December 15, 1974  Bruhat decomp for $GL_n$

$k = \mathbb{C}$, $F =$ manifold of flags
in $k^n$, $B = (\emptyset \updownarrow) =$ the stabilizers in $GL_n$ of
the flag $F_0 = ke_1 < ke_1 + ke_2 < \ldots < V_1 V_2$

Given $F^0 = (0 < F_1 < \ldots < F_n = k^n) \in F$ define
a sequence of integers $p(a), 1 \leq a \leq n$, by
$p(1) =$ least $p$ such that $ke_1 < F_p$
$p(2) =$ least $p$ such that $ke_2 \in ke_1 + F_p$
Choose $x_1 \in F_{p(1)}$ such that $e_1 = x_1$
$x_2 \in F_{p(2)}$ such that $e_2 \in ke_1 + x_2 = ke_1 + x_2$
$x_3 \in F_{p(3)}$ such that $e_3 \in ke_1 + ke_2 + x_3 = ke_1 + ke_2 + x_3$

Then $x_\in \in F_{p(0)} = (ke_1 + \ldots + ke_{a-1} + F_{p(a)-1})$

which shows that $\{x_\in \mid p(\in) = p\}$ have
independent images in $F_p / F_{p-1}$

$$n = \sum_p \text{card}\{\in \mid p(\in) = p\} \leq \sum_p \dim F_p / F_{p-1} = n$$

It follows that $\in \mapsto p(\in)$ is a permutation of
$\{1, \ldots, n\}$. As it depends only on $F$ and $F_0 = ke_1, \ldots$ it
is an invariant of the orbit $BF$ in $B \mid F$.

Also

$$F_p = \sum_{\sigma(p) \neq p} k x_\sigma.$$

Since there is a unique element in $B^u \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ carrying $[e_i]$ into $[x_i]$, $F$ is in the $B^u$ orbit of the flag

$$\{ p \mapsto \sum_{\sigma(p) \neq p} k e_{\sigma^{-1}(i)} \} = \{ p \mapsto \sum_{i \neq p} k e_i \}$$

$$= \sigma^{-1}\{ p \mapsto \sum_{i \neq p} k e_i \}$$

where $\sigma$ is the permutation matrix

$$\sigma = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} a_1^{-1} & & \\ & \ddots & \\ & & a_n^{-1} \end{pmatrix} \quad \sigma = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \sigma(i)$$

It follows that

$$B^u \mid F \sim B \mid F \sim \Sigma_n'$$

To compute the dimension of the orbit indexed by $\sigma \in \Sigma_n'$. This orbit is $B^u / B \cap B_\sigma^{\perp}$. 

Recall $\sigma^{-1} e_\sigma = e_{\sigma^{-1}(i)}$. If $a = (a_{ij})$ is a matrix,
then \( a(\frac{x_i}{x_n}) = (\frac{\Sigma a_{ix_i}}{\Sigma a_{nx_i}}) \) and \( a e_i = \sum_j a_{i,j} e_j \).

Thus \( (\sigma^{-1} a \sigma^2) e_i = a e_i \).

\[= \sigma (\sum_i a_{i\sigma} e_i) = \sum_i a_{i\sigma} e_{\sigma^{-1}i} = \sum_j a_{j\sigma} e_{\sigma j} \]

\[= (\sigma^{-1} a \sigma^2) e_j \]

So if \( a \in B \iff a_{ij} = 0 \) for \( i > j \), then

\[\sigma^{-1} a \sigma^2 \]satisfies \( a_{ij} = 0 \) if \( \sigma i > \sigma j \).

Hence \( b \in \sigma^{-1} B \sigma \iff b_{ij} = 0 \) for \( \sigma i > \sigma j \).

So

\[\begin{align*}
B_n \cap \sigma^{-1} B \sigma & = \left\{ (b_{ij}) \mid b_{ii} = 1, b_{ij} = 0 \text{ if } i > j \text{ or } \sigma i > \sigma j \right\} \\
\dim (B_n / B_n \cap \sigma^{-1} B \sigma) & = \text{card } \left\{ (i < j) \mid \sigma i > \sigma j \right\}
\end{align*}\]

Note: You want \( \tau e_i = e_{\tau(i)} \) so that

\((\tau \tau) e_i = e_{\tau \tau(i)} = \sigma e_{\tau(i)} = \tau(\tau e_i)\).
Normal structure to $B_0B$.

It is clear that if $H = \text{diagonal matrices in } \text{GL}_n$, then the flags $\tau F_0$ as $\tau$ runs over $\Sigma_n$ are the $H$-invariant flags exactly. Hence each $B$ orbit has a unique $H$-fixpoint.

Put $B_0 = \text{the opposite Borel to } B \text{ wrt } H$

$$B_0 = \begin{pmatrix} 0 & \ast \\ \ast & \ast \end{pmatrix}$$

so that $B_0B = H$. Because $\text{gl}_n = b_+ + b_-$ the $B_0$ and $B$ orbits intersect transversally. Let's see what happens at $\tau F_0$.

$$\tau B \tau^{-1} = \{ a \mid a_{ij} = 0 \iff \tau^{-1}(i) > \tau^{-1}(j) \}$$

$$B^u \cap \tau B \tau^{-1} = \{ a \mid a_{ii} = 1 \wedge a_{ij} = 0 \iff \{ \begin{array}{c} \tau^{-1}(i) > \tau^{-1}(j) \\ \text{or } i > j \end{array} \} \}$$

$$B_-^u \cap \tau B \tau^{-1} = \{ a \mid a_{ii} = 1 \wedge a_{ij} = 0 \iff \{ \begin{array}{c} \tau^{-1}(i) < \tau^{-1}(j) \\ \text{or } i < j \end{array} \} \}$$

$$\dim \frac{B^u}{B^u \cap \tau B \tau^{-1}} = \text{card } \{ (i \leq j) \mid \tau^{-1}(i) > \tau^{-1}(j) \}$$

$$\dim \frac{B^u}{B_-^u \cap \tau B \tau^{-1}} = \text{card } \{ (i > j) \mid \tau^{-1}(i) > \tau^{-1}(j) \}.$$
Thus these two orbits have complementary dimensions, and so they intersect in a point.

Alternatively we can work in the Lie algebra. The tangent space to $G/B$ at $TB$ may be identified with

$$\mathfrak{g}/\text{Ad}(B) = \mathfrak{g}/\mathfrak{tb}^{-1}$$

The tangent space to $B^uTB$ is

$$\mathfrak{b}^u/\mathfrak{b}^u\cap TB^{-1} \cong \mathfrak{b}^u + \mathfrak{tb}^{-1}/\mathfrak{tb}^{-1}$$

$$\mathfrak{b}_-^u/\mathfrak{b}_-^u\cap \mathfrak{tb}^{-1} \cong \mathfrak{b}_-^u + \mathfrak{tb}^{-1}/\mathfrak{tb}^{-1}$$

So the thing to see is $$(\mathfrak{b}_-^u + \mathfrak{tb}^{-1}) \cap (\mathfrak{b}_-^u + \mathfrak{tb}^{-1}) = \mathfrak{tb}^{-1}$$

which is clear from the usual basis for $\mathfrak{g}$.

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Homotopy type of poset of Schubert cells in $F$.

Let $B$ be a Borel subgroup and let $x \in F$. Can I recover $B$ from the orbit $Bx$?

Put

$$P = \{ g \in G \mid gBx = Bx \}$$

This is a subgroup containing $B$. If $x = TB$, then
$P_0 B = B_0 B$. Now $P = U B r B$, and if

$B r B_0 B = B_0 B$, we have $r_0 B = B_0 B$, hence

$r_0 = r$ because any double coset contains a unique element of the Weyl group. Thus $P = B$.

So from the orbit $O = B x$, we can recover $B$ as $\{g | g B x = B x\}$.

If $H$ is a maximal torus of $B$, there is a unique point of the orbit $B x$ fixed under $H$, which I may as well assume is $x$. In fact, identifying $F$ with $G/B$, then $x = r B$ for a unique $r \in W$. 

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$F = \text{manifold of flags in } \mathbb{P}^n$. A Borel sub-group $B$ of $GL_n$ may be identified with a point in $F$, same the unique point of $F_B$. The orbit of $B$ on $F$ are cells. Subsets of $F$ arising this way I call Schubert cells, and I would like to determine the homotopy type of the poset $\mathcal{S}$ of Schubert cells.

**Ex. $n=2$:** Here $F = \mathbb{P}_1(k)$ and the Schubert cells are either points or complements of points. So the poset $\mathcal{S}$ has elements $(l,0)$, $(l,1) \quad \forall \ l \in \mathbb{P}_1(k)$ with $(l,0) < (l',1) \iff l \neq l'$. The associated simp. $\mathcal{S}$ is my friend.

The diagram is included with all segments projecting monodromically.
Suppose $O, O'$ are in $S$. I have seen that one can recover a Borel from any of its orbit: $B = \{ g | g\delta = \delta \}$. Let $B$ belong to $O'$. I want to understand when $\delta' \leq O'$. 

A special case where $B, B'$ are opposite so that $B_0B' = \max$ torus $T$. Let $\sigma \in$ Weyl group be such that $\sigma \cdot B = (O')^T = O^T$. Thus

$$\delta' = B' \sigma B,$$

$$\delta = B \sigma B.$$

Now because $B, B'$ are opposite, I know these orbits $O, O'$ meet transversally at $\sigma B$, hence $O' \leq O$ imply that $O$ is open. Hence $O$ must be the open $B$-orbit which means $T$ is the Coxeter element and $B' = \sigma B \sigma^{-1}$. Thus the case where $B, B'$ are opposite corresponds to the case of a point sitting in an open cell.

In general start with $\delta' \leq O$, where $O, O'$ determine Borels $B, B'$. Choose $T \subseteq B_0 B'$, whence $(O')^T = \sigma B$, $\sigma \in W$. And we have

$$B' \sigma B \subseteq B \sigma B.$$

Now $\exists! \tau \in W$ such that $B' = \tau B \tau^{-1}$, in fact

$$(G/B)' = \tau B \quad (B' \tau B = \tau B \iff B' = \tau B \tau^{-1}).$$
Thus we have \( \tau B \tau^{-1} B \leq B \tau B \) or
\[
B \tau B \tau^{-1} \sigma B = B \tau B.
\]

Now in general, I believe that one has
\[
B \tau B \tau B = B \tau B \tau B \iff l(\alpha \beta) = l(\alpha) + l(\beta)
\]
where \( l \) is the length. Assuming this one gets

Assertion: Let \( 0' \subset 0 \) be Schubert cells belonging to Borels \( B' \) and \( B \). If \( x \in 0' \) then
\[
d(x, B) = d(x, B') + d(B', B)
\]
Conversely if this holds one has \( B' \subset B \).

Given two points \( x, y \in F \) one defines
\[
d(x, y) = \text{by either}
\]
i) \( \dim (B_x y) = \dim (B_y x) \)

ii) choose \( T \subset B_x \cap B_y \), whence \( B_x, B_y \) determine Weyl chambers in \( \text{Hom}(G_m, T) \otimes \mathbb{R} \). \( d(x, y) \) is the number of root hyperplanes crossed in going from one chamber to another.
Why these are the same: Can assume $B_x = B = \{0\}$ and $y = \sigma B < \mathbb{G}/B \cong \mathbb{F}$. So I want the dimension of $B y B/B$, which I computed to be

$$\text{card } \{i < j \mid \sigma_i > \sigma_j\}$$

which is exactly the number of hyperplanes crossed in going from the positive Weyl chamber to the one described by $\sigma$.

Suppose $x, y, z \in \mathbb{F}$, put $B = B_x$ and $x = B/B$.

$$y = fB y, \quad z = gB/B \in \mathbb{G}/B \cong \mathbb{F}.$$  Then $B_y = fB f^{-1}, B_z = gB g^{-1}$.

And

$$B x y = B f B / B$$

$$B x z = B g B / B$$

$$B y z = f B f^{-1} g B / B$$

$$B g B \subseteq B f B \times B f^{-1} g B$$

\[ \dim B g B / B \leq \dim B f B / B + \dim B f^{-1} g B / B \]

\[ \dim (B x z) \leq \dim (B x y) + \dim (B y z) \]

Thus if equality holds for these dimensions we have

$$B y B = B f B \bullet B f^{-1} g B$$

equality of double cosets.
So I want to be sure that if \( l(x') < l(x) + l(\beta) \), then \( B_x B \beta B \beta \) consists of more than one double cover.

We consider the first. It is true that \( l(x') + l(\beta) > l(x_\beta) \). We are not using the fact that \( x_\beta \) is an element of \( W \), since \( x_\beta \) is not a fixed point of \( \beta \). Use induction on \( l(x) \): write \( x = x' \beta' \), \( l(x') = l(x) - 1 \).

Then \( B_x B = B_x B \beta B \beta B \). If \( l(x') + l(\beta) > l(x_\beta) \), then already \( B_x' B_\beta B \beta B \) contains more elements of \( W \) than \( x_\beta \), so done. If \( l(x') + l(\beta) = l(x_\beta) \), then \( B_x' B_\beta B = B_x B \beta B \), and so we are reduced to showing 
\[ l(s_\beta) < l(\beta) + 1 \implies B_s B \beta B \beta B \] contains more element of \( W \) than \( s_\beta \). But one knows then that \( \beta = s_\beta \), so \( B_s B \beta B \beta B = B_s B \beta s_\beta B \beta B \), and \( B_s B \beta B \beta B = B_s B \beta B \beta B \).

So the simplicial complex associated to the point \( s \) of Schubert cells in \( F \) has for its \( p \)-simplices \( (O, B_0, \ldots, B_p) \) where \( B_i \) are Borels, \( O \) is a \( B_0 \)-orbit, and
\[ d(B'_i, B_0) + \ldots + d(B'_i, B_p) = d(B'_i, B_p) \]
for some (hence any) \( B'_i \in O \).

**Question**: Given a simplex \( (O, B_0, \ldots, B_p) \) does there exist a \( T \subset \bigcup B_i \)?
Question: Given Borels $B_0, \ldots, B_p$ such that
\[ d(B_0, B_1) + \ldots + d(B_{p-1}, B_p) = d(B_0, B_p), \]
does there exist $T \subset \cap B_i$?

Suppose $B_0, B_p$ are opposite, i.e. $B_0 \cap B_p = T$. Consider the set of Borels $B_1$ such that $d(B_0, B_p) = d(B_0, B_1) + d(B_1, B_p)$ with $d(B_0, B_1)$ a given integer $t$. For each $z \in \mathbb{Z}$,
\[ \dim(B_0z) = t \]
\[ \dim(B_pz) = d(B_0, B_p) - t \]
and $\mathcal{G}$ know the orbit $B_0z, B_pz$ are transversals. Thus the orbits $B_0z$ and $B_pz$ have a zero-dimensional intersection, which is $T$-stable; hence $T$ being connected, $z$ is a $T$-fixed point. This means that any $B_1$ with $d(B_0, B_p) = d(B_0, B_1) + d(B_1, B_p)$ contains $T$.

Now given $B_0, \ldots, B_p$ as above, choose a maximal torus $T \subset B_0 \cap B_p$, and let $B_{p+1}$ be the opposite Borel to $B_0$ containing $T$. Then
\[ d(B_0, B_p) + d(B_p, B_{p+1}) = d(B_0, B_{p+1}). \]
Thus $d(B_0, B_1) + \ldots + d(B_p, B_{p+1}) = d(B_0, B_{p+1})$, and so by what I have already done, I see that
Suppose I fix \( T \) and consider only cells fixed under \( T \). Each cell has a unique \( T \)-fixpoint, so a \( p \)-simplex is a sequence of Borels \( (B_{-1}, B_0, \ldots, B_p) \) containing \( T \) such that the distance condition holds. So this simplicial complex is clearly a union of cones, one for each \( T \)-fixpt, i.e. for each element of \( W \).

**Generalization:** Consider orbits of various Borels on all the simplices of the building as Schubert cells.

**Example:** \( n=2 \). Here the building is the set of lines in \( k^2 \) with the discrete topology. But if you "topologize" the set of lines using Schubert cells, you get something which is connected. You get a connected graph whose \( H_1 \) has rank \( q^2-q-1 \) for \( k=\mathbb{F}_q \).

\[
\begin{align*}
h_0-h_1 &= 2(q+1) - (q+1)q \\
1-h_1 &= 2q + 2 - q^2 - q = 2 - q^2 + q \\
h_1 &= q^2 - q - 1
\end{align*}
\]
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Let's first classify the $B$-orbits on the building $X$. Given a simplex $0 < F_1 < \ldots < F_q = V$, better to call this a flag, we can define a map $\alpha : \{1, \ldots, n\} \rightarrow \{1, \ldots, q\}$ as follows:

$\alpha(1) = \text{least } p \text{ such that } e_1 \in F_p$

$\alpha(2) = \text{least } p \text{ such that } e_2 \in ke_1 + F_p$

Choose $x_1 = e_1 \in F_0, e_2 \in ke_1 + x_2, x_2 \in F(x_2)$ etc.

Then by old arguments I know $F_p/F_{p-1}$ has as basis the images of the $x_0, \alpha(x) = p$. Hence

$$F_p = \sum_{\alpha(x) = p} kx_x$$

One sees that $\alpha : \{1, \ldots, n\} \rightarrow \{1, \ldots, q\}$ is surjective with $\text{card } \alpha^{-1}(p) = \text{dim } (F_p/F_{p-1})$. $\alpha$ is an invariant of the $B$-orbit of the flag, and determines the $B$-orbit. So if I denote by $D_{r_1, \ldots, r_q}(V)$ the flags with jumps $r_1, \ldots, r_q$, I see that

$$B \backslash D_{r_1, \ldots, r_q}(V) \cong \{ \alpha : \{1, \ldots, n\} \rightarrow \{1, \ldots, q\} \mid \text{card } \alpha^{-1}(p) = r_p \}$$

$B$-orbits are same as $B^u$-orbits. Also the $B$-orbit
indexed by \( \alpha : \{1, \ldots, n\} \rightarrow \{1, \ldots, p\} \). Contains the flag with
\[
F_p = \sum_{\alpha(a) \leq p} k \mathbf{e}_a
\]
which is \( T \)-invariant. This is the unique \( T \)-invariant flag in the orbit, so again each \( B \)-orbit has a center, the unique \( T \)-fixpt.

Thus each \( B \)-orbit of simplices in \( X \) contains a unique \( T \)-fixpt, which means that \( B \X \) is the subcomplex of \( T \)-fixpoints of \( X \). (In particular \( B \X \) is a simplicial complex.)

Recall that you have identified the simplicial complex of all subspaces of \( V \) with self-adjoint operators on \( V \) such that \( 0 \leq A \leq 1 \). If \( 0 < \lambda_1 < \cdots < \lambda_k < 1 \) are the eigenvalues of \( A \) except for 0, 1, then these are the simplicial coordinates, and the simplex has as vertices
\[
E^A_0 V < E^A_{\lambda_1} V < \cdots < E^A_{\lambda_k} V
\]
where \( A = \int \lambda dE^A_\lambda \). (Think \( E^A_0 V = \ker A \), \( E^A_{\lambda_i} V \) is where \( A \leq \lambda_i \), etc.) (Thought: The vertices are the projectors; every point is an average of commuting projectors. Maybe projectors are the extreme points of the convex set of \( A \), \( 0 \leq A \leq 1 \).)
So now let me count the B-orbits on the big building of all subspaces of V, same as the subcomplex of T-invariant simplices, i.e. operators A commuting with T. Such A are of the form \[ A = \sum_i \lambda_i e_i \quad 0 \leq \lambda_i \leq 1, \]
so we get \( I^n \) for the orbit space, triangulated in the standard way.

Next consider the proper building of proper subspaces, which I wish to sketch in the figure with the help of the space of real eigenvalues.

By the preceding formulas the simplex
\[
E_0^A V < E_{\lambda_1}^A V < \ldots < E_{\lambda_8}^A V
\]
will be a chain of proper subspaces, provided A has the eigenvalues 0 and 1. So after taking T-orbits we get the closed subspace of \( I^n \) consisting of points \((\lambda_i)\) such that \( \lambda_i = 0 \) \( \lambda_j = 1 \) for some \( i, j \).

So it should be a triangulation of a sphere.
Given \( \alpha : \{1, \ldots, n\} \to \{1, \ldots, g\} \), what is the dimension of the B-orbit corresponding to \( \alpha \)?

Compute the stabilizer of the flag \( \mathfrak{F} \):

\[
\mathfrak{F}_p = \sum_{i \leq p} F_{\alpha_i} = \{ x \in k^n | \alpha_i = 0 \text{ if } \alpha(i) > p \}.
\]

Stabilizer of \( \mathfrak{F}_p \):

\[
\text{Stabilizer of } \mathfrak{F}_p = \{ a \in \text{GL}_n | \alpha(i) > p, \alpha(t) < p \implies \alpha_i = 0 \}.
\]

Stabilizer of \( \left\{ \mathfrak{F}_p \right\}_{1 \leq p \leq g} \):

\[
\bigcap_{1 \leq p \leq g} \text{Stabilizer of } \mathfrak{F}_p = \{ a \in \text{GL}_n | \text{ if } i > t, \alpha(i) > \alpha(t) \implies \alpha_i = 0 \}.
\]

\( B_0 \): The dimension of \( B_0 \) stabilizer of \( \mathfrak{F}_p \) in \( B = \text{card } \left\{ (i < t) | \alpha(i) > \alpha(t) \right\} \).

Formula: The dimension of the B-orbit indexed by \( \alpha : \{1, \ldots, n\} \to \{1, \ldots, g\} \) is \( \text{card } \left\{ (i < j) | \alpha(i) > \alpha(j) \right\} \).

Addition to page 10. The

\[
\begin{align*}
|\text{all subspaces of } V| & = \{ \sigma \leq \mathbf{1} \} \quad \overset{\text{B-orbit}}{\longrightarrow} \quad \mathbf{I}^n \\
U & \\
|\text{subcomplexes of } & \overset{\text{B-orbit}}{\longrightarrow} \quad \mathbf{I}^n \\
\text{chain } V_p < \ldots < V_1 & \quad \text{of } \quad \left\{ \sigma \leq \mathbf{1} \right\} = \sigma \downarrow \leq \mathbf{1} \quad \overset{\lambda, \mu \in \text{sp}(A)}{\longrightarrow} \quad \mathbf{I}^n \\
|T(V)| & = \left\{ \sigma \leq \mathbf{1} \right\} \quad \overset{\lambda_i = 0}{\longrightarrow} \quad \left\{ (\lambda_i) \in \mathbf{I}^n \right\}.
\end{align*}
\]
So I can now identify a $B$-orbit on the big building with a point $(\lambda_i) \in I^n$. The associated $x$ is the surjection one gets by arranging the $\lambda_i$ in order. Thus if $0 \leq \mu_1 < \cdots < \mu_n \leq 1$ is the sequence arranged in order with repetitions, $\lambda_i = \mu(x(i))$.

Take $(\lambda_i)$ and any point $0 \leq z_1 < z_2 < \cdots < z_n \leq 1$ of the interior of the positive Weyl chamber. Consider the straight line segment

$$(x(1)) = (1-t)\lambda + tz \quad \text{in } I^n.$$ 

Take the root $x_j - x_i$, $i < j$ and consider the image of this straight line under this root function. It is the segment going from $\lambda_j - \lambda_i$ to $z_j - z_i \geq 0$. Thus the line segment $(x)$ crosses the hyperplane $X_i = x_j$ iff $\lambda_j - \lambda_i \leq 0$ i.e. $\mu(x(j)) < \mu(x(i))$ or $x(i) > x(j)$.

**Formula:** The dimension of the $B$-orbit, indexed by $x$, is the number of hyperplanes crossed in going along a straight line joining any point of the $x$-stratum of $R^n$ to the positive Weyl chamber.
Now the program will be to show the (proper) building $X$ has the homotopy type of a sphere by Serre's method. This method involves filtering $X$ according to the length of elements in the Weyl group.

So one lists the elements of $\mathcal{W}, w_0, w_1, w_2, \ldots \ldots \ldots$ such that $\alpha = l(w_1) > l(w_2) > \ldots \ldots \text{ etc.}$

Let $S_p \subset I^n$ be the closed subcomplex containing all the $n$-simplices associated to $w_0, \ldots, w_p$, and let $X_p$ be the inverse image of $S_p$ inside of $X$.

Now let's consider a specific example.

Suppose we have a point $(\lambda_i) \in I^n$ such that $\lambda_i = \mu_i \alpha(i)$, where $\alpha : \{1, \ldots, n\} \rightarrow \{1, \ldots, g\}$ and $0 < \mu_1 < \ldots < \mu_g \leq 1$. Then we can take all of the $i$ such that $\alpha(i) = p$ and perturb the $\lambda_i$ so as to get a new point $(\lambda'_i) \in I^n$ with all $\lambda'_i$ distinct and such that $\alpha(i) = \alpha(j), i < j \Rightarrow \lambda'_i < \lambda'_j$.

I can also arrange none of the $\lambda'_i$ to be 0 or 1. Let $\omega$ be the permutation associated to $\lambda'_i$, whence we see $\lambda$ is on the boundary of the $n$-simplex $I^n$.
described by \( \omega \). \( \omega \) is obviously the unique permutation refining \( \alpha \) of the same length. ("refine" means \( \omega(i) \leq \omega(j) \) \( \Rightarrow \) \( \alpha(i) \leq \alpha(j) \) or equivalent \( \alpha(i) > \alpha(j) \) \( \Rightarrow \) \( \omega(i) > \omega(j) \).) Thus I see that each stratum of \( \Pi^\alpha \) is in the boundary of a unique chamber of the same length. It's also clear that the \( \alpha \)-stratum is in the boundary of the \( \beta \)-stratum iff \( \beta \) refines \( \alpha \).

So let us now consider how \( X^p \) is obtained from \( X^{p-1} \). First consider the passage from \( S^{p-1} \) to \( S^p \), where we add on the closure of the stratum belonging to the permutation \( \omega \). This stratum is the chamber of \( (\lambda_i) \in \Pi^\alpha \) where the \( \lambda_i \) are distinct and arranged in the order given by \( \sigma \) where \( \lambda_i \leq \lambda_j \) iff \( \sigma(i) \leq \sigma(j) \).

This stratum is open except that \( \lambda_0^{-1}(1) \) can be 0 and \( \lambda_0^{-1}(n) \) can be 1. The boundary of this stratum will consist of strata refined by \( \sigma \). Such things are of the form

\[
\begin{array}{c}
\{1, \ldots, n\} \\
\mapsto \\
\{1, \ldots, n\}
\end{array}
\]

\( \sigma \) surjective, monotone

and we know \( l(\alpha) \leq l(\sigma) \), with equality iff \( \sigma^{-1} \)
preserves order on the fibres of \( \eta \). I now have to figure out which \( \alpha \) have already occurred in \( S_{\eta+1} \). Certainly all \( \alpha \) with \( l(\alpha) < l(\sigma) \) have occurred; if \( l(\alpha) = l(\sigma) \), then \( \alpha \) has not occurred before because then \( \alpha \) would be refined by some \( \delta_j \) with \( j < p \); hence \( l(\delta_j) \leq l(\sigma) = l(\alpha) \), so \( \delta_j \leq \sigma \) since we know there is a unique refining permutation of the same length. Thus I must see what sort of geometric object is the union of the \( \alpha \)-strata with \( \alpha \) refined by \( \sigma \) and \( l(\alpha) < l(\delta) \).

It is a closed subcomplex.

Suppose we have \( \eta \) such that \( \sigma^{-1} \) does not preserve the order on the fibres. Pick one such bad fibre, say the interval \( [a, b] = [u, v, \ldots, b] \), and choose \( i \) least such that \( \sigma^{-1}(u_i) > \sigma^{-1}(b) \). Then I find \( \eta \) is refined by the degeneracy 3-collecting \( a_i \) and \( a_i+1 \). This shows that the \( a_i \)'s involved form a union of the faces such that \( \sigma^{-1} \) reverses \( i \), \( i+1 \). This is the union of not all the faces, except when \( \sigma \) is the Coxeter element. So to make this all airtight, I have to take the closed \( \sigma \)-stratum, and to retract it onto the subcomplex with smaller length.

So next I try the same game on \( X \).
Changing notation let me denote by $X_p$ the union of all strata of length $\leq p$. Then in going from $X_{p-1}$ to $X_p$ I attach all strata with indices $\sigma$ of length $p$. Each such $\sigma$ is attached to a unique permutation of length $p$. So what I want to see is that if I put $Z_\sigma = \text{closure of } \sigma$\text{-stratum}, then $Z_\sigma$ deforms strongly down to $Z_\sigma \cap X_{p-1}$. Have to be careful that $Z_\sigma$ contains new stuff besides $\sigma$\text{-strata with } $\sigma < \sigma$ ($\sigma$ refines $\sigma$).

Let $U_\sigma = \sigma$\text{-stratum}. This will consist of $\sigma$-operators $A_\sigma$ with distinct eigenvalues $0 < \lambda_1 < \cdots < \lambda_n$, satisfying a condition relative to the flag $0 = V_0 \subset V_1 \subset V_2 \subset \cdots$. A typical $A_\sigma$ is of the form

$$A = \sum_{i=1}^n \lambda_i \otimes p^2_i x_i$$

where $L_1 \in PV_{\sigma(1)} - PV_{\sigma(1)-1}$, $L_2 \in PV_{\sigma(2)} - PV_{\sigma(2)-1}$, and $L_2 + L_1$, etc. Something is wrong.
December 22, 1974

Cohomology classes associated to Schubert cells.

Let $Y$ be the manifold of full flags in $V = k^n$, $Y = GL_n/B$ where $B = \langle e \rangle$ is the stabilizer of the basic flag $\{ V_p = ke, \ldots, k^n \}$. $Y$ is an iterated projective bundle

\[(x) \quad Y = \{0 < F_1 < \cdots < F_{n-1} < V \} \longrightarrow \{0 < F_1 < \cdots < F_{n-2} < V \} \longrightarrow \cdots \longrightarrow \{0 < F < V \} \longrightarrow \mathbb{P}^1 \]

Recall for $f : H \longrightarrow X$ the formula

$$f_\ast a(\xi) = res \left( \frac{a(T) dT}{T^{\sum c_i E} + \sum c_i E} \right)$$

where $\xi = e_1(\xi(1)) = res \left( \frac{a(T) (T^{p+q} Q \cdot T^{-q} + c_1 E) dT}{T^n} \right)$

if $E + Q = n, p + q = n$.

Applying this to each of the maps $(x)$, we get

\[
\int a(f_\ast, x) = \text{res} \left( \frac{a(T_1) dT_1}{T_1^n + \cdots + c_1 E_1} \right) + \text{res} \left( \frac{a(T_2) dT_2}{T_2^n + \cdots + c_1 E_2} \right)
\]

Put $Y_i = \{(F_1, \ldots, F_i)\}$. Over $Y_i$ we have
the bundle \( F_i \) with fibre \( F_i \) at \((F_i, \ldots, F_i)\), and \( F_i \subset Q_i \times V \). Denote all \( \pi_p \) by \( \tilde{\pi} \).

Put \( L_i = F_i / F_{i-1} \), \( \tilde{\xi}_i = \epsilon (\tilde{\xi}_i) \). For the map \( f : Y \rightarrow Y_{i-1} \), we have \( Y_i = \mathbb{P}(V / F_i) \), \( O(-1) = \xi_i \).

\[
(f_i)_* a(\tilde{\xi}_i) = \text{res} \frac{a(T_{\tilde{\xi}_i}) dT_{\tilde{\xi}_i}}{T^{n-1} + \cdots + c_{n-i+1}^{n-i+1} (V / F_{i-1})}
\]

\[
= \text{res} \frac{a(T_{\tilde{\xi}_i}) (T^{i-1} + \cdots + c_{n-i}^{n-i} (F_{i-1})) dT_{\tilde{\xi}_i}}{T_{\tilde{\xi}_i}^{n-1}}
\]

\[
= \text{res} a(T_{\tilde{\xi}_i}) \prod_{j < i} \frac{dT_{\tilde{\xi}_i}}{T_{\tilde{\xi}_i}^{n-1}}
\]

Thus we get the formula

\[
\int \gamma a(\tilde{\xi}_1, \ldots, \tilde{\xi}_n) = \text{coefficient of } (T_1 \cdots T_n)^{n-1} \text{ in } a(T_1, \ldots, T_n) \prod_{i < j} (T_j - T_i).
\]

Consider now the map

\[
Y = u/T \rightarrow BT
\]

which classifies the line bundles \( \tilde{\xi}_i \) on \( Y \).
Since $H^k(B^r) = \sum_i T_i$ \ldots \text{and} $T_i = c_i\left(p_i: T \rightarrow S^1\right)$, the formula \* tells us that
\[
\left\langle Y \left[\prod_{i \neq j} a(T_i, \ldots, T_n)\right] \right\rangle = \text{coeff. of } (T_1 \ldots T_n)^{n-1} \text{ in } a(T_1, \ldots, T_n) \prod_{i < j} (T_i - T_j)
\]

Now,
\[
\prod_{i < j} (T_j - T_i) = \left| \begin{array}{ccc}
1 & T_1 & T_1^2 & \ldots & T_1^{n-1} \\
1 & T_2 & T_2^2 & \ldots & T_2^{n-1} \\
1 & T_n & T_n^2 & \ldots & T_n^{n-1}
\end{array} \right| = \sum_{\sigma \in \Sigma_n} (-1)^{\sigma} T_{\sigma(2)} T_{\sigma(3)}^2 \ldots T_{\sigma(n)}^{n-1}
\]

Therefore,
\[
\left\langle Y \left[\prod_{i \neq j} T_i^{a_i} \ldots T_n^{a_n}\right] \right\rangle = \begin{cases} (-1)^{\sigma} & \text{if } n - \alpha = \sigma^{-1}(i) \\ 0 & \text{otherwise} \end{cases}
\]

**Question:** Can you construct resolutions of the Schubert cells?

**Example:** Take $G_p(k^n)$. A $B$-orbit on this is described by $\alpha: \{1, \ldots, n\} \rightarrow \{1,2\}$ such that $\alpha^{-1}(1)$ has $p$ elements. It is therefore described by a sequence $1 \leq \alpha_1 < \ldots < \alpha_p \leq n$, and specifically, is the subset consisting of $A$ in \( V \), dim $A = p$, such that
the induced filt. \( V_1 \in A \) has jumps at \( i = n_i, \ldots, n_p \). The closure of this cell consists of \( A \) such that \( \dim (V_j \cap A) \geq j \). I resolve the \( \Delta \) closure by considering the manifold of flags \( F_1 < \ldots < F_p \) in \( V \) such that \( F_j \subset V_j \).

But we can also get another resolution as follows. Let \( \Delta^{-1}(2) = \{ t_1, \ldots, t_q \} \) with \( k, t_1, \ldots, t_q \leq n \). Then the cell under consideration can be described as consisting of \( A \) such that \( V_i + A + A \) has jumps at \( t_1, \ldots, t_q \) that is \( V_i + A \subset V_j + A \) has \( \dim (p + g) \).

So the closure of the cell would seem to consist of \( A \) such that \( \dim (V_j + A) \leq p + g \). I should be able to resolve this by the manifold of flags \( F_1 < F_1^+ < F_2^+ < \ldots < F_n = V \) such that \( F_i^+ J \cap V_j \) for \( j = 1, \ldots, g \). Questions: Are these two resolutions the same or different?
December 23, 1974

\[ V \text{ is a vector space of dim. } n \text{ with a} \]
given full flag \( 0 < V_1 < \ldots < V_n, \quad B = \text{corres. Borel} \]
subgroup \( \leq G = \text{Aut}(V). \]

Given \( 1 \leq A_1 < \ldots < A_\mu = n, \) I let \( D_\mu (V) \) be the
manifold of flags \( 0 < F_1 < \ldots < F_\mu = V \) such that
\[ \dim (F_j) = s_j. \]
Here \( s = (s_1, \ldots, s_\mu). \) I know
that the \( B \)-orbits on \( D_\mu (V) \) are classified
by functions \( \alpha: \{1, \ldots, n\} \to \{1, \ldots, \mu\} \) such that
\[ \alpha^{-1}[\{i, \ldots, j\}] = s_j. \]
More precisely, suppose a
flag \( 0 < F_1 < \ldots < F_\mu = V \) in \( D_\mu (V) \) is given. Then for
each \( p, 1 \leq p \leq n, \) the quotient \( V_\mu / V_p \) "appears"
in one of the quotients \( F_j / F_{j-1} \), and then \( \alpha(p) = j. \)
\( \alpha \) being fixed, the corres. cell in \( D_\mu (V) \) consists of
all \( F \) such that
\[ \dim (F \cap V_p) = \text{card } \{ i \leq p \mid \alpha(i) \leq j \}. \]

Call this cell \( C_\alpha. \)

I have the following candidate for a resolution
of \( C_\alpha. \) Consider families \( (F_{\beta p}) \), \( j = 1, \ldots, \mu; \quad p = 1, \ldots, n \)
of subspaces monotone in both \( j \) and \( p \) such that
\[ F_{\beta p} \leq V_p \]
\[ \dim F_{\beta p} = \text{card } \{ a \leq p \mid \alpha(a) \leq j \}. \]
These families form a closed subvariety $\tilde{C}_a$ of a product of Grassmannians, hence $\tilde{C}_a$ is a complete variety. By sending $F_{ij}$ into $F_{jn}$ for $1 \leq j < \mu$, we get a map $\tilde{C}_a \to D_\mu(V)$.

If $(F_{jn})_{1 \leq j \leq \mu}$ is in $C_a$, then

$$\dim F_{jn} \cap V_p = \operatorname{card} \{ \alpha \leq p \mid \alpha(a) \in j \}$$

$$\dim (F_{ij})$$

hence $F_{ij} = F_{jn} \cap V_p$ showing that over $C_a$, $\tilde{C}_a$ has fibres reduced to a point.

**Examples.** Take $\alpha : \{1, \ldots, n\} \to \{1, 2, 3\}$. Then $\tilde{C}_a$ consists of flags in $A = F_{1n}$

$V_1 < \ldots < V_n = V$

$U \subset V$

$F_{1i} \subset \ldots \subset F_{1n} = A$

such that $F_{ij} = \{ \alpha \leq p \mid \alpha(a) = 1 \}$ has jumps at the points of $2^* (1)$. So this is one of the resolutions used before for Schubert cells in the Grassmannians.

2. Take $\alpha : \{1, \ldots, n\} \to \{1, \ldots, n\}$ to be the Coxeter
permutation $\alpha(i) = n - i + 1$. Take $n = 2$

$0 < V_1 < V \quad \downarrow \quad V_{2,3} \quad \downarrow \quad F_{1,3} \quad \downarrow \quad 0$

Try $n = 3$.

$V_1 < V_2 < V_3 \quad \downarrow \quad V_{2,3}$

$0 < F_{2,2} < F_{2,3} \quad \downarrow \quad V_{2,3}$

$0 = 0 < F_{1,3}$

$F_{2,2}$ can be any line in $V_2 \cap F_{2,3}$, hence it is not uniquely determined. Thus $\tilde{C}_x$ is not $V$ in this case, but bigger.

Is $\tilde{C}_x$ non-singular? Consider for example the case $\mu = 2$, where I have a filtration $F_p$ with $\dim F_p = \{i \leq p \mid \alpha(i) = 1\}$ and $F_p < V_p$. Start by mapping

$F_i \mapsto F_1$.

Let $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\} = \alpha^{-1}(1)$. Then $0 = F_1 = \cdots = F_{\mathcal{F}_n - 1}$ and one chooses $F_{\mathcal{F}_n}$ to be any line in $V_{\mathcal{F}_n}$. Then $F_\mathcal{F}_n = F_{\mathcal{F}_n - 1}$.
and the next jump occurs at $F_{t_2}$ which can be any 2-plane in $V_{t_2}$ containing $F_{t_1}$. Then $F_{t_3}$ can be any 3-plane in $V_{t_3}$ containing $F_{t_2}$. So we see that $C_t$ is non-singular (more or less). Its dimension is

$$(d_4 - 1) + (d_2 - 2) + \ldots + (d_y - y).$$

And

$$\left\{(i < j) \mid \chi(i) > \chi(j) \right\} = t_1 - 1$$

$$\left\{(i < j) \mid \chi(i) > \chi(j) \right\} = t_2 - 2$$

so its dimension is $d(x)$. (Pairs out of order are of the form $(a, t_j)$, where $a \neq t_1, \ldots, t_j - 1$. The number of these is $t_j - 1 - (j - 1) = t_j - j$.)

---

See if we can do the same computation without introducing $t_1, \ldots, t_3$. Consider the choice of $F_p$ once $F_{t_1}, \ldots, F_{t_2}$ have been chosen. The conditions are $F_{t_1} \subset F_p \subset V_p$ and that $F_p$ have dimension and $\chi(i) > 1$. So it's clearly non-singular, this fibre is, and its dimension is
The given equation is that of a linear algebra problem. The solution is that of finding the dimension of the vector space. If the space is isomorphic to another, then the dimension is the same. In a given context, if $F_{p}$ is a field, then $F_{p}$ is isomorphic to $F_{p}$, where $p$ is a prime. Therefore, the dimension of the vector space is equal to the order of the field.
So we choose $F_{11}, \ldots, F_{1n}$, then $F_{21}, \ldots, F_{2n}$, etc. Consider the possible choices for $F_{j,p}$, once $F_{j',p}$ has been chosen for all $j' < j$ or $j' = j$ and $p < p$. $F_{j,p}$ is a subspace of $V_p$ of $\dim = \text{card} \{ a \leq p \mid \alpha(a) < j \}$, subject to the conditions $F_{j,p} \supset F_{j-1,p}$, $F_{j,p} \subseteq F_{j-1,p}$. So the dimensions of the possible $F_{j,p}$ will depend on $F_{j-1,p} + F_{j,p-1}$, and this method won't work.

Try instead choosing the columns $F_{1p}, \ldots, F_{j,p}$ inductively starting with $F_{j,p} = V_p$. Assume the columns $F_{j,p}$ chosen for $p < p$, and that I have also chosen $F_{j+1,p}, \ldots, F_{j,p} = V_p$. Consider the possible choices for $F_{j,p}$. This is a subspace of $\dim = \text{card} \{ a \leq p \mid \alpha(a) < j \}$, such that $F_{j,p} \supset F_{j+1,p} - 1$, $F_{j,p} 
subseteq F_{j+1,p}$. This is the set of lines in $F_{j+1,p} / F_{j,p-1}$ whose dimension doesn't vary.

$$\dim (F_{j+1,p} / F_{j,p-1}) = \dim (F_{j+1,p} / F_{j,p}) + \dim (F_{j,p} / F_{j,p-1})$$

$$= \text{card} \{ a \leq p \mid \alpha(a) = j + 1 \} + \dim_{F_{j,p}}$$

Two cases: $\alpha(p) > j \Rightarrow F_{j,p} = F_{j,p-1}$, no choice for $F_{j,p}$.
\[ \alpha(p) \leq j \implies F_{j+p}/F_{j+p-1} \text{ is any line in } F_{j+1}/F_{j+p-1} \]

\[ -1 + \dim \left( F_{j+1}/F_{j+p-1} \right) = \dim \left( F_{j+1}/F_{j+p} \right) = \text{card } \{ a \leq p \mid \alpha(a) = j+1 \} \]

So we see now that \( \tilde{C}_\alpha \) is non-singular of dimension

\[ \sum_{1 \leq j < \mu} \sum_{1 \leq p \leq n} \left\{ \begin{array}{ll} 0 & \alpha(p) > j \\ \text{card } \{ a \leq p \mid \alpha(a) = j+1 \} & \alpha(p) \leq j \end{array} \right\} \]

\[ = \sum_{1 \leq k \leq p} \sum_{1 \leq j < \mu} \left\{ \begin{array}{ll} 1 & \alpha(p) \leq j, \alpha(a) = j+1 \\ 0 & \text{otherwise} \end{array} \right\} \]

\[ = \text{card } \{ a < p \mid \alpha(a) > \alpha(p) \}. \]

Next thing to understand is the image of \( \tilde{C}_\alpha \to D_2(V) \), which should be the closure of \( C_\alpha \).

In particular, I want to understand which \( \beta \) are such that \( C_\beta \subset \overline{C_\alpha} \).
Change notation and replace $s_\mu \to s_\mu$ by $d = (d_1, \ldots, d_\mu)$, $0 < d_1 < \ldots < d_\mu = n$.

Then $D_\mu(V) \cong G/P_d$ where $P_d$ is the stabilizer of the flag $F_j = \sum_{p \leq d_j} k e_p$.

![Diagram](image.png)

Can also describe $P_d$ as the subgroup of $Gln$ generated by $B$ and the simple roots $s_i$ ($s_i$ transposes $i$ and $i+1$, $1 \leq i < n$) such that $\alpha_i^*(i) = \alpha(i+1)$, where $\alpha_i$ is the monotone map $\{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $d_j = \text{card } \{p \mid \alpha(p) \leq j\}$.

\[ W = \text{prod}, \quad W_d = \text{prod} \times \text{prod} \times \text{prod} \]

$=$ subgroup gen. by $s_i$ such that $\alpha_i^* s_i = s_i \alpha_i$ as above. $W/W_d$ can be identified with the set of $\alpha : \{1, \ldots, n\} \to \{1, \ldots, \mu\}$ such that $\text{card } \{p \mid \alpha(p) \leq j\} = d_j$.

So I understand the Bruhat decomposition for
(Recall that $\sigma \in W$ is interpreted as the matrix such that $\sigma(e_p) = e_{\sigma^{-1}p}$, whence,

$$\sigma \left( \sum_{p \leq j} k e_p \right) = \sigma \left( \sum_{q \leq j} k e_{\sigma^{-1}q} \right) \quad \text{if } 1 \leq j \leq \mu$$

$$= \left( \sum_{p \leq j} k e_{\sigma^{-1}p} \right) \quad \text{if } 1 \leq j \leq \mu.$$ 

In other words, applying $\sigma$ to the basic flag $F_d$ gives the flag corresponding to $\sigma^{-1}$. Therefore the identification of $W/W_d$ with the set of $\alpha$ proceeds by making $\sigma$ act on $\alpha$ by $\sigma \cdot \alpha = \alpha \cdot \sigma^{-1}$.

Observe that if $s$ is a simple root, then $B_s B$ contains $B$ in its closure. Hence if $l(s\sigma) = 1 + l(\sigma)$, which is equivalent to

$$B_{s\sigma} B = BsB \cdot B_s B$$

then $B_{s\sigma} B$ contains $B_s B$. 
More generally, $B$ is contained in the closure of every cell $B_0 B$, hence one sees that

$$l(\sigma) + l(\tau) = l(\sigma \tau) \Rightarrow \boxed{B_0 B \subset \overline{B_0 \tau B}}.$$ 

If $\sigma \in W$, then $\sigma P_d$ is the flag

$$(\sigma P_d)_j = \sigma(P_d)_j = \sigma \sum_{s_j(p) \leq j} ke_p$$

$$= \sum_{s_j(p) \leq j} ke_{\sigma(p)}$$

$$= \sum_{(\sigma^{-1} r) s_j \leq j} ke_p$$

Now recall we saw that any $\alpha : \{1, \ldots, n\} \rightarrow \{1, \ldots, \mu\}$ factorized uniquely

$$\{1, \ldots, n\} \xrightarrow{\sigma} \{1, \ldots, n\}$$

$$\alpha \downarrow \alpha_d$$

monotone

where $l(\sigma) = l(\alpha)$ (meaning $\sigma$ preserves order on fibres over each $j$, $1 \leq j \leq \mu$). Thus if I convert $\alpha$ to a permutation $\sigma$, the orbit indexed by $\sigma$ is $B\sigma^{-1}P_d$. In particular, a flag in $B_0 B \subset G/B$ is indexed by $\sigma^{-1} : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. 
Formula: \[ G = \prod_{W_d \in W/W_d} \mathcal{B} \sigma P_d \]

where \( \mathcal{B} \sigma P_d \) is indexed by \( d^\sigma \).

Now suppose we consider a product \( \mathcal{B} \tau \mathcal{B} \cdot \mathcal{B} \sigma P_d \). This is a quotient of \( \mathcal{B} \tau \mathcal{B} \times \mathcal{B} \mathcal{B} \sigma P_d \)

which has \( \text{dim} = l(\tau) + l(\alpha_d^\sigma \sigma^{-1}) + \text{dim} P_d \)

Thus \( l(\alpha_d^\sigma \sigma^{-1}) = \text{dim}(\mathcal{B} \tau \sigma P_d) - \text{dim} P_d \)

\[ \leq l(\tau) + l(\alpha_d^\sigma \sigma^{-1}) \]

In particular:

\[ l(\tau) + l(\alpha_d^\sigma \sigma^{-1}) = l(\alpha_d^\sigma \sigma^{-1} \tau^{-1}) \]

\[ \Rightarrow \mathcal{B} \tau \mathcal{B} \cdot \mathcal{B} \sigma P_d = \mathcal{B} \tau \sigma P_d \]

To establish the converse it suffices to do so where \( \tau \) is one of the \( \alpha_i \). Use induction on \( l(\tau) \).

Assuming \( \text{dim} = 1 \) if \( l(\tau) + l(\alpha_d^\sigma \sigma^{-1}) \leq l(\alpha_d^\sigma \sigma^{-1}) \), write \( \tau = \theta \tau' \) with \( l(\tau') = 1 + l(\tau) \), whence...
Simpler proof: Can suppose $\sigma$ such that $l(\sigma) = l(\sigma_1 \sigma_1^{-1})$. Then $l(\sigma) + l(\sigma_1) = l(\sigma_1 \sigma_1^{-1}) < l(\sigma) \Rightarrow B \sigma B. B \sigma B = B \sigma B$ hence $B \sigma B. B \sigma B = B \sigma B$.

\[ B \sigma B = B \sigma B. B \sigma B. B \sigma B.\]

Converse isn't true.

Given $\sigma : \{i_1, \ldots, i_j\} \rightarrow \{i_1, \ldots, \mu_j\}$, let $s = s_i = (1, \ldots, i, \ldots, n, n, i, \ldots, 1)$, the transposition interchanging $i$ and $i+1$.

\[ l(\omega s) = \text{card} \{ a < p \mid \alpha(sa) > \alpha(sp) \} \]

Now $\{ (a, p) \mid a < p \}$ and $\{ (a, p) \mid sa < sp \}$ are the same except that $(i, i+1)$ has been lifted out of the replaced by $(i+1, i)$. Thus

\[ l(\omega s) - l(\omega) = \text{card} \{ (a, p) \mid sa < sp, \alpha(sa) > \alpha(sp) \} \]

\[ -\text{card} \{ (a, p) \mid a < p, \alpha(a) > \alpha(p) \} \]

\[ = \begin{cases} 1 & \alpha(i+1) > \alpha(i) \\ 0 & \alpha(i+1) \leq \alpha(i) \end{cases} - \begin{cases} 1 & \alpha(i) > \alpha(i+1) \\ 0 & \alpha(i) \leq \alpha(i+1) \end{cases} \]

\[ = \begin{cases} 1 & \alpha(i) < \alpha(i+1) \\ 0 & \alpha(i) = \alpha(i+1) \Rightarrow s = \alpha \end{cases} \]

Where $s = \alpha$.\]
So one can have \( B_s B \cdot C_\alpha = C_{\alpha s} \) when \( \alpha s = \alpha \), even though \( l(\alpha s) = l(\alpha) \neq l(\alpha s) = l(\alpha) \).

I would like to know when \( C_\beta \subset C_\alpha \). Recall that \( B \) is in the closure of any cell \( B \tau B \), hence \( C_\beta \) is in the closure of \( B \tau B \cdot C_\beta \).

\[
C_\beta \subset C_\alpha \quad \text{if} \quad l(\tau) + l(\beta) = l(\beta \tau^{-1}).
\]

Hence I see that \( C_\beta \subset C_\alpha \) when \( \alpha = \beta \tau^{-1} \) where \( l(\alpha) = l(\beta) + l(\tau) \). I conjecture this condition is also necessary.

Compute \( H_\alpha = \{ g \in G \mid g C_\alpha = C_\alpha \} \). This subgroup contains \( B \), hence it should be generated by \( B \) and those reflections \( s_\beta \), which it contains. Now \( s C_\alpha = C_{\alpha s} \) and \( \dim(C_{\alpha s}) = l(\alpha s) \). If \( C_{\alpha s} = C_\alpha \), then we have \( l(\alpha s) = l(\alpha) \), which by a preceding calculation shows that \( \alpha s = \alpha \).

Thus \( H_\alpha \) is generated by \( B \) and those transp. \( s_\beta \) such that \( \alpha(s_i) = \alpha(s_i+1) \).

Example: Take \( \alpha = (1, n) \), whence \( D_{n}(V) = PV \cdot \alpha \) is given by \( \alpha^{-1}(1) = \{ k \mid 1 \leq k \leq n \} \). The corresponding Schubert cell is \( PV_k - PV_{k-1} \).
and its stabilizer is the parabolic subgroup fixing the flag $0 \leq V_{k-1} < V_k < V_1$ which is indeed generated by $B$ and the transpositions $s_1, \ldots, s_{k-2}, s_{k-1}$.  

**Assertion:** Let $Y_d$ be flag manifold of type $d$, let $C$ be a Schubert cell in $Y_d$, let $P = \{g \mid gC = C\}$, and let $B'$ be any Borel subgroup of $P$. Then $B'$ acts transitively on $C$.

**Proof:** We know $C$ is an orbit of $Y_d$ for some Borel subgroup $B$ of $P$. Let $T$ be a maximal torus contained in $B \cap B'$. We know there is a unique point of $C$ fixed under $T$, because $C$ is a $B$-orbit. $C$ is a union of $B'$-orbits, each having a unique $T$-fixed point. Hence $C$ must be a single $B'$-orbit.

**Problem:** To understand the poset of Schubert cells fixed by a given maximal torus $T$.

The maximal torus $T$ is equivalent to a set of axes, independent lines spanning $V_1$. An element $F$ of $Y_d$ fixed under $T$ may be identified
with a map \( \alpha : \Phi \rightarrow \{1, \ldots, \mu\} \) such that
\[
\text{card } \left\{ p \in \Phi \mid \alpha(p) < \mu \right\} = d_i.
\]
To give a Borel subgroup of \( G \) containing \( T_f \) is the same as giving a linear ordering of \( \Phi \), i.e., an isomorphism \( \{1, \ldots, n\} \sim \Phi \).

\[\alpha : \Phi \rightarrow \{1, \ldots, \mu\} \quad \text{being fixed, we can now try to classify the different orbits } BF_x \text{ as } B \text{ runs through the different Borels containing } T.\]

\[
\dim (BF_x) = \max \left\{ (a, p) \mid \alpha(a) > \alpha(p) \right\}.
\]

This will be zero provided the \( B \)-ordering on \( \Phi \) refines the \( \alpha \)-ordering.

Suppose we choose \( B \) so that \( F_\alpha \) is a fixed point for \( B \). Then we can identify \( \Phi \) with \( \{1, \ldots, \mu\} \) and \( \alpha : \{1, \ldots, \mu\} \rightarrow \Phi \) is monotone, i.e., \( \alpha = \alpha_f \).

Also, \( Y_\alpha = G/P_{\alpha} \) where \( P_\alpha = \{ g \in G_f \mid \alpha(i) > \alpha(j) \Rightarrow gij = 0 \} \).

Consider the Borel subgroup \( B' = \sigma B \sigma^{-1} \) preserving the flag \( \sigma(F_\alpha) = (\Sigma_{i \in \alpha(p)} e_i) = (\Sigma_{i \in \alpha(p)} e_{p_i}) \). Then \( B' \) consists of matrices \( b \) such that
\[
\sigma^{-1}(a) > \sigma^{-1}(p) \implies b_{ap} = 0.
\]
And \( \mathcal{B}' \cap P_d = \{ b \mid \sigma^{-1}(a) > \sigma^{-1}(p) \quad \alpha \} \Rightarrow b_{ap} = 0 \} \)

so \( B'/B' \cap P_d \) can be represented by matrices \( (b_{ap}) \) with non-zero entries at positions \( (a, p) \) if \( \sigma^{-1}(a) < \sigma^{-1}(p) \), \( \alpha(a) > \alpha(p) \), and 1's on the diagonal.

Note that these representatives for the Schubert cell \( B' \cap P_d \) form a subgroup of the unipotent \( N \) group with non-zero entries in the positions \( (a, p) \), \( \alpha(a) > \alpha(p) \), and 1's on the diagonal.

\[
N = \begin{pmatrix}
0 & 1 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{pmatrix}
\]