
Nature of Schubert cells: \( G = G_n \), \( B = \text{Borel subgroup} \), \( Y = G/P \) a flag manifold. The \( B \)-orbits on \( Y \) are cells \( (\cong k^m) \) with unique \( T \)-fixpts. The theory classifies and parametrizes these \( B \)-orbits (essentially row-echelon forms). Schubert cycles are the closures of these orbits. Desingularization of Schubert cycles + their cohomology classes (paper of Gelfand + co).

\[ B \setminus T(V) \cong 2^\Delta(n-1). \]

Schubert decomposition.

Version for a field \( K \) with \( \text{d}.v. \) \( B = \text{Iwahori subgroup} \). One classifies and parametrizes the \( B \)-orbits on the simplices of the building. This interests me because \( \Omega U_n \sim \mathcal{L} = \text{space of lattices in } \mathbb{Z}^n \), and the cell decomposition gives the minimal cell complex structure for \( \Omega U_n \).

Poset of Schubert cells. I wanted to determine the homotopy type of this poset, but too hard.
December 28, 1974  Schubert cycles.

Let's examine carefully Schubert cells and cycles in the Grassmannian of d-planes in \( V \). Here \( \alpha : \{ i \} \rightarrow \{ 1, 2 \} \) is determined by \( \alpha^{-1}(1) = \{ a_1 < \ldots < a_d \} \). The cell \( C_\alpha \) in \( Y_d(V) \) consists of all \( d \)-dimensional subspaces \( F \) such that

\[
\dim (F \cap V_\alpha) = \card \{ a \leq p \mid \alpha(a) = 1 \}
\]

i.e. such that the filtration \( F \cap V_\alpha \) of \( F \) has jumps at \( p = a_1, \ldots, a_d \).

My candidate for a resolution of \( \overline{C}_\alpha \) is the manifold \( \widetilde{C}_\alpha \) consisting of all flags

\[
F^1 < \ldots < F^d \quad \text{in} \quad V
\]

such that \( \dim F^i = i \)

and \( F^i \subset V_\alpha \).

Better approach — define \( \widetilde{C}_\alpha = \) subset of \( Y_d \) consisting of \( F \) such that \( \dim (F \cap V_\alpha) \geq \card \{ a \leq p \mid \alpha(a) = 1 \} \), i.e. \( \dim (F \cap V_\alpha) \geq i \). Then \( \overline{C}_\alpha \) is closed and it contains \( C_\alpha \). Also \( \widetilde{C}_\alpha \) maps to \( \overline{C}_\alpha \) and fibres over \( C_\alpha \) are single points.

It's also clear that \( C_\alpha \) is open in \( \overline{C}_\alpha \).

It's clear that \( \overline{C}_\alpha = U \cup \beta \) where \( \beta \) is any map corresponding to a sequence \( b_1 < \ldots < b_d \) such
that $b_i < a_i$. But now I can see easily that $C_{b_i}$ is in the closure of $C_{a_i}$. For given $F \in C_{b_i}$ we write $F = L_1 \oplus \cdots \oplus L_d$ where $L_j \in PV_{a_i} - PV_{b_i}$. Now since $b_i < a_i$, I can find $L_j' \in PV_{a_i} - PV_{b_i}$ as close to $L_j$ as I like. Then $F = L_1' \oplus \cdots \oplus L_d'$ is in $C_{b_i}$ and it can be made as close to $F$ as I want.

To each such flag, I can associate a sequence $b_1 \leq b_2 \leq \cdots \leq b_d$ such that $L_j' \in \mathcal{F}_{b_j}$. Suppose $F' = \cdots = F^d$ is in $C_{a_i}$, i.e. $F^d \in V_{a_i}$ and choose $L_i$ such that $F^i = L_1 \oplus \cdots \oplus L_i$. Then $L_i$ can be approximated by $L_i' \in PV_{a_i} - PV_{b_i}$.

Put

$$F^i' = L_1' \oplus \cdots \oplus L_i'$$

Then $F^d' = \bigoplus_p^d$ has jumps at $p = a_1, \ldots, a_d$ and $F^d' \cap V_{a_i} = F^i'$, so $F' \in C_{a_i}$. Thus $C_{a_i}$ is dense in $C_{a_i}$. Note that $C_{a_i}$ is the Schubert cell in $Y_{i, \ldots, a_i}$ corresponding to the map $[1, \ldots, n] \to [1, \ldots, d+1]$ sending $a_1, \ldots, a_d$ to $1, \ldots, d$ and the rest to $d+1$. 
Here is how I have computed the cohomology class associated to $\overline{C}_x$. Since $\overline{C}_x \to C_x$ is birational I took the class is $f^*1$ where $f: \overline{C}_x \to Y_d$. I factor $f$ into

$$\overline{C}_x \xrightarrow{\pi} Y_{1, \ldots, d} \xrightarrow{f} Y_d$$

where $\pi$ is the pullback bundle of $\overline{F}_d$ on $Y_d$. Now I propose to compute $\pi^*1$ and then use my res. formula for $p^*$.

Recall the computation of $\pi^*1$. First choice in the construction of $E = (E'_1, \ldots, E'_d)$ in $\overline{C}_x$ is the choice of $E'_1$ which can be any element of $PV_{a_1}$. What this means is that I am looking at:

\[\overline{C}_x \xrightarrow{\pi} Y_{1, \ldots, d} \xrightarrow{\pi} \{F'_1, F'_2\} \xrightarrow{f^*C V_{a_1}} Y_{12} \xrightarrow{p^*} PV_{a_1} \subset Y_1 = PV\]

or better I should write a triangular array
\[ \{ F \leq F \} \subset \{ F \leq F \} \subset \{ F \leq F \} \subset \{ F \leq F \} \]

vanishing of a trans. section of \( \text{Hom}(F', V/V_a) \)

**Lemma:** Suppose one has \( Z \subset Y \subset X \) where \( Y \) is the zero-set of a section of a vector bundle \( F \) on \( X \), and where \( Z \) is the zero-set of a trans. section of \( i^* E \), where \( E \) is a v.b. on \( X \). Then

\[ i_* j_* 1 = e(E) e(F) \]

**Proof:** \( j_* 1 = e(i^* E) = i^* e(E) \)

\[ i_* j_* 1 = i_* (i^* e(E)) = i_* 1. e(E) = e(F) e(E) \]

Q.E.D.
As a consequence the class of $C_\alpha$ in $H^*(Y_{d-1})$ is
\[
\prod_{i=1}^d e\left(\text{Hom}\left(\mathcal{F}^i/\mathcal{F}^{i-1}, V/V_{a_i}\right)\right) = \prod_{i=1}^d \xi_i^{n-a_i}
\]

Residue formula: \( f : PE \to X \quad d = d_{\text{nil}(E)} \)
\[
\tilde{\xi} = e(\alpha(1)).
\]

\[
f_* a(\tilde{\xi}) = \text{res} \frac{a(T) dT}{T^{d+\ldots+c_d E}}
\]

Generalizes to
\[
\tilde{\xi}_j = e\left(\frac{\mathcal{F}^j}{\mathcal{F}^{j-1}}\right)
\]
\[
\tilde{\xi}_j = e\left(\frac{\mathcal{F}^j}{\mathcal{F}^{j-1}}\right)
\]
\[
f_d(a(\tilde{\xi}_1, \ldots, \tilde{\xi}_d)) = (f_{d-1})_* \text{res} a(\tilde{\xi}_1, \ldots, \tilde{\xi}_{d-1}, T_d) dT_d
\]

Call this $f_1$ and generalize to $f_d : Y_{d-1} d(E) \to X$.

Put \( \tilde{\xi}_j = e\left(\frac{\mathcal{F}^j}{\mathcal{F}^{j-1}}\right) \)

\[
(f_d)_* a(\tilde{\xi}_1, \ldots, \tilde{\xi}_d) = (f_{d-1})_* \text{res} \frac{a(\tilde{\xi}_1, \ldots, \tilde{\xi}_{d-1}, T_d)}{T_{d+\ldots+c_{d-1}(E/\mathcal{F}_{d-1})}} dT_d
\]

\[
= (f_{d-1})_* \text{res} \frac{a(\tilde{\xi}_1, \ldots, \tilde{\xi}_{d-1}, T_d) \prod_{j<d} (T_d - \tilde{\xi}_j) dT_d}{T_{d+\ldots+c_n(E)}}
\]
so iterating:

\[ (f_d)^* \alpha_{(i_1 \rightarrow \ldots \rightarrow i_d)} = \text{res} \left[ \frac{a(T_{i_1}, \ldots, T_d) \prod_{i < j \leq d} (T_j - T_i) \, dT_1 \cdots dT_d}{T_1^{n-1} + \cdots + c_n E} \right] \]

so taking \( E = F_d \) on \( Y_d(V) \) I get for the class of \( C_x \):

\[ \text{coeff of } (T_1 - T_d)^{n-1} \text{ in } \prod_{i \leq 1} T_i^{n-1} \prod_{j > d} (T_j - T_i) \frac{d}{d (T_i^{n-d} + c_{n-d} (V/F_d))} \]

Now \( \prod T_{j-i} = \begin{vmatrix} 1 & T_1 & T_1^{-1} \\ \pm T_d & T_d^{-1} \end{vmatrix} \)

and so calculation shows this coeff. to be

\[
\begin{pmatrix}
C_{n-d-a_1+1}(V/F_d) & C_{n-d-a_1+2}(V/F_d) & \ldots \\
C_{n-d-a_2+1}(V/F_d) & C_{n-d-a_2+2}(V/F_d) & \ldots
\end{pmatrix}
\]

Can we generalize this calculation to other flag manifolds.
We will now work with $Y = Y_{i, \ldots, n}(V)$, and now $\alpha : \{i, \ldots, n\} \rightarrow \{1, \ldots, n\}$. $C_{\alpha}$ consist of flags $F_{i} \subset \cdots \subset F_{n}$ such that

$$\dim F_{i} = i$$

$$\dim F_{i} \cap V_{p} = \text{card } \{a \leq p | \alpha(a) \leq i\}$$

$C_{\alpha}$ will consist of families $F_{\cdot p}$ monotone in $i$ and $p$ such that $F_{np} = V_{p}$ and

$$\dim F_{\cdot p} = \text{card } \{a \leq p | \alpha(a) \leq i\}.$$
possibility for $F_{j,n-1}$. But $F_{j,n-1} = F_{j,n}$ for $j < \alpha(n)$, the first choice we have comes with $j = \alpha(n)$, except that then $F_{\alpha(n)-1,n-1} = F_{\alpha(n),n-1}$. But the next time $F_{\alpha(n)+1,n-1}$ is a hyperplane in $F_{\alpha(n)+1,n}$ containing $F_{\alpha(n),n-1}$. Thus the possibilities form a projective line.

Thus $Z_\alpha$ will be the end of a succession of fibre bundles with $P^1$-fibres.

For every $j, p$ let $Z_{j,p}$ denote the manifold consisting of choices for $F_{j,p}$, for $p' > p$ or $p' = p$ and $j' < j$. Thus the fibres of $Z_{j,p} \rightarrow Z_{j-1,p}$ (or $Z_{j,p} \rightarrow Z_{n,p+1}$) are the choices for $F_{j,p}$. The conditions on $F_{j,p}$ are what?

$$\dim (F_{j,p+1}/F_{j,p}) = \begin{cases} 0 & \alpha(p+1) > j \\ 1 & \alpha(p+1) \leq j \end{cases}$$

$$\dim (F_{j,p}/F_{j-1,p}) = \begin{cases} 0 & \exists a \leq p, \alpha(a) = j \text{ or } \alpha^{-1}(j) \geq p \\ 1 & \exists a \leq p, \alpha(a) = j \text{ or } \alpha^{-1}(j) < p \end{cases}$$

We get two 1's when $\alpha(p+1) \leq j$, and $\alpha^{-1}(j) \leq p$.

Thus the pair $\alpha^{-1}(j) < p+1$ will have $j = \alpha(\alpha^{-1}(j)) > \alpha(p+1)$.

So we see the choice of $F_{j,p}$ matters exactly when $\alpha^{-1}(j) < p+1$. 

the manifold consisting of choices
Thus in the construction of $Z_x$ we count pairs $a < p$ such that $x(a) > x(p)$, first by decreasing $p$ and then increasing $x(a)$.

Basic line bundles on $Z_x$ will be the quotients $\tilde{F}_{j,p}/\tilde{F}_{j,p}$ for $\pi^{-1}(j) < p+1$, $j > x(p+1)$.

If I put $\tilde{\xi}_p = e(\tilde{\xi}_p)$, then for the map

$$ f : Z_{j,p} \to Z_{j,p} $$

one has $f_x(1) = 0$

$$ f_x(\tilde{\xi}_p) = \tilde{\xi}_p. $$

Thus it would seem that if we use the notation $\tilde{\xi}_p$ for any of the pairs $(j,p)$ as $a$, then $H^*(Z_x)$ has as basis over $H^*(\mathbb{Y})$, the monomials $\Pi \tilde{\xi}_p$ where $\tilde{\xi}_p = 0, 1$ and all of these but the top one are killed by the integration map to $H^*(\mathbb{Y})$.

So to finish I have to express the cohomology class of $\hat{\xi}_p \in Z_x$ in terms of the basis $\Pi \tilde{\xi}_p$ and integrate.
Now $\tilde{C}_x$ is where $F_{np} \subset V_p$ for all $p$, so its class is

$$\prod_{p=1}^n e(\text{Hom}(F_{np}/F_{np-1}, V/V_p))$$

Rest looks like fun!

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**Summary:** Given the permutation $\sigma$, I let $Z_\sigma$ be the manifold consisting of systems $F_{np}$, $1 \leq j \leq n$, of subspaces of $V$, which are monotone in $y$ and $p$, and such that $\dim (F_j) = \text{card}\{x \in \sigma^{-1}(y) \leq j\}$. On $Z_\sigma$ I have tautological vector bundles $\tilde{F}_{np}$.

Let $p_1: Z_\sigma \to Y$ send $(F_{np})$ to the flag $(F_{jn}, 1 \leq j \leq n)$, and $p_2: Z_\sigma \to Y$ send $(F_{np})$ to the flag $(F_{np}, 1 \leq p \leq n)$.

$\tilde{C}_\sigma$ is the fibre of $p_2$ over the base flag $(V)$. I know the restriction of $p_1$ to $\tilde{C}_\sigma$ is a birational map $\tilde{C}_\sigma \to C_\sigma$. Thus if I want to integrate a cohom class over $C_\sigma$ I can pull it back to $Z_\sigma$, multiply by the class of $\tilde{C}_\sigma$ and integrate over $Z_\sigma$.

The class of $\tilde{C}_\sigma$ in $Z_\sigma$ is the inverse image of the top class of $H^*(Y)$ under $p_2$.

$$\prod_{p=1}^n e(\text{Hom}(F_{np}/F_{np-1}, V/V_p)) = \prod_{p=1}^n \epsilon_{n-p}$$

where $\epsilon_{np} = e(F_{np}/F_{np-1})$; \( \ast \) as $\dim(F_{np}/F_{np-1}) = \{0, 1, \ldots, x(p)\}$. 

we consider $f_{jp}$ only when $j \geq \alpha(p)$.

A typical class in $H^*(Y)$ is of the form $\xi_1^{a_1} \cdots \xi_n^{a_n}$. Better: one gets a basis for $H^*(Y)$ of the form $\xi_1^{a_1} \cdots \xi_n^{a_n}$ with $0 \leq a_i < n-i$. When lifted up to $\mathbb{Z}_2$, via $p_1$, it becomes

$$\prod_{p=1}^{n} \xi_{p,n}^{a_p}$$

So the problem is to compute

$$\int_{\mathbb{Z}_2} \prod_{p=1}^{n} \xi_{n,p}^{a_p} \cdot \prod_{p=1}^{n} \xi_{p,a}^{a_p}$$

Alternate approach to integrating over $Y$:
Consider $Y$ as $U/T \delta Y$, i.e. perpendicular lines, i.e. I embed $Y$ inside of $(PV)^n$. I want to compute the class of $Y$.

$$Y = \text{subset of } \bigoplus_{i=1}^{n} L_i \text{ in } (PV)^n$$

where $L_i \cap L_j$ for each $i < j$. But if $H_{ij}$ is the set where $L_i \cap L_j$, then $H_{ij}$ is where

$$L_i \subset O \otimes V \overset{\text{orth proj.}}{\rightarrow} \mathbb{L}_j$$

vanishes, so $[H_{ij}] = c_1(L_i \otimes \mathbb{L}_j) = \xi_i - \xi_j$. These hyperplanes intersect transversally at $Y$, so the
cohomology class of $\gamma$ is $\prod_{i<j} (x_i - x_j)$ up to sign.

Let suppose given a $k$ might try to describe the image of the Schubert cell induced by $k$ inside of $(PV)^n$.

Review what I've understood about $Y_d(V)$. Here a Schubert cell $C_\alpha$ is described by $\alpha : \{1, \ldots, n\} \to \{1, 2\}$ and $\alpha^{-1}(i) = d_i$, or equivalently $\alpha^{-1}(1) = 1 < a_1 < \cdots < a_d < n$. $C_\alpha = \bigcup C_\beta$, where $\beta$ runs over sequences $b_1 < \cdots < b_d$ such that $\beta_i < \alpha_i$ for all $1 \leq i \leq d$. $C_\alpha$ consists of flags $F^1 < \cdots < F^d$ in $V_d$ such that $F^i \subset V_{\alpha_i}$. A nice way to describe an element of $C_\alpha$ is to give independent lines $L_1, \ldots, L_d$ with $L_i \subset V_{\alpha_i}$ and put $F^i = L_1 \oplus \cdots \oplus L_i$; this description is 1-1 if one requires the $L_i$ to be perpendicular.

Next I want to understand $Y_{12}(V)$, where $C_\alpha$ is described by $\alpha : \{1, \ldots, n\} \to \{1, 2, 3\}$, which may be identified with $\alpha^{-1}(1), \alpha^{-1}(2)$. $C_\alpha$ consists of $F^1 < F^2$ such that

$$\dim F^i \cap V_p = \text{card } \{ q \leq p \mid \alpha(q) \leq j \}.$$  Thus $F^i \cap V_p$ jumps at $\alpha(p) = 1$, or $p = \alpha^{-1}(1)$.
and $F^2 \cap V_p$ jumps at $p = \alpha^{-1}(1), \alpha^{-1}(2)$. There are two cases:

1) $\alpha^{-1}(1) < \alpha^{-1}(2)$. Here I can describe $C_\alpha$ as generated by lines $L_1, L_2$ such that $L_1 \in PV_{\alpha^{-1}(1)} - PV_{\alpha^{-1}(1)-1}$, $L_2 \in PV_{\alpha^{-1}(2)} - PV_{\alpha^{-1}(2)-1}$, and the description is 1-1 if I require $L_2$ to be perpendicular to $L_1$. This sort of cell I encountered before as the lift of the cell in $Y_2(V)$ corresponding to the sequence $\alpha^{-1}(1) < \alpha^{-1}(2)$.

2) $\alpha^{-1}(1) > \alpha^{-1}(2)$. Here elements of $C_\alpha$ are in 1-1 correspondence with pairs of lines $L_1, L_2$ such that $L_1 \in PV_{\alpha^{-1}(1)} - PV_{\alpha^{-1}(1)-1}$.

In the first case, the closure of $C_\alpha$ is the set of $F^1 < F^2$ such that $F^1 \subset V_{\alpha^{-1}(1)}$, $F^2 \subset V_{\alpha^{-1}(2)}$, and it is a non-singular subvariety whose cohomology class is $\frac{n - \alpha^{-1}(1)}{1} \cdot \frac{n - \alpha^{-1}(2)}{2}$. Here $\tilde{C}_\alpha = C_\alpha$.

In the second case, $C_\alpha$ consists of triples $F^1 < F^2 < L_2$, where $F^1 \subset V_{\alpha^{-1}(1)}$, $F^2 \subset V_{\alpha^{-1}(2)}$. In effect, an element of $C_\alpha$ is given by $(F_p)$, where $F_p$ has a jump at $\alpha^{-1}(1)$, hence $(F_p)$ is given by $F^1 = F_{p, \alpha^{-1}(1)}$; $F_{2p}$ has jumps at $\alpha^{-1}(2)$ and $\alpha^{-1}(1)$, hence is determined by $L_2 = F_{2p, \alpha^{-1}(2)}$, $F^2 = F_{2p, \alpha^{-1}(1)}$. 
Note that the image of $C_\alpha$ in $Y_2(V)$ consists of all $F^2$ such that $F^2 \cap V_p$ jumps at $p=a_1$ and $p=a_2$, where I put $a_1 = \alpha^{-1}(1) < a_2 = \alpha^{-1}(2)$. Thus the inverse image of the cell in $Y_2(V)$ described by $a_1 < a_2$ becomes 2-cells in $Y_{1,2}(V)$, namely given by $\alpha^{-1}(1,2) = (a_1, a_2)$ and $(a_2, a_1)$ respectively.

Change notation: Suppose we fix $1 \leq a_1 < a_2 \leq n$ and define $\alpha, \beta, \gamma$ by

$\alpha: \{1, \ldots, n\} \rightarrow \{1, 2, 3\}$ \quad $\alpha^{-1}(1) = a_2$ \quad $\alpha^{-1}(2) = a_1$

$\beta: \{1, 2\}$ \quad $\beta^{-1}(1) = a_1$ \quad $\beta^{-1}(2) = a_2$

$\gamma: \{1, \ldots, n\} \rightarrow \{1, 2\}$ \quad $\gamma^{-1}(1) = \{a_1, a_2\}.$

So that $C_\gamma \subset Y_2$ and $C_\gamma \cap C_\beta = C_\beta \subset Y_{1,2}$. I know

$\overline{C_\gamma} = \overline{C_\beta} = \overline{C_\beta} = \{ F^1 < F^2 \mid F^1 \cap V_{a_1}, F^2 \cap V_{a_2} \}$

has the cohomology class $\xi^{n-a_1}_{\beta 1} \xi^{n-a_2}_{\beta 2} \in H^*(Y_{1,2}).$

On the other hand $\overline{C_\alpha}$ consists of $F^1 < F^2 > L_2$ as above where $F^2 > L_2$ is a point of $\overline{C_\beta}$. Thus

$\overline{C_\alpha} = \text{projective bundle } \mathbb{P}(F^2) \text{ over } \overline{C_\beta},$

which means that $\overline{C_\alpha}$ is the inverse image of $\overline{C_\gamma}$. 
To calculate the cohom class of $C_\alpha$, we want to use

\[ \tilde{C}_\alpha \to \tilde{C}_\beta \to Y_{12} \to Y_2 \]

Better - we want to use

\[ (F^1 < F^2 > L_1) \]

\[ L_1 \subset V_{a_1} \]

\[ f \circ p \]

\[ Y_{12} \times Y_2 \]

\[ Z \]

\[ [C_\alpha] = f_* \cdot 1 = p_* \cdot i_* \cdot 1. \]

Now,

\[ i_* \cdot 1 = e(L_1^{\vee})^{n-a_1} \cdot e(F_2/L_1^{\vee})^{n-a_2}. \]

and $p$ is the projective fibre bundle $PF_2$ over $Y_{12}$ where $\mathcal{O}(-1) = \mathbb{L}_2$.

Put $z_1 = e(L_1^{\vee})$, $z_2 = e(F_2/L_1^{\vee})$ where
\[ z_1 + z_2 = e_2(\tau_1) = c_1(\tau_1) + c_1(\tau_2) + c_1(\tau_2) = i_1 + i_2 \]
\[ z_1 \cdot z_2 = e_2(\tau_2) = i_1i_2. \]

Now, \[ \lambda_x 1 = z_1^{n-a_1} z_2^{n-a_2} \] we should rewrite this in the form \[ M(i_1, i_2) + N(i_1, i_2) \cdot z_1 \] so we can compute \[ p_x. \]

\[ \lambda_x 1 = (z_1z_2)^{n-a} \cdot z_1^{-a_1-a_2} \]

Working with universal polynomials, we can write

\[ f(z_1, z_2) = \frac{g}{z_2} + \frac{h}{2} (z_1 - z_2) \]

where \( g, h \) are symmetric:

\[ g = f(z_1, z_2) + f(z_2, z_1) \]
\[ h = \frac{f(z_1, z_2) - f(z_2, z_1)}{z_1 - z_2} \]

Then integrating and using that \[ p_x(z_1) = 1 \quad p_x(z_2) = -1 \]

one gets

\[ p_x f = h \]

So for example if \[ f = \frac{z_k}{z_2} \], then

\[ h = \frac{z_1^{k-1} - z_2^{k-1}}{z_1 - z_2} = + \left( z_2^{k-1} + \cdots + z_1^{k-1} \right) \]
\[ = + \left( i_2^{k-1} + \cdots + i_1^{k-1} \right) \]

Thus

\[ p_x \lambda_x 1 = (i_1i_2)^{n-a} \cdot \left( i_2^{q_2-1} + \cdots + i_1^{q_2-1} \right) \]
So we find

\[ [C_2] = \frac{\xi^{n-a_1}}{2} + \frac{\xi^{n-a_2}}{2} + \cdots + \frac{\xi^{n-a_d}}{2} \]

Next try to generalize to \( Y_{1,2,\ldots,d} \) A Schubert cell in \( Y_{1,2,\ldots,d} \) is described by \( \alpha: \{1, \ldots, n\} \to \{1, \ldots, d+1\} \)
such that \( \alpha^{-1}(1), \ldots, \alpha^{-1}(d) \) have one element. Let
\( a_1 < \cdots < a_d \) be the set \( \alpha^{-1}(1), \ldots, \alpha^{-1}(d) \) arranged in order.
Then if \( (F^1 \cdots F^d) \in C_\alpha \) one has

\[ \dim (F^d n V_p) = \text{card} \{ a \leq p \mid \alpha(a) \leq d \} \]

hence \( F^d n V_p \) has jumps at \( p=a_1, \ldots, a_d \). Note that
if \( F^d n V_p = F^d n V_{p+1} \Rightarrow F^d n V_p = F^d n V_{p+1} \) for \( j \leq d \).
Thus once the jumps in \( F^d n V_p \) are given, one has the flag \( F^d n V_{a_1} \cdots \) in \( F^d \), and then the orbit type of \( (F^1 \cdots F^d) \) is determined by the relation of these two flags in \( F^d \).
December 31, 1974

1. Problem: Find formulas for the cohomology classes belonging to the Schubert cells in $Y_1 \ldots d$.

Given $\alpha: \{1, \ldots, n\} \rightarrow \{1, \ldots, d+1\}$ such that $\text{card } \alpha^{-1}(j) = 1$ for $1 \leq j \leq d$, one has the Schubert cell

$$C_\alpha = \left\{ (F_1, \ldots, F_d) \in Y_1 \ldots d \mid \text{dim} \left( F_{\alpha^{-1}(p)} \right) = \text{card} \left\{ a \in \alpha^{-1}(p) \mid a \leq j \right\} \right\}.$$

I propose to resolve $C_\alpha$ using the manifold $C_\alpha$ consisting of systems of $d$ subspaces $F_{\alpha^{-1}(p)}$, $1 \leq j \leq d$, $1 \leq p \leq n$, monotone in $j$ and $p$,

$$\text{dim} = \text{card} \left\{ a \in \alpha^{-1}(p) \mid a \leq j \right\},$$

such that $F_{\alpha(p)} \cap \cap_{j=1}^{d} F_{\alpha^{-1}(p)}$. That $C_\alpha$ is a manifold I have shown by constructing the system $(F_{\alpha(p)})$ by decreasing $j$, and for $j$ fixed by increasing $p$;

Once $F_{\alpha^{-1}(p)}$ is given the construction of $F_{\alpha(p)}$ proceeds by induction on $p$.

I want to embed $C_\alpha$ in the fibre bundle $Z$ over $Y_1 \ldots d$ whose fibre over $F_1 \ldots F_d$ is $Y_1(F_1) \times Y_2(F_2) \times \ldots \times Y_d(F_d)$. The point is that for each $j$, $(F_{\alpha(p)})$ is essentially a full flag in $F_{\alpha^{-1}(p)}$ except that it has been indexed strangely.

A point of $Z$ will be a system $F_1 \ldots d$. 

\[ \text{\[} \] }
$1 \leq i \leq j \leq d$ of subspaces, such that $F_{s_i}$, $1 \leq i \leq j$ is a full flag in $F^{s_j} = F_j$. Given $(F_{s_j})$ in $\tilde{C}_d$ I send it to the element $(F_{ij})$ of $\mathcal{Z}$ defined as follows:

$$F_{ij} = F_{s_p} \quad b_j \leq p < b_{i+1}$$

where $b_{ij}, \ldots, b_{ij}$ is the set $\alpha^{-1}(1), \ldots, \alpha^{-1}(j)$ arranged in order.

I want next to find the image of $\tilde{C}_d$ inside of $\mathcal{Z}$. The condition $F_{s_p} \subset V_p$ becomes simply

$$F_{id} \subset V_{a_i} \quad a_i = b_{id}$$

where $a_1, \ldots, a_d$ is the sequence $\alpha^{-1}(1), \ldots, \alpha^{-1}(d)$ arranged in order.

Condition next the condition $F_{s_{j-p}} \subset F_{s_p}$. Let $b_1, \ldots, b_j$ be the sequence $\alpha^{-1}(1), \ldots, \alpha^{-1}(j)$ arranged in order, so that

$$F_{s_p} = F^{-s_j} \quad b_j \leq p < b_{i+1}$$

Put $b_k = \alpha^{-1}(j)$, so that $b_1, \ldots, b_k, \ldots, b_j$ is the sequence $\alpha^{-1}(1), \ldots, \alpha^{-1}(j-1)$ arranged in order. Then
Hence the conditions are:

\[ F_{i,j}^{i,j-1} = F_{i,j} \quad i < k \]
\[ F_{i,j}^{i,j-1} \subseteq F_{i+1,j} \quad i \geq k \]

where \( k = \text{card} \{ a \leq \alpha^{-1}(j) \mid \alpha(a) \leq j \} \)

To compute \([Z_{\alpha}]\) in \(H^*(Y_1, \ldots, d)\) I have two steps: 1) integral formula for \(Z \rightarrow Y_1, \ldots, d\) 2) class of \(Z_{\alpha} \subseteq Z\).

It seems desirable to change \(Z\) at this point. The point is that the choice of \(F_{i,j}^{i,j-1}\) involves at most choosing a point in a projective line. Because we have already chosen \(F_{i-1,j}^{i-1,j-1}\) \(F_{i,j}^{i,j-1}\)
I think we will get a simpler integration formula using the following.

Let $W$ be the set of systems of subspaces $F^i,j$ $1 \leq i < j \leq d$ in $V$ such that

$$\dim F^i,j = i$$

$$F^i,j \subset F^{i+1,j}$$

$$F^i,j \subset F^{i,j+1}$$

**Picture:**

$$\begin{align*}
&< F^{d-1,d} < F^{dd} \\
&V \\
&< F^{d-2,d-1} < F^{d-1,d-1} \\
&V \\
&V
\end{align*}$$

Map this to $Y_{1, \ldots , d}$ by $(F^i,j) \mapsto (F^{i,j})$.

**Example:** $d = 2$, it is the gadget $(F^2 F^2 > 2)$ encountered before.

$$\dim (W/Y_{1, \ldots , d}) = \frac{d(d-1)}{2}$$

In effect the choice of $F^i,j$ for $1 \leq i < j \leq d$ is a projective line.
Write $W \rightarrow Y_1 \cdots d$ as an iteration of $p^d$-bundles. Denote by $W_b \delta$ the set of $(F^{ij})$ of the preceding type with $j \leq b$, whence $W = W_d$. To lift an element of $W_b \delta$ into $W_b$ we choose $F_a^b$, $1 \leq a < b$, starting with $F_{bb}$ which is given. Let $W_{ab}$ be the set of systems $(F_i^{ij})$ given $\bigcirc$ for $j \leq b$ and $1 \leq i \leq b$, so that the map

$$T_{ab} : W_{ab} \rightarrow W_{a+1,b} \quad 1 \leq a < b$$

forgets $F_{ab}$. The fibre is the set of $F_{ab}$ of dim $a$ such that

$$F_a^{a-1,b-1} \subset F_a^b \subset F_{a+1,b}$$

hence the fibre is isomorphic to $P^1$. Denote by

$$L_{ab} = F_a^b / F_{a-1,b-1}$$

Then $H^*(W_{ab})$ is a free module over $H^*(W_{a+1,b})$ with basis $1, \xi_{ab}$; moreover

$$(T_{ab})_* : H^*(W_{ab}) \rightarrow H^*(W_{a+1,b})$$

is given by

$$(T_{ab})_* \{1\} = \{0\}, \quad (T_{ab})_* \{\xi_{ab}\} = \{1\}.$$
\[ W_{1b} = W_b \]

\[ W_{bb} = W_{b-1} \times Y_{1, \ldots, b} \]

What I am interested in are the bundles

\[ W_{ab} = W_{ab-1} \times Y_{1, \ldots, d} \text{ over } Y_{1, \ldots, d} \]

Then

\[ W_{bb} = W_{1b-1} \]

Consists of systems \( F_{ij} \), \( 1 \leq i \leq j < b \), \( F_j \), \( 1 \leq j \leq d \), such that \( F_{dd} = F_j \) for \( 1 \leq j \leq b \).

Anyway it is now clear that \( H^*(W_{ab}) \) admits as a \( H^*(Y_{1, \ldots, d}) \) module the simple system of generators \( F_{ij} \) with \( j < b \), or \( j = b, i \geq 2 \).

**Review:** Over \( Y_{1, \ldots, d} = \{(F_1, \ldots, F_d)\} \) I introduce the fibre bundle \( W \) whose fibre over \( F_1, \ldots, F_d \) consists of systems \( (F_{ij}) \) \( 1 \leq i \leq j \leq d \) of subspaces such that

\[ \dim F_{ij} = i \]
\[ F_{id} \subset F_{i+1, d} \]
\[ F_{i,j} \subset F_{(i+1),d+1} \]
If \( 1 \leq a \leq b \leq d \), I let \( W_{ab} \) be the quotient of \( W \) which forgets \( F_{ij} \) for \( j > b \) and for \( j = b, i < a \). Thus the fibre of

\[
W_{ab} \to W_{a+1,b} \quad 1 \leq a < b
\]

consists of all \( \mathcal{F}_{ab} \) of dimension \( a \) such that

\[
\mathcal{F}_{a-1,b-1} \leq \mathcal{F}_{ab} \leq \mathcal{F}_{a+1,b} \quad i
\]

the fibre \( \mathcal{F}_{ab} \) is therefore \( \in \mathbb{P}^i \). The corresp. \( O(-1) \)

is \( \mathcal{L}_{ab} = \mathcal{F}_{ab} / \mathcal{F}_{a-1,b-1} \), so if we put \( \mathcal{L}_{ab} = c \left( \mathcal{L}_{ab} \right) \)

then \( H^*(W_{ab}) \) is a free \( H^*(W_{a+1,b}) \) module with

basis \( 1, \mathcal{F}_{ab} \). If \( a = b \), we have

\[
W_{bb} = W_{1,b-1}
\]

Now suppose \( \alpha : \{1, \ldots, n\} \to \{1, \ldots, d\} \) is given

with \( \text{card} \ \alpha^{-1}(j) = 1 \) for \( 1 \leq j \leq d \). Let

\[
C_\alpha = \{ F_1 \leq \cdots \leq F_d \in W_{b-d} \mid \text{ dim } (F_j \cap V_p) = \text{card } \alpha^{-1}(j) \}
\]

This is one of the Schubert cells in \( Y_{b-d} \) whose coh.

class I want to compute. I want to find inside of

\( W \) a manifold \( C_\alpha \) which maps birationally to \( C_\alpha \).