April 18, 1974

Γ' subgroup of finite index with Γ without p'-torsion. $Y = \{\sigma \mid \Gamma'_\sigma \neq 1\}$; Y is a subcomplex of X. For each σ in Y I can consider the subcomplex

$$C_\sigma = T(V)^{\Gamma'_\sigma}$$

of subspaces invariant under Γ'_σ. This subcomplex is contractible. (see below)

There seem to be many ~~ways to maps Y to T(V)~~ maps $f: Y \longrightarrow T(V)$ with the carrier $\sigma \mapsto C_\sigma$, i.e. such that $f(\sigma)$ is in the open star of C_σ. Examples:

i) $f(\sigma) = V^{\Gamma'_\sigma}$

ii) $f(\sigma) = \sum_{E \in \sigma} V^{\Gamma'_E}$

also can take the invariant part of the first slope of E.

Let G act on a finite dimensional vector space V, and consider the subcomplex $T(V)^G$ of G-invariant subspaces.

More generally let V be an object in an Artinean category, and consider the ordered set T of subobjects neither 0 nor V. According to Jordan-Hölder every simplex of T is contained in a maximal simplex which has dimension $l-2$. If V is not semi-simple, then $\mathrm{soc}(V) \in T$, (socle of V), and T is contractible by

$$W \supset \underset{\neq 0}{soc(W)} \subset soc(V).$$

Otherwise V is a direct sum of primary parts V_i, and $T(V)$ = join of the $T(V_i)$ and $T(V_i)$ is isomorphic to the building belonging to a vector space over a skew-field, so one sees that $T(V)$ is a bouquet of spheres of dim. $l-2$.

<u>Question:</u> Can the notion of canonical filtration be defined for a interior point of a simplex in X?

Yes. Given a point x of $\tilde{X}$ write its barycentric coordinates

$$x = \sum t_i E_i$$

and put

$$\mu(x) = \frac{\deg(x)}{r} = \sum t_i \mu(E_i)$$

$$\deg(x) = \sum t_i \deg(E_i).$$

Then from the formulas

$$\deg(E \cap (W_1+W_2)) + \deg(E \cap W_1 \cap W_2) = \deg(E \cap W_1) + \deg(E \cap W_2) + \deg\left(\frac{E \cap (W_1+W_2)}{E \cap W_1 + E \cap W_2}\right)$$

we get

$$\deg(x \cap (W_1+W_2)) + \deg(x \cap W_1 \cap W_2) = \deg(x \cap W_1) + \deg(x \cap W_2) + \sum t_i \deg\left(\frac{E_i \cap (W_1+W_2)}{E_i \cap W_1 + E_i \cap W_2}\right)$$

So if we define $\mu_{max}(x) = \sup\{\mu(x \cap W) \mid 0 < W \leq V\}$, and have $\mu(x \cap W_i) = \mu_{max}(x)$ $i = 1, 2$, then we can conclude as before that $\mu(x \cap (W_1+W_2)) = \mu_{max}(x)$, and that $E_i \cap (W_1+W_2) = E_i \cap W_1 + E_i \cap W_2$ for all vertices of x.

So it is clear that all of the old ideas – canonical filtration, maximal slope subobjects form a lattice, – go through for an arbitrary point of $\tilde{X}$, hence also X.

I will need an estimate which will show that for any τ in $T(V)$, the set of x such that τ is part of the canonical filtration of x is open in X. Enough to do where $\tau = \{W\}$.

~~Let $x = \sum_{i=0}^{p} t_i E_i$ and put~~

$$\cancel{\alpha = \mu_{min}(x \cap W) - \mu_{max}(x / x \cap W)}$$

~~which we are assuming > 0. Assume $E_1 \supset E_2 \supset \cdots \supset E_p$ and denote this simplex by σ. A point in the open star of~~ Fix x so that W is part of the canonical filtration of x, i.e.

$$\alpha = \mu_{min}(x \cap W) - \mu_{max}(x / x \cap W) > 0.$$

Let y be a point in the open star of x and, write

$$x = \sum t_i E_i \qquad y = \sum t_i' E_i$$

where $E_0 \supset \cdots \supset E_p$, $m_\infty E_0 \subset E_p$. What I want to do is show that if the t_i' are sufficiently close to the t_i, then W is also part of the canon. filt. of y.

So choose $W_0 < W$ and $W < W_1$ such that

$$\mu_{min}(y \cap W) = \mu(y \cap W / y \cap W_0)$$

$$\mu_{max}(y / y \cap W) = \mu(y \cap W_1 / y \cap W_0)$$

Now

const and $\sum t_i = \sum t_i' = 1$

$$\mu(y\cap W/y\cap W_0) - \mu(x\cap W/x\cap W_0)$$

$$= \sum_{i=0}^{p} (t_i' - t_i)\{\mu(E_i\cap W/E_i\cap W_0) - \mu(E_0\cap W/E_0\cap W_0)\}$$

Now $p \leq r$ and $|\mu(E_i\cap W/E_i\cap W_0) - \mu(E_0\cap W/E_0\cap W_0)| \leq d_\infty$.
So [redacted] no matter how y [redacted] approaches x
$\mu(y\cap W/y\cap W_0) \longrightarrow \mu(x\cap W/x\cap W_0)$ $\geq$ [redacted] μ_{min} ([redacted] $x\cap W$).
Similarly $\mu(y\cap W_1/y\cap W_0) \longrightarrow \mu(x\cap W_1/x\cap W_0) \leq \mu_{max}(x/x\cap W)$.
So

$$\mu_{min}(y\cap W) - \mu_{max}(y/y\cap W)$$

approaches something $\geq \alpha$. Thus clear.

April 19, 1974

Question: Let x be a semi-stable point of X. Then does $\mathrm{Aut}(x)$ (mod scalars) acts faithfully on the link of x?

Write $x = \sum_0^p t_i E_i$ where $E_0 > E_1 > \cdots > E_p > E_0(-1)$, $\sum t_i = 1$, $t_i > 0$. Thus x is in the interior of the simplex $\{E_0, \dots, E_p\}$. The link of x is a suspension of the subspaces of $E_0/E_0(-1) = E_0 \otimes k(\infty)$ which refine the flag $E_0/E_0(-1) > E_1/E_0(-1) > \cdots > E_p/E_0(-1) > 0$. Thus to say that an automorphism θ of x acts trivially on the link means that θ acts trivially on the subspaces of these quotient spaces, which means that θ induces a scalar on each E_i/E_{i+1}.

Assume that θ is unipotent. Then $(\theta - 1)$ carries E_0 into E_1, E_1 into E_2, $\dots$, E_p into $E_0(-1)$. Because x is semi-stable, I believe I know that the semi-stable points of a given slope form an abelian artinean category. In particular if $W = (\theta-1)V$, then what I proved yesterday showed that $E_i \cap W = (\theta - 1)E_i$ for all i. Thus I have $E_i \cap W \subset E_{i+1}$, hence $E_i \cap W = E_{i+1} \cap W$, so

$$E_0 \cap W = E_0(-1) \cap W$$

which is possible only if $W = 0$. $\therefore \theta = 1$.

In general, if θ acts trivially on the link of x

$$\theta^{q-1} = 1.$$

Localization: A ring, S mult. system in A consisting of central regular elements, $S^{-1}A = A[S^{-1}]$, $\mathcal{P}(A,S)$ = full subcat of $\mathcal{P}_1(A)$ consisting of $M \ni S^{-1}M = 0$.

Thm. One has an exact sequence

$$\to K_1A \to K_1 S^{-1}A \to K_0\,\mathcal{P}(A,S) \to K_0A \to K_0\,S^{-1}A$$

I will now give a proof using the following generalities:

① Segal's approach to exact sequence K-theory:

Let $\mathcal{M}$ be an exact category, $F_p\mathcal{M}$ the exact cat of objects of $\mathcal{M}$ equipped with an admissible filtration of length p:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_p = M.$$

Then $p \mapsto F_p\mathcal{M}$ is a simplicial exact cat with

$$d_i(0=M_0\subset\cdots\subset M_p) = \begin{cases} (0=M_0\subset\cdots\subset \hat{M}_i\subset\cdots\subset M_p) & 0<i\le p \\ (0=M_1/M_1\subset M_2/M_1\subset\cdots\subset M_p/M_1) & i=0\end{cases}$$

(Grain of salt about subobjects and quotients).

If $p\mapsto \mathcal{C}_p$ is a simplicial cat, denote by $\mathcal{O}(p\mapsto\mathcal{C}_p)$ the corresp. fibred category over Ord.

Prop. 1: $\mathcal{O}(p\mapsto \mathrm{Iso}\,F_p\mathcal{M})$ is heq to $Q(\mathcal{M})$.

Proof. One has functor $f: \mathcal{O}(p\mapsto \mathrm{Iso}\,F_p\mathcal{M}) \to Q(\mathcal{M})$ given by

$$f(0\subset M_1\subset\cdots\subset M_p) = M_p$$

on objects. One sees f/M is equivalent to the fibred cat over Ord assoc. to the simplicial set which is the nerve of the poset of admissible subobjects of M. This nerve is contractible as the poset has an initial element.

② Waldhausen's relative K-theory:

$\mathcal{A}$ exact cat. $\mathcal{B}$ = full subcat. closed under extensions. Put $F_p(\mathcal{A}, \mathcal{B})$ = exact cat. consisting of objects of $\mathcal{A}$ equipped with an admissible filtration of length $p+1$

$$0 \subset A_0 \subset A_1 \subset \cdots \subset A_p$$

such that $A_i / A_{i-1} \in \mathcal{B}$ for $i = 1, 2, \ldots, p$. Then $p \mapsto F_p(\mathcal{A}, \mathcal{B})$ is [redacted] a simplicial exact cat. with

[redacted] $$d_i(A_0 \subset \cdots \subset A_p) = (A_0 \subset \cdots \subset \hat{A}_i \subset \cdots \subset A_p)$$

~~Prop. 2. One has a homotopy-cartesian square~~

[redacted]
$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{O}(p \mapsto F_p \\ \mathcal{A} \arrow[r] & \mathcal{O}(p \mapsto F_p(\mathcal{A}, \mathcal{B})) \end{tikzcd}$$

Prop. 2. One has a fibration (up to homotopy)

$$Q(\mathcal{B}) \longrightarrow Q(\mathcal{A}) \longrightarrow \mathcal{O}(p \mapsto Q(F_p(\mathcal{A}, \mathcal{B})))$$

Proof. [redacted] It follows from Segal theory that one has a fibration

$$Q(\mathcal{B}) \longrightarrow Q(\mathcal{A}) \longrightarrow \mathcal{O}(p \mapsto Q(\mathcal{A}) \times Q(\mathcal{B})^p).$$

where on the [redacted] right is the nerve of the bicategory obtained by [redacted] letting $Q(\mathcal{B})$ act on $Q(\mathcal{A})$ by direct sum.

Thus one needs only to observe that one has an simplicial exact functor

$$\cancel{F_p(\mathcal{a},\mathcal{B}) \to \mathcal{a}\times\mathcal{B}^p}$$

$$\mathcal{a} \times \mathcal{B}^p \longrightarrow F_p(\mathcal{a},\mathcal{B})$$

$$(A, B_1, \ldots, B_p) \longmapsto (A \subset A\oplus B_1 \subset \cdots \subset A\oplus B_1 \oplus \cdots \oplus B_p)$$

which induces heq's on the Q-cats by the exactness thm.

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③ Waldhausen's cofinality thm.

Prop. 3: $\mathcal{a}$ exact cat, $\mathcal{B}$ full subcat. closed under extensions. Assume $\mathcal{B}$ cofinal in $\mathcal{a}$, in the sense that $\forall A, \exists A' \ni A\oplus A' \in \mathcal{B}$. Then

$$\begin{cases} K_i\mathcal{B} = K_i\mathcal{a} & i \geq 1 \\ K_0\mathcal{B} \subset K_0\mathcal{a} \end{cases}$$

Now for the localization thm. ~~assume~~ Put

$\mathcal{a} = \mathcal{P}_1(A)$ $\qquad\qquad K_i\mathcal{a} = K_i A$

$\mathcal{B} = \mathcal{P}(A,S)$ $\qquad\qquad \cancel{K_0\mathcal{B} = K_0\mathcal{B}}$

$\mathcal{V}$ = full subcat. of $\mathcal{P}(S^{-1}A)$ consisting of modules of the form $S^{-1}P$ with $P \in \mathcal{P}(A)$.

then

$$K_i\mathcal{V} = \begin{cases} K_i(S^{-1}A) & i \geq 1 \\ \operatorname{Im}\{K_0A \to K_0(S^{-1}A)\} & i = 0 \end{cases}$$

and we have to establish a heq between

$$\mathcal{O}(p \mapsto QF_p(\mathcal{a},\mathcal{B})) \quad\text{and}\quad Q(\mathcal{V}).$$

in order to prove thm. 1.

Let $\mathcal{A}' = \mathcal{P}(A) \subset \mathcal{A}$, and extend previous notation by putting $F_p(\mathcal{A}';\mathcal{B})$ = full subcat of $F_p(\mathcal{A},\mathcal{B})$ consisting of $A_0 \subset \cdots \subset A_p \ni A_i \in \mathcal{A}'$.

Lemma 1: $F_p(\mathcal{A}';\mathcal{B}) \subset F_p(\mathcal{A},\mathcal{B})$ induces an equiv of K-theories.

Proof. Immediate from resolution thm. If one has

$$0 \longrightarrow \underset{\substack{\cap \\ F_p(\mathcal{A}',\mathcal{B})}}{\{A_i'\}} \longrightarrow \{A_i\} \longrightarrow \underset{\substack{\cap \\ F_p(\mathcal{A},\mathcal{B})}}{\{A_i''\}} \longrightarrow 0$$

then one has $0 \to \underset{\substack{\cap \\ \mathcal{P}(A)}}{A_i'} \to A_i \to \underset{\substack{\cap \\ \mathcal{P}_1(A)}}{A_i''} \to 0 \implies A_i' \in \mathcal{P}(A)$.

Also given $\{A_i\}$, I can find $P \twoheadrightarrow A_p$ and put $P_i = P \times_{A_p} A_i$. Then I get an exact sequence

$$0 \longrightarrow \{P_i'\} \longrightarrow \{P_i\} \longrightarrow \{A_i\} \longrightarrow 0$$

$$\begin{tikzcd} P_0' \arrow[r, phantom, "="] \arrow[d] & P_1' \arrow[d] \\ P_0 \arrow[r] \arrow[d] & P_1 \arrow[d] \\ A_0 \arrow[r, phantom, "\subset"] & A_1 \end{tikzcd}$$

(where $P_0' = P_1' = \cdots = P_p'$ in fact), where the first objects are in $F_p(\mathcal{A}';\mathcal{B})$.

Key point: The localization functor

$$\mathcal{O}(p \mapsto QF_p(\mathcal{A}',\mathcal{B})) \longrightarrow Q(\mathcal{V})$$

is a heq.

Proof: Take apart the Q-cats vertically.

$$\mathcal{O}(p \mapsto \mathcal{O}(q \mapsto {}^{\mathrm{Iso}}F_q(F_p(\mathcal{A}';\mathcal{B})))) \longrightarrow \mathcal{O}(q \mapsto \mathrm{Iso}\,F_q(\mathcal{V})).$$

identify:

$$\mathcal{O}(p \mapsto \mathcal{O}(q \mapsto \mathcal{C}_{pq})) = \mathcal{O}(p,q \mapsto \mathcal{C}_{pq})$$
$$= \mathcal{O}(q \mapsto \mathcal{O}(p \mapsto \mathcal{C}_{pq}))$$

whence one is reduced to proving that $\forall q$

$$\mathcal{O}(p \mapsto \mathrm{Iso}\, F_q(F_p(\mathcal{A}';\mathcal{B}))) \longrightarrow \mathrm{Iso}(F_q \mathcal{V})$$

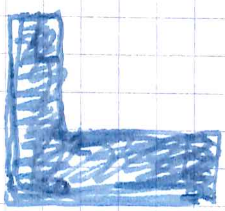
is a heq. Now $\mathrm{Iso}\{F_q(F_p(\mathcal{A}';\mathcal{B}))\}$ consists of bifiltered ~~objects~~ A-modules of finite type

$$\begin{array}{ccccccc} L_0^q & < & \cdots & \cdots & < & L_p^q \\ \cup & & & & & \cup \\ \vdots & & & & & \vdots \\ \cup & & & & & \cup \\ L_0^1 & < & L_1^1 & < \cdots < & & L_p^1 \end{array}$$

$<$ means cokernel is S-torsion

$\subset$ admissible in $\mathcal{P}(A)$

which à la Waldhausen come from the bicategory whose bimorphisms are those squares



$$\begin{array}{ccc} L_0' & < & L_1' \\ \cup & & \cup \\ L_0 & < & L_1 \end{array}$$

which are admissible monos in the cat $F_1(\mathcal{A},\mathcal{B})$ (in particular $L_0 = L_1 \cap L_0'$)

<u>Lemma 2</u>: $\mathcal{O}(p \mapsto \mathrm{Iso}\, F_q(F_p(\mathcal{A}';\mathcal{B}))) \to \mathrm{Iso}\, F_q \mathcal{V}$ is a homotopy equivalence.

<u>Pf</u>: Denoting this functor by g one sees that $g/\{V^j\}_{j=1,\dots,q}$ is equivalent to the ~~nerve~~ fibred cat/Ord belonging to the nerve of the posets $\mathrm{Lat}(\{V^j\})$

defined as follows. A lattice in $\{V^j\}$ consists of $P(A)$-submodules $L^j \subset V^j$ such that ~~[illegible]~~ ~~[illegible]~~ $S^{-1}L^j = V^j$ and such that L^j/L^i is a $P(A)$-submodule of V^j/V^i for each $0 \le i \le j \le q$. One orders $\{L^j\}$ by inclusion. Thus everything comes to proving

<u>Lemma 3.</u> Lat $\{V^j\}$ is a non-empty directed set:

Proof. By assumption I have

$$\{V^j\}: \quad V^1 \subset V^1 \oplus V^{2,1} \subset V^1 \oplus V^{3,2} \oplus \overset{V^{3,2}}{\ } \subset \cdots$$

where $V^{i,i-1} = S^{-1}P_i$ for some P_i. Put

$$L^j = P_1 \oplus \cdots \oplus P_j \subset V^j$$

so that $\{L^j\} \in \mathrm{Lat}(V)$. Then as s ranges over S, one has

$$V^j = \bigcup_{s \in S} s^{-1} L^j$$

~~whence the filtered system $s^{-1}\{L^j\}$ exhausts.~~ ~~By fin. type ... done ... since every~~. If $\{L^j\}' \in \mathrm{Lat}(\{V^j\})$, as each L^j is fin. type $\Rightarrow \{L^{j'}\} \le s^{-1}\{L^j\}$ some s. Follows that $\mathrm{Lat}(\{V^j\})$ is directed.

The preceding raises many questions:

1). The use of simplicial and bisimplicial groupoids is inherently ugly. To what extent can things be improved using bicategories?

2). What is the relation with buildings and their compactifications?

3) Take a scheme X, an eff. Cartier divisor Y, such that $X - Y = U$ is not affine. Denote by $\mathcal{V}$ the category of vector bundles on U which extend to vector bundles on X. It seems that the preceding proof might show there is an exact seq

$$\longrightarrow K_i(\mathcal{P}(X,Y)) \longrightarrow K_i(X) \longrightarrow K_i(\mathcal{V}) \longrightarrow \cdots$$

although this isn't clear immediately. In any case we have to get from $\mathcal{V}$ down to $\mathcal{P}(U)$. Thus

a) Is $\mathcal{V}$ closed under extensions in $\mathcal{P}(U)$?

b) Is

$$\begin{array}{lll} K_i\mathcal{V} = K_i U & & i \geq 1 \\ K_0\mathcal{V} \hookrightarrow K_0 U & & i = 0 \end{array} \quad ?$$

Further discussion of 3): X scheme, Y effective Cartier divisor, $U = X - Y$. Define $\mathcal{P}(X,Y) = \{F \in \mathcal{P}_1(X) \mid F|U = 0\}$. Then it seems that the arguments given above allow us to identify the "quotient of the K-theory of X by the action of $\mathcal{P}(X,Y)$" with the following simplicial groupoid:

$\mathcal{V}_q =$ full subcat. of $\mathrm{Iso}\{F_q\, \mathcal{P}(U)\}$ consisting of those

$$0 \subset V_1 \subset \cdots \subset V_q$$

which comes via restriction from something in $\mathrm{Iso}\{F_q\, \mathcal{P}(X)\}$

Note that if $\mathcal{V}$ = full subcategory of $\mathcal{P}(U)$ consisting of those V which lift to $\mathcal{P}(X)$, then $\mathcal{V}$ doesn't seem to be closed under extensions. In effect

$$\begin{array}{ccccc}
\mathrm{Ext}^1_X(E'', E') & \longrightarrow & \mathrm{Ext}^1_U(E''|U, E'|U) & & \\
\| & & \| & & \\
H^1(X, \underline{\mathrm{Hom}}(E'', E')) & \longrightarrow & H^1(U, \underline{\mathrm{Hom}}(E'', E')) & \xrightarrow{\delta} & H^2_Y(X, \underline{\mathrm{Hom}}(E'', E')) \\
& & & & \| \\
& & & & H^1_Y(Y, \underline{H}^1_Y(\underline{\mathrm{Hom}}\ldots))
\end{array}$$

and $\underline{H}^1_Y(E) = E(Y)/E$ where

$$E(Y) = \varinjlim E \otimes L^n \qquad \mathcal{O}_Y = \mathcal{O}_X/L^{-1}.$$

Next question: Can the simplicial groupoid $q \mapsto \mathcal{V}_q$ be obtained from a suitable exact category? ~~In fact~~ ~~what is the story~~ Can I even form a Q-style category out of $q \mapsto \mathcal{V}_q$ such that the functor from $\mathcal{O}(q \mapsto \mathcal{V}_q)$

to the Q-category is a homotopy equivalence?

Suppose then I want to ~~make the functor~~ define $Q(\mathcal{V})$ so that one has a functor

$$\theta(g \mapsto \mathcal{V}_g) \longrightarrow Q(\mathcal{V})$$

which, to a filtered object $0 \subset V_1 \subset \cdots \subset V_g$ in $F_g\mathcal{P}(\mathcal{U})$ which is liftable, assigns V_g. ~~Thus I should~~
Thus it appears that given $V', V \in \mathcal{P}(\mathcal{U})$ which are liftable a morphism in $Q(\mathcal{V})$ from V' to V has to be an isom. of V' with a subquotient V_2/V_1 of V where $0 \subset V_1 \subset V_2 \subset V$ is liftable. If this idea is going to work, then I have to be able to compose such morphisms, in other words, if V'' is a liftable subquotient of V', and if V' is a liftable subquotient of V, then V'' is a liftable subquotient of V. Let's try special cases.

Assume I have $0 \subset V_1 \subset V_2 \subset V$ and that $0 \subset V_1 \subset V_2$ and $0 \subset V_2 \subset V$ are liftable. Does it follow that $0 \subset V_1 \subset V_2 \subset V$ is liftable?

So I know $\exists$ $E_1 \subset E_2$ rest. to $V_1 \subset V_2$ and $E_2' \subset E$ restricting to $V_2 \subset V$. ~~There is no problem with showing that~~ replacing $E_1 \subset E_2$ by $E_1(n) \subset E_2(n)$ I can assume that $E_2 \supset E_2'$. Now define E^+ by pushout:

$$\begin{tikzcd}
0 \arrow[r] & E_2' \arrow[r] \arrow[d, phantom, "\cap" description] & E \arrow[r] \arrow[d, phantom, "\cap" description] & E/E_2' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_2 \arrow[r] & E^+ \arrow[r] & E/E_2' \arrow[r] & 0
\end{tikzcd}$$

I claim E^+ is a vector bundle on X. This is a local question on X, hence we can assume $E = E_2' \oplus E/E_2'$, whence $E^+ = E_2 \oplus E/E_2'$ is a vector bundle.

Thus I see that I can suppose $E_2 = E_2'$, whence $0 \subset E_1 \subset E_2 \subset E$ lifts $0 \subset V_1 \subset V_2 \subset V$.

In general suppose given

$$0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset V$$

and assume $(0 \subset V_1 \subset V_4 \subset V)$, $(0 \subset V_2/V_1 \subset V_3/V_1 \subset V_4/V_1)$ liftable. Can suppose $V_4 = V$ by preceding. Assume the dual assertion to above (i.e. $0 \subset V_1 \subset V$, $V_1/V_1 \subset V_2/V_1 \subset V/V_1$ liftable $\Rightarrow$ $0 \subset V_1 \subset V_2 \subset V$ liftable). Better check this: I am given $0 \subset E_1 \subset E$, $0 \subset \bar{E}_2 \subset \bar{E}$ reducing to $V_1 \subset V$, $V_2/V_1 \subset V/V_1$. I can assume that $E/E_1 \supset \bar{E}$. Then define E^+ so that $E^+/E_1 = \bar{E}$. Since $0 \to E^+ \to E/E_1 \to (E/E_1)/\bar{E} \to 0$ it follows that E^+ is in $\mathcal{P}(X)$. Thus can suppose $\bar{E} = E/E_1$, whence if E_2 is the inverse image of $\bar{E}_2$, one has $0 \subset E_1 \subset E_2 \subset E$ lifts $0 \subset V_1 \subset V_2 \subset V$.)

So it is clear to me now that one has a Q-category.

Question: $\mathcal{V}$ = full subcategory of liftable objects of $\mathcal{P}(U)$, with usual notion of exact sequence. Is $\mathcal{V}$ an exact category? ~~(Note $\mathcal{V}$ not closed under extensions in $\mathcal{P}(U)$.)~~

~~Check the axioms.~~

$$\begin{array}{c} \\ \end{array}\cancel{0 \to V_1 \to W \to V'' \to 0 \quad \text{liftable}}$$

First: Is any exact sequence of liftable objects liftable? Thus given $0 \to V' \to V \to V'' \to 0$ with V', V, V'' liftable. If E lifts V, and E' lifts V', and E'' lifts V'' we can suppose that $E' \overset{?}{\subset} E$ and that $\mathrm{Im}(E \to V'') \supset E''$ (It may be necessary to assume things worth so that $E \cap V'$ fin. type, hence contains in $E'(n)$ some n).

Wait: $V'' = \bigcup E''(n)$. Consider induced ext.

$$0 \to V' \to V \times_{V''} E'' \to E'' \to 0$$

it is represented by an element of

$$H^1(X, Hom(E'', V'))$$

$$= \varinjlim_n H^1(X, Hom(E'', E'(n))).$$

Thus it seems clear that the exact sequence is liftable. In fact before we had the question of

$$\begin{array}{ccc} H^1(X, \underline{Hom}(E'', E')) \to & H^1(U, \underline{Hom}(E'', E')) & \\ & \downarrow \delta & \\ & H^2_Y(X, F) & F = \underline{Hom}(E'', E') \\ & \| & \\ & H^1(Y, \underline{H}^1_Y(X, F)) & \\ & \| & \\ & F(\infty)/F & \end{array}$$

So we can kill this element by quasi-compactness using ~~$E' \mapsto E'(n)$~~ the embedding of E' in $E'(n)$ for some n.

Conclude that $\mathcal{V}$ = liftable bundles in $P(\mathcal{U})$ is closed under extensions, hence it is an exact category, and we have ~~hope of~~ an exact seq.

$$\longrightarrow K_i(P(X,\mathcal{U})) \longrightarrow K_i X \longrightarrow K_i \mathcal{V} \longrightarrow \cdots$$

Thus one has this:

<u>Question:</u> What is the relation between $K_i \mathcal{V}$ and $K_i \mathcal{U}$?

So it is necessary to ~~hav~~ review the proof of Waldhausen's cofinality thm. Thus assume $\mathcal{B}$ is a full subcat ~~[illegible]~~ closed under extensions in $\mathcal{A}$. One compares

$$\Omega B(\mathrm{Iso}(F_p\mathcal{A})) \supset \Omega B(\mathrm{Iso}(F_p\mathcal{B})).$$

and wants to show that they have the same zero ~~[illegible]~~ components, but that ~~[illegible]~~ one has cokernel $(K_0^+\mathcal{A}/\mathrm{Im}\, K_0^+\mathcal{B})^p$. Here $K_0^+\mathcal{A} = \oplus$ Groth grp. and one then ~~[illegible]~~ has to show $K_0^+\mathcal{A}/\mathrm{Im}\, K_0^+\mathcal{B} = K_0\mathcal{A}/\mathrm{Im}\, K_0\mathcal{B}$.

It will now be necessary to understand ~~the~~ Waldhausen's proof. ~~He starts with $Q(\mathcal{B}) \to Q(\mathcal{A})$ and forms over $Q\mathcal{A}$~~ Put $G = K_0\mathcal{A}/\mathrm{Im}\, K_0\mathcal{B}$ and $F : Q(\mathcal{A}) \to G$ the functor assoc. to ~~[illegible]~~

$$\left(A' \overset{p}{\twoheadleftarrow} A_0 \hookrightarrow A\right) \longmapsto cl(\mathrm{Ker}(p)) \in G.$$

An object of $F/*$ is (M, g) $\quad M \in \mathcal{A},\ g \in G.$

$(M, g) \longrightarrow (M', g')$ $\qquad M \overset{p}{\twoheadleftarrow} N \longrightarrow M'$

$\qquad\qquad \mathrm{Ker}\, p + g' = g.$

$\mathcal{C}$ = subcat of $F/*$ such that $g = 0$. and whose map $\ni$ $\mathrm{Ker}(p) \in \mathcal{B}$.

$$Q(\mathcal{B}) \xrightarrow{k} \mathcal{C} \xrightarrow{j} F/*$$

$k/(M, 0)$

$k(M') \dashrightarrow M$

\begin{tikzcd}
 & \mathcal{B}/\mathcal{C} \arrow[dl] \arrow[d] & \\
\mathcal{B} \arrow[r, dashed, "u"] & M
\end{tikzcd}
$\qquad \mathrm{Ker}(u) \in \mathcal{B}$

$(M', 0) \longrightarrow (M, 0)$.

$(M, g) \backslash j$. $\qquad \underline{M \overset{p}{\twoheadleftarrow} N \hookrightarrow M'}$ $\qquad \mathrm{Ker}(p) \longmapsto g$

\begin{tikzcd}
N \arrow[d] \arrow[dr, "u"] & \\
M & N' \arrow[l]
\end{tikzcd}

Thus what these deformations do is get you down to the cat $\mathcal{D}(M, g)$ consist. of $N \twoheadrightarrow M$ with kernel rep. $g \in G$, and whose maps are
\begin{tikzcd}
N \arrow[rr, two heads, "u"] \arrow[dr, two heads] & & N' \arrow[dl, two heads] \\
 & M &
\end{tikzcd}
$\ni \mathrm{Ker}(u) \in \mathcal{B}$.

For example, take $M = 0$. Then I consider all bundles N whose class in $K_0\mathcal{A}$ lifts to $K_0\mathcal{B}$, modulo epis where the kernel is in $\mathcal{B}$.

<u>Question:</u> Let $\mathcal{V} \subset \mathcal{P}(\mathcal{U})$ be as above. What can one say about the relative K-groups?

April 25, 1974.

Summary: X scheme (noetherian), Y eff. Cartier divisor, $U = X - Y$. Let $\mathcal{V}$ = full subcat of $\mathcal{P}(U)$ consisting of bundles V which lift to X; one shows $\mathcal{V}$ is closed under ext. in $\mathcal{P}(U)$. $\mathcal{P}(X, \mathcal{U}) = \{F \in \mathcal{P}_1(X) \mid \operatorname{Supp}(F) \subset Y\}$. My local. thm. gives a fibration

$$Q(\mathcal{P}(X,U)) \longrightarrow Q(\mathcal{P}(X)) \longrightarrow Q(\mathcal{V}).$$

Question: Is $\begin{matrix} K_i \mathcal{V} = K_i U & i \geq 1 \\ K_0 \mathcal{V} \hookrightarrow K_0 U & \end{matrix}$?

True when U is affine, for then $\mathcal{V}$ is cofinal in $\mathcal{P}(U)$ in the sense that for every $E \in \mathcal{P}(U)$ $\exists$ $E \oplus E' = V \in \mathcal{V}$. One can apply Waldhausen's cofinality result: $\mathcal{A}$ exact, $\mathcal{B}$ closed under extensions + full in $\mathcal{A}$, $\mathcal{B}$ cofinal in $\mathcal{A}$ $\Longrightarrow$ $Q(\mathcal{B}) \simeq$ covering of $Q(\mathcal{A})$ corresp. to the subgroup $\operatorname{Im}\{K_0\mathcal{B} \to K_0\mathcal{A}\}$ of $\pi_1 Q(\mathcal{A}) = K_0 \mathcal{A}$.

The proof reduces formally to showing the following categories $\mathcal{D}(A, g)$ are contractible. Here $A \in \mathcal{A}$, $g \in K_0\mathcal{A}/K_0\mathcal{B}$; an object of $\mathcal{D}(A,g)$ is an epi $N \overset{p}{\twoheadrightarrow} A$ $\ni$ $\operatorname{Ker}(p)$ represents g, and the maps are

$$\begin{tikzcd} N \arrow[rr, two heads, "u"] \arrow[dr, two heads, "p"'] & & N' \arrow[dl, two heads, "p'"] \\ & A & \end{tikzcd}$$

where $\operatorname{Ker}(u) \in \mathcal{B}$. If $A = 0 = g$, $\mathcal{D}(A, g)$ is the cat of objects $N \in \mathcal{A}$ whose class in $K_0\mathcal{A}$ lifts, with the action of $\mathcal{B}$ for maps.

This suggests I consider the relative K-theory of $(\mathcal{P}(U), \mathcal{V})$, and try to show that if $\mathcal{P}(U)'$ = subcategory of E whose image in $K_0 U$ lifts, then $\mathcal{V} \subset \mathcal{P}(U)'$ induces an equivalence of K-theory.

Question: Does every element of $\mathcal{P}(U)'$ have a resolution $\quad 0 \to V_1 \to V_0 \to P \to 0 \quad$ with $V_i \in \mathfrak{V}$?

<u>Example</u>: Suppose X non-singular, and Y non-singular subvar. of codim ≥ 3, $U = X - Y$. Then for any v.b. E on X I have

$$0 \to H^0_Y(X,E) \to H^0(X,E) \to H^0(U,E) \to H^1_Y(X,E) \to H^1(X,E) \to H^1(U,E) \to H^2_Y(X,E)$$

and
$$H^n_Y(X,E) \Leftarrow H^p(Y, \underline{\mathcal{H}}^q_Y(E)) \qquad \text{deg.}$$

$$H^n_Y(X,E) = H^{n-c}(Y, \underline{\mathcal{H}}^c_Y(E)) \qquad c = \operatorname{codim} Y \text{ in } X.$$

Thus when $c \geq 2$, one has $\quad H^0(X,E) \xrightarrow{\sim} H^0(U,E)$

and $c \geq 3$, ———— $\quad H^1(X,E) \xrightarrow{\sim} H^1(U,E)$.

Applying the first to $\underline{Hom}(E', E'')$, one sees that the restriction functor

$$\mathcal{P}(X) \longrightarrow \mathcal{P}(U)$$

is fully faithful, $c \geq 2$, and to the second that the image is closed under extensions, $c \geq 3$. By regularity, I know that the relative K-theory is that of Y:

$$\cdots \longrightarrow K_* Y \longrightarrow K_* X \longrightarrow K_* U \longrightarrow \cdots$$

One might ask whether there is any way to directly relate $\mathcal{P}(U)$ mod $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ in this situation.

$K_0 \mathcal{A}$. An element of this can be represented in the form $[E]-[F]$ with E, F objects of $\mathcal{A}$. One has

$$[E]-[F]=0 \quad \text{in} \quad K_0\mathcal{A}$$

when one can find exact sequences

$$0 \longrightarrow E' \longrightarrow \bar{E} \longrightarrow E'' \longrightarrow 0$$
$$0 \longrightarrow F' \longrightarrow \bar{F} \longrightarrow F'' \longrightarrow 0$$

such that $[E]-[F] = [\bar{E}]-[E']-[E''] - ([\bar{F}]-[F']-[F''])$

~~[illegible]~~ i.e. s.t.

$$E \oplus (E' \oplus E'') \oplus \bar{F} \simeq F \oplus \bar{E} \oplus (F' \oplus F'').$$

In other words $[E]=[F]$ in $K_0\mathcal{A}$, when $\exists$ an object $U = (\bar{E}\oplus\bar{F})$ and two filtrations of U:

$$0 \to E' \oplus \bar{F} \longrightarrow \bar{E}\oplus\bar{F} \longrightarrow E'' \to 0$$

$$0 \to \bar{E} \oplus F' \longrightarrow \bar{E}\oplus\bar{F} \longrightarrow F'' \to 0$$

such that $E \oplus gr_1 U \simeq F \oplus gr_2 U$. Actually ~~these are transverse filtrations~~ these two filtrations intersect nicely, in fact transversally $(E'\oplus\bar{F}) \oplus (\bar{E}\oplus F') = \bar{E}\oplus\bar{F}$.

Again: Every element of $K_0\mathcal{A}$ ~~be represented by~~ can be represented in the form $[E]-[F]$. One has $[E]-[F]=0$ if $\exists\, U$ with admissible subobjects U_1, U_2 $\ni$ $U_1+U_2=U$ such that

$$E \oplus U_1 \oplus U/U_1 \quad \text{isom. to} \quad F \oplus U_2 \oplus U/U_2.$$

$$\begin{array}{ccc} \parallel & & \parallel \\ gr_{(1)}(E\oplus U) & & gr_{(2)}(F\oplus U) \end{array}$$

Thus suppose I try first to take pairs (E,F) and to make them into a category whose maps are isomorphisms after subdivision. This should be possible, and then to obtain the good category I must localize by adding diagonally.

Return to $\mathcal{V} \subset \mathcal{P}(\mathcal{U})$. Suppose P is such that $[P] \in K_0(\mathcal{U})$ comes from $\mathcal{V}$. Thus I get a $V \in \mathcal{V}$, $U \in \mathcal{P}$ such that

$$P \oplus gr_{(1)}(U) \simeq V \oplus gr_{(2)}(U)$$

for two filtrations on U. I would like to prove, if I could, that P can be resolved by objects of $\mathcal{V}$. The problem with going any further is that I don't know anything about U. However I do know that U is a quotient of an object of $\mathcal{V}$.

April 25, 1974. On ζ for an elliptic curve.

Let C be an elliptic curve over $\mathbb{F}_q$. Then

$$\zeta(s) = \sum_L{}' \; z^{\deg(L)} \frac{q^{h^0(L)}-1}{q-1}$$

and one has

$$
\begin{aligned}
h^0(L) &= \deg L && \text{if } \deg L > 0 \\
&= \begin{cases} 1 \\ 0 \end{cases} && \begin{matrix}\text{if } \deg L = 0 & L = \mathcal{O} \\ \quad " & L \neq \mathcal{O}\end{matrix} \\
&= 0 && \deg L < 0 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\zeta &= 1 + \frac{h}{q-1}\Big\{\sum_{n\geq 1} q^n z^n - z^n\Big\} \\
&= 1 + \frac{h}{q-1}\left\{\frac{qz}{1-qz} - \frac{z}{1-z}\right\} \\
&= 1 + \frac{h}{q-1}\,\frac{qz - \cancel{qz^2} - z + \cancel{qz^2}}{(1-qz)(1-z)} \\
&= 1 + \frac{hz}{(1-qz)(1-z)} \\
&= \frac{1+(-1-q+h)z + qz^2}{(1-z)(1-qz)}
\end{aligned}
$$

~~If the quadratic $1+(-1-q+h)z+qz^2$ has complex conjugate roots, then they must be of abs. value $\sqrt{q}$. Thus the Riemann hypothesis in the present case [illegible] is equivalent~~

The Riemann hypothesis in the present case says that the two roots of the numerator are conjugate complex of abs. value $\sqrt{q}$. Since their product is q, one sees that the RH follows from (and is equiv. to) the discriminant being ≤ 0, i.e.

$$(-1-q+h)^2 \leq 4q$$

or $\quad *$

But on the other hand, if α_1 and α_2 are the roots, one has for general reasons

$$\alpha_1^n + \alpha_2^n = \operatorname{card}\left(C(\mathbb{F}_{q^n})\right) - 1 - q^n$$

by looking at the curve extended to $\mathbb{F}_{q^n}$. Thus one sees that if RH were false i.e. $|\alpha_1| > q^{1/2}$, say then we would not have an estimate

$$\operatorname{card}\{C(\mathbb{F}_{q^n})\} - q^n = O(q^{n/2})$$

$*$

$$-2\sqrt{q} \leq -1-q+h \leq 2\sqrt{q}$$

$$(\sqrt{q}-1)^2 \leq h \leq (1+\sqrt{q})^2$$

$$\boxed{-1+\sqrt{q} \leq h^{1/2} \leq 1+\sqrt{q}}$$

$$\boxed{|h-q-1| \leq 2\sqrt{q}}$$

Moral: I had originally the feeling that the RH for a curve was a matter of understanding the peculiarities of $h^0(L)$ on the family of line bundles of small degree $0 \leq \deg L \leq 2g-2$. But for an elliptic curve I understand what $h^0(L)$ is precisely, and RH comes down to the estimate $|h-q-1| \leq 2\sqrt{q}$. To prove it, it seems one must use a different method.

April 2?, 1974

Lang problem again:

k alg. closed field of char. p. One wants to establish an exact sequence

$$\cdots\cdots \longrightarrow K_i\mathbb{F}_q \longrightarrow K_i k \xrightarrow{\Phi^q - 1} K_i k \longrightarrow \cdots\cdots$$

One idea I had was to consider the ring $k[F, F^{-1}]$ of non-commutative Laurent polynomials. One gets an exact sequence of localization

$$\longrightarrow K_{i+1} k[F, F^{-1}] \to K_i k \xrightarrow{\Phi^q - 1} K_i k \longrightarrow \cdots\cdots$$

as follows. One considers $k[F]$ which is a PID (non-commutative). Inside of $\mathrm{Modf}(k[F])$ one has the modules on which F is nilpotent.

$$K_i\left(\bigcup \mathrm{modf}(k[F]/(F^n))\right) = K_i k$$

$$K_i(k[F]) = K_i k \qquad \text{homotopy axiom.}$$

And one computes the transfer

$$0 \longrightarrow k[F]F \longrightarrow k[F] \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow k[F]F \otimes_k V \longrightarrow k[F] \otimes_k V \longrightarrow V \longrightarrow 0$$

$$k[F]F \otimes_k V = k[F] \otimes_k (kF \otimes_k V)$$

and

$$kF \otimes_k V \longleftarrow V$$
$$F \otimes v \longleftarrow\!\shortmid\; v$$

is bijective but of degree q: $\quad (F \otimes \lambda v) = \lambda^q (F \otimes v)$.

$\therefore\ kF \otimes_k V = \Phi^q(V)$.

So the problem now comes down to showing that k algebraically closed $\Longrightarrow$

$$K_{i+1}(k[F,F^{-1}]) = \begin{cases} K_i\mathbb{F}_q & i \geq 0 \\ \mathbb{Z} & i = -1. \end{cases}$$

Possibility: One has the suspension of a ring $Susp(A)$ such that

$$K_1(Susp(A)) = K_0A$$

$$K_{i+1}(Susp(A)) = K_iA. \quad \text{all } i$$

Thus one method of proving this would be to relate $k[F,F^{-1}]$ with the suspension of $\mathbb{F}_q$, in fact one might have a map

$$Susp(\mathbb{F}_q) \longrightarrow k[F,F^{-1}]$$

which would induce isos. on all K_i, $i \geq 1$. Actually it is more likely one has a map

$$k[F,F^{-1}] \longrightarrow Susp(\mathbb{F}_q).$$

Recall that $Susp(\mathbb{F}_q)$ is roughly infinite matrices over $\mathbb{F}_q$ modulo finite rank ones ($\approx$ bdd/compact operators).

This I don't understand at all. But in any case I have $k_0 = \mathbb{F}_q$

$\cancel{Mod(\mathbb{F}_q[T])}$

$$\begin{tikzcd} Mod_f(k_0[F]/(F)) \arrow[r] \arrow[d] & Mod_f(k_0[F]) \arrow[r] \arrow[d] & Mod_f(k_0[F,F^{-1}]) \arrow[d] \\ Mod_f(k[F]/(F)) \arrow[r] & Mod(k[F]) \arrow[r] & Mod_f(k[F,F^{-1}]) \end{tikzcd}$$

hence I get a map of exact sequences

$$\begin{tikzcd}
{} \arrow[r] & K_{i+1}k_0 \arrow[r] & K_{i+1}(k_0[F,F^{-1}]) \arrow[r] \arrow[d] & K_i k_0 \arrow[r, "0"] \arrow[d] & K_i k_0 \arrow[r] \arrow[d] & {} \\
 & {} \arrow[r] & K_{i+1}(k[F,F^{-1}]) \arrow[r] & K_i k \arrow[r, "F^b-1"] & K_i k &
\end{tikzcd}$$

Thus what I seem to be able to prove is that I have a map

$$k_0[F,F^{-1}] \rightarrow k[F,F^{-1}]$$

which will carry $K_{i+1}k_0[F,F^{-1}]$ onto $K_{i+1}k[F,F^{-1}]$, with kernel $K_{i+1}k_0$, except for $i+1=0$.

Recall in the "Karoubi" theory, one has a map

$$A[T,T^{-1}] \rightarrow \mathrm{Susp}\,A \qquad \text{ring homo.}$$

in which one lets T act on $Ae_1 + Ae_2 + \cdots\cdots$ by the shift $Te_i = e_{i+1}$, $T^{-1}e_i = \begin{cases} 0 & i \le 0 \\ e_{i-1} & i \ge 1. \end{cases}$

Now in analogy with this I can try to find something that elts of $k[F,F^{-1}]$ act on analogous to a residues. In particular can I find a k_0-vector space

At the moment I know that

$$\mathrm{Susp}(A) = \mathrm{Ops}/\mathrm{Comp}.$$

$$K_{i+1}(\mathrm{Susp}(A))$$

April 27, 1974. Lang problem

For $k = \bar{\mathbb{F}}_q$ I have proved that

$$K_{i+1}(k[F,F^{-1}]) \simeq \begin{cases} K_i \mathbb{F}_q & i \geq 0 \\ \mathbb{Z} & i = -1 \end{cases}$$

in particular there is a close relation between $k[F,F^{-1}]$ and the suspension of $\mathbb{F}_q$. In fact at the moment one has maps

$$\begin{tikzcd} \tilde{K}_{i+1}(\mathbb{F}_q[F,F^{-1}]) \arrow[r, "\sim"] \arrow[d, "s"] & K_{i+1}(k[F,F^{-1}]) \\ K_{i+1}(\mathrm{Susp}\, \mathbb{F}_q) & \end{tikzcd}$$

Recall Susp is like operators mod compact operators. Thus it would be nice if I knew how to associate to any element of $k[F,F^{-1}]$ an operator modulo compact operators.

Suppose one by analogy considers the ring

$$k_0[x, \tfrac{d}{dx}] \qquad \sim \qquad k[F].$$

Then I would like to have something like the symbol of a differential operator.

Recall how you used to consider $\mathbb{C}[z, \frac{d}{dz}]$ acting on holomorphic functions with

$$\int |f(z)|^2 e^{-\pi|z|^2} dz\, d\bar{z} < \infty$$

and that this gave a natural problem of index 1.

Recall what you looked at before:

To every nice function $p(x, D)$ I consider the operator $p(x, D)$ on functions $f(x)$. If I remember correctly I wanted to consider the line compactified analytically in some sense, meaning probably that I work in L^2 of the line. actually in L^2 with respect to Gaussian measure. Then to each function $\varphi(z)$.

f holomorphic $\qquad \|f\|^2 = \int |f(z)|^2 e^{-|z|^2} \, dV$

$$\left\langle \frac{d}{dz} f, g \right\rangle = \int \frac{d}{dz} f \, \bar{g} \, e^{-z\bar{z}} \, dV$$

$$= + \int f \, \bar{g} \, \bar{z} \, e^{-z\bar{z}} \, dV$$

$$= \langle f, zg \rangle$$

$dV \ni \|1\| = 1.$

Now to each functor $\varphi(z)$ which is elliptic in some sense, I get a Fredholm operator by multiplying and projecting onto the holomorphic functions.

~~Thus in the example I get a map of can extend the $k[F, D] \to$ Operators homomorphism have a map of~~

$$
\begin{tikzcd}
K_{i+1}(k_0) \arrow[r] \arrow[d] \arrow[dr, bend left, "0"] & K_{i+1}(k_0[F,F^{-1}]) \arrow[r] \arrow[d] & K_i k_0 \arrow[d] & \\
K_{i+1}(k) \arrow[r] & K_{i+1}(k[F,F^{-1}]) \arrow[r] & K_i k \arrow[r, "F^i - 1"] & {}
\end{tikzcd}
$$

The map is zero by my computations, for k arbitrary.

So look. I have this ring homo.

$$
\begin{tikzcd}
k_0 \arrow[r] \arrow[d] & k_0[F,F^{-1}] \arrow[d] \\
k \arrow[r] & k[F,F^{-1}]
\end{tikzcd}
$$

<u>which</u> kills K_i for $i \geq 1$ but not K_0.

$k_0[F,F^{-1}]$ acts on $\bigoplus_{i=0}^{\infty} k_0 e_i$ as shift.

can this be extended to $k[F,F^{-1}]$.

So suppose $k_0[F,F^{-1}]$ acts on $\bigoplus_{i=0}^{\infty} k_0 e_i$. Can this be extended to $k[F$

Situation: Would like to find a natural map from $k[F,F^{-1}]$ to the suspension of $\mathbb{F}_q$ which generalizes the known map from $\mathbb{F}_q[F,F^{-1}]$ to $k[F,F^{-1}]$.

Idea: Interpret an element $f \in k[F,F^{-1}]$ as a map from k into itself and take its fixpoints which should be a finite dimensional k_0 subspace

Given $f = \sum_{i=0}^{n} a_i F^i$ $\qquad a_0 \neq 0$.

Then $f(x) = \sum a_i x^{q^i}$ is a map from k to k.
and if $f \neq 0$, then it is onto, and the kernel is a finite diml. space over k_0. Clearly of dimension n.

Maybe this is the map I seek. Maybe I can identify

This is obviously the map I want
~~$\alpha+\beta F$~~ because when applied to $K_1(k[F,F^{-1}])$
I see that $F-\lambda \mapsto$
F

April 28, 1974 Lang problem.

k alg. closed char. $p>0$, $k_0 \subset k$ subfield with q elements. I want to prove the square of K-theories assoc to

$$\begin{tikzcd} \mathcal{P}(k_0) \arrow[r] \arrow[d] & \mathcal{P}(k) \arrow[d, "\Delta"] \\ \mathcal{P}(k) \arrow[r, "\Gamma"'] & \mathcal{P}(k) \times_{\mathbb{N}} \mathcal{P}(k) \end{tikzcd}$$

is homotopy cartesian.

Previous work reduces this to showing

$$f: \langle \mathrm{Iso}\,\mathcal{P}(k), \mathrm{Iso}\,\mathcal{P}(k_0) \rangle \longrightarrow \langle \mathrm{Iso}\,\mathcal{P}(k) \times_{\mathbb{N}} \mathcal{P}(k), \Delta\,\mathrm{Iso}\,\mathcal{P}(k) \rangle$$

is a homotopy equivalence, where f is the functor induced by Γ.

Let (V, V') be two vector spaces of same dim over k. ~~Then $f/(V,V')$ is equivalent to a poset.~~ An object of $f/(V,V')$ consists of $A \in \mathcal{P}(k)$ + map $(A, FA) \to (V, V')$, i.e. a pair of isoms

$$A \oplus B \simeq V$$
$$FA \oplus B \xrightarrow{\sim} V'$$

Thus it is clear that $f/(V,V')$ is equivalent to a poset whose elements ξ are made up of direct sum decomps

$$V = A \oplus B$$
$$V' = A' \oplus B'$$

together with isoms $FA \xrightarrow{\sim} A'$, $B \xrightarrow{\sim} B'$. Given another $\bar{\xi}$ with bars, one has $\xi \leq \bar{\xi}$ iff

$$V = A \oplus L \oplus \bar{B} \qquad L = \bar{A} \cap B$$

and $$V' = A' \oplus L' \oplus \overline{B'} \quad \text{where } L' = \overline{A'} \cap B'.$$

and the iso. $F\overline{A} \xrightarrow{\sim} \overline{A}'$ splits as a sum of $FA \xrightarrow{\sim} A'$ and an isom $FL \xrightarrow{\sim} L'$, and finally the isom $B \xrightarrow{\sim} B'$ splits as a sum of $\overline{B} \xrightarrow{\sim} B'$ + and an isom. $L \xrightarrow{\sim} L'$.

<u>April 29, 1974.</u>

I know that for $k = \overline{\mathbb{F}}_p$, one has

$$K_{i+1}(k[F,F^{-1}]) = \begin{cases} K_i\mathbb{F}_q & i \geq 0 \\ \mathbb{Z} & i = -1 \end{cases}$$

but I have never been able to see the homotopy equivalence

$$BGL(k[F,F^{-1}])^+ \longrightarrow Q(\mathcal{P}(\mathbb{F}_q))$$

that has to be there.

<u>Idea</u>: Waldhausen has a description of the K-theory of Laurent rings such as $k[F,F^{-1}]$ which somehow builds in the "linearization" to $\alpha + F^{-1}\beta$. Since yesterday I saw a relation between the ordered set of decompositions $(V,V') = \Gamma(A) \oplus \Delta(B)$ and such linearized autos, it might be worthwhile to work out his theory in detail for $k[F,F^{-1}]$.

~~The basic object~~

In general, he considers two ring homos.

$$A \underset{v}{\overset{u}{\rightrightarrows}} B$$

and lets R be the result of adjoining an invertible element t to B such that $u(a)t = tv(a)$ $\forall a$. To analyze $K_*(R)$, he considers diagrams

$$M_A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M_B$$

where α is a dihom wrt u, β is a dihom wrt v. To such a diagram one can associate the map of R-mods

$$R_u \otimes_A M_A \longrightarrow R \otimes_B M_B$$

$$1 \otimes m \longmapsto 1 \otimes \alpha(m) - t \otimes \beta(m)$$

$$\left(\underline{\text{Check:}} \quad t \otimes \beta(am) = t \otimes v(a)\beta(m) = t v(a) \otimes \beta(m) = u(a)\, t \otimes \beta(m) \right).$$

and one will call the diagram a <u>presentation</u> of M if the associated R-module homo. is injective with cokernel given isomorphic to M via $\theta : M_B \to M$.

Prolongation: Define M_B' by cocartesian square

$$\begin{tikzcd}
B_u \otimes_A M_A \arrow[r, "1\otimes\beta"] \arrow[d, "\tilde\alpha"'] & B_u \otimes_{A\,v} M_B \arrow[d, "\tilde\alpha'"] \arrow[ddr, bend left] \\
M_B \arrow[r, "\beta'"] \arrow[drr, bend right, "\theta"'] & M_B' \arrow[dr, "\theta'"] \\
& & M
\end{tikzcd}$$

$$b \otimes m \longmapsto b t \theta(m)$$

should draw square other way: $B_u \otimes_A M_A \xrightarrow{\tilde\alpha}$, $1\otimes\beta \downarrow$ etc.

Define $M_A' = {}_v M_B$ i.e. with $a \in A$ acting thru v. Then β' becomes the arrow $M_B \to M_B'$ and α' becomes the arrow ${}_v M_B \to M_B'$.

Assuming β is a split mono. over A, then we will get exact sequences.

$$
\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
R \otimes_{u,A} M_A \arrow[rr, "1\otimes\alpha - t\otimes\beta"] \arrow[d, "1\otimes\beta"] & & R \otimes_B M_B \arrow[d, "1\otimes\beta'"] \\
R_u \otimes_{A} {}_v M_B \arrow[rr, "1\otimes\alpha' - t\otimes\beta'"] \arrow[d] & & R \otimes_B M'_B \arrow[d] \\
R \otimes_A ({}_v M_B / \beta M_A) \arrow[rr, "1\otimes\alpha'", "\sim"'] \arrow[d] & & R \otimes_B (M'_B / M_B) \arrow[d] \\
0 & & 0
\end{tikzcd}
$$

because square was cocart.

Similarly there has to be a prolongation process in the other direction.

Specialize to the case of interest $A = B = k$, $t = F$, $v = id$, $u = x \mapsto x^q$. Then we are concerned with diagrams of k-modules

$$V_{-1} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V_0$$

where β is linear and $\alpha(xv) = x^q \alpha(v)$. To such a diagram we assoc.

$$k[F, F^{-1}] \otimes_k \Phi V_{-1} \xrightarrow{1\otimes\alpha - F\otimes\beta} k[F,F^{-1}] \otimes_k V_0 .$$

If this is a presentation of M we have

$$\begin{tikzcd}
\Phi V_{-1} \arrow[r, "\alpha"] \arrow[d, "\Phi\beta"'] & V_0 \arrow[dd, "\theta"] \\
\Phi V_0 \arrow[dr, "F\Theta"'] & \\
& M
\end{tikzcd}$$

When α, β are injective, so that θ induces an isom of V_0 with its image, we can identify V_0 with θV_0, whence

$$\theta\alpha = \theta(F \cdot \beta)$$

hence ... Too confusing

Work instead with

$$V_{-1} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V_0 \qquad \begin{array}{l} \alpha \text{ linear} \\ \beta(\lambda x) = \lambda^\sigma \beta(v). \end{array}$$

and assoc. to its

$$R \otimes_k V_{-1} \xrightarrow{1\otimes\alpha - F^{-1}\otimes\beta} R \otimes V_0$$

Then from

$$\begin{tikzcd}
V_{-1} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & V_0 \arrow[d, "\theta"] \\
F^{-1}\otimes V_0 \arrow[r, "\theta"'] & M
\end{tikzcd} \qquad \theta(\alpha v) = F^{-1}\theta(\beta v)$$

i.e. if α is thought of as the inclusion, then

$$V_{-1} = V_0 \cap F^{-1} V_0$$

and β is induced by multiplication by F.

Now in general given M over R, I can consider k-subspaces V_0 which generate M, and put

$$V_{-1} = V_0 \cap F^{-1}V_0$$

Involutivity in this case probably amounts to

$$(F^{-1}V_0 + V_0) \cap (V_0 + FV_0) = V_0 .$$

Given V_0 one defines its prolongation in the F-direction to be

$$V_0' = V_0 \underset{FV_{-1}}{\amalg} FV_0 \quad \text{``}=\text{''} \quad V_0 + FV_0$$

and in the F^{-1}-direction

$$V_0'' = F^{-1}V_0 \underset{V_{-1}}{\amalg} V_0 \quad \text{``}=\text{''} \quad F^{-1}V_0 + V_0$$

Assuming everything works we can make a simplicial complex out of these involutive subspaces by allowing intervals $V_0 \subset V_0'$ such that $V_0' \subset F^{-1}V_0 + V_0 + FV_0$. In this case I should have that V_0'/V_0 splits into $P \oplus Q$, corresponding to the two prolongation directions.

~~Moreover one has the foll~~

Recall that by localization one gets a fibration of K-theories

$$Q(\mathrm{Nil}) \longrightarrow Q(\mathrm{Pres}) \longrightarrow Q(R)$$

where things get computed as follows.

$$\mathrm{Nil} \sim P(k) \times P(k).$$

because to $(P,Q) \in \mathcal{P}(k)^2$ we get the presentations

$$P \overset{id}{\underset{0}{\rightrightarrows}} P$$

$$\Psi^{-1}Q \overset{0}{\underset{F}{\rightrightarrows}} Q$$

and

$$\begin{aligned} \mathcal{P}res &\sim \mathcal{P}(k) \times \mathcal{P}(k) \\ (V_{-1} \rightrightarrows V_0) &\mapsto (V_{-1}, V_0). \end{aligned}$$

Thus the map

$$\mathcal{N}il \longrightarrow \mathcal{P}res$$

is

$$(P, Q) \longmapsto (\Delta P \oplus \Gamma Q)$$

in a very canonical fashion.

Now all I have to compute is why the map

$$\begin{tikzcd} B(GL_n(R)) \arrow[r] & Q(\mathcal{N}il) \arrow[r] & Q(\mathcal{P}res) \arrow[d] \\ & & Q(k)^2 \end{tikzcd}$$

is homotopic to zero.

The idea is that I replace $GL_n(R)$ by the category consisting of injections $\Lambda' \subset \Lambda$ of involutive presentations of R^n, such that $\Lambda/\Lambda' \in \mathcal{N}il$. Then the map

$$[\Lambda' \subset \Lambda] \longmapsto \Lambda/\Lambda' \longmapsto \Lambda/\Lambda'$$

lifts to the category $\tilde{Q}(\mathcal{P}res)$.

<u>Question:</u> In what sense is Λ/Λ' killed by $\Delta\oplus\Gamma$?

Thus inside of M I [redacted] have $\Lambda' \subset \Lambda$. Then $\Lambda/\Lambda' \in Q(\mathrm{Nil}) \subset Q(\mathrm{Pres})$. Here Pres refers to the exact category of diagrams $V_{-1} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V_0$ such that [redacted] $\alpha - F^{-1}\beta$ is injective, and I have the exact sequence

$$0 \longrightarrow \Lambda' \longrightarrow \Lambda \longrightarrow \Lambda/\Lambda' \longrightarrow 0$$

in Pres which should give me a canonical path of Λ/Λ' to the basepoint, namely

$\Lambda' \subset \Lambda_1' \subset \Lambda_1 \subset \Lambda$

$$\begin{tikzcd}
\Lambda/\Lambda' \arrow[r, dashed, "{[redacted]}"] & \Lambda & 0 \arrow[l, dashed, leftrightarrow] \arrow[dl] \\
\Lambda_1/\Lambda_1' \arrow[u, dashed] \arrow[r, dashed] & \Lambda_1 \arrow[u, dashed] &
\end{tikzcd}$$

Thus it would seem that a trivialization of $\Delta P \oplus \Gamma Q$ would be given by an epimorphism

$$\Lambda \twoheadrightarrow \Delta P \oplus \Gamma Q$$

in the category of presentations, with maybe Λ involutive.

Now Λ will be a [redacted] diagram

$$V_{-1} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V_0$$

with α, β both injective. Now ~~[illegible]~~ the epimorphism will be a map

$$
\begin{tikzcd}
V_{-1} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, two heads] & V_0 \arrow[d, two heads] \\
P \oplus Q \arrow[r, shift left, "id + 0"] \arrow[r, shift right, "0 + F"'] & P \oplus \underline{\Phi} Q
\end{tikzcd}
$$