Problem: The space $L = \text{GL}_n(C[z, z^{-1}]) / \text{GL}_n(C[z])$, we have seen is a union of algebraic varieties, hence it has an etale homotopy type. The problem is to give an algebraic proof that $L \sim \Omega \text{GL}_n$.

Put $\Omega = \text{GL}_n(C[z, z^{-1}])$, so that we have a homotopy equivalence $\Omega \to L$. Then we have algebraic maps

$$\Omega \times \mathbb{G}_m \to \text{GL}_n \quad (\Omega \times 1) \to 1.$$ 

So if we give $S^1 \to (\mathbb{G}_m)_{et}$, then we ought to get a map

$$\Omega_{et} \times S^1 \to (\text{GL}_n)_{et}$$

which should give the desired homotopy equivalence $\Omega_{et} \to \Omega(\text{GL}_n)_{et}$.

Slight problem: Consider $n = 1$, whence $L = \mathbb{C}$. Then we would be trying to prove that $\Omega_{et} = \mathbb{C}$ which isn't true in the algebraic context. What in fact one has is

$$\mu_m \to \mathbb{G}_m \to \mathbb{G}_m$$

together with $\Omega = \mathbb{Z} \to \mathbb{Z}/m \to \mu_m$ given by a choice of primitive $m$-th root of unity.

The principal bundle of $A(\omega) = e^{2\pi i \omega} F(z)$ over $\text{GL}_n$ I have constructed is not algebraic over $\text{GL}_n$. The problem is to somehow approximate
by an inverse system of algebraic gadgets.

Let us determine where

\[(*) \quad GL_n \rightarrow GL_n, \quad A \mapsto A^m\]

is étale. A tangent vector to $GL_n$ at $A$ is of the form $(I + \epsilon B)A$, $B \in M_n$.

\[
(I + \epsilon B)A]^m = (A + \epsilon BA)^m
= A^m + \epsilon \left[ (BA)A^{m-1} + A(BA)A^{m-2} + \ldots + A^{m-1}(BA) \right]
\]

Right translate by $A^{-m}$ to get a tangent vector at $0$

\[
(I + \epsilon B)A]^mA^{-m} = I + \epsilon \left[ B + ABA^{-1} + \ldots + A^{m-1}BA^{-(m-1)} \right]
= I + \epsilon \left[ \frac{Ad(A)^m - 1}{Ad(A)} (B) \right]
\]

Thus we see by reasoning analogous to before that \((*)\) is étale at $A$ if and only if no eigenvalue of

\[Ad(A)\]

is an $m$-th root of 1 $\neq 1$, hence if and only if no two eigenvalues of $A$ have as ratios an $m$-th root of $1 \neq 1$.

Let $A$ have eigenvalues \($\lambda_1, \ldots, \lambda_n$\) so that $A^m$ has the eigenvalues \($\lambda_1^m, \ldots, \lambda_n^m$\). If

\[\lambda_i^m = \lambda_j^m\]

for some $i \neq j$, then $(\lambda_i^m \lambda_j^{-m})^m = 1$, and so if \((*)\) is étale at $A$, we must have $\lambda_i = \lambda_j$. Thus when we have an étale solution of $A^m = B$, the..
multiplicities of the eigenvalues \( \lambda \) of \( B \) equal the multiplicity of the eigenvalues \( \lambda^m \) of \( A_m \).
Moreover one has the following possibilities for \( A_m \).
If one breaks up \( \mathbb{C}^n \) according to the eigenspaces of \( B \), so that \( B \xi = \mu \xi \), then \( A_\xi = \lambda \xi \) where \( \lambda \) is one of the \( m \)-th roots of \( \mu \). Of course \( A_\xi = B_\xi^m \).
So we see any two etale solutions of \( A_m = B \) commute.

Summarizing.

\[
\text{Proposition:} \quad (i) \text{ The map } A \mapsto A^m \text{ from } GL_n \to GL_n \\
\text{is etale at } A \text{ if and only if for any two eigenvalues } \lambda, \lambda' \text{ of } A \text{ one has } \lambda^m = \lambda'^m \Rightarrow \lambda = \lambda'.
\]

\[(ii) \text{ Given } B \in GL_n, \text{ let } B \text{ have } k \text{ distinct eigenvalues } \mu_1, \mu_2, \ldots, \mu_k, \text{ say}
\]

\[
B_\xi = \sum_{i=1}^{k} \mu_i E_i
\]

Then any solution of \( A_m = B \) which is etale (i.e., a "simple" root) is of the form \( A = A_\xi A_n \) where

\[
A_n = (B_n)^{1/m}
\]

\[
A_\xi = \sum_{i=1}^{k} \lambda_i E_i \quad \text{where } \lambda_i^m = \mu_i
\]

In particular there are \( mk \) etale solutions and all these solutions commute with each other.
Notation: \( \Omega = \text{GL}_n(\mathbb{C}[z, z^{-1}])', \quad \Omega^{(m)} = \text{GL}_n(\mathbb{C}[z^m, z^{-m}])'. \)

\( \mathcal{P} = \) holomorphic maps \( \omega \mapsto A(\omega), C \rightarrow \text{GL}_n \)
of the form \( e^{2\pi i \omega} X F(e^{2\pi i \omega}) \)
where \( X \in \text{gl}_n \), \( F \in \Omega \).

\( \mathcal{P} \) is a principal \( \Omega \)-bundle over \( \text{GL}_n \)
the map \( \mathcal{P} \rightarrow \text{GL}_n \) being \( A \mapsto A(1) \).

Introduce an equivalence relation in \( \mathcal{P} \).
Call \( A_1(\omega) \) and \( A_2(\omega) \) equivalent if \( A_1(\omega)A_2(\omega)^{-1} \in \Omega^{(m)} \).
Because \( \Omega^{(m)} \) is a group, this is an
equivalence relation. Let \( \mathcal{P}_m \) denote the set of
equivalence classes. Observe that

\[ A_1 \sim A_2 \implies A_1 F \sim A_2 F \text{ for any } F \in \Omega \]

hence \( \Omega \) acts to the right on \( \mathcal{P}_m \).
Observe also that \( A_1 \sim A_2 \) means

\[ A_1(\omega) = \prod F(z^m) A_2(\omega) \quad \text{with} \quad z = e^{2\pi i \omega} \]

\[ \implies A_1(1) = F(1) A_2(1) = A_2(1) \]

hence \( A_1 \sim A_2 \implies A_1 A_2 \) lie in the same fibre over \( \text{GL}_n \).
Thus we have an induced map

\[ \mathcal{P}_m \rightarrow \text{GL}_n \]

Let \( C \in \text{GL}_n \), and choose \( X \in \text{gl}_n \) so that

\[ e^{2\pi i \frac{1}{m}} X = C \]
If \( e^{2\pi i \frac{1}{m} Y} = C \) also, then we have seen that
\[
e^{2\pi i w X} e^{-2\pi i w Y} = e^{2\pi i (mw) \frac{1}{m} X} e^{-2\pi i (mw) \frac{1}{m} Y}
\]
Thus \( e^{2\pi i w X} \sim e^{2\pi i w Y} \). Thus we have a section

\[
\begin{array}{c}
\text{GL}_n \\
\downarrow m \\
\rho_m \longrightarrow \text{GL}_n
\end{array}
\]

\[
\begin{array}{c}
\text{e}^{2\pi i \frac{1}{m} X} \\
\downarrow m \\
\text{e}^{2\pi i X}
\end{array}
\]

I next want to compute the fibres of \( \rho_m \) over \( \text{GL}_n \). Fix \( C \in \text{GL}_n \), and choose \( X \) so that \( e^{2\pi i w X} = C \). Then since \( \Omega \) acts transitively on the fibre of \( \rho_m \) over \( C \), this fibre is \( \sigma \backslash \Omega \) where \( \sigma \) is the stabiliser of a point in the fibre, e.g. \( e^{2\pi i w X} \).

But \( F \in \sigma \iff \)
\[
e^{2\pi i w X} \sim e^{2\pi i w X} F
\]
\[
\iff F(e^{2\pi i w X}) = e^{2\pi i w X} F
\]
\[
\iff F \in e^{-2\pi i w X} \backslash \Omega(m) e^{2\pi i w X}
\]

Thus \( \sigma = \Omega \cap e^{-2\pi i w X} \backslash \Omega(m) e^{2\pi i w X} \) which shows that \( \rho_m \) isn't a fibre bundle over \( \text{GL}_n \), probably i.e. if \( f(w) = e^{-2\pi i w X} F(z) e^{2\pi i w X} \) then
\[
f(w+1) = C^{-1} f(w) C
\]
won't usually be periodic, so this makes me lose confidence in \( F_m \).

Let \( V \) be the open subset of \( \text{GL}_n \) consisting of matrices such that no two eigenvalues have as ratio an \( m \)-th root \( \lambda \neq 1 \); equiv., \( V \) is the open set where the \( m \)-th power map \( \text{GL}_n \to \text{GL}_n \) is etale.

The analogous subset \( U \subset \text{GL}_n \) is defined to be where \( e^{2\pi i t} \) is etale, thus on \( U \) no two eigenvalues differ by an integer \( \neq 0 \). Given two points in \( U \) with the same image \( C \in \text{GL}_n \), their difference is a semi-simple matrix with integer coeffs.

Given \( A, B \in V \) with \( A^m = B^m = C \), we know that \( A, B \) commute and that \( A^{-1}B \) is a matrix of order \( m \), i.e. a homomorphism \( \mathbb{Z}/m\mathbb{Z} \to \text{GL}_n \).
November 22, 1974.

Problem: Find the universal group $G$ equipped with a continuous map $\varphi: \{ A \in GL_n \mid A^m = 1 \} \to G$ such that $\varphi(AB) = \varphi(A) \varphi(B)$ if $A, B$ commute. As an example of such $G$, we have

$$G = \prod_{i=1}^{n-1} GL_n \quad \varphi(A) = (A^i)$$

but the hope is that the universal $G$ is somehow richer.

Analogous problem: Find the universal map

$$\varphi: \{ X \in oGL_n \mid \exp(2\pi i X) = 1 \} \to G$$

such that $\varphi(X+Y) = \varphi(X) \varphi(Y)$ if $(X,Y) = 1$

Example: $G = G_n \left( \mathbb{C} I_3 2^{-1} \right)'$, $\varphi(X) = z^X$. I hope this is universal.

$$\{ X \mid e^{2\pi i X} = 1 \} \to G$$

whence we find that $\Omega$ is a normal subgroup generated by $z^X$, $e^{2\pi i \frac{1}{m} X} = 1$ is the universal group for the mod $m$ problem. It seems fairly certain that the subgroup of $\Omega$ generated by $z^X$ with $e^{2\pi i \frac{1}{m} X} = 1$, is $\Omega^{(m)} = GL_n (C[z, z^{-m}])$, and that $\Omega^{(m)}$ would have to
Let $N$ be the normal subgroup of $\Omega$ generated by $\Omega^{(m)}$. Then since $\Omega^{(m)}$ is stable under conjugation by constant matrices, so must $N$ be, hence $N$ is normal in $\Omega \rtimes \text{GL}_n = \text{GL}_n(\mathbb{C}[z, z^{-1}])$. Now Bass has proved, I think, that normal subgroups of $\text{GL}_n(\mathbb{C}[z, z^{-1}])$ are all congruence subgroups essentially, hence it should follow that $N = \ker \{ \text{GL}_n(\mathbb{C}[z, z^{-1}]) \to \text{GL}_n(\mathbb{C}[z]/(z^{n-1})) \}$. Therefore the mod $m$ universal group should be

\[
\text{image of } \text{GL}_n(\mathbb{C}[z, z^{-1}]) \rtimes_\mu \prod_{i=1}^{m-1} \text{GL}_n = \text{GL}_n(\mathbb{C}[z, z^{-1}]/(z^m))
\]

functions $\{ \mu_n \to \mathbb{C} \}$

Conclusion: This isn't going to work.

So far we have tried to form over $\text{GL}_n$ a fibre bundle with fibre $\Omega$ mod $m$ which would be algebraic. One might also examine the restriction of $P$ to the points of order $M$ in $\text{GL}_n$. This restriction is the subset of $P$ consisting of $A(\omega)$ such that $A(1)^m = A(\omega)^m = 1$, i.e. $A(\omega + m) = A(\omega)$, which means $A(\omega) = B(e^{2\pi i \frac{\omega}{2^n}}) = B(2^k \omega)$. Call this restriction $P^{(m)}$. Thus

$P^{(m)} \subseteq \Omega^{(m)}$

and for $B(y) \in \Omega^{(n)}$,$y^{2^n}$
to be in $\mathfrak{p}^{(m)}$ means that $B(5y) = B(5)B(y)$,

Assertion: $\Omega/\Omega^{(m)}$ is fiber of $m: GL_n \rightarrow GL_n$.

Proof: We have a map of fibrations

$\Omega \rightarrow P \rightarrow GL_n$

$\downarrow \quad \downarrow \quad \downarrow m$

$\Omega \rightarrow P \rightarrow GL_n$

$\alpha_m : A \mapsto A(m \omega)$

and $\Omega/\Omega^{(m)} = \Omega^{(m)}$. Thus one has a fibration

$\Omega/\Omega^{(m)} \rightarrow P/\Omega^{(m)} \rightarrow GL_n$

$\beta^2$ leg as $P$, $\alpha_m P$ are contractible

$s_m P/\Omega^{(m)}$

$P/\Omega = GL_n$

and the induced map lifts $C \in GL_n$ to $A \in P$, $A(\omega) = C$, which goes into $s_m A = A(m \omega)$, which projects to $A(m) = C^m$ in $GL_n$. Q.E.D.
This raises the question of whether I can algebraically map \( \Omega/\Omega^{(m)} \) into \( GL_n \), and homotop the composition to the basepoint.

\[
\begin{array}{ccc}
\Omega & \to & P \\
\alpha & \mapsto & ? \\
\alpha & \mapsto & GL_n
\end{array}
\]

Example: Suppose we have an element of \( \Omega \) given by a 1-parameter subgroup \( \alpha: \mathbb{G}_m \to GL_n \). Then the element of \( GL_n \) I want is \( \alpha(1) \), \( \alpha = \exp(2\pi i \frac{m}{n}) \). Yes, because the image of \( \alpha \) actually sits in \( \alpha \cdot GL_n \). \( \alpha \cdot GL_n \) is the set of \( A \in GL_n \) such that \( A(\alpha) = A(\alpha') \cdot A(\beta) \) and this includes 1-parameter subgroups.

Problem: To define "algebraically" a map \( \Omega/\Omega^{(m)} \to GL_n \) whose comp. with \( \alpha \) is homotopically trivial. In the case \( n=1 \) this amounts to giving a \( \mathbb{G}_m \) generating \( \mu_m \).

Example: If \( F \in \Omega \) satisfies \( F(\alpha^2) = CF(\alpha) \) (so \( C = F(1) \)), then we want to map \( F \) to \( F(\alpha) \).

One thing you might try is to take \( F \in \Omega \) send it to \( F(\alpha) \), then use the universal homotopy \( F(\alpha^m) \simeq F(\alpha)^m \).
Another thing to try is the following. As we've done before let us identify a path $I \to \text{GL}_n$ with a continuous map $\omega \mapsto f(\omega)$ from $I$ to $\text{GL}_n$ such that $f(\omega + 1) = f(1)f(\omega)$. Then for any $\lambda \in \mathbb{R}$, we have

$$\omega \mapsto f(\omega + \lambda)f(\lambda)^{-1}$$

satisfies

$$f(\omega + 1 + \lambda)f(\lambda)^{-1} = f(1)f(\omega + \lambda)f(\lambda)^{-1}$$

$$= [f(1 + \lambda)f(\lambda)^{-1}]f(\omega + \lambda)f(\lambda)^{-1}$$

hence $f \mapsto f(1 + \lambda)f(\lambda)^{-1}$ is a map of the path space of $\text{GL}_n$ to itself, which covers the identity map of $\text{GL}_n$. Thus the induced map on the fibre $\text{BGL}_n$ must be homotopic to the identity. Thus we have proved:

**Assertion:** The map $F(\zeta) \mapsto F(1 + \zeta)F(\zeta)^{-1}$ from $I$ to itself is homotopic to the identity map. Here $\zeta \in \mathbb{C}^*$.

**Direct proof:** Pick a path $\zeta_t$ joining $\zeta$ to 1. Then clearly $F(1 + \zeta)F(\zeta)^{-1}$ is the path we need. Another version:

$$A(\zeta) \mapsto A(\zeta + \lambda)A(\lambda)^{-1} \quad \lambda \in \mathbb{C}$$

maps $P$ into itself:

$$(e^{2\pi i (1 + \zeta)}F(e^{2\pi i (1 + \zeta)}F(1 + \zeta)^{-1}2\pi i \zeta)$$

$$e^{2\pi i \zeta}F(e^{2\pi i \zeta})$$

and covers the identity map of $\text{GL}_n$. 
Thus we know that $F \Rightarrow F(sz)F(s)^{-1}$ is homotopic to the identity. Hence
\[ F \rightarrow F(z), \ F(sz)F(s)^{-1}, \ F(s^2z)F(s)^{-2}, \ldots, \ F(s^mz)F(s)^{-m} \]
are homotopic, giving a canonical homotopy of $F(s)^m$ to $1$.
Nov. 24, 1979 - Review.

Over $\mathbb{C}$ I can show that $\Omega = \text{GL}_n(\mathbb{C}[x, x^{-1}])$ is homotopy equivalent to $\Omega \text{GL}_n$ by exhibiting a principal bundle:

$$\Omega \rightarrow P \rightarrow \text{GL}_n$$

with $P$ contractible. $P$ is a holomorphic gadget which is trivialized by $\exp 2\pi i : \text{GL}_n \rightarrow \text{GL}_n$. ($P$ can probably be described algebraically, but not the map $P \rightarrow \text{GL}_n$.)

(Alg. description of $P$: Take pairs $(X, F)$ with $X \in \text{gl}_n$, $F \in \Omega$. Introduce an equivalence relation $(X, F) \sim (X_1, F_1) \iff e^{2\pi i \omega X} F(z) = e^{2\pi i \omega X} F_1(z)$ as functions, i.e. if their Taylor power series at $\omega = 0$ coincide. For example, $e^{2\pi i \omega X} F(z) = 1$ \iff 

$$\frac{1}{2\pi i} \frac{d}{d\omega} (e^{2\pi i \omega X} F(z)) = e^{2\pi i \omega X} (XF(z) + z F'(z)) = 0 \iff XF(z) + z F'(z) = 0.$$}

Basic problem: give an algebraic proof that $(\Omega)_{et}$ and $\Omega(\text{GL}_n)_{et}$ have the same profinite completions, or maybe even that $\Omega(\text{GL}_n)_{et}$ is the profinite completion of $(\Omega)_{et}$.

I know that the map $\Omega \rightarrow \Omega$, $F(x) \mapsto F(z^n)$ corresponds to looping the map $\text{GL}_n \rightarrow \text{GL}_n$, $A \mapsto A^n$. 
If I write $\Omega^{(m)}$ for the image, then I know that $\Omega/\Omega^{(m)}$ is homotopy equivalent to the fibre of $\text{Gl}_n \to \text{Gl}_n$. So the point therefore is to show that one has $(\Omega/\Omega^{(m)})_\ast = \text{fibre of } m : (\text{Gl})_{\ast \text{et}} \to \mathbb{C}$.

**Action of $G_m$ on $\Omega$:**

$$(\varphi_\lambda F)(z) = F(\lambda z)F(\lambda)^{-1}$$

$$(\varphi_\lambda \varphi_\mu F)(z) = (\varphi_\mu F)(\lambda z)(\varphi_\mu F)(\lambda)^{-1}$$

$$= F(\mu \lambda z) F(\mu)^{-1} \left[ F(\mu \lambda) F(\mu)^{-1} \right]^{-1}$$

$$= (\varphi_{\mu \lambda} F)(z).$$

So I am now interested in the action of $\mu_m$ on $\Omega$.

**Change notation:**

$$(\varphi_\lambda F)(z) = F(\lambda)^{-1} F(\lambda z).$$

Now what are the fibres of $\mu_m$?

$F(z) = (\varphi_\mu F)(\lambda z) = F(\lambda)^{-1} F(\mu z) = F(\lambda)^{-1} F(\mu z) = F(1)^{-1} F(z) \mu_m.$

This means:

$$0 \leq L \leq 1 \implies (\varphi_{1/L}) = F(1)^{-1} F(z).$$

Anyway, I know that the map $\Omega^{(m)} \to \Omega \to \Omega/\Omega^{(m)} \to \text{Gl}_n \to \text{Gl}_n$ is given by sending $F$ to $F(1)$. Of course you
want to understand why composing with \( F(\delta) \to F(\delta)^m \) is null-homotopic.

Problem: Show: \( \Omega/\Omega(\mathfrak{m}) \xrightarrow{\delta} GL_n \to GL_n \) is null-homotopic.

Method:

\[
\Omega/\Omega(\mathfrak{m}) \xrightarrow{\delta} \mathcal{P}/\mathcal{P}(\mathfrak{m}) \xrightarrow{\text{hug}} GL_n \nabla GL_n
\]

So we have \( F \in \Omega \) and want to homotop it to something in \( \mathcal{P}(\mathfrak{m}) = \{ A \mid A(\omega + \frac{1}{m}) = A(\frac{1}{m}) A(\omega) \} \) and then take \( A(\frac{1}{m}) \).

Problem: I define a map \( \Omega/\Omega(\mathfrak{m}) \to GL_n \) in the homotopy category using

\[
\Omega/\Omega(\mathfrak{m}) \xrightarrow{\delta} \mathcal{P}/\mathcal{P}(\mathfrak{m}) \xrightarrow{\text{hug}} GL_n
\]

Show this map is given by \( F(\varepsilon) \Omega(\mathfrak{m}) \to F(\delta) \).

Prop: i) The action of \( C \) on \( \mathcal{P} \) by

If \( \alpha \in C \)

\[ (\xi_\alpha A)(\omega) = A(\alpha)^{-1} A(\alpha + \omega) \]

ii) \( \mathcal{P}(\mathfrak{m}) = \{ A \mid \xi_\alpha A = A \} \).