

The product \( D^n \) has \( \mathbb{Z}^n \) for vertices and a simplex is a subset of \( \mathbb{Z}^n \) of the form \( \{v_0, \ldots, v_i\} \) where

\[
v_0 \leq v_1 \leq \cdots \leq v_i \leq v_i + 1
\]

for \( i = 1, \ldots, n \).

By associating to \( (d_1, \ldots, d_n) \in \mathbb{Z}^n \) the lattice \( \sum R^{d_i} e_i \), it is easily seen that we obtain an isomorphism \( D^n \) with \( A' \). Thus

\[
|A'| = |D^n| = \mathbb{R}^n.
\]

Picture of \( A' \) when \( n = 2 \):

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Note that each vertex in $A$, i.e., homothety class of lattices, contains a unique lattice of the form $\Sigma R^{d_i}e_i$, where $d_n = 0$, so that $A$ is isomorphic to the subcomplex of $A'$ made up out of lattices with $LnFe_n = Re_n$. Thus we get an isom. of $D^{n-1}$ with $A$ by associating to $(d_1, \ldots, d_{n-1})$ the class $\Sigma R^{d_i}e_i$, where $d_n = 0$. So $|A| \cong R^{n-1}$.

It is customary to identify $|A|$ with the space $E = \{x \in R^n | \sum x_i = 0\}$ as follows. One sends the vertex $\text{cl}(\Sigma R^{d_i}e_i)$ to the point $(d_1, \ldots, d_n) - (\frac{1}{n} \sum x_i)(1, \ldots, 1)$ of $E$ and extends linearly over simplices to get a map $|A| \rightarrow E$. This map is a homeomorphism because its composition with the homeo above is the isomorphism $$(x_1, \ldots, x_n-1) \rightarrow (x_1, \ldots, x_{n-1}, 0) - (\frac{1}{n} \sum x_i)(1, \ldots, 1)$$ of $R^{n-1}$ with $E$.

This picture has the advantage of being compatible with the action of $\Sigma'$. Noting that $L$ special $\leftrightarrow L = \Sigma LnFe_i$. In general, given a splitting $V = W_1 \oplus \cdots \oplus W_n$ of $V$ into 1-dimensional spaces, we get apartments in $X_1 X$ made up of lattices $L \cong L = \Sigma LnW_i$. 
Suppose $V = F^n$ with basis $e_i$, $1 \leq i \leq n$. We identify automorphisms $g$ of $V$ with invertible matrices $(g_{ij}) \in \text{GL}_n F$ by the formula

$$g e_i = \sum g_{ij} e_j$$

Define lattices $V_p$ for $p \in \mathbb{Z}$ by putting

$$V_p = \{ \sum_{i=1}^n t_i e_i | t_i \in F, \sum t_i = p \}$$

and setting $V_p = V$ if $p = 0$. Then $\{V_p\}$ is a chamber of $X$, which we call the fundamental chamber; $V_p / V_{p-1}$ is one-dimensional space over $R/1R$ with basis $e_p$. The subgroup of $SL_n(F)$ preserving the chain $\{V_p\}$ will be denoted $B$ and called the fundamental Iwahori subgroup of $SL_n(F)$. We have

$$B = \{ g \in SL_n(R) | g e_j \in \mathbb{Z}R, j > j \}.$$ 

We propose now to compute the $B$-orbits on $X'$ and $X$.

Let $\{L_j, j \in \mathbb{Z}\}$ be an increasing sequence of lattices such that $\mathbb{Z}L_j = L_{j-k}$ for all $j$. Define a map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ by
\[ V_p \cap L_j + V_{p-1} = \begin{cases} V_{p-1} & j < \phi(p) \\ V_p & j \geq \phi(p) \end{cases} \]

Then \( \phi(p+n) = \phi(p+k) \) for all \( p \), and the function \( \phi \) is an invariant of the \( \mathbb{B} \)-orbit of the sequence \( L_j \).

**Lemma:** Let \( \mathbb{Z} \) be a set of representatives for the \( \mathbb{R} \)-cosets in \( \mathbb{R} \). Then there exists a unique element \( x_p \in L_{\phi(p)} \) of the form

\[ x_p = e_p + \sum_{a < p} \gamma_a p e_a \]

with \( \gamma_a \in \mathbb{Z} \).

**Proof:** By (*) above can find \( y_p \in V_p \cap L_{\phi(p)} \) such that \( y_p - e_p \in V_{p-1} \). By induction on \( N \) we can find \( \gamma_a \in \mathbb{Z} \) for \( -N < a < p \) such that

\[ y_p - e_p - \sum_{-N < a < p} \gamma_a e_a \in V_{-N}. \]

Choose \( N \) large enough so that \( V_{-N} \subset L_{\phi(p)} \), and let \( a^* \) be largest such that \( \phi(a^*) = \phi(p) \) and \( \gamma_{a^*} \neq 0 \). Then \( y_p - \gamma_{a^*} p e_a \) is another choice for \( y_p \) having same coefficients in the places above \( a^* \), but \( 0 \) coefficient at the
Continuing we get an element \( x_p \in \Lambda_\phi(p) \) of the form (**). If \( x'_p \) is another element with the coefficients \( z'_{ap} \), let \( a \) be largest such that \( z_{ap} \neq z'_{ap} \). Then \( x'_p - x_p \in V_a \) but not \( V_{a-1} \), so

\[
\begin{align*}
V_a \oplus L_p(p) \oplus V_{a-1} &= V_a \oplus \phi(p)
\end{align*}
\]

contradicting the fact that \( \phi(a) > \phi(p) \). This proves the uniqueness of \( \phi \), the elements \( x_p \).

In virtue of the uniqueness we have:

\[
\Pi x_p = x_p - \kappa
\]

and hence if \( g \) is the \( \phi \)-endo. of \( V \) with \( ge_i = x_i \) for \( i = 1, \ldots, n \), then \( g(x_p) = x_p \)

for all \( p \).

Now \( g(e_i) = \sum gji e_j \), where from (***) we have:

\[
gji = \delta ji + \sum_{j-t n < i} \frac{\pi^t}{j-t}
\]

where \( t \) runs over integers such that \( j-t n < i \) and \( \alpha(j-t n) = \alpha(j) - \alpha(i) \).

That is

\[
\frac{j-i}{n} < t < \frac{\alpha(j) - \alpha(i)}{k}
\]
By virtue of the canonical isomorphisms
\[
\frac{V_p \cap L_j + V_{p-1}}{V_p \cap L_{j-1} + V_p - 1} \leftrightarrow \frac{V_p \cap L_j}{V_p \cap L_{j-1} + V_{p-1} \cap L_j} \rightarrow \frac{V_p \cap L_j + L_{j-1}}{V_p \cap L_{j-1} + L_j - 1}
\]
and \((*)\), the filtration \(\{V_p \cap L_j + L_{j-1}\}\) of the layer
\(L_{j-1} \subset L_j\) has 1-dimensional jumps generated by \(x_p\)
for those \(p\) such that \(\phi(p) = j\). Thus \(L_j / L_{j-1}\)
has a basis over \(R/\pi R\) given by the images of
the \(x_p\) with \(\phi(p) = j\). Similarly, \(L_j / \pi L_j = L_j / L_j - \pi\)
has the basis given by the \(x_p\) with \(j - k < \phi(p) \leq j\).
As \(L_j\) is a free \(R\)-module, these \(x_p\) form
a basis for \(L_j\) over \(R\):

\[
L_j = \bigoplus_{j - k < \phi(p) \leq j} R x_p.
\]

(1)

By the uniqueness assertion in the lemma one has
\(\pi x_p = x_p - \pi\), hence if \(g\) is the endomorphism of \(V\)
with \(g e_i = x_i\) for \(1 \leq i \leq n\), then \(g e_p = x_p\) for
all \(p\) in \(\mathbb{Z}\). Let \(g(e_i) = \sum g_{ji} e_j\), whence from \((**)\)
we have

\[
g_{ji} = \delta_{ji} + \sum_{t} \frac{\mathbb{Z}_{j - t n, i}}{\pi t}^t
\]

(2)

Where \(t\) runs over integers such that \(j - \pi n < i\)
and \(\phi(j - \pi n) = \phi(j) - \pi k > \phi(i)\), that is,

\[
\frac{j - i}{\pi} < t < \frac{\phi(j) - \phi(i)}{\pi}
\]
Clearly \( g_{ji} = s_{ji} \) if \( \phi(j) \leq \phi(i) \), hence \( \text{det}(g) = 1 \), so \( g \in B \).

We are now in a position to classify \( B \)-orbits on the set of increasing sequences of lattices \( L_j, j \in \mathbb{Z} \) such that \( \pi L_j = L_{j-k} \) for all \( j \). Call this set \( S_k \).

To each such sequence we have associated a function \( \phi : \mathbb{Z} \to \mathbb{Z} \) such that \( \phi(p+n) = \phi(p) + k \) for all \( p \), which depends only on the \( B \)-orbit of the sequence. On the other hand, to each such \( \phi \) we can associate the sequence of special lattices:

\[
L^\phi_j = \bigoplus_{p \leq \phi(j)} \text{Rep}_p.
\]

It is easily checked that the function assoc. to \( (L^\phi_j) \) is \( \phi \), hence we obtain a 1-1 correspondence between sequences in \( S_k \) made up of special lattices and the functions \( \phi \). From the above formulas (1), (3) we see that \( (L_j) \) is in the \( B \)-orbit of \( (L^\phi_j) \) if \( \phi \) is associated to \( (L_j) \). Thus we have:

\[
S^\phi_k \xrightarrow{\sim} B \backslash S_k \xrightarrow{\sim} \{ \phi : \mathbb{Z} \to \mathbb{Z} | \phi(p+n) = \phi(p) + k \}
\]

where \( S^\phi_k \subset S_k \) is the subset of "special" sequences.
Let's now consider the $B$-orbit of $(l_j^φ)$. From the lemma we have associated to a sequence $(l_j)$ in this orbit a set of $z_{ap} \in \mathbb{Z}$ for $a < p$ and $x(a) > x(p)$ satisfying $z_{ap} = z_{a+p}$. Satisfying $z_{a+n}, p+n = z_{ap}$.

From the $z_{ap}$ we can construct a $g \in B$ (see (2)) such that $g(l_j^φ) = (l_j)$. So I get a map $B(l_j^φ) \rightarrow (\mathbb{Z}/\mathbb{N})^l(φ)$, $l(φ) = \text{card} \{ (a, p) \mid 1 \leq p \leq n, x(a) > x(p) \}$

$(l_j) \mapsto (z_{ap})$; and also a map the other way sending $(z_{ap})$ to $g(l_j^φ)$, where $g$ is given by (2). It is clear from the preceding that these two maps are inverses of each other, so we get a bijection:

$$B(l_j^φ) \rightarrow (\mathbb{R}/\pi\mathbb{R})^l(φ)$$
Geometric interpretation of \( \ell(\phi) \): According to the calculation made at the bottom of page 9, \( \ell(\phi) \) is the sum over pairs \((i,j)\) such that \(1 \leq i < n\) of the number of integers \(t\) such that
\[
\frac{i-1}{n} < t < \frac{\phi(j) - \phi(i)}{k}
\]
This is the same as the sum over \(1 \leq i < n\) of the number of integers strictly between \(\frac{i-1}{n}\) and \(\frac{\phi(j) - \phi(i)}{k}\), so we have the following.

**Assertion:** \( \ell(\phi) \) is the number of hyperplanes in \(\mathbb{R}^n\) of the form \(x_i - x_j = t\) with \(1 \leq i < j \leq n\) and \(t \in \mathbb{Z}\) which are crossed in going along the straight line joining \((-\frac{1}{n}, \ldots, -\frac{1}{n})\) and \((-\frac{\phi(1)}{k}, \ldots, -\frac{\phi(n)}{k})\).

If \(p = i + tn > 1 \leq i < n\), satisfies \(j - k < \phi(p) < j\), then \((j - \phi(i))/k - 1 < t \leq (j - \phi(i))/k\), so
\[
t = \left[\frac{j - \phi(i)}{k}\right]
\]
where \([x]\) = greatest integer \(\leq x\). Hence (3) page 10 can be written
\[
\ell_j = \sum_{i=1}^{n} R_{\pi} \left[\frac{j - \phi(i)}{k}\right] e_i
\]

Under the identification of special lattices with integral points of \(\mathbb{R}^n\), the lattice \(L_j\) corresponds to the sequence \(i \mapsto \left[\frac{j - \phi(i)}{k}\right]\). Now we have the
\[ \sum_{j=0}^{k-1} \left[ j \phi(i)/k \right] = -\phi(i) \]

hence the point \((-\phi(1)/k, \ldots, -\phi(n)/k)\) is the average of the integral points of \(\mathbb{R}^n\) corresponding to the lattices \(L_0, \ldots, L_{k-1}\). This is an interior point of the simplex in \(\mathbb{R}^n\) (for the triangulation \(\mathbb{R}^n = \left\{(t) \mid \sum t_i = 1 \right\}\)) having these lattices for vertices.

Next note that the functions \(x_i - x_j\) from \(\mathbb{R}^n\) to \(\mathbb{R}\) maps any simplex to a simplex, so the set of hyperplanes of the form \(x_i - x_j = t \in \mathbb{Z}\) crossed in going along the line joining two points \(\xi, \eta \in \mathbb{R}^n\) is the same as for \(\xi', \eta\) where \(\xi'\) is another point with the same support at \(\xi\) (meaning \(\xi, \xi'\) are interior points of the same simplex). Consequently the point \((-\phi(1)/k, \ldots, -\phi(n)/k)\) can be replaced by any point with the same support in so far as computing \(l(\phi)\).

Next note that the \(\phi_i\) functions \(x_i - x_j\) are defined on \(\mathbb{R}^n/\mathbb{R}S\) where \(S = (1, -1, \ldots, -1)\), where I recall \(1A1 \cong \mathbb{R}^n/\mathbb{R}S\), this homeomorphism being induced by sending a special lattice \(L = \sum_{i=1}^{n} \mathbb{R}e_i\) to the class.
of \((d_1, \ldots, d_n)\) mod \(\mathbb{R}^d\). So we get the following:

**Assertion:** Given the sequence \((l_j^\phi)\) of special lattices assoc. to \(\phi: \mathbb{Z} \to \mathbb{Z}\) such that \(\phi(p+n) = \phi(p) + k\). Then, the dimension \(l(\phi)\) of the \(B\)-orbit of this sequence can be computed as follows: Choose any interior point in the simplex in \(|A| = \mathbb{R}^n/\mathbb{Z}^n\) with vertices \(l(\phi)\). The number of hyperplanes of the form \(x_i - x_j = t\), \(1 \leq i < j \leq n\), \(t \in \mathbb{Z}\) crossed in going along the straight line joining \(\xi\) to an interior point of the fundamental chamber.

Now we apply what precedes to determine the \(B\)-orbits in \(X'\) and \(X\). Given \(\sigma: L_0 < \cdots < L_{k-1}\) a \((k-1)\)-simplex of \(X'\), we can extend it to a sequence \(L_j\) by putting \(L_{i+k} = \pi^{-1}L_i\). In this way \((k-1)\)-simplices of \(X'\) become identified with sequences of lattices \((L_j)\) satisfying \(\pi L_j = L_{j-k}\) and such that \(L_0 < L_1 < \cdots < L_{k-1}\) (note that \(L_{k-1}\) can be equal to \(\pi^{-1}L_0 = L_k\)).

According to what has been shown, the \(B\)-orbit of \((L_j)\) contains exactly one sequence made up of special lattices, hence the orbit \(B\sigma^+\) contains exactly one simplex \(\overline{\sigma}\) in the apartment \(A'\). Therefore we have...
an isomorphism

\[ A' \sim B \setminus X' \]

and we get a retraction \( p : X \to A' \) by associating to \( \sigma \) the unique simplex in \( B_0 \) which is in \( A' \). This map \( p \) is essentially the one sending \( \sigma \) to the function \( \phi \) associated to \( (L_j) \).

Given a \((k-1)\)-simplex \( \sigma \) of \( X \), we have \( \sigma = \{(L_j), j \in \mathbb{Z}\} \), where \( (L_j) \) is an increasing sequence of lattices such that \( \prod L_j = L \) for all \( j \) and \( L_0 < L_1 < \ldots < L_k = \prod L_0 \). If \( (L'_j) \) is another sequence giving \( \sigma \), then \( L'_j = L_j + c \) for some integer \( c \).

Thus \((k-1)\)-simplices of \( X \) can be identified with equivalence classes of sequences \( (L_j) \) in \( S_k \) which are strictly increasing \( (L_j < L_{j+1}) \) and where sequences are considered equivalent when they coincide after a translation of the indices.

According to what's been shown, the \( B \) orbit of \( \sigma \) contains a unique simplex in the apartment \( A \). Thus

\[ A \sim B \setminus X \]

and we have a retraction \( p : X \to A' \) whose fibres are the \( B \)-orbits.
Note that these two results in boxes imply:

\[ |A| \sim B \setminus |X| \quad \text{and} \quad |A| \sim B \setminus |X| \]

In effect the map \( |A| \rightarrow B \setminus |X| \) is onto, and \( p \) induces a map \( B \setminus |X| \rightarrow |A| \) which is a retraction for the preceding map.

Given \( x \in |X| \) let \( \sigma \) be the support of \( x \) so that \( x = \sum_{v \in \sigma} \lambda_v v \) where \( \lambda_v > 0 \) and \( \sum_{v \in \sigma} \lambda_v = 1 \). Then the stabilizers of \( x \) and of \( \sigma \) in \( B \) are the same, because any \( b \in B \) such that \( b \sigma = \sigma \) satisfies \( b v = v \) for all \( v \) in \( \sigma \).

To see this I can apply the retraction \( p: X \rightarrow A \); or I can argue that if \( b \) preserves \( Lj \), then \( bLj = Lj \) because \( \det(b) = 1 \) and either \( bLj \subset \sigma \subset Lj \).

From the assertion on page 19, the dimension of the \( B \)-orbit of a point \( x \) is the number of hyperplanes in the apartment crossed in going along the line joining \( px \) to an interior point of the fundamental chamber.
Chambres in $A$, the Weyl group, and the Bruchet decomposition:

Let $N' = \sum_n \times (F^*)^n$, $\mathcal{H}' = (R^*)^n$ be considered as subgroups in $\text{GL}_n(F)$ in the standard way. Put $W' = N'/\mathcal{H}' = \sum_n \times \mathbb{Z}^n$. $N'$ is the subgroup of $\text{GL}_n(F)$ preserving the splitting $V = \text{Fr}_1 \oplus \cdots \oplus \text{Fr}_n$ into lines. Hence $N'$ acts on special lattices. Since $\mathcal{H}'$ carries a special lattice into itself, we get an action of $W'$ on special lattices, hence an action of $W'$ on $A'$ and $A$.

Recall we have identified increasing sequences $(L_j)$ of special lattices such that $\pi L_j = L_j - k$ with functions $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying $\phi(p+n) = \phi(p) + k$. Hence $W'$ acts on such functions $\phi$.

Let $\alpha \in N'$. There is a permutation $\tau \in \Sigma_n$ and a sequence of integers $t_1, \ldots, t_n$ such that

$\alpha(e_i) = u_i \prod e_{\sigma_i}$

$u_i \in R^*$.

Hence $\alpha$ determines a map $\nu: \mathbb{Z} \rightarrow \mathbb{Z}$ by $\alpha(e_p) \in R^* e_{\nu(p)}$. The map $\nu$ is an isomorphism satisfying $\nu(p+n) = \nu(p) + n$. It depends only on the image of $\alpha$ in $W'$. 
On one hand we have identified $W'$ with $\Sigma_n \times \mathbb{Z}^n$ as follows: $w$ corresponds to $(\sigma, t)$, $\sigma \in \Sigma_n, t \in \mathbb{Z}^n$, iff for any $\alpha \mapsto w$ we have

$$\alpha(e_i) \in \mathbb{R}^* \cap -t_i e_{\sigma_i}$$

On the other hand we have identified $W'$ with permutations of $\mathbb{Z}$ commuting with translation by $n$ as follows: $w$ corresponds to $v$ iff for any $\alpha \mapsto w$ we have

$$\alpha(e_p) \in \mathbb{R}^* e_{\exp(p)}$$

Thus $v$ corresponds to $(\sigma, t)$ iff

$$v(i) = \sigma i + t_i \cdot v$$

Suppose now $(L_j)$ is an increasing sequence of special lattices corresponding to $\phi_j$. This means $\phi(p) = j$ iff $e_p$ is in the natural basis of $L_j/L_{j-1}$. Apply $\alpha$ to get a new sequence $(\alpha L_j)$ and let $\psi$ be the corresponding function. Then $\phi(p) = j$ iff $e_p$ is a basis elt of $L_j/L_{j-1}$ iff $e_p$ is a basis elt of $\alpha L_j/\alpha L_{j-1}$ iff $\psi(e_p) = j$. Thus $\phi = \psi \cdot v$ or $\psi = \phi \cdot v^{-1}$. Therefore we have:

Assertion: If we identify $W'$ with perms. of $\mathbb{Z}$ commuting with translation, then the action of $W'$ on functions $\phi: \mathbb{Z} \to \mathbb{Z}$ such that $\phi(p+n) = \phi(p) + k$ is
given by $v(p) = \phi \circ v^{-1}$.

Consider the set of chambers in $A'$. These are classified by functions $\phi$ which are isos. such that $\phi(p+n) = \phi(p)+n$. Consequently $W'$ acts simply transitively on the set of chambers in $A'$.

Let $B' = \{g \in \text{GL}_n(F) | g_{ij} \in \mathbb{R} \text{ if } i \neq j\}$. $B'$ is the stabilizer in $\text{GL}_n(F)$ of the fundamental chamber $V_0 < \cdots < V_n$ of $A'$. Thus $\text{GL}_n(F)/B'$ is essentially the set of chambers of $X'$. Since each $B'$-orbit on $X'$ meets $A'$ at exactly one point, $B' \backslash \text{GL}_n(F)/B'$ may be identified with the set of chambers in $A'$. Thus we get the Bruhat decomposition

$$W' \sim \to B' \backslash \text{GL}_n(F)/B'$$

Next let's discuss the modifications for $\text{SL}_n(F)$. Put $N = N' \cap \text{SL}_n(F)$, $H = H' \cap \text{SL}_n(F)$, and $W = N/H$.

Then $W \subset W'$. Let $v \in W$ be interpreted as an auto of $\mathbb{Z}$ commuting with translation by $n$, and also as a pair $(\sigma, t)$, so that $v(x) = \sigma x + t \cdot n$.

If $x \mapsto v$ with $x \in N$, then the lattice

$$<R^n = \sum R \Pi t_i e_{\sigma_i}$$

must have same volume as $R^n$, hence $\sum t_i = 0$. 
Conversely given \( D \leftrightarrow (s, t) \) with \( \sum t_i = 0 \), the monomial matrix sending \( e_i \) to \( \Pi^{-i} e_0 \) has determinant \( \text{sgn}(s) \), and we can multiply by a diagonal matrix: \( e_i \rightarrow t e_i \) to get an element of \( W \) mapping to \( v \). Thus

\[
W = \sum_n \times \{ t \in Z^n \mid \sum t_i = 0 \}.
\]

I can think of elements of \( W \) as being perms of \( Z \) commuting with \( p \rightarrow p + n \) which preserve volume.

\[
p + N = \text{card} \{ p \mid -N < a, \nu(a) < p \}
\]

for \( N \) large enough so that \( a + N \Rightarrow \nu(a) < p \), \( a < p \).

Consider chambers in \( X \). These are chains \( a \sim \{ l_j \} \), \( \Pi l_j = l_j - n \), \( \text{dim}(l_j/l_{j-1}) = 1 \). We can normalize the indexing of the chain so that \( l^n j = l^n j' \) for each \( j \). In this case the function \( \phi \), describing the \( B \)-orbit of \( T \) is a volume-preserving auto of \( Z \).

As before the \( W \)-action on \( \phi \) will be given by \( \nu(\phi) = \phi \circ \nu^{-1} \), hence \( W \) acts simply-transitively on the chambers in \( A \). As in the case of \( \text{SL}_n \) we get the Bruhat decomposition:

\[
W = B / \text{SL}_n F / B.
\]

(This map is induced by inclusion of \( N \) in \( \text{SL}_n F \).)
Link of a simplex in $X$.

Let $\sigma$ be a $(k-1)$-simplex of $X$, say $\sigma = \{L_j\}$, where $(L_j)$ is an increasing sequence $\Rightarrow L_j = L_{j-k}$ and $L_0 < L_1 < \ldots < L_k = L_0$. A vertex in $\text{Link}(\sigma)$ is of the form $c(h)$ where $\exists j$ such that $L_{j-1} < L < L_j$, and $\exists$ we can suppose $j < k$.

A simplex $\tau$ in $\text{Link}(\tau)$ is a simplex of $X$ such that each vertex is in $\text{Link}(\tau)$, because any two lattices in our $\tau$ are comparable.

If $W$ is a vector space over $R/\pi R$, let $T(W)$ denote the simplicial complex associated to the poset of proper subspaces of $W$. Any simplex $\tau$ of $\text{Link}(\tau)$ is a join $\tau = \tau_0 \ast \tau_1 \ast \ldots \ast \tau_k$, where $\tau_j$ is either empty or a simplex of $T(L_j/L_{j-1})$, and not all $\tau_j$ are empty. Thus

$$\text{Link}(\tau) = T(L_1/L_0) \ast \ldots \ast T(L_k/L_{k-1}).$$

(T$_j$/L$_{j-1}$) is a bouquet of $(\dim(L_j/L_j)-2)$-spheres, hence this join is a bouquet of spheres of dimension $(\dim(L_1/L_0)-2) + 1 + (\dim(L_2/L_1)-2) + 1 + \ldots + 1 + (\dim(L_k/L_{k-1})-2)$

$$= n - 1 - k.$$

The local homology $H_*(X, X - \{x\})$ is therefore concentrated in degree $n-1$.\)
Suppose \( \sigma \) is a \((k-1)\)-simplex of \( X \) given by \( (l_i^j) \), and let \( \mathcal{M}_j \) be a chamber containing \( \sigma \). Let \( \phi, \psi \) be the functions associated to \( \sigma \) and \( \mathcal{M}_j \) respectively. Thus \( \phi(p) = j \) means that the quotient \( V_p/V_{p-1} \) of the composition series \( \{V_p\} \) appears in \( l_j/l_{j-1} \) in the sense of the Jordan-Hölder-Schreier theorem.

Let \( R_\phi = \{ (a,p) \mid a < p, \phi(a) > \phi(p) \} \). Then \( \mathbb{Z} \) acts on \( R_\phi \) by \( (a,p) \mapsto (a+t, p+t) \) and we have

\[
\ell(\phi) = \text{card } \mathbb{Z} \backslash R_\phi
\]

\[
= \text{card } \{ (a,p) \mid a < p, 1 \leq s \leq n, \phi(a) > \phi(p) \}
\]

\[
= \text{card } \{ (a,p) \mid a < p, \phi(a) > \phi(p), 1 \leq \phi(p) < k \}
\]

We put \( \ell(\sigma) = \ell(\phi) \); \( \ell(\sigma) \) is well-defined since \( \phi \) is determined up to translation by \( \sigma \).

Now since \( \{M_j^\phi \} \) refines \( \{L_j^\phi \} \), it is clear that \( \phi(a) > \phi(p) \Rightarrow \psi(a) > \psi(p) \). Thus \( R_\phi \subset R_\psi \); so \( \ell(\sigma) \leq \ell(\psi) \). If \( \psi(a) > \psi(p) \), then \( \phi(a) > \phi(p) \), so \( R_\psi - R_\phi \) consists of pairs \( (a,p) \) with \( \phi(a) = \phi(p) \). Thus

\[
\ell(\sigma) - \ell(\phi) = \sum_{j=1}^{k} \text{card } \{ (a,p) \mid \phi(a) = \phi(p) = j, a < p, \psi(a) > \psi(p) \}.
\]
Let us consider the \( j \)-th term of this sum. This is the number of pairs in \( \varphi^{-1}(j) \) whose order is reversed by \( \varphi \). Translating the indexing of the sequence \( (M_j) \), I can suppose that \( M_0 = L_{j-1} \) whence I get a composition series:

1) \( L_{j-1} = M_0 < M_1 < \ldots < M_k = L_j \)

and \( \varphi^{-1}(j) = \{ \varphi^{-1}(a) \mid 1 \leq a < b \leq k \} \). Then \( S_j = \text{card } \{ (a,b) \mid 1 \leq a < b \leq k \text{ and } \varphi^{-1}(a) > \varphi^{-1}(b) \} \). Note that the fundamental chamber \( [V_p] \) induces a composition series:

2) \( V_p \cap L_j = L_{j-1} \)

in the layer \( L_{j-1} \subset L_j \).

When the two composition series (1) and (2) are compared via Jordan–Hölder, the \( p \)-th quotient of (2) is isomorphic to the \( \varphi(p) \)-th quotient of (1).

Suppose \( \ell(a) = \ell(b) \), that is, \( a < b \Leftrightarrow \varphi^{-1}(a) < \varphi^{-1}(b) \). One sees easily then that the series (1) and (2) coincide, so we obtain:

**Lemma:** Given a simplex \( \sigma = \{ L_j \} \) then is a unique chamber \( \mathcal{T} \) containing \( \sigma \) such that \( \ell(\sigma) = \ell(\mathcal{T}) \). \( \mathcal{T} \) is the chamber obtained by taking the filtration in each layer \( L_j / L_{j-1} \) induced by \( [V_p] \).
Suppose next that \( l(\sigma) < l(\tau) \). Then for some \( j \), the \( \Psi \)-ordering on \( \phi^{-1}(j) \) is different from the usual ordering. Supposing without loss of generality that (i) hold, we see there exists an integer \( i \) with \( 1 \leq i, i+1 \leq s \) such that \( \psi^{-1}(i) > \psi^{-1}(i+1) \). It is easily seen that if \( \mathcal{P} \) is the panel (codimension 1 face) of \( \gamma \) obtained by deleting \( cl(M_i) \) from \( \gamma \), then \( l(\mathcal{P}) < l(\gamma) \).

**Lemma 2:** If \( \sigma \) is contained in a chamber \( \tau \) with \( l(\sigma) < l(\tau) \), then there exists a panel \( \mathcal{P} \) of \( \tau \) such that \( \mathcal{P} \supset \sigma \) and \( l(\mathcal{P}) < l(\tau) \).

**Corollary of Lemma 1:** Let \( \sigma \subset \tau \) be as in Lemma 1. If \( \tau \) is any simplex containing \( \sigma \), then \( l(\sigma) < l(\tau) \) with equality iff \( \tau = \gamma \).

For if \( \gamma' \) is the unique chamber containing \( \tau \) with \( l(\gamma') = l(\gamma) \), then \( \gamma' \subset \gamma \) so \( l(\sigma) \leq l(\gamma') = l(\gamma) \). If \( l(\sigma) = l(\gamma) \), then \( l(\sigma') = l(\sigma) \Rightarrow \gamma' = \gamma \).

Suppose now \( \gamma \) is a chamber \( \gamma = \{ M_j \} \) associated to the isomorphism \( \phi \). Let \( \mathcal{P}_i \) be the panel of \( \gamma \) obtained by deleting \( cl(M_i) \) from \( \gamma \).
for $i = 0, \ldots, n-1$. Then $l(p_i) = l(y)$ iff $\varphi^{-1}(i) < \varphi^{-1}(i+1)$, otherwise $l(p_i) = l(y) - 1$.

If $l(p_i) = l(y)$ for all panels of $\Omega$, then

$\varphi^{-1}(0) < \varphi^{-1}(1) < \ldots < \varphi^{-1}(n) = \varphi^{-1}(0) + n$

which is possible only if $\varphi^{-1}(i) = j + \varphi^{-1}(0)$. In this case $\varphi$ preserves ordering on $\mathbb{Z}$, so $l(y) = 0$, which means $y \in A$ since its $B$-orbit is trivial, hence $y$ is a fundamental chamber. Finally if $l(p_i) < l(y)$ for all panels of $\Omega$, then we would have

$\varphi^{-1}(0) > \ldots > \varphi^{-1}(n) = \varphi^{-1}(0) + n$

which is impossible. Thus we have

Lemma 3: Given a chamber $\Omega$, there exist panels of the same length. There exist panels of smaller length iff $\Omega \neq$ fundamental chamber.

Contractibility of $X$:

Let $F_n X$ be the subcomplex consisting of the closures of simplices of length $\leq n$; it is the union of all the chambers of length $\leq n$. We show by induction that $F_n X$ is contractible; thus as $X = \bigcup F_n X$, this will show $X$ is contractible. $F_n X$ is the closure of the fundamental chamber which is an $(n-1)$-simplex hence is contractible.
$F_n X$ is the union of $F_{n-1} X$ and each chamber of length $n$. By Lemma 1 any two chambers of length $n$ intersect inside $F_{n-1} X$. It suffices therefore to show that for any chamber $F_n$ that $|X/n| F_{n-1} X | = \{ \sigma \in S \mid \ell(\sigma) \leq n-1 \}$ as a strong deformation retract. But by Lemmas 2 & 3 this subcomplex of $|X|$ is a union of panels, not all the panels of $X$, so this point is okay.

Further topics to be explored.

1) Case of a field $F$ with maximal order $R$ and $R^{\circ}$ unique maximal ideal in $R$

2) $W$ generated by reflections, $W$ is a Coxeter group exchange condition

3) $B \setminus \hat{G} / \hat{P} \sim W / \hat{W} \quad I \subseteq S$
Scattering theory is concerned with the following situation. A Hilbert space $\mathcal{H}$ is given with two 1-parameter unitary groups $U_0(t)$ and $U(t)$. $U_0(t)$ is understood, and $U(t)$ is a perturbation of $U_0(t)$ which one wants to understand. The scattering operator $S$ arises as follows: One starts with $x \in \mathcal{H}$ and lets $J_x$ be the element of $\mathcal{H}$ such that $U_0(t)x$ and $U(t)J_x$ are asymptotic as $t \to -\infty$ (assuming this exists). Then $U(t)J_x$ will be asymptotic to $U_0(t)(J_x^*J_x)$ for $t \to +\infty$. One sets $S = J_x^*J_x$. The problems of the theory consist in proving these operators are well-defined, etc. (see Kato's address at NICE congress.)

I propose to discuss a simplified example of the theory in order to understand the basic phenomena. First I consider the discrete case: where one is given two unitary operators $U_0, U$ on $\mathcal{H}$. I suppose $U_0$ given explicitly as multiplication by $z$ on $\mathcal{H} = L^2(S^1)^n$. Put $W = Ce_1 + \ldots + Ce_n \subset \mathcal{H}$. The unitary operator $U$ will be required to coincide with $U_0$ on $\mathcal{H}$ for $|z|$ sufficiently large. Thus $U = U_0\Theta$ where $\Theta$ is a unitary operator on the finite dimensional space.
extended by the identity on the orthogonal complement.

Let \( \Delta = \mathbb{C}[z, z^{-1}] \), \( \Delta_\pm = \mathbb{C}[z^{\pm 1}] \); let \( D_\pm \) be the closure of \( \Delta_\pm \) in \( L^2(S^1) \). Note that \( U \) carries the dense subspace \( \Delta^n \subset \mathcal{H} \) into itself. (Thus the situation at hand is very algebraic).

For large \( N \), \( U = U_0 \) on \( z^{-N} \Delta_\pm \), so \( U(z^{-N} \Delta_\pm) = z^{-N+1} \Delta_\pm \). Taking perpendicular spaces (in \( \Delta \)), one gets \( U^{-1}(z^{-N+1} \Delta_\pm) = z^{-N} \Delta_\pm \) or \( U(z^{-N} \Delta_\pm) = z^{-N+1} \Delta_\pm \). Put \( L_2 = z^{-N} \Delta_+ \) and \( L_1 = z^{N+1} \Delta_+ \); these are stable under \( U \).

Let \( P \) be the orthogonal projection of \( L_2 \) onto \( Y = L_2 \oplus L_1 = \bigoplus z^i \mathbb{C}W \). The operator \( PU \) is a contraction operator on \( Y \).

Let \( v \in Y \) be an eigenvector for \( PU \) with eigenvalue \( \lambda \) of absolute value 1. Then as \( P \) is a projection, for to have the same norm as \( Uv \) means that \( PUv = Uv \); hence \( v \) is an eigenvector for \( U \).
Let $v$ be an eigenvector for $U$ in $\mathcal{H}$ with eigenvalue $\lambda \in S^1$. If $v = \Sigma v_i$ with $v_i \in Z^iW$, then for $i > N$, $uv$ has the projection $v_i$ in $Z^{i+1}W$, hence $Z v_i = \lambda v_{i+1}$. This implies $\|v_i\| = \|v_{i+1}\| = \ldots$, which is impossible unless $v_i = 0$ for $i > N$. Similarly $v_i = 0$ for $i < -N$. Thus any eigenvector of $U$ in $\mathcal{H}$ is contained in $Y = \bigoplus_{-N \leq i \leq N} Z^iW$. And $PLv = \sum_{-N \leq i \leq N} v_i Z^iW$.

Let $T = \text{subspace of } Z$ spanned by the eigenvectors of $U$. It is the subspace of $Z$ consisting of the eigenvectors of $Z$ whose eigenvalues are on $S^1$. Put $\mathcal{H} = \mathcal{H} \oplus T$, $L_2 = L_2 \oplus T$. Then on $L_2 \oplus L_1$, $Z$ has all eigenvalues inside $S^1$.

Let $p$ be the minimal polynomial of $Z$ acting on $L_2 \oplus L_1$. The roots of $p$ are inside of $S^1$. Since $p(Z) = 0$, we have

$$p(U) L_2 \subseteq L_1.$$
where the \( S_i \in \text{End}(W) \).

Now let us compute the scattering operator \( S \). First we note that for any \( x \in \Delta^n \), \( U^{-t} U_0^t x \) is constant for \( t \ll 0 \) and \( t \gg 0 \), so we get operators on \( \Delta^n \) defined by

\[
J_\pm x = \lim_{t \to \pm \infty} U^{-t} U_0^t x
\]

such that \( U J_\pm = J_\pm U \).

Since \( ||J_\pm x|| = ||x|| \) and \( \Delta^n \) is dense in \( \mathcal{H} \), these operators extend to isometries on \( \mathcal{H} \); and the formula (2) holds for any \( x \in \mathcal{H} \). (Proof: Given \( \varepsilon > 0 \), let \( y \in \Delta^n \) be such that \( ||x-y|| < \varepsilon \). Then

\[
|| U^{-t} U_0^t x - U^{-s} U_0^s x || \leq || U^{-t} U_0^t (x-y) || + || U^{-s} U_0^s (x-y) ||
\]

\[
< 2\varepsilon
\]

for \( s, t \) sufficiently close to \( \pm \infty \).

Now wish to show there is an operator such \( \text{Id} = \sum \text{Id}_i + \sum S_i \) for all \( x \in \mathcal{H} \).

Given \( w \in W \), let's try to compute \( Sw \). Then

\[
J_- w = U^{N} \overline{z} - Nw
\]

so applying 1) we get
\[ p(u) J_w = U^N \sum_{0 \leq i < m} z^{N+i} s_i(u) \]

\[ = J_+ \left( \sum z^{2N+i} s_i(u) \right) \]

But because \( p \) has no roots on \( S^1 \), \( p(u) \) is an isomorphism of \( \mathbb{H} \), so

\[ J_w = J_+ \left( \frac{\sum z^{2N+i} s_i}{p(z)} w \right). \]

Actually it might be clearer to argue that we have

\[ J_+ p(z) w = J_+ \left( \sum z^{2N+i} s_i \right) w. \]

Applying \( U^k \) to both sides of this formula we see it holds with \( w \) replaced by \( z^k w \).

Hence one has for any \( x \in \mathbb{H} \) that

\[ J_+ x = J_+(Sx) \]

where \( S \) is multiplication by the matrix function \( \frac{1}{p(z)} \sum z^{2N+i} s_i \). Thus we see the scattering operator exists. Because things are symmetric with respect to \( U, U_0 \) versus \( U^{-1}, U_0^{-1} \) the preceding argument can be modified so as to show that there exists \( S' \) with \( J_+ S' = J_+ \). Thus we have that
$J_+ \mathcal{H} = J_- \mathcal{H}$ and that $S$ is unitary.

Since $J_+ = \text{id}$ on $L^1$, $J_+ \mathcal{H} \supset L^1 \ominus p(u) L^2$, so $J_+ \mathcal{H} = L^2 \oplus L^2$. It follows that $J_+ \mathcal{H} = J_- \mathcal{H} = \mathcal{H}' = \text{orthogonal complement of } T$.

---

Here is a simple way to think of the scattering operator $S$. Suppose that the perturbation is supported in $u W \otimes \cdots \otimes z^{NW}$ $N \geq 1$ in the following sense:

Thus I want $U_0 = U^{-1}$ on $z^i W$, $i < 0$.

$U_0 = U$ on $z^i W$, $i \geq N$.

whence $J_+ = \text{id}$ on $u^{-N} D^+_+$.

$J_- = \text{id}$ on $D^-_-$.

Let us now consider the $U$-path $U^t w$ as it moves through the scatterer. As soon as $U^t w$ has a component in $z^{NW}$, this component translates along without change. Thus
$U^0 w = w$

$U^1 w = \text{elt in box} + z^{N-1} \phi_{N-1}(w) + z^{N-2} \phi_{N-2}(w) + \cdots + z^1 \phi_1(w)$

$U^2 w = \text{elt in box} + z^{N} \phi_{N-2}(w) + z^{N+1} \phi_{N-1}(w)$

$U^3 w = \text{elt in box} + z^{N} \phi_{N-3}(w) + z^{N+1} \phi_{N-2}(w) + z^{N+2} \phi_{N-1}(w)$

$S w = \lim_{t \to +\infty} U_0^{-t} U t w$

$= \sum_{i \leq N-1} z^i \phi_i(w)$

Notation from Schweber: The generalized eigenvectors for $U_0$ are of the form

$\delta_a(z) w = \sum_{i \in \mathbb{Z}} a^{-i} z^i w$

where $a \in S^1$, $w \in W$, and $\delta_a$ is a $S^1$-function at the point $a$. One has

$U_0(\delta_a(z) w) = a \cdot \delta_a(z) w$. 

This operator $S$ transforms it into a generalized eigenvector for $U$:

$\psi_a w = \sum_{i \in \mathbb{Z}} a^{-i} z^i w$
The operators $J_\pm$ transform this eigenvector for $U_0$ into ones for $U$ with the same eigenvalue $\lambda$

$$J_-(\delta_a \omega) = \sum a^{-i} U^i \omega$$
$$J_+(\delta_a \omega) = \sum a^{-i} U^i J_\omega.$$

The scattering operator $S$ applied to $\delta_a \omega$ is

$$S(\delta_a \omega) = \sum_i \delta_i S(\omega) \varphi_i(\omega)$$

$$= \delta_a \cdot S(a) \omega$$

where $S(a) = \sum a^i \varphi_i \in \text{End}(W)$. We have

$$J_-(\delta_a \omega) = J_+(S \delta_a \omega) = J_+(\delta_a S(a) \omega)$$

so we get the following interpretation for the scattering operator:

Take the generalized eigenfunction of $U$ of the form $J_-(\delta_a \omega) = \sum a^{-i} U^i \omega$ which has the component $a^{-i} \omega^i$ in degrees $\leq 0$. Then its components in degrees $\geq N$ are $a^{-i} \omega^i (S(a) \omega)$, where $S(a)$ is the value of the scattering matrix at $z=a$. 
Let's reconcile the two expressions for $S$ given on pages 5 and 7. Recall the picture

\[
\begin{array}{c}
\xrightarrow{\text{W}} \quad \xrightarrow{\text{z}^N\text{W}}
\end{array}
\]

It is clear from this picture that

\[ T \subset z^W \oplus \cdots \oplus z^{N-1}W. \]

We take \( L_1 = z^N\Delta^+_1 \), \( L_2 = \Delta^+_2 \), and let \( p(t) \) be the minimal polynomial of \( z \) on \( L_2 \ominus (L_1 \oplus T) \). As \( W \subset L_2 \ominus (L_1 \oplus T) \), we have \( p(u)W \subset L_1 \), hence

\[
p(u)z^j = \sum g_{ji}(u) z^N e_j, \quad g_{ji} \in \mathbb{C}[u].
\]

By an analogous argument there is a poly \( q(t) \) with roots inside \( S^1 \) such that

\[
q(u^{-1})(z^N e_j) = \sum h_{ji}(u^{-1}) e_j, \quad h_{ji} \in \mathbb{C}[u^{-1}].
\]

Applying \( J^*_+ \) to (*) we get

\[
p(z) J^*_+ e_i = \sum g_{ji}(z) z^N e_j
\]

or

\[
S(z) e_i = \sum \alpha_{ji}(z) z^N e_j
\]

where \( \alpha_{ji}(z) = \frac{g_{ji}(z)}{p(z)} \) is a rational function with poles inside \( S^1 \). The inverse of \( (\alpha_{ji}) \) is the matrix \( \frac{h_{ji}(z^{-1})}{\xi(z^{-1})} \) which has \( \xi \) poles.
outside of $S^1$. Thus the scattering matrix $S(z)$ is a rational function of $z$, holomorphic on $S^1$ and its exterior while $S(z)^{-1}$ is holomorphic on $S^1$ and the interior. ($0$ and $\infty$ are excluded.)

The above argument shows that the scattering matrix $S(z)$ is such that the matrix $U^{-N} S(U) = \alpha(U)$ transforms the sequence $\{z^n e_j\}$ into the sequence $e_i$. In fact looking at $H^\oplus T$ as isomorphic to $L^2(S^1)^n$ with $z \leftrightarrow U$ and $e_i \leftrightarrow z^n e_i$, the matrix $\alpha$ is a scattering matrix carrying the outgoing space $L^+_1 = z^N D^+_n$ into the outgoing space $L^+_2 \oplus T$. Thus I know that

$\text{degree}(\det \alpha: S^1 \to S^1) = \text{index of } L^+_2 \oplus T \text{ wrt } L^+_1$

$= \dim (L^+_2 \oplus T \oplus L^+_1)$

$= nN - \dim T$.

So

\[\text{degree} (\det S: S^1 \to S^1) = \dim T\]

(This degree is the number of zeroes minus poles for $\det(S)$ in the unit circle.)
Examples: Take \( n = 1 \).

1) \[ \ldots \rightarrow \begin{array}{c} z^{-1} \ 1 \ z \ z^2 \end{array} \rightarrow \begin{array}{c} \lambda \end{array} \rightarrow \begin{array}{c} \vdots \end{array} \rightarrow \begin{array}{c} 1 \ \vdots \ z \ \vdots \ z^2 \end{array} \rightarrow \]

Thus \( U1 = \lambda z \)

where \( \lambda I = 1 \). Then we have

\[ \begin{align*}
U1 &= \lambda z \\
U2 &= \lambda z^2 \\
\vdots & \\
U^k &= \lambda z^k \\
\therefore S(z) &= \lambda.
\end{align*} \]

2) \[ \ldots \rightarrow \begin{array}{c} z^{-1} \ 1 \ z \ z^2 \end{array} \rightarrow \begin{array}{c} \vdots \end{array} \rightarrow \begin{array}{c} Q \end{array} \rightarrow \begin{array}{c} \vdots \end{array} \rightarrow \begin{array}{c} z \ z^2 \end{array} \rightarrow \]

Then

\[ \begin{align*}
U1 &= z^2 \\
U2 &= z^3 \\
\vdots & \\
U^k &= z^{k+1} \\
\therefore S(z) &= z.
\end{align*} \]

3) Let \( \Theta z = (\cos \alpha) z - (\sin \alpha) z^2 \)
\( \Theta z^2 = (\sin \alpha) z + (\cos \alpha) z^2 \)
\( \Theta z^m = z^m \) if \( m \neq 1, 2 \).

and put \( U = \Theta U_0 \).

\[ \begin{align*}
U1 &= \Theta z = (\cos \alpha) z - (\sin \alpha) z^2 \\
U^2 &= \Theta z^2 = (\sin \alpha) z + (\cos \alpha) z^2 \\
Uz^2 &= \Theta z^3 = z^3 \\
U^2z^2 &= (\cos \alpha) \left[ (\sin \alpha) z + (\cos \alpha) z^2 \right] - \sin \alpha z^3 \\
&= \cos \alpha \sin \alpha z + \cos^2 \alpha z^2 - \sin \alpha z^3
\end{align*} \]
\[ U^1 = \cos^2 \alpha \sin \alpha \sin z - \cos \alpha \sin^2 \alpha z^2 + \cos^2 \alpha z^4 - \sin z^6 \]

\[ U^31 = (\cos \sin \alpha \sin z + \cos \alpha \sin z^2) + \cos^2 \alpha z^3 - \sin z^6 \]

\[ U^41 = \cos \sin^3 \alpha z + \cos^2 \alpha \sin^2 \alpha z^2 + \cos^2 \sin \alpha z^3 + \cos^2 \alpha z^4 - \sin z^6 \]

\[ U^51 = \cos \sin^4 \alpha z + \cos^2 \alpha \sin^3 \alpha z^2 \]

Thus

\[ S = (-\sin \alpha) z + \cos^2 \alpha \left(1 + z^{-1} \sin \alpha + z^{-2} \sin^2 \alpha + \cdots\right) \]

\[ = (-\sin \alpha) z + \frac{\cos^2 \alpha}{1 - z^{-1} \sin \alpha} \]

\[ = \frac{\sin^2 \alpha - (\sin \alpha) z + \cos^2 \alpha}{1 - z^{-1} \sin \alpha} \]

\[ S = \frac{1 - z \sin \alpha}{1 - z^{-1} \sin \alpha} \]