Recall that if $X$ is the tree of extensions of a rank 2 bundle $M$ over $C \to C$, and $\Gamma = \text{Aut}(M)$, then we have seen that

$$I(M) = H_1(X, X_{\text{inst}})$$

where $X_{\text{inst}}$ is the subcomplex consisting of vertices of $X$ which are instable bundles. When the ground field is finite, this implies that one can define the Euler characteristic $\chi(\Gamma, I)$ as follows.

Since the isom. classes of stable and semi-stable bundles is finite, one can find a normal subgroup of finite index in $\Gamma$ which acts freely on $X - X_{\text{inst}}$. Thus in

$$0 \to I(M) \to C_1(X, X_{\text{inst}}) \to C_0(X, X_{\text{inst}}) \to 0$$

the $C_i(X, X_{\text{inst}})$ are free $Z[\Gamma']$-modules of finite type, hence $I(M)$ is $Z[\Gamma']$-projective of finite type, and also

$$H_0(\Gamma', I(M)) = H_1(X/\Gamma', X_{\text{inst}}/\Gamma').$$

So it is clear how to define $\chi(\Gamma', I(M))$:

$$\chi(\Gamma', I(M)) = -\chi(X/\Gamma', X_{\text{inst}}/\Gamma')$$

and then one can define

$$\chi(\Gamma, I(M)) = \frac{-1}{[\Gamma : \Gamma']} \chi(X/\Gamma', X_{\text{inst}}/\Gamma').$$

Alternative version:
\[ \chi(\Gamma, I(m)) = \sum_{\sigma \in X/\Gamma} \frac{(-1)^{\text{deg}(\sigma)}}{\text{aut}(\sigma)} \]

This infinite sum converges.

In effect, if I take \(\Gamma\) to be the sum over the vertices of even degree (resp. odd degree), this series converges by theiegel formula. As for the one-simplices, way out in the tree the autom group is the same as for one of its vertices.

It is clear that there is a lot of cancellation in the above sum, because once in the unstable region, the term for a vertex is cancelled by the unique simplex in the cusp which follows this vertex.

I can compute this Euler characteristic using theiegel formula. In effect, having chosen \(\Gamma\) acting freely on the region defined by \(\mu_{\max} \leq n\) (\(d_1 = 0\), \(d_2 = 1\)), i.e. \(X_n\) is bounded by vertices \(L(n) \oplus L(n-1)\), let \(Y_n = X_n/\Gamma\). Note that everywhere except at \(\partial Y_n\), there are \((g+1)\) edges coming into each vertex. At each point of \(\partial Y_n\), there are \(g\) edges coming in. Thus one finds that

\[ 2(\text{number of edges in } Y_n - \partial Y_n) = (g+1)(\text{no. of vertices in } Y_n - \partial Y_n) \]

(count pairs \(\sigma \in \Gamma\) in two ways). Not quite for each edge going to \(\partial Y_n\) belongs to only one vertex. Thus
\[(q+1) \left( \frac{\text{no. of vertices}}{\text{in } Y_n - \partial Y_n} \right) = 2 \left( \frac{\text{no. of edges}}{\text{in } Y_n - \partial Y_n} \right) - \text{card}(\partial Y_n)\]

Too complicated. Return to your picture.
Recall one has the following picture of \( X/\Gamma \):

![Diagram showing sets \( B_{-2}, B_{-1}, B_0, B_1, B_2 \) with bounding stable, semi-stable, and unstable bundles.]

Call these sets
\[ B_{-2}, B_{-1}, B_0, B_1, B_2 \]

Note from each element of \( B_n \), \( n \geq 1 \) there are 8 edges going to \( B_{n-1} \) and 1 edge going to \( B_{n+1} \) of the sort that
\[ \chi(B_n, \Gamma) = 8 \chi(B_{n+1}, \Gamma) \quad n \geq 1. \]

The Euler char I want is
\[
\chi(B_2, \Gamma) - (q+1) \chi(B_2, \Gamma) \\
+ \chi(B_{-1}, \Gamma) \\
\chi(B_0, \Gamma) - (q+1) \chi(B_0, \Gamma) \\
= -8 \chi(B_2, \Gamma) + \chi(B_1, \Gamma) - \chi(B_0, \Gamma) \\
0 \left\{ \begin{array}{l}
\chi(B_1, \Gamma) \\
\chi(B_2, \Gamma) - (q+1) \chi(B_2, \Gamma) \\
0 \left\{ \end{array} \right. \]
The Siegel formula gives
\[ \chi(B_{-2}, \Gamma) + \chi(B_{-1}, \Gamma) + \chi(B_0, \Gamma) + \ldots = \frac{1}{g-1} Z_c(g) \]
\[ \chi(B_{-1}, \Gamma) + \chi(B_0, \Gamma) + \chi(B_1, \Gamma) + \ldots = \frac{1}{g-1} Z_c(g) \]

Thus one gets

\[ \chi(\Gamma, \text{I}(M)) = + Z_c(g) \]

Notice also that we can easily compute the number of stable bundles of degree \(-1\) up to isom. Since \(B_{2n+1}\) for \(n\) large consists of iso-classes \(L(n) \oplus L^*(n-1)\) and

\[ \text{Aut}(L(n) \oplus L^*(n-1)) = \left( \frac{k^*}{k^*} H^0(L^*(2n+1)) \right) \]

\[ h^0(L^2(2n+1)) = 2n+1 + 1 - g \]

\[ \text{Aut}(L^2(2n+1)) = (g-1)^2 \frac{q}{g}^{2n+2-g} \]

\[ \chi(B_{2n+1}, \Gamma) = \frac{h}{(g-1)^2 \frac{q}{g}^{2n+2-g}} \quad h > 0 \]

\[ \chi(B_i, \Gamma) + \chi(B_3, \Gamma) + \ldots = \frac{h}{(g-1)^2 \frac{q}{g}^{2n}} \sum_{n=0}^{\infty} \frac{1}{\frac{q}{g}^{2n}} \]

\[ = \frac{h}{(g-1)^2} \frac{1}{\frac{q}{g}^{2-g}} \frac{1}{1 - \frac{1}{\frac{q}{g}}} \]

\[ = \frac{h \frac{q^g}{(g-1)^2}}{(g^2-1)} \]

\[ = \frac{h \frac{q^g}{(g-1)^2}}{(g^2-1)} \]
Therefore
\[ \chi(B_{-1}, \Gamma) = \frac{1}{g-1} \ Z_C(g) = \frac{h \ g^3}{(g-1)^2} \ \frac{1}{g^2-1} \]
or
\[ \text{(no. of stable $g^2$)} \]
\[ \text{(bundles of deg 1)} \text{ (with given det.)} \]
\[ = Z_C(g) = \frac{h \ g^3}{(g-1)(g^2-1)} \]

Check: \( g = 0 \) \quad \text{get} \quad 0
\[ g = 1 \quad \frac{1 + (-1-g + \Gamma)g + g^3 - h}{(g-1)(g^2-1)} = 1 \]

Next try to compute the number of stable bundles of degree 0. It is necessary to divide the set \( B_0 \) of semi-stable non-stable vertices into the decomposable and non-decomposable groups:
\[ B_0 = B_0^{\text{dec}} \sqcup B_0^{\text{ind}} \]
and to divide \( J \) into \( J_2^{\text{dec}} + J_+ + J_- \) where
\[ J_2 = \{ L \mid L = L^* \} \quad \text{and} \quad J_- = J_+^{-1} \]
corresponding to this we have \( h = h_0 + 2h_+ \). We know that through each element of \( B_0^{\text{ind}} \) there are \( g \) vertices going toward \( B_{-1} \), and 1 toward \( B_1 \). From something of the form
\[ \text{Le} J_2 \text{ Le} L \text{ have} \ g+1 \text{ edges to} \ B_{+1} \text{, more to} \ B_{-1} \]
\[ \text{Le} J_+ \text{ Le} L^* = 2 \quad \text{B_1, g-1 to B_-} \]
So now count the number of edges from $B_0$ to $B_1$.

On one hand there are

$$\chi(B_1, \Gamma) = \frac{h}{(q-1)^2 q^{1-q}}.$$

On the other we have

$$\begin{align*}
\frac{\text{card}(J_2)}{(q^2-1)(q^2-q)} (q+1) + \frac{\text{card}(J_2)}{(q-1)^2} 2 + \chi(\beta_0, \Gamma)
\end{align*}$$

But

$$\begin{align*}
0 = \chi(B_0^{\text{dec}}, \Gamma) = \frac{\text{card}(J_2)}{(q^2-1)(q^2-q)} - \frac{\text{card}(J_2)}{(q-1)^2} 2 + \chi(\beta_0, \Gamma)
\end{align*}$$

Now

$$\begin{align*}
\chi(B_{-2}, \Gamma) + \chi(B_0, \Gamma) + \frac{h}{(q-1)^2} \sum_{n=1}^{\infty} \frac{1}{q^{2n+1-q}} &= \frac{1}{q-1} \frac{h \cdot q^{2-1}}{(q-1)^2(q^2-1)}
\end{align*}$$

But

$$\begin{align*}
\frac{h}{(q-1)^2 \cdot q^{1-q}} = \frac{\text{card}(J_2)}{(q^2-1)(q^2-1)} + \frac{\text{card}(J_2)}{(q-1)^2} + \chi(B_0, \Gamma)
\end{align*}$$

so

$$\begin{align*}
\chi(B_{-2}, \Gamma) + \frac{h \cdot q^{1+q}}{(q-1)^2(q^2-1)} = \frac{\text{card}(J_2)}{(q^2-1)(q-1)} + \frac{\text{card}(J_2)}{(q-1)^2} + \frac{1}{q-1} \frac{h \cdot q^{2-1}}{(q^2-1)}
\end{align*}$$

Check: take $q = 0$

$$\begin{align*}
\chi(B_{-2}, \Gamma) + \frac{0}{(q-1)^2(q^2-1)} = \frac{1}{(q^2-1)(q-1)} + 0 + \frac{1}{(q-1)^2(q^2-1)} \Rightarrow \chi(B_{-2}, \Gamma) = 0
\end{align*}$$
Now what I really want to get at is the Euler char of the stable part which is

\[ \chi(B_{-2}, \Gamma) = -8 \chi(B_{-1}, \Gamma) \]

\[ = -\frac{\frac{1}{2} + \frac{3}{2}}{8^2 - 1} \cdot \frac{1}{(8^2 - 1)(8 - 1)} + \frac{\text{card}(T_2)}{(8^2 - 1)(8 - 1)} + \frac{\text{card}(T_+)}{(8 - 1)^2} + \frac{1}{8 - 1} \cdot Z_C(8) \]

\[ + \frac{\frac{1}{2} + \frac{3}{2}}{8^2 - 1} \cdot \frac{1}{(8^2 - 1)(8 - 1)} - \frac{1}{8 - 1} \cdot Z_C(8) \]

\[ = -Z_C(8) + \frac{\text{card}(T_2)}{(8^2 - 1)(8 - 1)} + \frac{\text{card}(T_+)}{(8 - 1)^2} \]

I want to find a formula for \( \chi(X/\Gamma) \). Clearly

\[ \chi(X_{\text{inst}}/\Gamma) = \text{card } \Gamma \]

because it is clear that the deformation to the wrops proceeds \( \Gamma \)-equivariantly. On the other hand

\[ \chi(X_{\text{inst}}/\Gamma) = 0 \]

as we have seen.

\[
\frac{1}{8 - 1} \chi(X/\Gamma, X_{\text{inst}}/\Gamma) = \chi(X, X_{\text{inst}}; \Gamma) + \sum_{\sigma} \left( \frac{1}{8 - 1} - \frac{1}{\text{aut}(\sigma)} \right) c^{(-1)} \alpha_{\sigma} \]

What are the \( \tilde{\omega} \) simplifies whose auto groups exceed \( k^* \)?

a) Semi-stable stable ind. bundles of degree 0

\[ + \text{edge leading to } B_1. \]

b) \( L \oplus L, L \in T_2; \text{ aut } = (8^2 - 1)(8^2 - 8) \)
edge leading from \( L \oplus L \) to \( B_1 \), ant. \((g-1)^2\)

\[
\begin{align*}
\left[ 1 - \frac{(g-1)}{(g^2-1)(g+1)} \right] - \left[ 1 - \frac{(g-1)}{(g-1)^2} \right] &= \frac{1}{g-1} \left( \frac{1}{g-1} - \frac{1}{(g+1)^2} \right) = \frac{1}{g-1}
\end{align*}
\]

Thus get

\[
\text{card} (J_2) = \frac{1}{g-1} \left( \frac{1}{g-1} - \frac{1}{(g+1)^2} \right).
\]

\( c) \) \( L \oplus L^* \), \( L \in J_4 \). ant. \((g-1)^2\)

2 edges leading from \( L \oplus L^* \) to \( B_1 \) with ant. \((g-1)^2\)

\[
\text{contribution}
\]

Thus get

\[
\text{card} (J_4) = \frac{1}{g-1} \left( \frac{1}{g-1} - \frac{1}{(g+1)^2} \right).
\]

\( d) \) indecomposable bundles of degree 0 which decompose into \( L \oplus L^* \) over \( \mathbb{F}_g \). Here ant. \( g^2 - 1 \).

Let \( \alpha \) be the number of these. Then we have the contribution

\[
\alpha \left( 1 - \frac{1}{g+1} \right) = \frac{\alpha \cdot g}{g+1}
\]

Therefore it seems that

\[
\chi (X/\Gamma, X_{\text{int}}/\Gamma) = -(g-1) Z_{c}(g) + \alpha \frac{g}{g+1} + \text{card} (J_2) \frac{(g-2)}{g-1} + \text{card} (J_4) \frac{g}{g-1}
\]
Put

$h = \# \ker \{ 1 + \frac{1}{2} \}$

and one gets

$x(1 \mid T \cdot x_{\text{int}}(1)) = 1 - h$

and we do have the formula on the bottom of page 8.

Then

$[g + 1 - \frac{1}{2}(h + a)] \frac{g}{g+1} + \frac{1}{2}(h+a)$

$= \frac{8}{8 + 1 - \frac{1}{2}(h+a)}$.

To check for an elliptic curve, put $a = \text{and } (T)$,
Then
\[
\begin{align*}
& h + \frac{a}{g^2-1} + \frac{b}{g+1} + a \left( \frac{1}{g^2-1} \right) + \left( \frac{1}{g+1} \right) \\
& = h \left[ 1 + \frac{1}{2} \frac{2-g}{g} \right] + a \left( \frac{1}{2} + \frac{1}{2} \frac{1}{g+1} + \frac{1}{g^2-1} + \frac{1}{2} \frac{2-g}{g+1} \right) \\
& \quad + b \left( \frac{1}{2} - \frac{1}{2(g+1)} \right) \\
& = \frac{h}{2} \frac{g}{g-1} + \frac{b}{2} \frac{g}{g+1}
\end{align*}
\]

so we have
\[
\chi(X/\Gamma) = \frac{-(-g-1)Z_c(g)}{\frac{h}{2} \frac{g}{g-1} + \frac{b}{2} \frac{g}{g+1}}
\]

but if the numerator of \( Z_c(g) \) is \( 1 + c_1 z + \ldots + c_2 z^2 \)
then \( h = 1 + c_1 + \ldots + c_2 \), \( b = 1 - c_1 + c_2 + \ldots + c_2 \)

\[
\frac{h \cdot \frac{g}{g-1} + b \cdot \frac{g}{g+1}}{2(g^2-1)} = \frac{g}{2(g^2-1)} \left[ (g+1)(1+c_1+\ldots+c_2) + (g-1)(1-c_1+\ldots) \right]
\]

\[
= \frac{g}{2(g^2-1)} \left( 1 + c_2 + \ldots + c_2 \right) + (c_1 + \ldots + c_2 - 1)
\]

so
\[
\chi(X/\Gamma) = \frac{-\frac{1}{g^2} + \frac{1}{g^2} 2^g + \frac{1}{g^2} 3^g}{\frac{g}{g^2-1}} + \frac{g^2}{g^2-1} (1+c_2+c_2 + \ldots + c_2 - 1)
\]

\[
\chi(X/\Gamma) = 1 - \frac{1}{g^2} - \frac{1}{g^2} c_3 - \frac{1}{g^2} c_4 - \frac{1}{g^2} (g^2+1) c_5 - \ldots
\]
April 6, 1974. Euler characteristic formula

In the case of symmetric spaces the Euler measure is a multiple of the volume. Want the p-adic analogue.

So let \( X \) be a building belonging to a \( F \)-vector space of dimension \( n \), \( F \) field with a discrete valuation. Let \( \Gamma \) be a group acting freely on \( X \) and such that \( X/\Gamma \) is compact. I want a formula for \( \chi(X/\Gamma) \) in terms of \( \text{vol}(X/\Gamma) \), number of vertices in \( X/\Gamma \) of a given type. Put \( \mathbf{V} = X/\Gamma \).

Example: \( n = 2 \). Here \( \mathbf{V} \) is of dim. 1

\[
\# \{ \text{pairs } (\sigma_0, \sigma_1) \mid \sigma_0 \subset \sigma_1 \} = 2 \# \{ \sigma_1 \} = (g + 1) \# \{ \sigma_0 \}
\]

Hence

\[
\chi(\mathbf{V}) = \# \{ \sigma_0 \} - \# \{ \sigma_1 \} = \left(1 - \frac{g+1}{2}\right) \# \{ \sigma_0 \}
\]

= \left(1 - \frac{g}{2}\right) \# \{ \sigma_0 \}

Example:

\[
\# \{ \sigma_1 \mid \sigma_0 \subset \sigma_1 \} = \# \mathbf{G}_1(\mathbf{V}) + \cdots + \# \mathbf{G}_{n-1}(\mathbf{V})
\]

\[
\# \{ \sigma_p \mid \sigma_0 \subset \sigma_p \} = \sum_{i_1 + \cdots + i_p < p} \# \mathbf{G}_{i_1, \cdots, i_p}(\mathbf{V})
\]

\[
\# \{ \sigma_p \} = \frac{1}{p+1} \sum_{i_1 + \cdots + i_p < p} \# \mathbf{G}_{i_1, \cdots, i_p}(\mathbf{V}) \cdot \# \{ \sigma_0 \}
\]

\[
\sum (-1)^p \# \{ \sigma_p \} = \left[ \sum_{p=0}^k \frac{(-1)^p}{p+1} \sum_{i_1 + \cdots + i_p < p} \# \mathbf{G}_{i_1, \cdots, i_p}(\mathbf{V}) \right] \# \{ \sigma_0 \}
\]
For \( k = 3 \) one gets
\[
\left[ 1 - \frac{1}{2}(1+\varphi^2 + 1+\varphi^2) + \frac{1}{3}(1+\varphi)(1+\varphi^2) \right] \# \{ \sigma_0 \} = (1-\varphi)(1-\varphi^2) \# \{ \sigma_0 \}
\]

Conjecture. For general \( k \) one gets
\[
\sum (-1)^{p} \# \{ \sigma_p \} = (1-\varphi). (1-\varphi^{k-1}) \frac{\# \{ \sigma_0 \}}{k+1}.
\]

To prove this it will be necessary to do the combinatorics intelligently.

So now return to curves, where \( X \) represents extensions of a given vector bundle \( \mathcal{E} \) on \( \mathbb{C}^* \) and \( \Gamma = \text{Aut}(\mathcal{E}) \). Here I want to compute the sum
\[
\sum_{\sigma \in X/\Gamma} \frac{(-1)^{\sigma}}{\text{aut}(\sigma)}
\]
which I claim converges. To see this it suffices to note that the sum over the vertices converges by the Siegel formula, absolutely.

Hence because one has maps
\[
\begin{align*}
\{ (\sigma_0, \sigma_i) \mid \sigma_0 \prec \sigma_i \} & \rightarrow \{ \sigma_0 \} \\
\text{each fibre has} & \begin{array}{c} \sigma_0 \in X/\Gamma \\
\text{2 + 1 els.} \end{array} & \text{each fibre has} & \begin{array}{c} \text{m elements} \end{array}
\end{align*}
\]

one finds
\[
(i+1) \sum_{\sigma_i \in X/\Gamma} \frac{1}{\text{aut}(\sigma_i)} = m \sum_{\sigma_0 \in X/\Gamma} \frac{1}{\text{aut}(\sigma_0)}
\]
So it is now clear to me that I will get in this way the formula

$$\sum_{\sigma \in \text{Aut}(\mathcal{G})} \frac{(-1)^\sigma}{\text{aut}(\sigma)} = (1 - g_1 \cdots (1 - g_{N-1}) \sum_{N \in \text{ex}} \frac{1}{\text{aut}(e)} \]$$

This part is only combinatorial.

But the next thing to will be to identify

$$\sum_{\sigma \in \text{Aut}(\mathcal{G})} \frac{(-1)^\sigma}{\text{aut}(\sigma)} \rightarrow \chi(\Gamma; X, X_{\text{int}})$$

which means one has lots of cancellation in the infinite sum.
For an honest arithmetic group $\Gamma$ Serre has defined $X(\Gamma)$ as $\frac{1}{[\Gamma: \Gamma]} X(\Gamma')$ where $\Gamma'$ is a torsion-free subgroup of finite index and $X(\Gamma') = X(B\Gamma')$ is defined because $B\Gamma'$ has the homotopy type of a finite complex.

The question arises of how to make sense of this when $\Gamma$ is arithmetic in the function field sense. To fix the ideas let $\Gamma = GL_2(A)$, $A = \mathbb{R}(C, \mathfrak{q}^{-\infty})$, and let $X$ be the associated tree.

The good number $\mathfrak{o}$ to call $X(\Gamma)$ seems to be

$$X(\Gamma; X \mod X_{\text{inst}})$$

for the following reasons.

1) $X(\Gamma; X \mod X_{\text{inst}}) = \sum_{\sigma \in \chi(\Gamma)} (-1)^{\sigma} \frac{1}{\# \Gamma_{\sigma}}$

2) $X(\Gamma', X \mod X_{\text{inst}}) = [\Gamma': \Gamma] X(\Gamma; X \mod X_{\text{inst}})$

3) $X(\Gamma; X \mod X_{\text{inst}})$ is given by a $1$ function value

4) If $\Gamma'$ acts freely on $X - X_{\text{inst}}$, then

$$X(\Gamma', X \mod X_{\text{inst}}) = X(\Gamma', X \mod X_{\text{inst}}) = X_{c}(X|\Gamma', X_{\text{inst}})$$

(This last formula shows that we should think of $X(\Gamma)$ as depending on $X(\Gamma)$ with compact support.)
\[ \chi(x/\Gamma') = \chi_c(x/\Gamma') + h' \]

where \( h' = \prod \frac{p_i(e)}{\Gamma'} \) is the number of \( \Gamma' \)-cusps. But \( h' \) does not depend multiplicatively in \( \Gamma' \), e.g. if

\[ \prod_i \Gamma_i = \prod_j \Gamma_j \]

and if \( \Gamma' \) is normal of finite index in \( \Gamma \), then

\[ \prod_i \Gamma_i/\Gamma' = \prod_i \Gamma_i/\Gamma' \Gamma_i \]

Thus

\[ \frac{h'}{[\Gamma : \Gamma']} = \sum_{i=1}^n \frac{1}{[\Gamma_i : \Gamma_i \cap \Gamma']} \]

This changes as \( \Gamma' \) does. Notice however that as \( \Gamma' \) gets smaller and smaller, then the denominators in the right get larger. Thus

\[ \lim_{\Gamma'' \to \Gamma'} \frac{\chi(x/\Gamma')}{[\Gamma : \Gamma']} = \chi(\Gamma, x \mod \chi_{\text{num}}) \]

Compare with the arithmetic case \( \Gamma = \text{SL}_2(\mathbb{Z}) \).

Here it should be true that

\[ \chi(x/\Gamma') = \chi_c(x/\Gamma') + h' \chi(S^1) \]

because the cusps are circles. Also by the Borel-Serre
duality thm. one has

$$H_i(\Gamma', I) = H^{d-i}(\Gamma', \mathbb{Z})$$

so the Steenborg Euler characteristic should be the same as the Euler char.
April 6, 1974. Upper half plane and quadratic forms.

I want the formulas giving the isomorphism of $H$ with \textit{pos. def.} quadratic binary forms.

\[
\begin{align*}
\mathbf{SL}_2(\mathbb{R})/\mathbf{SO}_2(\mathbb{R}) & \sim H \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{SO}_2(\mathbb{R}) & \mapsto \frac{a}{a + b} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{SO}_2(\mathbb{R}) & \mapsto \frac{a + ib}{a + d} \\
\end{align*}
\]

On the other hand

\[
\begin{align*}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto (ax + by)^2 + (cx + dy)^2 \\
& = (a^2 + c^2)x^2 + 2(ab + cd)xy + (b^2 + d^2)y^2 \\
\end{align*}
\]

gives an isomorphism

\[
\mathbf{SL}_2(\mathbb{R})/\mathbf{SO}_2(\mathbb{R}) \sim \text{pos. def. forms on } \mathbb{R}^2 \text{ of discriminant 1}
\]

\[
\begin{align*}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) \right|^2 = (x, y) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T (x, y) \\
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} & = 1
\end{align*}
\]

So now if $z = \alpha + i\beta \in H$, $\beta > 0$, then

\[
\begin{align*}
\begin{pmatrix} \beta^{\frac{1}{2}} & \alpha \beta^{-\frac{1}{2}} \\ 0 & \beta^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} & = \begin{pmatrix} \beta^{\frac{1}{2}} i + \beta^{-\frac{1}{2}} \\ \beta^{\frac{1}{2}} \end{pmatrix} = \alpha i + \beta
\end{align*}
\]

and the corresponding quadratic form is

\[
\begin{align*}
(\beta^\frac{1}{2}x + \alpha \beta^{-\frac{1}{2}}y)^2 + (\beta^{-\frac{1}{2}}y)^2 & = \beta x^2 + 2\alpha xy + \frac{\alpha^2 + 1}{\beta} y^2
\end{align*}
\]

Rule

\[
\begin{pmatrix} \alpha & i \beta \\ 0 & \beta \end{pmatrix} \leftrightarrow \begin{pmatrix} \beta & \alpha \\ \alpha & \alpha^2 + 1 \end{pmatrix}
\]
$k = \mathbb{F}_q$ of large. C curve over $k$, $\infty$ a rational point $A = \Gamma(C, \infty, \mathcal{O}_C)$. $X$ the building associated to a proj. module $M$ of rank $n$ over $A$. $\Gamma = \text{Aut}(M)$. If $X_{\text{inst}}$ is the sub-complex of $X$ containing those vertices whose corresponding bundles are unstable then I believe I can show that

$$H^i(X, X_{\text{inst}}) = \begin{cases} 0 & i \neq n-1 \\ I & i = n-1 \end{cases}$$

where $I = \text{Steinberg module associated to } \mathbb{F}_q M$.

But if now $\Gamma'$ is a subgroup of finite index of $\Gamma$ such that $\Gamma'$ acts freely on the semi-stable region, then each of the complexes $C^i(X, X_{\text{inst}})$ is a free $\Gamma'$-module of finite type. It follows from the exact sequence

$$0 \to I \to C_{n-1}(X, X_{\text{inst}}) \to \cdots \to C_0(X, X_{\text{inst}}) \to 0$$

that $I$ is a projective $\mathbb{Z}[\Gamma']$-module of finite type, which is even stably-free.

Hence

$$0 \to I_{\Gamma'} \to C_{n-1}(X, X_{\text{inst}})_{\Gamma'} \to \cdots \to C_0(X, X_{\text{inst}})_{\Gamma'} \to 0$$

is exact:

$$C_{n-1}(X/\Gamma' X_{\text{inst}}/\Gamma') \to \cdots \to C_0(X/\Gamma' X_{\text{inst}}/\Gamma') \to 0$$

And so we see that we have
\[ H_i(X/\Gamma, X_{\text{inst}}/\Gamma') = \begin{cases} 0 & i \neq n-1 \\ \mathbb{I}_{\Gamma'} & i = n-1, \end{cases} \]

We even know the rank of \( \mathbb{I}_{\Gamma'} \) over \( \mathbb{Z} \) since we have the Harer formula for \( \chi(\Gamma, X \mod X_{\text{inst}}) \).
For example, in the rank 2 case I have
\[ \chi(\Gamma, X \mod X_{\text{inst}}) = -Z_0(q). \]
April 8, 1977

It will be necessary to find proofs of various things I believe. It is probably enough to understand completely bundles of rank 3.


Maybe I should think of a point of $X$ as an equivalence class of $C'$-lattices in $M$, where $\Lambda$ and $\Lambda'$ are equivalent iff $\Lambda = \Lambda'(n)$ for some $n$.

$x_{\text{ins}}$ = full subcomplex whose vertices are the unstable $\Lambda$. From the theory of canonical filtrations of vector bundles I know already that $x_{\text{ins}}$ has a covering whose nerve is the Tate complex $T(V)$ as follows. For each $W$ proper subspace of $V$, I let $X_{XW}$ be the full subcomplex of $X$ whose vertices are those $\Lambda$ such that $\mu_{\min}(\Lambda \cap W) > \mu_{\max}(\Lambda \cap \Lambda \cap W)$, or equivalently such that $\Lambda \cap W$ is part of the canonical filtration of $\Lambda$.

Then

$$x_{\text{ins}} = \bigcup W$$

and

$$X_{XW_0 \cap \cdots \cap XW_p} \neq \emptyset \iff \{W_0, \cdots, W_p\} \text{ is a simplex of } T(V).$$

---

Put

$$x_{\mathcal{T}} = \bigcap X_{\mathcal{W}_i} \quad \text{if} \quad \mathcal{T} = \{W_0, \cdots, W_p\}$$

The first thing I want to understand well is why $x_{\mathcal{T}}$ is contractible. Also I want to show that the weighted Euler characteristic

$$\sum_{\sigma \in \mathcal{F}} (-1)^{\sigma} \frac{1}{\varepsilon(\sigma)}$$
is zero. Also it would be nice to be able to enlarge $X_{in}$ so that it includes semi-stable indecomposable bundles.

First recall what happens in rank 2 case. Here $W$ is a line in $V$ and I can normalize my lattice $\Lambda$ such that $\Lambda + W/W$ is a fixed lattice in $V/W$. Then I have a basic operation which increases the size of $N/W$ by 1.

$$
0 \longrightarrow N/W \longrightarrow \Lambda \longrightarrow N/W \longrightarrow 0
$$

$$
0 \longrightarrow N/W(1) \longrightarrow \Lambda^* \longrightarrow N/W \longrightarrow 0.
$$

This is really the only possibility and gives the picture:

$$
\xrightarrow{\text{map intersection with } W \text{ after normalization}} X(W)
$$

**Why does this prove $X(V)$ is contractible?**

Because if I take a finite subcomplex $K$ of $X(V)$, then I can contract.
Let $X$ be the Tits building of lattices in an $F$-vector space $V$, lattices for a d.v.r. $O$. Let $\text{rank } V = r$, whence $\dim X = r - 1$. I recall that the link of any point of $X$ is a bouquet of spheres of dimension $r - 2$. For more detail, let $\sigma$ be a $p$-simplex of $X$. Realize $\sigma$ by a chain of lattices $L_0 < L_1 < \cdots < L_p$, where $L_p \subseteq \pi^{-1} L_0$. Any vertex $L$ in the link of $L_0$ may be represented by a unique lattice $L$ such that $L_0 < L < \pi^{-1} L_0$, and the vertex is in the link of $\sigma$ if $L_0 \not\equiv \pi^{-1} L_0$ and is not in this chain. One concludes that

$$\text{Link } \sigma = \left\{ L \mid L_0 < L < \pi^{-1} L_0 \right\} \times \cdots \times \left\{ L \mid L_0 < L < \pi^{-1} L_0 \right\}$$

where $|S|$ denotes the simplicial complex belonging to the point $S$. Hence if $r_0 = \dim (L_1/L_0), \ldots, r_p = \dim (\pi^{-1} L_0/L_p)$, $\sum r_i = r$, and

$$\text{Link } \sigma = T(L_1/L_0) \times \cdots \times T(L_p/\pi^{-1} L_0)$$

$$= \left( V S^{r_0 - 2} \right) \times \cdots \times \left( V S^{r_p - 2} \right)$$

$$= V S^{\sum r_i - 2p - 2} = V S^{r - p - 2}$$

since

$$\text{Link } (b_0) = \partial S \star \text{Link } \sigma$$

$$= S^{p - 2} \times V S^{r - p - 2}$$

$$= V S^{r - 2}$$

as claimed. Note also that the number of these spheres is

$$\sum_{i=0}^{p} \frac{1}{2} r_i (r_i - 1)$$
Dual complex. Recall that there is something called the dual cell complex to a PL manifold $X$.

To obtain this, consider the ordered set of simplices $S$ of $X$, but with the reverse ordering, so that now one has the the dimension function

$$c(\sigma) = n - \dim(\sigma).$$

Then one attaches to $\sigma$ the ordered set

$$S_{<\sigma} = \{ \tau \mid \tau > \sigma \}$$

which is the link of $\sigma$, and which by the assumption that $X$ is a PL-manifold, is homotopy equivalent to a sphere of dimension $c(\sigma) - 1$. Thus by filtering $S$ by

$$S_{\leq p} = \{ \sigma \in S \mid c(\sigma) \leq p \}$$

one gets the skeletal filtration of a CW complex. In effect in passing from $S_{\leq p-1}$ to $S_{\leq p}$, I add those $\sigma$ with $c(\sigma) = p$, and $\sigma$ is attached by putting a cone on $\{ \tau \mid \tau > \sigma \}$ which is a $(p-1)$-sphere.
Now consider the case where $X$ is a simplicial complex such that the link at each point is (homologically) a bouquet of $(n-1)$-spheres. Then again I can let $S$ be the set of simplices with the ordering reverse to inclusion, and $c(\sigma) = n - \dim(\sigma)$.

\[ S_{\geq p} = \{ \sigma \in S \mid c(\sigma) \geq p \} \]

Again I get a spectral sequence

\[ E_{pq}' = H_{pq}(S_{\geq p}, S_{\geq p}) = \bigoplus_{c(\sigma) = p} H_{pq}(\text{link}(\sigma), \text{link}(\sigma)) \]

\[ \implies H_{p+1}(X) \]

\[ H_{pq}(\{ \tau \geq \sigma \}, \{ \tau > \sigma \}) = \begin{cases} 0 & \delta \neq 0 \\ I(\sigma) & \delta = 0 \end{cases} \]

so we get the complex

\[ \cdots \rightarrow \bigoplus_{d(\sigma) = n-1} I(\sigma) \rightarrow \bigoplus_{d(\sigma) = n} I(\sigma) \]

whose homology is that of $X$.

Now I wanted to apply this to show that the Steinberg module is finite type projective over $\Gamma'$ when $\Gamma$ is met (met means that every element of finite order has order a power of $p$). The idea is to use the fact that for each $\sigma$, $\Gamma_{\sigma}$ is a $p$-group, hence $I(\sigma)$ is a $\mathbb{Z}[\Gamma'_{\sigma}]$-free module of finite type. There are
Some problems at the boundary

Example: $n = 2$

Here

$I = H_1(X, X_{\text{int}})$

One has

$0 \rightarrow I \rightarrow \bigoplus I(\sigma) \rightarrow \bigoplus \{ Z \}$

$\sigma \in X - X_{\text{int}}$

$d(\sigma) = 0$

if $\text{det} \neq 0$

if not,

$\rightarrow 0$

Picture:

So what we want works here:

Conjecture: If $\Gamma'$ is meto (every element of finite order has order a power of $p$), then the Steinberg module is projective of finite type over $\mathbb{Z}[\Gamma']$.

I know the conclusion is true when $\Gamma'$ acts freely on $X = X_{\text{int}}$, because then $C_\ast(X, X_{\text{int}})$ is free fin. type over $\mathbb{Z}[\Gamma']$.

In the case $n = 2$, I can enlarge $X_{\text{int}}$, and split every decomposable semi-stable bundles of degree 0, in which case $\Gamma'$ will act freely. : The conjecture is true for $n = 2$. 


April 10, 1974

F = function field of a curve C, V vector space rank r
over F, we consider vector bundles E over C with
generic fibre V; we can think of E as a lattice inside of V.

I have to recall first the canonical filtration of
E which is the unique subbundle filtration 0 < E₁ < ... < Eₚ = E
such that the quotients are semi-stable with
strictly decreasing slopes. The uniqueness comes as
follows. Let F ⊂ E, and consider the induced filtration
F = F ∩ E. Since F/F ∩ E embeds in E/E ∩ E, its
slope (when defined, i.e. F/E ∩ E is < μ = slope (E/E ∩ E))

\[ \text{deg}(F) = \sum \text{deg}(F/F ∩ E) \]
\[ < \sum \mu_i \text{ rank}(F/F ∩ E) \]

\[ \text{rank}(F) = \sum \text{rank}(F/F ∩ E) \]

Thus if μ(F) = μ max (E) ⇒ F ⊂ E.

Assume F is a subbundle of E such that
μ min (F) > μ max (E/F). Let

0 < F₁ < ... < Fₚ₋₁ < Fₚ = F
μ₁ > ... > μₚ

and F < Fₚ₊₁ < ... < Fₘ₋₁ < E
μₚ₊₁ > ... > μₘ

be the canonical filtrations. Then μₚ₊₁ = μ max (E/F)
and μₚ = μ min (F). Thus μₚ > μₚ₊₁ and it
follows that F is part of the canonical filtration of
E.
Next assume only that $\mu_{\min} = \mu_{\max}(E/F)$ i.e. $\mu_p = \mu_{p+1}$. Then the canonical filtration of $E$ is

$$0 < F_1 < \ldots < F_{p-1} < F_p < \ldots < F_m = E$$

with the sequence of slopes $\mu_1 > \ldots > \mu_{p-1} = \mu_p > \ldots > \mu_m$. And one sees that $F$ corresponds to a division of the semi-stable chunk $F_{p+1}/F_{p-1}$ of $E$ into two semi-stable bundles of the same slope.

Given $E$ then, I want to consider these proper subsheaves $W$ of $V$ such that

$$\mu_{\min}(E/W) > \mu_{\max}(E/E/W).$$

(These will be the faces of a convex body?). These subsheaves form a simplicial complex.

**Philosophy:** For each simplex $\sigma$ in $T(V)$ I will associate a contractible subcomplex $X_\sigma$ of $X$, such that $\sigma \subset \tau \Rightarrow X_\tau \subset X_\sigma$. Then one will have a category over $X$ homotopy equivalent to $T(V)$. Over most of the points of $X$, the fibre will be contractible.
Let $W$ be a proper subspace of $V$. Define $X_W$ to be the full subcomplex of $X$ whose vertices $E$ are nice with respect to $W$ in the sense that

$$\mu_{\min}(E \cap W) \geq \mu_{\max}(E/E \cap W).$$

Given a simplex $\sigma$ in $X$, I wish to consider the full subcomplex of $T(V)$ consisting of those $W$ such that $\sigma \in X_W$. Call this complex $S_\sigma$. Clearly

$$S_\sigma = \bigcap_{E \in \sigma} S_E$$

where $S_E = \{W \mid W \text{ nice w.r.t. } E\}$.

Consider the case where $\sigma$ is a vertex $E$, in which case we want to know about the full subcomplex $S_E$ of $T_E$ whose vertices are those $W$ nice with respect to $E$. If $E$ is unstable, let its canonical filtration be $0 < E^1 \subseteq \cdots \subseteq E^{p+1} \subseteq E$. Then we have seen that any $W$ in $S_E$ is compatible with $E^1, \ldots, E^{p+1}$; hence $S_E$ is contractible.

If $E$ is stable, then there are no $W$ nice with respect to $E$, hence $S_E$ is empty.
Lemma: Let $E' \subseteq E$ be a $1$-simplex, and $W$ a proper subspace of $V$ such that

$$\mu_{\text{min}}(E \cap W) \geq \mu_{\text{max}}(E/E_{\cap W}) + d_{\infty}$$

Then

$$\mu_{\text{min}}(E' \cap W) \geq \mu_{\text{max}}(E'/E'_{\cap W}).$$

Proof: Choose $W_2 < W_1$ and $W_3 > W_1$ such that

$$\mu(E'_{\cap W_2}/E'_{\cap W_1}) = \mu_{\text{min}}(E'_{\cap W_1})$$

$$\mu(E'_{\cap W_3}/E'_{\cap W_1}) = \mu_{\text{max}}(E'/E'_{\cap W_1})$$

The because $E'_{\cap W_2}/E'_{\cap W_1}$ embeds in $E_{\cap W_1}/E_{\cap W_2}$ and the cokernel is killed by $m_{\infty}$, I know that

$$\mu(E'_{\cap W_2}/E'_{\cap W_1}) \geq \mu(E_{\cap W_1}/E_{\cap W_2}) - d_{\infty}$$

$$\geq \mu_{\text{min}}(E_{\cap W_1}) - d_{\infty}$$

And because $E'_{\cap W_3}/E'_{\cap W_1}$ embeds in $E_{\cap W_3}/E_{\cap W_1}$, I know that

$$\mu(E'_{\cap W_3}/E'_{\cap W_1}) \leq \mu_{\text{max}}(E/E_{\cap W_1}).$$

Then subtracting (2) from (1) and using (3), I get

$$\mu_{\text{min}}(E'_{\cap W_1}) - \mu_{\text{max}}(E'/E'_{\cap W_1})$$

$$\geq \mu_{\text{min}}(E_{\cap W_1}) - \mu_{\text{max}}(E/E_{\cap W_1}) - d_{\infty}$$
To each $W$ proper, let $X_w^\circ$ be the region of $X$ consisting of those open simplices $\sigma$ such that $W$ is nice with respect to each vertex of $\sigma$, and such that at least one vertex $E_{\sigma}$ of $\sigma$ has $W$ as part of its canonical filtration. Thus $X_w^\circ$ is the open part of $X_w$ obtained by removing those $E_{\sigma}$ such that $\mu_{\min}(E_{\sigma} \cap W) = \mu_{\max}(E_{\sigma} \cap W)$; equiv. such that $W$ is not part of the canonical filtration of $E_{\sigma}$.

Given a simplex $\sigma$ of $X$, let $\sigma \in X_w^\circ$, $\sigma \in X_w$, where $\dim(W) \leq \dim(W')$. By assumption $\sigma$ contains a vertex $E$ such that $W$ is part of the canonical filtration of $E$. But then because $W'$ is nice with respect to each vertex of $E_{\sigma}$ in part, $\sigma$, I know that $W \cap W'$. Hence the set of $W$ such that $\sigma \in X_w^\circ$ is either empty or contractible.

Assuming I can prove for each simplex $\tau$ of $T(V)$ that $\bigcap_{W \in \tau} X_w^\circ$ is contractible, this implies that $\bigcup_{W \in \tau} X_w^\circ$ has the homotopy type of $T(V)$.

Now the question arises as to what simplices are contained in $\bigcup_{W} X_w^\circ$. In particular I want to show that there are only finitely many $T$-classes of simplices not in this union. However if $\sigma$ is a simplex containing a vertex $E$ such that $E \in W$ with $\mu_{\min}(E \cap W) - \mu_{\max}(E \cap W) > d_\infty$, then the preceding lemma shows that $W$ is nice with respect to all vertices of $E$, hence $\sigma \notin X_w^\circ$. 


Case of $\mathbb{P}^1$, $d_\infty = 1$: Here I know, because there are no stable bundles besides line bundles, the slope of any semi-stable bundle is integral. Hence there are no simplices between two semi-stable bundles. Thus in this case $\bigcup W X_w^0$ contains all simplices except for vertices isomorphic to $\mathbb{P}^0$. Thus one finds my old formula

$$I(F^n) = H_{X,1}(X, \bigcup W X_w^0) = \bigoplus N I(N)$$

where $N$ runs over all unimodular subspaces of divisors in $V$. 
April 12, 1974. Pointing stable bundles (after Harder and Nara ---).

One starts with the Siegel formula
\[ \sum_{\Lambda^E \cong L_0} \frac{1}{\operatorname{aut}(E)} = \frac{1}{\mu} \]
where \( L_0 \) is a given element of \( \text{Pic}(C) \), and the sum is taken over isomorphism classes of \( E \) of rank \( r \) such that \( \Lambda^E \cong L_0 \). Here \( \frac{1}{\mu} \) is a constant \( \left( = \frac{1}{\delta - 1} \frac{1}{\Gamma(2)} \right) \) (I think).

\( \Lambda = 2 \), rank = 2. Remove the unstable bundles from the above sum. These are exactly
\[ \sum_{\substack{E \cong E_1 \\Lambda^E \cong L_0 \\mu(E_1) > \mu(E) \\mu(\mu(E)) \equiv 1 \\mu(E_1) \equiv 1 \\mu^2 \equiv 1\}} \frac{1}{\operatorname{aut}(E \ast E_1)} \]

In effect, \( E \) is unstable \( \Rightarrow E \) contains a unique sub-line bundle \( E_1 \) with \( \mu(E_1) > \mu(E) \), and \( \operatorname{aut}(E) = \operatorname{aut}(E \ast E_1) \).

Better: The map \( (E \ast E_1) \mapsto E \) from the groupoid of \( (E \ast E_1) \Lambda^E \cong L_0, \mu(E_1) > \mu(E) \) to the groupoid of \( \Lambda^E \cong L_0 \) is fully faithful with image the unstable \( E \).

Thus
\[ \sum_{\Lambda^E \cong L_0} \frac{1}{\operatorname{aut}(E)} = \sum_{\Lambda^E \cong L_0} \frac{1}{\operatorname{aut}(E_1)} - \sum_{\substack{E \cong E_1 \\mu(E_1) > \mu(E) \\mu^2 \equiv 1 \\mu \equiv 1}} \frac{1}{\operatorname{aut}(E \ast E_1)} \]
To evaluate this last sum we use the functor

\[(E \Rightarrow E_1) \mapsto (E_1, E/E_1)\]

which is cofibred in groupoids. Thus if I fix \(A, B\), then

\[\sum_{\substack{E \approx A \\ E \leq E_1 = B}} \frac{1}{\text{aut}(E \Rightarrow E_1)} = \frac{1}{\text{aut}(A) \cdot \text{aut}(B)} \sum_{\text{Ext}(B, A)} \frac{1}{\text{aut}(E)}\]

Now

\[\text{Ext}^1(B, A) = H^1(\text{Hom}(B, A))\]

and if \(E\) is any extension of \(B\) by \(A\)

\[\text{Aut}(E) = H^0(\text{Hom}(B, A))\]

Thus

\[\sum_{\text{Ext}^1(B, A)} \frac{1}{\text{aut}(E)} = \frac{\# H^1(\text{Hom}(B, A))}{\# H^0(\text{Hom}(B, A))} = \frac{\partial}{\partial(\text{deg} \text{Hom}(B, A) + \text{rank} \text{Hom}(B, A) (1 - \gamma))}\]

In the case at hand, \(B, A\) are line bundles. So

\[\sum_{\substack{E \Rightarrow E_1 \\ \mu(E) > \mu(E) \\ A^2 E \leq L}} \frac{1}{\text{aut}(E \Rightarrow E_1)} = \left(\frac{1}{\gamma - 1}\right)^2 \sum_{L \in \text{Pic}(C)} \frac{1}{\gamma - 2 \deg(L) - \deg E}\]

\[\deg(\text{Hom}(E \oplus L, L)) = 2 \deg L - \deg E\]
\[
\begin{align*}
\frac{h}{(\delta-1)^2} \sum_{n \geq 1} \frac{q^{-2n}}{\delta-1} &= \frac{h}{(\delta-1)^2(\delta^2-1)} \\
\mu(E) &= \frac{1}{2} \\
\mu(E) &= 0
\end{align*}
\]

\(\delta = 3\). Here we count the unstable bundles according to the canonical filtration. Category:

\[
(E \rightarrow E_1) \quad \begin{cases} 
E_1 \text{ line bundle} \\
E_1 \text{ semi-stable} \\
\mu(E_1) > \mu(E/E_1)
\end{cases}
\]

Then the functor

\[
(E \to E_1) \mapsto E
\]

is fully faithful. With image those unstable bundles whose canonical filtration reduces to a line. So if \(S\) denote by

\[
f_2(\mu) = \sum_{E \text{ semi-stable}} \frac{1}{\text{aut}(E)}
\]

the answer obtained in the rank 2 case, then

\[
\sum_{E \in E_1} \frac{1}{\text{aut}(E \to E_1)} = \sum_{E_1 \text{ line bundle}} \sum_{\text{deg}(E) > \mu(E)} \frac{1}{\text{deg}(E)} \sum_{E \in E_1 \text{ semi-stable}} \frac{1}{\text{deg}(E/E_1)}
\]

\(\text{deg}(E) > \mu(E), \text{deg}E \approx L_0(3\mu)\)
Let $\Gamma'$ be a net subgroup of $\Gamma$ of finite index in $\Gamma$, i.e., every torsion element of $\Gamma'$ is a $p$-torsion element. Then $I$ is a projective $\mathbb{Z}[\Gamma']$-module.

Let $Y = \{ \sigma \in X \mid \Gamma_0' \neq 1 \}$. Since $\sigma < \tau \Rightarrow \Gamma_0' \supseteq \Gamma_1'$, $Y$ is a sub-complex of $X$. Since $\Gamma'$ is of finite index in $\Gamma$, and since there are only finitely many $\sigma$ in $X/\Gamma$ with $\text{ant}(\sigma) \leq N$, it follows that $Y/\Gamma'$ contains all but finitely many simplices of $X/\Gamma'$.

**Conjecture 1:** $Y$ has the homotopy type of $\text{BH}(V)$.

Observe that if this is true then $I = H_{r-1}(X, Y)$ and $H_j(X, Y) = 0$ $j < r-1$, hence we will have a resolution

$$0 \to I \to C_{r-1}(X, Y) \to \cdots \to C_0(X, Y) \to 0.$$ 

But because every $\sigma$ in $X - Y$ has $\Gamma_0' = 1$, $C_j(X, Y)$ is a free $\mathbb{Z}[\Gamma']$-module of finite type, hence $I \in \mathcal{P}(\mathbb{Z}[\Gamma'])$.

**Proof of Conjecture 1 for $r=2$:** If $\sigma \in Y$, then we consider the action of $\Gamma_0'$ on $\text{BH}(V)$. As $\Gamma_0'$ is a $p$-group, there is a line $L$ in $V$ invariant under $\Gamma_0'$, in fact a unique line as $\Gamma_0' \neq 1$. Thus one has

$$Y = \bigsqcup_{L \in \text{BH}(V)} Y_L$$

where $Y_L = \{ \sigma \in Y \mid \Gamma_0' \text{ leaves } L \text{ invariant} \}$. But now if $\Gamma_0'$ leaves $L$ fixed, then the shift in the $L$-direction
will carry $\sigma$ to $\sigma^*$ such that $\Gamma'_r \subset \Gamma_{r^*}$. ✗

Thus $Y_L$ is stable under the $L$-shift. As $Y_L$ is non-empty for each $L$ it is contractible.

---

* Incomplete. Not clear that $\cap_r Y_L$.

In the general case, we might try to cover $Y$ by

$$Y_W = \{ \sigma \mid \Gamma'_r \leq W \text{ invariant} \}$$

or more generally for each $\sigma \in T(V)$, we can put

$$Y_{\sigma} = \{ \sigma \in \sigma \mid \Gamma'_r \leq \sigma \text{ invariant} \}$$

whence $Y_{\sigma}$ is open in $\sigma Y$. This leads to the following question:

**Question.** Let a $p$-group $G$ act non-trivially on $V$. Is the subcomplex of $T(V)$ consisting of the invariant subspaces contractible?

Yes. Since the action is non-trivial $V^G$ is a proper subspace. Now given $0 < W < V$ stable under $G$, $W^G = W \cap V^G$ is non-zero. Thus we get a deformation

$$W \Rightarrow W^G \leq V^G$$

of $T(V)^G$ to a point.

So at this point it is clear that I ought to be able to prove the net conjecture.
study of the net conjecture.

\[ \Gamma' \] is a subgroup of \( \Gamma \). To simplify suppose \([\Gamma : \Gamma'] = \infty\) and we are over a finite field. Put \( Y = \{ \sigma | \Gamma_0' \neq 1 \} \). Then \( Y \) is a subcomplex of \( X \).

Proposition: \( Y \) has the homotopy type of \( \mathbb{T}(V) \).

Proof: \( \Box \) Let \( Z \) be the set of pairs \((\sigma, \tau)\) where \( \tau \in Y \) and \( \tau \in \mathbb{T}(V) \) and where \( \Gamma_0' \) leaves \( \tau \) invariant. We will consider the two projections

\[
Y \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} \mathbb{T}(V)
\]

\( \tau \xrightarrow{} (\sigma, \tau) \xrightarrow{} \tau \).

Equip \( Z \) with the ordering \((\sigma', \tau') \leq (\sigma, \tau)\) if \( \sigma' \leq \sigma \) and \( \tau' \leq \tau \).

Note that if \( \sigma' \leq \sigma \), then \( \Gamma_0' \supseteq \Gamma_0' \), hence

\((\sigma', \tau) \in Z \Rightarrow (\sigma, \tau) \in Z \). Thus

\[
\{ (\sigma, \tau) \in \pi_1^{-1}(\sigma) \mid (\sigma', \tau') \leq (\sigma, \tau) \}
\]

has a least element, namely \((\sigma, \tau')\). Thus \( \pi_1 \) is cofibred with base-change \((\tau', \tau) \mapsto (\sigma, \tau')\).

If \( \tau' \leq \tau \), then \((\sigma, \tau) \in Z \Rightarrow (\sigma, \tau') \in Z \). Thus

\[
\{ (\sigma', \tau') \in \pi_2^{-1}(\tau) \mid (\sigma' \tau') \leq (\sigma, \tau) \}
\]

has a largest element, namely \((\sigma, \tau')\). Hence \( \pi_2 \) is fibred with base-change \((\sigma, \tau) \mapsto (\sigma, \tau')\).

To prove the proposition, it suffices therefore to show that the fibres of \( \pi_1, \pi_2 \) are contractible.
\[ \pi^{-1}_1(\mathcal{S}) = \{ \tau \in \mathcal{N}(V) \mid \Gamma_\tau \text{ fixes } \tau \} \]

and we have seen that because \( \Gamma_\tau \) is a p-group acting non-trivially on \( V \), this is contractible. Now

\[ \pi^{-1}_2(\mathcal{S}) = \{ \tau \in X \mid \Gamma_\tau \neq 1 \text{, } \Gamma_\tau \text{ fixes } \tau \} \]

ordered by inclusion. (This is open in \( Y \) but not in \( X \).)

Let \( \tau = 0 < \omega_1 < \ldots < \omega_p < V \). 

To simplify notation put

\[ Y_\tau = \{ \sigma \in Y \mid \Gamma_\sigma \text{ fixes } \tau \} = \pi^{-1}_2(\mathcal{S}) \].

I want to show that translation \( \circ \) with respect to \( \omega \) maps \( Y_\tau \) into itself, and that this translation is homotopic to the identity of \( Y_\tau \). Let \( \sigma = (\mathcal{E}_0 \ldots \mathcal{E}_{p}) \in \pi^{-1}_2(\mathcal{S}) \).

Then

\[ T_\omega(\sigma) = \mathcal{E}_0 + \mathcal{E}_0 \omega(1) \leq \mathcal{E}_1 + \mathcal{E}_1 \omega(1) \leq \ldots \leq \mathcal{E}_p + \mathcal{E}_p \omega(1) \].

If \( \Gamma_\tau \) fixes \( \omega \), then it is clear that \( \Gamma_\tau' \) fixes \( \mathcal{E}_j + \mathcal{E}_j \omega(1) \)

hence

\[ \Gamma_\tau' \subseteq \Gamma_\tau(\sigma) \]

and therefore \( T_\omega(\sigma) \in Y \). But it is not clear that any auto. of \( T_\omega(\sigma) \) leaves \( \omega \) fixed.

\[ ? \]

It seems to be important to know that \( \omega \) is part of the canonical filtration of \( T_\omega(\sigma) \).
In view of the above problems, it is essential that I understand the case of rank 2 and \( d_{\infty} > 1 \), and that I check the 
conjecture carefully in this case.

For example, take \( C = P^1_k \), and let \( \infty \) be a point of degree \( d_\infty \), \( A = \Gamma(C, \infty, \mathcal{O}_C) \). Then one has

\[
\kappa_0, \kappa(\infty) \to \kappa_0 C \to \kappa_0 A \to 0
\]
hence \( \text{Pic} A = \mathbb{Z}/d_\infty \mathbb{Z} \), \( h = d_\infty \). Consider rank 2 bundles reducing to \( A^2 \) over \( C-\infty \). The isomorphism classes are

\[
\mathcal{O}(a) \oplus \mathcal{O}(b) \quad a > b \quad , \quad a+b \equiv 0 \quad (d_\infty)
\]
as usual. Here \( \mathcal{O}(\infty) = \mathcal{O}(d_\infty) \), so modulo homothety the iso. classes are

\[
\mathcal{O}(a) \oplus \mathcal{O}(a) \quad a > 0
\]

\[
\mathcal{O}(d_\infty+a) \oplus \mathcal{O}(-a) \quad a \geq \frac{d_\infty}{2}
\]

Now there are \( d_\infty \)-chains represented by the chains (take \( d_\infty = 3 \))

\[
\mathcal{O}(1) \oplus \mathcal{O}(-1) \subset \mathcal{O}(4) \oplus \mathcal{O}(-1) \subset \mathcal{O}(4) \oplus \mathcal{O}(-4) \subset \ldots \ldots
\]

\[
\mathcal{O}(2) \oplus \mathcal{O}(1) \subset \mathcal{O}(2) \oplus \mathcal{O}(-2) \subset \mathcal{O}(5) \oplus \mathcal{O}(-2) \subset \mathcal{O}(5) \oplus \mathcal{O}(-5) \subset \ldots \ldots
\]

\[
\mathcal{O}(3) \oplus \mathcal{O}(3) \subset \mathcal{O}(6) \oplus \mathcal{O}(-3) \subset \mathcal{O}(6) \oplus \mathcal{O}(-6) \subset \ldots \ldots
\]

\[
\text{Aut}(\mathcal{O}(1) \oplus \mathcal{O}(-1)) = \begin{pmatrix} k^* & H^1(\mathcal{O}(2)) \\ 0 & k^* \end{pmatrix}
\]

\[
\text{Aut}(\mathcal{O} \oplus \mathcal{O}) = \text{Gl}_2(k)
\]

Question: how does \( \text{Gl}_2(k) \) act on \( P^1_k(\mathcal{O}(\infty)) \)? The action is transitive, so \( \mathcal{O} \oplus \mathcal{O} \) is no longer extremal in the quotient graph.
New idea: Define a map from the set of $\sigma$ in $X$ such that $\Gamma_{\sigma} = \Gamma_0$ to the set of $W$, $0 < W < V$ by

$$f(\sigma) = H^0(\Gamma_{\sigma}, V).$$

Since $\Gamma_{\sigma}$ is a $p$-group acting non-trivially on $V$, this is indeed a proper subspace of $V$. Also if $\sigma' \subseteq \sigma$, then $\Gamma_{\sigma'} \supset \Gamma_{\sigma}$ hence

$$H^0(\Gamma_{\sigma'}, V) \subseteq H^0(\Gamma_{\sigma}, V)$$

and so $f$ is a map of posets.

Conjecture: $f$ is a homotopy equivalence (assuming $\Gamma'$ is of finite index in $\Gamma$).

This is true for $r = 2$. In effect we showed that $f^{-1}(L) = \mathcal{V}_L$ is stable under the deformation $T_L$. The point was that if $\bigcirc \Gamma_{\sigma}'$ fixes $L$, then $L$ is unique. Thus since $\Gamma_{\sigma}' < \Gamma_T^{-1}(\sigma')$ the latter must also fix $L$.

The analogue of this argument for $r = 3$, would be to consider for any fibre $L$ the fibre $f^{-1}(L)$ which is a subcomplex; that is if $L$ is the invariant subspace of $\Gamma_{\sigma}'$, it must also be for any $\sigma' \subseteq \sigma$, as well as for $\Gamma_T^{-1}(\sigma')$. Thus $f^{-1}(L)$ is stable under $T_L$. 
What can I say about the quotient complex, when $C$ is an elliptic curve, $r=3$, $d_\infty = 1$. I can try to classify vertices according to slopes. Thus we get the following scheme:

- **canonical filtration is a flag**: here $x_1, x_2$ are pos. integers
- **is a line**: here $x_1 > 0, x_2 = 0$ and $x_1$ is half-integral when the quotient is stable.
- **plane**: $x_1 = 0, x_2 > 0$ and $x_2$ is half-integral when the plane bundle is stable.
- **no canonical filtration**: $x_1 = 0 = x_2$. Here the bundle is semi-stable. There are two stable bundles of slopes $\frac{1}{3}, \frac{2}{3}$.

**Picture:**

I have drawn the dots corresponding to stable parts as slightly negative.
I have drawn the stuff that is definitely attached to some part of the building at $\infty$.

The problem begins with the point $x_1 = 0$, $x_2 = \frac{1}{2}$ which looks like $\cdots$. Its neighbors:

\[
\begin{align*}
down & \quad x_1 = 0, \quad x_2 = \frac{9}{2} \\
 & \quad x_1 = x_2 = 0.
\end{align*}
\]

\[
\begin{align*}
up & \quad x_1 = 0, \quad x_2 = 1
\end{align*}
\]

Now if $E \subset E'$ has cokernel killed by $k(\infty)$, and $E \to L$ induces $E' \to L(1)$, then one has

\[
\begin{align*}
0 & \to F \to E \to L \to 0 \\
& \quad \downarrow \quad \Downarrow f \\
0 & \to F \to E' \to L(1) \to 0
\end{align*}
\]

where $\deg(E') = 2$, $\mu(E') = \frac{2}{3}$. If $E'$ has a subbundle of slope 1, it must map $\to L(1)$, so $E' = F \oplus L(1)$, and $E = F \oplus L$. The sequence splits for
an elliptic curve. Thus we get for $E'$ the diagram

$$i.e. \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = 0.$$ \[ \]

This gives me an example of a 1-simplicial made out of unstable bundles.

Another possibility is that $E'$ doesn't contain a subbundle of slope 1, hence it is stable of slope $\frac{2}{3}$ with diagram. E.g.

$$\text{Ext}^1(L(1), F^1) = H^1(\text{Hom}(L, F^1(-1))) = k$$

by R-R since $\text{Hom}(L(1), F^1) = 0$.

Next. Take a stable $E$ of slope 2. The possible $E'$ of colength 1 in $E$ are

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2}$$

$$\mu = \frac{1}{3} \quad \text{stable.}$$

And the possible $E''$ of colength 2 in $E$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0$$

Observe that the is impossible because the quotient bundle jumps in degree by 2.

So if I am not mistaken there is a map of the quotient complex to the simplicial complex I have drawn on page 2, at least for elliptic curves.
For a general curve \( C \) but still with \( a_0 = 1 \), we perhaps get the same classification. Again we can classify the vertices according to slope.

**semi-stable:** slopes \( 0, \frac{1}{3}, \frac{2}{3} \), \( \alpha_1 = \alpha_2 = 0 \)

**canonical line:** \( \alpha_1 \in \mathbb{Z}_{\frac{1}{2}} > 0, \alpha_2 = 0 \)

**canonical 2-plane:** \( \alpha_1 = 0, \alpha_2 \in \mathbb{Z}_{\frac{1}{2}} > 0 \)

**canonical flag:** \( \alpha_1, \alpha_2 \in \mathbb{Z} > 0 \).

Suppose we are in an interior point \( E \) with \( \alpha_1, \alpha_2 > 0 \).

Let the canonical filtration be \( 0 < L < W < V \).

Let \( E' \subset E \) be of colength one. Then we have exactly one spot in the canonical filtration where \( E' \) first differs from \( E \). Better to denote by \( 0 < E_1 < E_2 < E \) the canonical filtration.

Then if \( E' \subset E \) is of colength one, there is exactly one \( i \) for which \( E' \cap E_{i-1} = E_{i-1}, E' \cap E_i < E_i \).

**Diagram:**

\[
\begin{array}{c}
\text{i = 1} \\
(\alpha_1-1, \alpha_2) \\
\text{i = 2} \\
(\alpha_1+1, \alpha_2) \\
\text{i = 3} \\
(\alpha_1, \alpha_2+1)
\end{array}
\]

If \( E'' \subset E \) is of colength 2, then there are two places where \( E/E'' \) gets distributed.

\[
\begin{array}{c}
(\alpha_1, \alpha_2-1) \\
(\alpha_1-1, \alpha_2+1) \\
(\alpha_1+1, \alpha_2)
\end{array}
\]

\[
\begin{array}{c}
(1, 2) \\
(1, 3) \\
(2, 3)
\end{array}
\]
and so we get the typical interior hexagon.

Next consider the situation where the canonical filtration is a 2-plane $0 < E_2 < E$, i.e. $x_1 = 0, x_2 \in \mathbb{Z} \frac{1}{2} > 0$. Suppose also $x_2 > 1$.

Given $E'$ of colength one, there are two possibilities according to whether $E'$ contains $E_2$ or not.

\[(0, x_2 + 1) \quad (0, x_2 - \frac{1}{2}) \quad (1, x_2 - 1) \]

when $x_2$ is integral.

\[(0, x_2 + 1) \quad (0, x_2 - \frac{1}{2})\]

And if $E''$ is colength two, there are two possibilities.

\[(0, x_2 - 1) \quad (0, x_2 + \frac{1}{2}) \quad (1, x_2) \quad (0, x_2 - 1) \quad (0, x_2 + \frac{1}{2}) \]
So for \((0, x_2)\), \(x_2\) integral \(> 0\) I get the hexagon

And for \((0, x_2)\), \(x_2\) half integral \(> \frac{3}{2}\) I get

Now for the interesting vertices:
\((0, \frac{1}{2})\)

For \(E'\), we have the possibilities:

\((0, \frac{3}{2})\)
\((0, 0)\)

For \(E''\), we have the possibilities:

\((0, 1)\)
\((\frac{1}{2}, 0)\)

Thus we get the picture:
stable \( \mu = \frac{2}{3} \): possibilities for \( E' \):

\( \frac{1}{2} \)

\( \frac{1}{2} \)

\( \frac{2}{3} \)

\( 0 \)

\( \frac{1}{2} \)

\( \mu = \frac{1}{3} \) or \( (0, \frac{1}{2}) \)

possibilities for \( E'' \):

\( (0, 0) \)

Conclusion: The complex on page 3 should be the same for any curve.

What is the region I know has the homotopy type of the building at \( \infty \)? I am happy about any simplex containing a vertex with either \( x_1 \) or \( x_2 \geq 1 \).
Example: Take the simplex $\mathcal{O}(1) \oplus \mathcal{O}(-1) < \mathcal{O}(1) \oplus \mathcal{O}(2)$

This joins two different cusps. This is an example of what one might call a stable simplex, because there is no line nice with respect to the vertices.

Now I should check the net conjecture in the case that $d_\infty > 1$. The net subgroup of $\Gamma$ of finite index, $\mathcal{Y} = \{ \gamma \mid \Gamma_\gamma \neq 1 \}$. I saw that $\Gamma_\gamma$ fixes a unique line $L$ in $\mathcal{V}_\gamma$ hence

$$\mathcal{Y} = \bigsqcup_{L} \mathcal{Y}_L \quad \mathcal{Y}_L = \{ \gamma \in \mathcal{Y} \mid \Gamma_\gamma \text{ fixes } L \}$$

Now I have to show that translation with respect to $L$ carries $\mathcal{Y}_L$ into itself. It is clear that if $\sigma \in \mathcal{Y}_L$, then $\Gamma_\sigma \mathcal{L}(0) \in \mathcal{Y}$. Moreover, the unique line fixed by $\Gamma_\sigma \mathcal{L}(0)$ must be $L$ clearly.
The net conjecture for \( n=3, d_0 = 1 \).

Denote by \( Z \) the subcomplex of \( X \) obtained by removing the open stars of the following simplices:

1. stable vertices
2. direct sum of stable bundles of degree zero.
3. edges of the form \( F(-1) \oplus L < F \oplus L \) where \( L \) is of degree 0 and \( F \) is stable of rank 2 and deg 1.

I want to show that \( Z \) has the homotopy type of \( T(V) \). So for each proper subspace \( W \) of \( V \) put

\[
Z_W = \{ \sigma \in Z \mid \text{W nice wrt } \sigma \}
\]

where nice means nice with respect to each vertex \( E \) of \( \sigma \), i.e. \( \mu_{\min}(E/W) \geq \mu_{\max}(E/EW) \). Observe that \( Z_W \) is closed in \( Z \).

Check

\[
(*) \quad \bigcup_W Z_W = Z
\]

Let \( \sigma \) belong to \( Z \). Then \( \sigma \) has no stable vertices

If some vertex, \( E \) of \( \sigma \), has a root \( \geq 1 \), and if \( W \) is the corresponding subspace, I know that \( \sigma \) is nice with respect to each of the other vertices, so \( \sigma \) is in \( Z_W \), so I can suppose no root is \( \geq 1 \).

If \( \sigma \) consists of a vertex \( E \), then \( E \) is not stable, so \( \exists W \ni W \text{ nice wrt } E \), so \( \sigma \in Z_W \).

So I can suppose \( \text{card}(\sigma) \geq 2 \).

Then at least one vertex \( \sigma \) has deg \( \equiv 1 \) or 2 \( \pmod{3} \), so as this vertex is not stable, it must be non-semi-stable.
Call this vertex $E$, so that either $\alpha_1(E) = -1$ or $\alpha_2(E) > 0$. Take the case $\alpha_1(E) > 0$, whence we have a line $\xi$ subbundle $E_1$ of $E$ as part of the canonical filtration of $E$; suppose $E_1 = E \otimes L$, $L$ line in $V$, and $\deg E_1 = 0$. Since $\alpha_2(E) < L$, geometry shows $E/E_1$ is stable of degree $-1$.

If $L$ is not nice for $\sigma$, there exists another vertex $E'$ such that $L$ is not nice with respect to $\sigma$. Assuming $E' \in E$ with cohomology killed by $m(x)$, then it must be that $E' \otimes L = E_1(-1)$. In effect $\mu(E' \otimes L) \leq \mu_{\max}(E') \leq \mu_{\max}(E)$, so if $E' \otimes L$ had degree 0, it would have to be part of the canonical filtration of $E$.

Since $L$ is not nice with respect to $E$ and $E'$ is not stable, the only possibility is $\mu_{\max}(E') = \frac{-1}{2}$ (i.e., $\mu_{\max}(E') > -1$), so the possibilities are $\frac{-1}{2}, \frac{-2}{3}$. Let $W$ be a $\mu(E' \otimes W) = \frac{1}{2}$. Then $E' \otimes W = E \otimes W$. So now consider the map

$$E \otimes W \mapsto E \mapsto E/E \otimes L$$

of stable bundles of slope $\frac{1}{2}$. Has to be an isom, hence $E = E \otimes L \oplus E \otimes W$. Hence $E' = (E \otimes L)(-1) \oplus E \otimes W$ and we have conveniently reached that $(E' \otimes E) = \sigma$ is of the type III of things we have removed.

The case $\alpha_2(E) > 0$ is similar.

Thus (a) has been proved.
Next one must show that $V\circ c Z$

$$S_\sigma = \{ W \mid W \text{ nice w.r.t } \sigma \}$$

is contractible.

As before there is no problem if some root is $\geq 1$, because then I get a subspace $W$ nice with respect to $\sigma$ such that any other $W'$ nice with respect to $\sigma$ is either included or is included in $W$. In fact, once I know that $\exists W$ nice w.r.t $\sigma$ such that $W$ belongs to the canonical filtration of some vertex, the contractibility is clear.

First worry about $S_\sigma$ being empty. This can only happen if $\sigma$ contains a stable vertex, or if $\sigma$ contains the example $F(4) \oplus L \subset F \oplus L$.

Thus the operating lemma appears to be:

**Lemma:** For $r=3$, $d_\infty = 1$, the only $\sigma$ for which

$$S_\sigma = \{ W \mid W \text{ nice w.r.t } \sigma \}$$

is empty, are the $\sigma$-s containing a stable vertex, or an edge $F(-1) \oplus L \subset F \oplus L$ where $F$ is stable of $\mu = \frac{1}{2}$, $L$ of deg 0.
Review the situation. I want to prove that if $\Gamma'$ is a subgroup of finite index which is not then $I$ is a projective finite type $Z[\Gamma']$-module. To do this I want to enclose the set of $\sigma$ such that $\Gamma' \neq 1$ into a complex $Y$, having the homotopy type of $T(V)$. It will then follow that $I = H_{n-1}(X, Y)$, and $H_2(X, Y) = 0$ for $i < k - 1$, and from the other hand the groups $C_i(X, Y)$ will be $Z[\Gamma']$-free.

Now I can start by letting $Y$ be the region consisting of all simplices $\sigma$ such that some vertex has a root $\geq 1$. Better, I call a proper subspace $W$ of $V$ nice with respect to $E$ if $\mu_{\min}(W \cap E) \geq \mu_{\max}(E/E \cap W)$. Then I let $Y$ consist of all $\sigma$ such that there exists at least one space $W$ with $W$ nice wrt all $E$ in $\sigma$ and $W$ part of the canonical filtration of some vertex of $\sigma$. Such $\sigma$ I can call unstable, and it determines a simplex in $T(V)$, namely its canonical filtration = those $W$ nice with respect to all $E$ very nice with respect to me.

In the case $r = 3, d_x = 1$, I have classified all simplices which are not unstable.

i) semi-stable vertices

ii) simplices containing a stable vertex

iii) simplices containing the edge $F + L = F(-1) + L$

\[2 \text{simplices to count} \quad \begin{array}{c}
\text{1 simp} \\
\text{0 simp}
\end{array} \]
Thus one adds to those simplices where some root is always \( \geq 1 \) the region

\[ \bigcap \]

**Digression:** In the general situation where \( d \) is not necessarily one, the natural thing to consider is \( U = \{ 0 \mid 0 \text{ contains a vertex with a root } \geq d \} \). More precisely, such that the canonical filtration has a slope change of at least \( \geq d \). Check this: let \( \sigma \) be \( E_0 < \cdots < E_p = E \), and suppose that

\[ \mu_{\min}(E \cap W) \geq \mu_{\max}(E/E \cap W) + d \]

Then one has for \( E' = E_1 \) that

\[ \mu_{\min}(E' \cap W) \geq \frac{\mu_{\min}(E \cap W)}{d} \]

\[ \mu_{\max}(E'/E' \cap W) \leq \mu_{\max}(E/E \cap W) \]

\[ \mu_{\min}(E' \cap W) \geq \mu_{\max}(E'/E' \cap W) \]

and so \( W \) is nice w.r.t. \( E' \).

**Good object:** \( U = \text{open region containing those vertices having a slope change } \geq d \).