Notes on construction:

Let \( f: X \to Y \) be a map of cell complexes. Suppose \( Y \) connected \( \Rightarrow \) simplify, let \( y_0 \) be a base point of \( Y \). Let \( F = \text{homm fibre of } f \text{ over } y_0 = \text{space of pairs } (x,a) \) \( \Rightarrow \) a path joining \( f(x) \) to \( y_0 \). Let \( \tilde{Y} \) be the universal covering of \( Y \).

Proposition: TFAE

(i) \( F \) acyclic (i.e., \( \tilde{H}_*(F, \mathbb{Z}) = 0 \))

(ii) \( \forall \) local system \( L \) on \( Y \), we have \( \tilde{H}_*(X, f^*L) = H_*(Y, L) \)

(iii) \( X \times \tilde{Y} \to \tilde{Y} \) induces injective, integral homology.

---

Proof: (i) \( \Rightarrow \) (ii). Consider spectral sequence

\[
E_{pq}^2 = H_p(Y, H_q(F, E)) \Rightarrow H_{pq}(X, E)
\]

for \( E \) any local system on \( X \). Take \( E = f^*L \), where \( f^* \) is trivial on \( F \), so \( \tilde{H}_*(F f^*L) = \mathbb{Z} L \) (use univ. coeffs.)
Thus spec. seq. degenerates yielding (ii).

(ii) \( \Rightarrow \) (iii). Have

\[
\begin{array}{ccc}
\tilde{X} \times \tilde{Y} & \to & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}
\]

where vertical maps are principal covering groups \( \pi_1 Y \).
Thus have

\[
\begin{align*}
H_*(\tilde{X} \times \tilde{Y}, \mathbb{Z}) & \to H_*(\tilde{Y}, \mathbb{Z}) \\
H_*(X, \mathbb{Z}[\pi_1 Y]) & \to H_*(Y, \mathbb{Z}[\pi_1 Y])
\end{align*}
\]

so clear.

(iii) \( \Rightarrow \) (i). Since homotopy fibre doesn't change under pulling back via \( Y \to Y \), we can suppose \( Y \) simply-connected.

Now observe at the spectral seq.

\[
E_{pq}^2 = H_p(Y, H_q(F, \mathbb{Z})) \to H_{pq}(X, \mathbb{Z}).
\]
Def: such a map will be called acyclic.

Corollary: Acyclic maps are stable under composition (use ii) homotopy base change, & homotopy colimit change.

Proof: As for last suppose have cocart.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

with \( f \) acyclic + a cofibration. Then \( (ii) \Rightarrow H_*(Y, X; L) = 0 \) for all local systems on \( Y \). Since \( H_*(Y, X; L) \cong H_*(Y, X; F^*) \) (consider cellular chains), one gets \( f' \) is acyclic.

Cor. 2: If \( f \) is acyclic, then \( \pi_1(f) : \pi_1 X \to \pi_1 Y \) is onto and its kernel is perfect. \( f \) is a hop \( \Rightarrow \pi_1(f) \) is an isomorphism.

Proof: \( \pi_0(f) = 0 \Rightarrow \pi_1(f) \) onto. \( F \) acyclic \( \Rightarrow H_*(F) = \pi_* F^* = 0 \Rightarrow \pi_1(f) \) perfect \( \Rightarrow \ker \pi_1(f) \) perfect. If \( \pi_1(f) \) is an isom, then \( X_y Y = X \), so by Whitehead \( \Rightarrow X \to Y \) is a hop \( \Rightarrow \pi_1(X) \to \pi_1(Y) \) all \( \geq 2 \). Thus \( f \) is a hop.

From now on, simplify, suppose.

From now on, work with connected ptsd CW eks. and put \([X, Y]\) for ptsd. homot. classes.

Proposition: Let \( f : X \to Y \) be acyclic. Then

\[
f^*: [Y, Z] \to \{ \text{deg} [X, Z] \mid \ker \pi_1(f) \subset \ker \pi_1(f) \}
\]
Proof: Surjectivity. Can suppose \( f \) is a cofibration.

Given \( g: X \to Z \) and \( \text{ Ker } \pi_1(f) \subseteq \text{ Ker } \pi_1(g) \).

![Diagram]

By van Kampen, \( \pi_1(Y \cup Z) = \pi_1 Y \times_{\pi_1 X} \pi_1 Z \subseteq \pi_1 Z \). But + acyclic \( \Rightarrow f' \) acyclic. \( \Rightarrow f' \) has and so \( g \) factors thru \( f' \).

Injectivity. Assume we have \( g_1, g_2: Y \to Z \Rightarrow g_1 f' = g_2 f' \).

By HET can homotop \( g_2 \) until \( g_1 f = g_2 f \); call this map \( g' \).

\( g_1, g_2 \) induce maps \( h_1, h_2: Y \cup Z \Rightarrow g_1 = h_1 g_{\ast} \). But \( f' \) is a bug and \( h_1 f' = id \Rightarrow h_4 = h_2 \Rightarrow g_1 = g_2 \).

Core: Given \( N \) perfect \( \subseteq \pi_1(X) \), \( f: X \to Y \) acyclic with \( \text{ Ker } \pi_1(f) = \text{ Ker } \pi_1(f') \), then \( \exists \) hom \( h: Y \to Y' \Rightarrow hf = f' \).

This is clear. \((Y \text{ and } Y' \text{ both represent the same functor})\).

Proph. Given \( N \) perfect \( \subseteq \pi_1(X) \), \( \exists f: X \to Y \) acyclic with \( \text{ Ker } \pi_1(f) = N \).

Proof: First suppose \( \pi_1(X) \) is perfect. Then choose
element $a_i \in H_1(X)$ which normally generate $H_1^2(X)$ and let $X'$ be the result of attaching 2 cells to kill the $a_i$. Then $X'$ is simply-connected by van Kampen and

$$V S^1 \xrightarrow{\alpha} X \xrightarrow{\alpha} X' \xrightarrow{\alpha} V S^2$$

$$H_i(X) \rightarrow H_i(X') \rightarrow 0 \rightarrow H_i(X)$$

Thus $H_2(X')$ is free with base $a_i$, $i \in I$. Since

$$H_2(X') \rightarrow H_2(X, X)$$

is onto, we can find $b_i \in H_2(X')$ such that $\beta_i$ goes to $a_i$. Thus define $Y$ by

$$V S^2 \xrightarrow{\beta} X' \xrightarrow{\beta} Y$$

$$H_2(X) \rightarrow H_2(X') \rightarrow H_2(Y) \rightarrow 0$$

so $H_2(X) \cong H_2(Y)$. It follows then that $H_n(X) \rightarrow H_n(Y)$ for all $n$. This proves the proof when $N = H_1(X)$.

Now in the general case, let $X'$ be the covering space of $X$ with $\pi_1 X' = N$, let $X' \xrightarrow{f} Y'$ be acyclic with $\pi_1 Y' = 0$, and form the pushout

$$\xymatrix{ X' \ar[r]^{f'} \ar[d] & Y' \ar[d] \\
X \ar[r]^{f} & Y }$$

Then $f'$ acyclic $\Rightarrow f$ acyclic. Also Van Kampen $\Rightarrow$

$$\pi_1(Y) = \pi_1(X) \ast_N e = \pi_1(X)/N.$$

done.
Remark: Clear from the proof that if $N$ is normally generated by a finite number of elements, it is necessary to attach only a finite number of $2 + 3$ cells to get $Y$. Actually if $N$ has a finite no. of gen. as a normal subgroup of $G$, this remains true. One has to go back to the proof but use cellular chains on the universal covering. (Thus set $X = \text{covering corrept. to } N$. Then $0 \to H_2(X) \to H_2(X') \to \bigoplus \mathbb{Z} \left[ \pi_1 X/N \right] \to 0$ so again can find $\beta_i \in \pi_2(X')$ mapping onto a basis for $H_2(X, X')$. etc.)

Let $N$ be the largest perfect subgroup of $\pi_1 X$. (A group gen. by perfect subgps. is perfect - consider a homo. to any abelian group.) The acyclic map in this case will be denoted $X \to X^+$. It is universal for maps to spaces having no perfect subgps. etc.

Formula:

$$(X \times Y)^+ = X^+ \times Y^+$$

because the product of two acyclic maps is acyclic.

Deleq's question: Given a fibration, can the plus construction be performed fibrewise.
Let $F 	o E 	o B$ be a fibration (of comm. lptd. spaces as always) and suppose we have a map of fibrations

\[
\begin{array}{ccc}
F & \to & E \\
\downarrow & & \downarrow \\
F' & \to & E' \\
\end{array}
\]

with $F \to F'$ acyclic. Then from homology spectral sequences, one can see $E \to E'$ is acyclic, hence it is determined by a perf. normal subgps $N$ of $\pi_1(E)$ which goes to 0 in $B$. Now we know $\pi_2 B$ maps into the center of $\pi_1(F)$, hence $\pi_1(F)$ is a central extension of its image in $\pi_1(E)$, and one knows there is a unique perf. subgp. $M$ of $\pi_1(F)$ mapping onto $N$, namely the commutator subgp. of the inverse image of $N$.

Diagram chasing shows that $\text{Ker}(\pi_1 F \to \pi_1 F')$ maps onto $N$. Hence $\text{Ker}(\pi_1 F \to \pi_1 F') = M$.

Thus we see that we can kill any perfect normal subgp. of $\pi_1 F$ whose image in $\pi_1(E)$ is normal, or equiv. which is stable under the action of $\pi_1(B)$ on $\pi_1(F)$ mod. inner auto.

Proof: Given $F \to E \to B$ and a perf. normal $M \subset \pi_1 F$ stable under the $\pi_1 B$-action, there exists a map of fibrations $\square$ over $B$. 
where \( f: F \to F' \) is acyclic with \( \ker \pi_1(f) = N \), and \( g \) is acyclic with \( \ker \pi_1(g) = \text{Image of } N \) in \( \pi_1(E) \).

(Clearer: Given \( E \to B \) with fibre \( F \), those acyclic maps \( E \to E' \) over \( B \) are classified by perf. normal subgroups \( N \) of \( \ker \pi_1(E) \to \pi_1(B) \).

If \( F' = \text{Fibre of } E' \) over \( B \), then \( E \cong F \otimes_{E'} E \) and as we can suppose \( E \to E' \) is a fibre, we have

\[
\begin{array}{c}
F \\
\downarrow f \\
F' \\
\downarrow g \\
E' \\
\end{array}
\]

is h-cart. \( \therefore f \) is acyclic and \( \pi_1(F) \to \pi_1(F') \times_{\pi_1(E')} \pi_1(E) \),

so \( \ker \pi_1(f) \to \ker \pi_1(g) \). But as \( \pi_1(F) \) is a central extension of \( \ker(\pi_1(E) \to \pi_1(B)) \), \( \ker \pi_1(f) \) is the unique perf. subgroup of \( \pi_1(F) \) with image \( N \). Conversely, given \( M \) perf. \( \triangleleft \) in \( \pi_1(F) \) stable under \( \pi_1(B) \) action, taking \( N = \text{Im } M \) in \( \pi_1(E) \), this process \( \triangleright \) kills \( N \) in the fibre.)
We begin by recalling a standard construction in homotopy theory introduced in Serre's thesis.

Let $K(A,n)$ be an Eilenberg-MacLane space with $n > 2$. From the Hurewicz theorem we have the formulas:

$$H_i(K(A,n)) = \begin{cases} 
0 & i = 0 \\
0 < i < n \\
A & i = n \\
0 & i = n+1
\end{cases}$$

(last follows because $\pi_{n+1} K(A,n) \to H_{n+1} K(A,n)$ is onto.)

**Lemma 1:** There exists a map $X \to K(H_n X, n)$ which induces the canonical isomorphism $\theta : H_n X \cong H_n (K(H_n X, n))$. If $H_{n-1} X = 0$, this map is unique.

**Proof:** Start with the U.C. formula

$$0 \to \text{Ext}^1(H_{n-1} X, A) \to H^n(X, A) \to \text{Hom}(H_n X, A) \to 0$$

where $[X, K(A,n)] \sim H^n(X,A)$ and $f \mapsto f^*(u_n)$

where $u_n \in H^n(K(A,n), n)$ is the unique class such that $\varphi(u_n)$ is the canonical isomorphism $\theta : H_n (K(A,n)) \cong A$. It follows that $\varphi$ is isomorphic to the map

$$[X, K(A,n)] \to \text{Hom}(H_n X, A)$$

and so the latter is always surjective and injective when $H_{n-1} X = 0$. Taking $A = H_n X$, the lemma follows.

**Lemma 2:** Assume $H(X) = H_{n-1}(X) = 0$, $n > 2$, and let $F$ be the homotopy fibre of the map $v : X \to K(H_n X, n)$ of Lemma 1. Then $H_i(F) \cong H_i(X)$ for $i \leq n-1$, and $H_n F = 0$. 
Proof: Put \( B = K(H_nX, n) \) and consider the spectral
\[
E^2_{pq} = H_p(B, H_qF) \Rightarrow H_{p+q}X
\]

which gives \( H_iF \xrightarrow{\sim} H_iX \) \( i \leq n-2 \) and an exact sequence

\[
0 \leftarrow H_{n-1}X \leftarrow H_{n-1}F \leftarrow H_nB \leftarrow H_nX \leftarrow H_nF \leftarrow 0
\]

(the last 0 results from the fact that all \( E^2 \) terms of total degree \( n \) are zero except for \( E^2_{0n} = H_nF \)). Now using the fact that \( H_{n-1}X = 0 \), and that \( H_nX \xrightarrow{\sim} H_nB \) the lemma follows.

Now use this lemma as follows. Given a space \( X \) such that \( H_1X = 0 \) and an integer \( n \geq 2 \) such that \( H_{n-1}X \) we construct recursively a tower of spaces

\[
\rightarrow X_{n+2} \rightarrow X_{n+1} \rightarrow X_n = X
\]

such that

\[
H_i(X_p) \xrightarrow{\sim} H_i(X) \quad \text{for} \quad i < n
\]

\[
= 0 \quad \text{for} \quad i \geq n
\]

by letting \( X_{p+1} \) be the fibre of the canonical arrow

\[
X_p \rightarrow K(H_pX_p, p) \quad \text{of lemma 1.}
\]

Put \( X = \text{hull}(X_p) \). Since

fibre of \( X_\infty \rightarrow X_p \) becomes increasingly connected with \( p \), we have

\[
H_i(X_\infty) \rightarrow H_i(X) \quad \text{for} \quad i < n
\]

\[
= 0 \quad \text{for} \quad i \geq n-1
\]
\[ X_{p+1} \rightarrow X_p \rightarrow K(H_pX_p, p) \]

\[ \Rightarrow \pi_i(X_{p+1}) \rightarrow \pi_i(X_p) \]

iso \( i < p-1 \)

onto \( i = p-1 \)

(\text{meaning: fibre begins in dim } p-1). \quad \text{(system)}

Put \( X' = \text{holim} \ x \leftarrow \ X_p \). Then because the \( \pi_i(X_p) \) is essentially constant, one has for each \( p \)

\[ \pi_i(X') \rightarrow \pi_i(X_p) \]

iso \( i < p-1 \)

onto \( i = p-1 \)

\[ \Rightarrow H_i(X') \Rightarrow H_i(X_p) \]

\[ \text{so we have proved:} \]

\[ \text{Proposition: Let } X \text{ be a space with } H_1X = 0 \]

and \( n \) an integer \( \geq 2 \) such that \( H_{n-1}X = 0 \). Then

there exists a map \( X' \rightarrow X \) such that

\[ i) \quad \pi_i X' \rightarrow \pi_i X \]

iso \( i < p-1 \)

onto \( i = n-1 \)

\[ ii) \quad H_i X' = 0 \quad i \geq n-1 \]

\[ \Rightarrow H_i X \quad i \leq n-1 \]

\[ \text{Examples: 1) If } \pi_1X = 0 \text{ and } n = 2, \text{ then the tower } \]

\( X_p \) is just the Postnikov tower of \( X \), so \( X' \sim pt. \)

2) \( X = K(A, m), \ m \geq 2, \ n = m+2. \quad \text{Then } X' \text{ is a Moore space } M(A, m). \quad (H_{m+1}(K(A, m)) = 0) \]

Now consider the general case \( n = 2 \):
Proof: Let $X$ be a space with $H_n(X) = 0$ (i.e. $\pi_n X$ perfect). Then the map $X' \to X$ constructed above starting with $n = 2$ is a universal map from an acyclic space to $X$.

Proof: Clearly the space $X'$ is acyclic. If now $Y$ is an acyclic space, one has $[Y, X_{n+1}] \to [Y, X_n]$ as $[Y, K(A, p)] = 0$ for all $p \geq 1$, and $A$ abelian. Thus passing to the limit $[Y, X'] \to [Y, X]$ proving the assertion.

Cor. $X' \to X$ is the fibre over $X \to X^+$. Proof: If $F$ is this fibre, then we know it is acyclic (first prop.), and $[F, F'] = 0$. Also for $Y$ acyclic we have $[Y, X^+] = 0$ (universal property for $Y \to pt$). Thus from $[Y, X^+] \to [Y, F] \to [Y, X^0] \to [Y, X^+]$ one concludes that $[Y, F] \to [Y, X]$. Thus $F \to X$ has the universal property of $X' \to X$.

So now here is Dror method for proving the existence of an acyclic map $f: X \to Y$ killing $\text{perf. } N \triangleleft \pi_n X$. He constructs $\tilde{X} = \text{covering corresponding to } N$ and then the universal acyclic space $A(\tilde{X}) \to \tilde{X}$ which has the property that its $\pi_n$ maps on $N$. Then be forms pushouts

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \tilde{X}/A(\tilde{X}) = \tilde{X}^+ \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]
Lemma 1: \( H_n X = 0, \quad H_{n-1} X = 0. \)

i) \( \exists ! \) map \( X \rightarrow K(H_n X, n) \) inducing the canon. iso.
\( H_n X \cong H_n K(H_n X, n) \).

ii) If \( F \) is fibre of \( u_0 \), then
\( \pi_i F \rightarrow \pi_i X \)
iso. \( i \leq n-1 \)
onto \( i = n-1 \)

\[ H_{n-1} F = H_n F = 0 \]

Dror's tower of a space \( X \quad H_1 X = 0. \) Let
\[
X_{p+1} \rightarrow X_p \rightarrow \cdots \rightarrow X_4 \rightarrow X_3 \rightarrow X_2 = X
\]

Has to be generalized!!

Properties:
(i) \( \tilde{H}_i(X_p) = 0 \quad i < p \)
(ii) \( [Y, X_p] \rightarrow [Y, X] \) if \( \tilde{H}_i(Y) = 0 \quad i < p \).
(iii) \( \pi_i(X_{p+1}) \rightarrow \pi_i(X_p) \)
iso. \( i < p-1 \)
onto \( i = p-1 \).

Put: \( X_\infty = \varinjlim X_p \)
(i) \( \tilde{H}_i(X_\infty) = 0 \)
(ii) \( [Y, X_\infty] \rightarrow [Y, X] \) if \( \tilde{H}_i(Y) = 0 \)

Thus \( X_\infty \twoheadrightarrow X \) universal map from an acyclic space to \( X \)

Cor: \( X_\infty = \) fibre of \( X \twoheadrightarrow X^+ \).
Example: \( X = BG \) \( G \) perfect.

Claim: \( X_3 = BG \) \( \widetilde{G} \) - covering group of \( G \).

Proof: From homotopy sequence finds \( \pi_3 X_3 = \pi_3 BG = 0 \) \( g \geq 2 \)
hence \( X_3 = BG' \) \( G' = \pi_1 X_3 \). Also get central ext.
\[
1 \rightarrow H_2 G \rightarrow \pi_1 X_3 \rightarrow G \rightarrow 1
\]

Finally \( H_1(G') = H_2(G') = 0 \) \( \Rightarrow G = \widetilde{G} \) by theory of Shur mult...

Next one sees inductively: since \( \pi_2 X_2 = 0 \) \( g \geq 2 \) that
\[
\pi_i X_p = \begin{cases} 
G' & i = 1 \\
H_{i-1} X_{i+1} & 2 \leq i \leq p-2 \\
0 & p-1 \leq i
\end{cases}
\]

so
\[
\pi_i X_\infty = \begin{cases} 
G' & i = 1 \\
H_{i-1} X_{i+1} & i \geq 2
\end{cases}
\]

Now look at fibration
\[
X_\infty \rightarrow BG \rightarrow BG^+
\]
and one gets
\[
\pi_8(BG^+) = H_8 X_8
\]

In particular
\[
\pi_2(BG^+) = H_2 G \\
\pi_3(BG^+) = H_3 \widetilde{G}
\]

Proposition: Let \( \widetilde{G} \) \( G \) be a perfect group and
let \( \{X_g\} \) be the Shur tower over \( BG \).

(i) \( X_3 = BG \) where \( \widetilde{G} \) is the universal covering group in the sense of the Shur mult. theory.

(ii) \( \pi_0(BG^+) = H_0(X_0) \).

(iii) \( \pi_2(BG^+) = H_2(G) \), \( \pi_3(BG^+) = H_3(G) \).
Complements: Given X connected but \( H_1X \) not necessarily zero, let \( N \) be the maximal perfect subgroup of \( \pi_1X \), and put 
\[ X_2 = X \]
\[ \pi_1X_2 = N \]
\[ X_2 = \text{covering of } X \text{ with } \pi_1X_2 = N. \]

Then extend the

\[ X_\infty \longrightarrow X \]

universal map from an acyclic space to \( X \)

\[ X_\infty = \text{Fibre} \{X \longrightarrow X^+\} \]

\[ X^+ = \text{Cone} \{X_\infty \longrightarrow X\} \]

Last formula gives another construction of \( X^+ \).
I will begin by proving a basic result on infinite matrix groups which has many applications in algebraic K-theory.

Recall the following result: Let $H, G$ be the subgroups

$$
\begin{pmatrix}
I_n & 0 \\
0 & G_{n\times n}(\mathbb{R})
\end{pmatrix},
\begin{pmatrix}
I_n & M_{n\times n}(\mathbb{R}) \\
0 & G_n(\mathbb{R})
\end{pmatrix},
$$

of $G_{n\times n}(\mathbb{R})$. Then $BH \rightarrow BG$ is a homotopy equivalence. Indeed $BH$ is hom. equiv. to the assoc. fibre space over $BG$ with fibre $G/H$, and $G/H$ is contractible.

The analogue of this result in alg. K-theory goes as follows. Let $A$ be a ring (always supposed assoc. with 1) and let $GL_n(A) = UGL_n(A), \ M_{n\times n}(A) = U \ M_{n\times n}(A)$ under the standard inclusions.

**Thm:** The inclusion

$$
\begin{pmatrix}
I_n \\
0
\end{pmatrix} \subset
\begin{pmatrix}
I_n & M_{n\times n}(A) \\
0 & GL_{n\times n}(A)
\end{pmatrix}
$$

induces isomorphisms on homology with coefficients in any abelian group $\Lambda$ equipped with trivial action.

(Improvement: Further inclusion $\mathbb{A}$ is satisfied

If $\Lambda$ is an abelian gp, let $H_\ast(G, \Lambda)$ denote the hom. of $G$ with coefficients in $\Lambda$ equipped with the trivial $G$-action. Say that $H \rightarrow G$ induces iso. in homology with constant coefficients if $H_\ast(H, \Lambda) \rightarrow H_\ast(G, \Lambda)$ for every abel. gp. $\Lambda$.

It is enough to check this for $\Lambda = \mathbb{Z}$, or also for each of the fields $\mathbb{Q}, \mathbb{F}_p$ (prime).
Before beginning the proof, recall that $H_*(GL_n(A), \Lambda)$ has a ring structure when $\Lambda$ is a ring defined as follows. One starts with the homomorphism

$$GL_p(A) \times GL_q(A) \to GL_{p+q}(A)$$

$$x \otimes \beta = (x \otimes 1) \cdot (1 \otimes \beta)$$

which induce pairings

$$\mu_{pq}: H_*(GL_p(A), \Lambda) \otimes H_*(GL_q(A), \Lambda) \to H_*(GL_{p+q}(A), \Lambda)$$

Example to show $n=\infty$ is necessary.

$$H_*(\left( \frac{1}{F_p}, F_p \right) = H_*(\left( \frac{1}{F_p}, F_p \right), F_p)$$

trivial as finite order $\text{im}(p)$ prime to $p$.

and since inner auto's of a group are trivial in homology, one gets that $\mu_{pq}$ is commutative (assuming $\Lambda$ commutative). Precisely, one has a comm. diag.

$$H_*(G_p) \otimes H_*(G_q) \to H_*(G_p \times G_q) \to H_*(G_{p+q})$$

where $\tau(x \otimes y) = \tau(x) \otimes \tau(y)$, so $\mu_{pq} \tau = \mu_{pq}$. The point is that because of this commutativity $\mu_{pq}$ is compatible with passing from $p$ to $p+1$, $q$ to $q+1$.

$$G_p \times G_q \to G_{p+q} \Delta$$

$$(a \otimes \epsilon) \otimes \beta \sim (a \otimes \beta) \otimes \epsilon$$
Have associativity

\[(\alpha \# \beta) \# \gamma = \alpha \# (\beta \# \gamma)\]

and unity

\[\alpha \# \text{id} = \text{id} \# \alpha = \alpha\]

and commutativity up to conjugacy

\[\alpha \# \beta \sim \beta \# \alpha\]

Thus if \(\varepsilon = (1)\), we have

\[
\begin{align*}
G_p \times G_q & \longrightarrow G_{p+q} \\
(\alpha, \beta) & \longmapsto \alpha \# \beta \\
G_{p+1} \times G_q & \longrightarrow G_{p+q+1} \\
(\alpha \# \varepsilon, \beta) & \longmapsto (\alpha \# \varepsilon) \# \beta
\end{align*}
\]

so we get that \(\{\mu_{pq}\}\) are compatible with stabilization and define in the limit a map

\[\mu: H_\ast(G_\infty) \otimes H_\ast(G_\infty) \longrightarrow H_\ast(G_\infty)\]

\(\mu\) has the following properties:

i) associativity \(\mu(\mu \otimes \text{id}) = \mu(\text{id} \otimes \mu)\)

ii) unity \(\mu(1 \otimes x) = x\), where 1 denotes the image of the basepoint in \(H_0(G_\infty)\), a canonical generator

iii) commutativity \(\mu \tau = \mu\).

Demonstration of these. Put \(G_n = (\text{In Mon})\) and define

\[G_p \times G_q \stackrel{\tau}{\longrightarrow} G_{p+q}\]

\[
\begin{pmatrix} 1 & u \\ x & \beta \end{pmatrix} \circ \begin{pmatrix} 1 & v \\ \alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & u + \alpha v \\ x & \beta + \gamma \end{pmatrix}
\]
This is assoc. + comm. up to conjugacy, so it induces a product on $H_*(G_\infty, \Lambda)$ as before.

\[
\begin{align*}
GL_n & \xrightarrow{\kappa_n} G_n & & \xrightarrow{\kappa_n} GL_n \\
\begin{pmatrix} \alpha & \end{pmatrix} & \mapsto \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} & & (1, \alpha) \mapsto \alpha \\
\end{align*}
\]

compatible with $\Theta, \perp$ hence induce alg. homos.

\[
\begin{align*}
H_*(G_\infty) & \xrightarrow{\kappa_*} H_*(G_\infty) & & \xrightarrow{\kappa_*} H_*(G_\infty) \\
\end{align*}
\]

\[
\begin{align*}
\eta_{\kappa_*} & = \text{id}. \\
\end{align*}
\]

Thus have to show that $\eta_{\kappa_*} \kappa_* = \text{id}$. Let $\eta_* \kappa_n = \psi_n$

\[
\begin{align*}
\psi_n \begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} & = \begin{pmatrix} 1 & \psi_n(u) \\ x & \psi_n(x) \end{pmatrix}
\end{align*}
\]

so that $\psi_* = \eta_* \kappa_*$. Identify:

\[
\begin{align*}
\begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} & \begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} & \begin{pmatrix} 1 & u \\ x & 1 \end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
H_*(G) & \xrightarrow{\Delta} H_*(G) \otimes H_*(G) \\
\end{align*}
\]

\[
\begin{align*}
\xrightarrow{\text{id}} H_*(G) \times H_*(G) & \xrightarrow{\text{id}} H_*(G_\infty) \times H_*(G_\infty) \\
\end{align*}
\]

\[
\begin{align*}
\xrightarrow{\text{id}} H_*(G_\infty) \\
\end{align*}
\]

Reduce to case $\Lambda = \text{field}$ so that $H_*(X \times Y) = H_*(X) \otimes H_*(Y)$

and such that

\[
\Delta(x) = 1 \otimes x + \sum_i x_i \otimes x_i^{\text{deg}(x)}
\]

1 denoting a basepoint in $H_0(X)$.

Can pass to limit by yet

\[
\mu \Delta = \mu (\text{id} \otimes \psi_\ast) \Delta \quad \text{on} \quad H_*(G_\infty)
\]

so now can show $x = \psi_\ast(x)$ for $x \in H_*(G_\infty)$ by induction.
\[ \chi + \sum \chi_i \mu(x_i^w) = \varphi(x) + \sum \chi_i \varphi(x_i^w) \]

**Induction:** \[ \chi = \varphi(x) \]

**Questions:** Would this work for \( E(A) = U E_n(A) \)? Better \( SL_n(A) \)? Permutative category is a bit tricky.

Define exact sequence to consist of a normal exact sequence

\[ 0 \to V' \to V \to V'' \to 0 \]

such that

\[ \Lambda^p V = \Lambda^p V' \otimes \Lambda^q V'' \]

\[ \text{commutes.} \]

This is not a perm cat., because

\[ V \oplus W \nRightarrow W \oplus V \]

is not compatible with orient.

\[ v \to v^p \wedge \omega_5 \wedge \omega_6 \Lambda^p (V \oplus W) = \Lambda^p (W \oplus V) \]

\[ v_i \to v_i^p \wedge \omega_5 \wedge \omega_6 \Lambda^p V \]

\[ \omega \cdot \omega \wedge \Lambda^p V \]

\[ \omega \cdot \omega \wedge \Lambda^p V \]

\[ \omega \cdot \omega \wedge \Lambda^p V \]

\[ (\omega \cdot \omega)_{(v_1 \wedge \cdots \wedge v_p \wedge \omega_5 \wedge \omega_6 \wedge \omega_7)} = (-1)^p \omega \cdot (v_1 \wedge \cdots \wedge v_p \wedge \omega_5 \wedge \omega_6 \wedge \omega_7) \]

\[ (-1)^p \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega \cdot \omega
Maybe there is a moral here. It seems that the cat of vector spaces with volume is not a perm.

cat.

$SL_p \quad SL_q \quad SL_{p+q}$

What this example shows is that one cannot kill the $K_1$ without first killing $K_0$.

$K_2F \quad K_1F \quad K_0F$

$F^* \quad \mathbb{Z}$

Thus the problem is to construct a model for the theory beginning in dim $2$.

---

**Applications:**

**Cor. 1:**

$H_\ast(Gl_\ast(A)) \simeq H_\ast(Gl_\ast(A)) \quad \text{mod } \quad M_{r\infty}(A)$

**Proof:**

Let $G'$

$\begin{array}{c}
1 \rightarrow H' \rightarrow G' \rightarrow Gl_r \rightarrow 1 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
1 \rightarrow H \rightarrow G \rightarrow Gl_r \rightarrow 1
\end{array}$

$E_2^{pq} = H_p(Gl_r, H_q(H')) \Rightarrow H_{p+q}(G')$

$E_1^{pq} = H_p(Gl_r, H_q(H)) \Rightarrow H_{p+q}(G)$

Preceding $\Rightarrow H_\ast(H') \simeq H_\ast(H)$.

**Cor. 2:**

$H_\ast(Gl_\ast) \Rightarrow H_\ast(Gl_\ast)$
Remark: \[ \text{GL}_n\left(\begin{array}{cc} A & A \\ 0 & A \end{array}\right) \cong \left(\begin{array}{cc} \text{GL}_n(A) & M_{m \times n}(A) \\ 0 & \text{GL}_n(A) \end{array}\right) \]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} e & f \\ g & h \end{pmatrix}
\]

\[ \text{GL}_n(\mathbb{C}) = \text{Aut}(A \otimes A^*) \cong \text{Aut}(A^* \subset A^* \otimes A^*) \]

\text{Cor. 3: } H_*(\text{GL}_n(\begin{array}{cc} A & A \\ 0 & A \end{array})) \leftarrow H_*(\text{GL}_n(A))

Claim that to \( G \to \text{Aut}(P) \) \( P \in P(A) \), one has \( f^*: H_*(G) \to H_*(\text{GL}(A)) \),

such that \( \psi_1^* = \psi_2^* \quad \psi_1 \otimes \psi_2 = \psi_2 \otimes \psi_2 \)

\( \mathbb{E}_1, \mathbb{E}_2 \) trivial reps. In effect

\[
\text{rep}(G, A) = \bigoplus_{P} \text{Hom}(G, \text{Aut}(P))_{\text{iso. classes.}}
\]

\[
\text{proj}(G, A) = \lim_{\longrightarrow} \text{Hom}(G, \text{Aut}(P))_{\text{limit is taken over trans. cat.}}
\]

\[
= \lim_{\longrightarrow} \text{Hom}(G, \text{GL}(A))_{\text{cofinality}}
\]

\[
\longrightarrow \text{Hom}(\text{GL}(G), H_*(\text{GL}(A)))
\]

In down-to-earth terms, one chooses \( Q + P \otimes Q \to A^* \) and lets \( G \) act trivially on \( Q \), hence getting \( G \to \text{GL}(A) \).
**Cor. 4:** Let $\rho$ be a repn. of $G$ on $P$ and suppose $G$ leaves stable a flag $0 = P_0 \subset P_1 \subset \cdots \subset P_n = P$ s.t. $P_i/P_{i-1} \in \mathcal{P}(A)$. Let $f_i$ be the ind. repn. of $G$ on $P_i/P_{i-1}$. Then $f_* = (f_1 \oplus \cdots \oplus f_n)_*$.

Proof: Induction reduces me to $n=2$: if $f' = \text{id. repn. on } P_{n-1}$, then $f_* = f'_*(f_1 \oplus f_n)_* = (f_1 \oplus \cdots \oplus f_{n-1} \oplus f_n)_*$. Can suppose $n=2$, $P_1 = A^\lambda$, $P_2 = A^{\lambda + n}$.

**Cor. 5:** $H_i(G_{\infty}(F_q), F_p) = 0$, $q = p^d$, $i > 0$.

Proof: Sylow subgroup of $G_{\infty}(F_q)$ enough to show this is zero.