October 11, 1974:

Given \( U_1 < U_2 < U_3 < U_4 \) with \( \dim U_2/U_1 = \dim U_4/U_3 = 1 \), we associate a Schubert cell

\[ C(U_1,\ldots,U_4) = \{ A \in G_2 V \mid A \cap U_1 < A \cap U_2 = A \cap U_3 < A \cap U_4 = A \} \]

If \( U_2 = U_3 \), then this cell depends only on \( (U_1, U_4) \) and we use the notation

\[ C(U_1, U_4) = \{ A \in G_2 V \mid A \cap U_1 = U_4 \} \].

I want to determine the various inclusions which hold between these cells.

Suppose \( (U_1,\ldots,U_4) \in L_1(V) \) and \( (W_1, W_2) \in L_2(V) \) and \( C(U_1,\ldots,U_4) \subset C(W_1, W_2) \). \( C(U_1,\ldots,U_4) \) is set of planes of the form \( A = L_1 \oplus L_2 \), where \( L_1 \in PU_2 - PU_1 \) and \( L_2 \in PU_4 - PU_3 \). Such lines \( L_2 \) span \( U_4 \) so \( U_4 \subset W_2 \). Since any \( A \) in \( C(W_1, W_2) \) contained in \( U_4 \) belongs to \( C(U_1,\ldots,U_4) \), we have

\[ C(U_1,\ldots,U_4) \subset C(U_4 \cap W_1, U_4) \]

Put \( V = U_4 \cap W_1 \).

If \( V < U_3 \), then we can find \( L_2 \in PU_2 \cap V - PU_3 \) whence \( A = L_1 \oplus L_2 \) (any \( L_1 \) is not ind. of \( V \)). Thus \( V < U_3 \).
\[(PU_2 - PU_1) \cap PV = \emptyset \Rightarrow PU_1 = PU_2 \cap PV\]

\[\Rightarrow U_1 \supset U_2 \cap V, \text{ hence } U_1 = U_2 \cap W \text{ as the intersection is at most of codim 1. Thus we get the picture}\]

\[
\begin{array}{c}
W_2 \\
\downarrow \quad U_2 \\
\downarrow \\
U_1 = U_4 \cap W_1 \\
\end{array}
\]

\[
\text{Prof. of } \quad C(U_1, U_4) \subset C(W_1, W_2), \text{ then } U_1 \subset W_1, \quad U_4 \subset W_2, \quad \text{and } U_4 \cap W_1 \subset U_3
\]

\[
(U_1, U_2) \preceq (U_4 \cap W_1, U_3).
\]

Suppose now that \(C(W_1, W_2) \subset C(U_1, \ldots, U_4)\), then for every choice of \(V\),

\[
\begin{array}{c}
W_2 \\
\downarrow \quad U_2 \\
\downarrow \\
U_1 \subset V
\end{array}
\]

\(W_1, W_2 \subset (V \cup U_4)\). It follows that \(W_1 \cup U_1\).
Thus what's happening is that we take \((W_1, W_2)\) and break it into \((W_1, W_2 \cup U_3)\) and \((W_2 \cup U_3, W_2)\) and map these to \((U_1, U_2)\) and \((U_3, U_4)\). So

\[
\text{Prop: If } C(W_1, W_2) < C(U_1, U_2, U_3, U_4), \text{ then there is a unique } W_1 < H < W_2 \text{ such that } (W_1, H, H, W_2) < (U_1, U_2, U_3, U_4) \text{ in } L_{11}(V).
\]

Finally suppose \(C(W_1, W_2, V_3, V_4) < C(U_1, U_2, U_3, U_4)\).

Assuming \(V_2 < V_3\), \(U_2 < U_3\), I want to prove \((V_1, V_2, V_3, V_4) \leq (U_1, U_2, U_3, U_4)\) in \(L_{11}(V)\). We know already that \(V_4 < U_4\), \(V_1 < U_1\), so we can assume \(V_1 = 0\), \(U_4 = V\). This means that

\[
C(V_1, V_2, V_3, V_4) = \{ V_2 \oplus L_2 \mid L_2 \in P(V_4) - P(V_3) \}.
\]

Now for any such thing \(V_2 \oplus L_2 \rightarrow U_4 / U_3\), I want to show \(V_2 < U_3\). Assume not, i.e.

\[
V_2 \geq U_4 / U_3.
\]

Then

\[
(V_2 \oplus L_2) \cap U_2 \geq U_2.
\]

Try to show \(V_3 < U_3\). Note that \(V_4 \rightarrow U_4 / U_3\) and \(V_3, V_4 \cap U_3\) are hyperplanes in \(V_4\). If \(V_3 \neq V_4 \cap U_3\) can find \(L_2 \in P(V_4) - P(V_3)\), whence \(V_2 \oplus L_2 \rightarrow U_4 / U_3\).
showing that \( V_2 \to U_4/U_3 \). But then we know \( (V_2 \oplus L_2) \cap U_3 = L_2 \), so \( L_2 \) must be a complement for \( U_1 \) in \( U_2 \). Thus

\[
P(V_4 \cap U_3) - PV_3 \subset PU_2 - PU_1
\]

\[\text{Check: } V_4 \to U_4/U_3 \Rightarrow V_4 \cap U_3 \text{ is a hyperplane in } V_4. \text{ If this hyperplane differs from } V_3, \text{ then I can find } L_2 \in P(V_4 \cap U_3) - PV_3. \text{ Then taking } A = L_1 \oplus L_2, \text{ any } L_1 \in PV_2 - PV_1 \text{ I get}
\]

\[L_1 \oplus L_2 \to U_4/U_3\]

\[\text{So } L_1 \to U_4/U_3 \Rightarrow PV_2 - PV_1 \subset PU_4 - PU_3 \Rightarrow (V_1, V_2) \leq (U_3, U_4)\]

But more: I know \( A \cap U_2 = A \cap U_3 = L_2 \) is a complement for \( U_4 \) in \( U_2 \). Thus

\[
P(V_4 \cap U_3) - PV_3 \subset PU_2 - PU_1
\]
\( (V_3 \cap U_3, V_4 \cap U_3) \leq (U_1, U_2) \)

\[ \Rightarrow V_4 \cap U_3 = V_4 \cap U_2 \quad V_3 \cap U_3 = V_3 \cap U_1 = V_4 \cap U_1 \]

Somehow (assuming \( V_1 = 0 \) again so \( L_1 = V_2 \)), the point is that if \( V_2 \Rightarrow U_4/U_3 \), then any plane \( A = V_2 \oplus L_2 \) has a canonical choice for \( L_2 \), namely \( A \cap U_3 = A \cap U_2 \) which is a line in \( V_4 \cap U_2 \).

Normally on \( C(V_1, V_4) \) there is no canonical way of splitting the exact sequence

\[ 0 \rightarrow V_2/V_1 \rightarrow A \rightarrow V_4/V_3 \rightarrow 0. \]

But the complement \( U_3 \) for \( V_1 \) does this. But then the A's I get will be in \( V_2 \oplus V_4 \cap U_2 \) so we will have to have

\[ V_2 \oplus (V_4 \cap U_2) = V_4 \]

\[ V_4 \cap U_2 = V_4 \cap U_3 \quad \text{codim 1} \]

So what I have managed to finish is

\[ C(V_1, V_2, V_3, V_4) \subset C(V_4 \cap U_1, \leq V_4 \cap U_2 = V_4 \cap U_3 \leq V_4) \]
So \( C(V_1, V_2, V_3, V_4) \subseteq C(V_4 \cap U_1, V_4) \subseteq C(U_1, U_4) \).

This inclusion results from

\[ V_4 \]

\[ V_2 \quad V_3 \]

\[ V_1 \quad V_4 \cap U_1 \]

So the first inclusion implies \( V_4 \cap U_1 \) is comp. to \( V_2 \) in \( V_3 \). Can this happen? Seems so.

Example of the inclusion.

\[
\begin{array}{ccc}
V_4 & V_4 & U_4 \\
V_2 & V_3 & U_2 - U_3 \\
V_1 & \Gamma & \Gamma = U_1
\end{array}
\]
Start again: Assume \( C(V_1, V_4) \subset C(U_1, U_4) \).

Then choosing \( T \subset V_4 \) suitably I have 
\( C(V_1, T) \subset C(V_1, V_4) \), showing by my earlier work that \( V_1 \subset U_1 \).
Similarly, choosing \( W \subset U_3 \) suitably I have 
\( C(U_1, U_4) \subset C(W, U_4) \), so again by earlier work I will have \( V_4 \subset U_4 \).

Since \( V_4 \rightarrow U_4/U_3 \), \( V_4 \cap U_3 \) is a hyperplane in \( V_4 \).

Case 1: \( V_3 \neq V_4 \cap U_3 \).

Then we can choose \( L_2 \in P(V_4 \cap U_3) - PV_3 \). If \( L_1 \) in any elt of \( PV_2 - PV_1 \), then \( L_1 + L_2 \in C(V_1, V_4) \), hence \( L_1 + L_2 \rightarrow U_4/U_3 \), \( L_1 \rightarrow U_4/U_3 \). Thus one has:

\[
\begin{align*}
V_2 &- V_4 - U_4 \\
V_1 &- V_4 \cap U_3 - U_3
\end{align*}
\]

In addition, we know 
\( L_2 = (L_1 + L_2) \cap U_3 = (L_1 + L_2) \cap U_2 \).
This mod \( V_1 \), \( V_2 \) and \( V_4 \cap V_3 \) are complementary in \( V_4 \). This means that for every \( A \) in \( C(V_1, V_4) \)
\[
A = (V_4 \cap A) \oplus (U_3 \cap A) = (V_4 \cap A) \oplus (U_2 \cap A)
\]
where \( U_2 \cap A \in (PU_2 - PU_1) \cap P(V_4 \cap U_3) = P(U_2 \cap V_4) - P(U_1 \cap V_4) \)
Summary: Suppose that
\[ C(V_1, \ldots, V_4) \subset C(U_1, \ldots, U_4). \]

Then \( V_1 \subset U_1, V_4 \subset U_4. \)

Case 1. \( V_3 \neq V_4 \cap U_3. \) In this case the inclusion factors
\[ C(V_1, \ldots, V_4) \subset C(V_4 \cap U_1, V_4) \subset C(U_1, \ldots, U_4) \]

\[
\begin{pmatrix} V_4 \\ V_2 \\ V_3 \\ V_1 \end{pmatrix} \preceq \begin{pmatrix} V_4 \\ V_4 \cap U_1 \\ V_4 \cap U_3 \\ V_4 \cap U_1 \end{pmatrix} \preceq \begin{pmatrix} U_4 \\ U_2 - U_3 \\ U_1 \end{pmatrix}
\]

So what I would like to say is that in this case there is a unique interval in \( L_2(V) \) consisting of layers \( (W_1, W_2) \) such that
\[ C(V_1, \ldots, V_4) \subset C(W_1, W_2) \subset C(U_1, \ldots, U_4). \]

The least layer is \( (V_4 \cap U_1, V_4) \), the largest \( (U_1, V_4 + U_4) \).
Case 2: $V_3 = V_4 \cap U_3$. In this case one should have also that $V_2 + U_1 = U_2$, so that $(V_1, \ldots, V_4) \leq (U_1, \ldots, U_4)$ in $L_{11}(V)$.

I further hope that the poset of Schubert cells described the following category. Objects are of two kinds:

i) two lines $L_1, L_2$

ii) a 2-plane $M$

Following maps:

- isoms. $M \xrightarrow{\sim} M'$
- maps $(L_1, L_2) \xrightarrow{\sim} M$ for any exact sequence $0 \rightarrow L_1 \rightarrow M \rightarrow L_2 \rightarrow 0$

Finally a map $(L_1, L_2) \rightarrow (L'_1, L'_2)$ for any isomorphism $L_1 \oplus L_2 \rightarrow L'_1 \oplus L'_2$. 
October 13, 1977.

Idea: The poset $S_{k_2}(V)$ of Schubert cells in $G_2(V)$ should be the classifying space of some category made up out of vector spaces, which I could construct as follows. 

$$\mathcal{C}(W_1, W_2) \hookrightarrow W_2/W_1$$ 2 dim $v.s.$

$$\mathcal{C}(u_1, u_2, u_3, u_4) \hookrightarrow (u_2/u_1, u_4/u_3)$$ pair of lines

Then I want to define morphism so that I get a functor.

$$\mathcal{C}(W'_1, W'_2) \leq \mathcal{C}(W_1, W_2) \hookrightarrow W'_2/W'_1 \cong W_2/W_1$$

$$\mathcal{C}(u_1', ..., u_4') \leq \mathcal{C}(u_1, ..., u_4) \hookrightarrow \circ \rightarrow u_2/u_1 \rightarrow W_2/W_1 \rightarrow u_4/u_3 \rightarrow \circ$$

$$\mathcal{C}(W_1, W_2) \leq \mathcal{C}(u_1, ..., u_4) \hookrightarrow \circ \rightarrow u_2/u_1 \rightarrow W_2/W_1 \rightarrow u_4/u_3 \rightarrow \circ$$

$$\mathcal{C}(V_1, V_4) \leq \mathcal{C}(u_1, ..., u_4') \hookrightarrow \text{either } (V_2/V_1, V_4/V_3') = (u_1'/u_1, u_4'/u_3) \text{ or } (V_2/V_1, V_4/V_3) = (u_1, u_3, u_2, u_4)$$

A problem is that if I have $\mathcal{C}(W'_1, W'_2) \subset \mathcal{C}(u_1, u_2, u_3, u_4)$ and $\mathcal{C}(W_1, W_2)$, I then get

$$\circ \rightarrow u_2/u_1 \quad \rightarrow \quad W'_2/W'_1 \rightarrow \quad u_4/u_3 \rightarrow \circ$$

$$\circ \rightarrow W_2/W_1 \quad \rightarrow \quad u_4/u_3 \rightarrow \circ$$
and I can't seem to deduce the isomorphism \( W_2/W_1 \approx W_3/W_1 \). So something here doesn't work.

**Question:** I know that the subset \( \Sigma_2 \times (F^*)^2 \) of the classifying space for \( GL_2 \) is consisting of cells \( C(U_1, \ldots, U_4) \) with \( U_1 < U_2 < U_3 < U_4 \). Is the complement subset a classifying space for \( \Sigma_2 \times (F^*)^2 \)?

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**Alternative approach.**

Try classifying Schubert cells close to a fixed one.

Every Schubert cell gives us a pair of integers \((i, j)\) with \(0 \leq i \leq j\). The integers associated to \( U_1 < U_2 < U_3 < U_4 \) are

\[
\begin{align*}
  i &= \dim U_1 = \dim PU_2 - PU_1 \\
  j &= \dim U_3 - 1 = \dim PU_4 - PU_3 - 1
\end{align*}
\]

and the dimension of the Schubert cell is \(i + j\). We have seen that \( C(U_1, \ldots, U_4) \subset C(V_1, \ldots, V_4) \Rightarrow U_4 \subset V_4 \).
hence \((i, j) \leq (i', j')\) for the product ordering.

It is natural to ask if by using simplices \(C_0 < \cdots < C_k\) with small distance we get a homotopy equivalent complex.
October 15, 1974

Still trying to prove the following conjecture:

\[
\begin{align*}
L_1^1(V) & \rightarrow L_2^2(V) \\
\downarrow & \downarrow \\
L_1^1(V) & \rightarrow S_h^2(V)
\end{align*}
\]

\[\dim V = \infty\]

is homotopy-cocartesian. Here \(L_2^2(V)\) is poset of layers \((W_1, W_2)\) in \(V\) with \(\dim (W_2/W_1) = 2\), \(L_1^1(V)\) is poset of layers \((U_1, U_2, U_3, U_4)\) with \(\dim U_2/U_1 = \dim U_4/U_3 = 1\), and \(L_1^1(V)\) is subposet with \(U_2 = U_3\).

What this conjecture says is

\[
B S_h^2(V) = B T_2(k) \cup B GL_2(k) \cup B B_2(k)
\]

\((k = \text{field under consideration})\).

So to prove this conjecture, it is undoubtedly necessary to understand more about inclusions between Schubert cells.

Case 1: \(C(W'_1, W'_2) \subset C(W_1, W_2)\). Then \\
\(W'_2 \subset W_2\).
are of codim 2 in \( W_1 \). If \( W' \neq W_2 \cap W_1 \), then \( \exists L \in (PW_2' \cap PW_1) \setminus PW_1 \). \( L \) can be extended to \( A \in C(W_1' W_2') \) not in \( C(W_1, W_2) \). Thus \( W_1' = W_2' \cap W_1 \).

Therefore
\[
C(W_1', W_2') \subset C(W_1, W_2) \iff (W_1', W_2') \subset (W_1, W_2)
\]

**Case 2:** \( C(U_1, U_2, U_3, U_4) \subset C(W_1, W_2) \). Again \( U_4 \subset W_2 \). Since \( U_4 \to W_2 / W_1 \), one has \( (U_4 \cap W_1, U_4) \subset (W_1, W_2) \) and \( C(U_1, U_2, U_3, U_4) \subset C(U_4 \cap W_1, U_4) \).

If \( U_4 \cap W_1 \neq U_3 \), then we can find \( L_1 \in PU_2 \cap PW_1 - PU_1 \) so for any \( L_1 \in PU_2 - PU_1 \), we have \( L_1 \oplus L_2 \subset C(U_4 \cap W_1, U_4) \) but \( L_1 \oplus L_2 \in C(U_4 \cap W_1, U_4) \). Thus \( U_4 \cap W_1 \subset U_3 \).

If \( U_4 \cap W_1 = U_3 \), then taking any \( L_1 \) and \( L_2 \), we would have \( L_1 \oplus L_2 \subset C(U_4 \cap W_1, U_4) \). Thus \( U_2 \cap U_4 \cap W_1 \) is a hyperplane in \( U_3 \).

If \( U_2 \cap U_4 \cap W_1 \), then \( \exists L_1 \in PU_2 \cap PW_1 - PU_1 \) so for any \( L_2 \), we have \( L_1 \oplus L_2 \subset C(U_4 \cap W_1, U_4) - C(W_1, W_2) \). Thus \( U_2 \cap W_1 \subset U_1 \). Since \( U_2 \cap W_1 = U_2 \cap (U_4 \cap W_1) \) and \( U_4 \cap W_1 \) is a hyperplane in \( U_3 \), it follows that \( \text{cod}(U_2 \cap W_1 \subset U_1) \leq 1 \). \( \therefore U_2 \cap W_1 = U_1 \). So we find the picture.
Case 3: \( C(W_1, W_2) \subseteq C(U_1, U_2, U_3, U_4) \). Again, consider the filtration \( W_2 \cap U_1 \subseteq W_2 \cap U_2 \subseteq W_2 \cap U_3 \subseteq W_2 \cap U_4 \). As \( W_2 \to U_4/U_3 \), \( W_2 \cap U_3 \) is a hyperplane in \( W_2 \), since for \( A \in C(W_1, W_2) \), \( A \cap U_2 \to W_2 \cap U_2/ U_2 \cap U_1 \to U_2/ U_1 \) is an isomorphism, it follows \( W_2 \cap U_1 \) is a hyperplane in \( W_2 \cap U_2 \).

As \( W_2 \cap U_3 \) is a hyperplane in \( W_2 \), if \( W_2 \cap U_4 \), then we can find \( A \in C(W_1, U_3, W_2 \cap U_3) \subseteq C(W_1, W_2) \), and then \( A \to U_4/U_3 \) is zero, contradiction. Thus \( W_1 \subseteq W_2 \cap U_3 \), so \( W_1 \) is a hyperplane in \( W_2 \cap U_3 \).

Fix \( L \in P(W_2) - P(W_2 \cap U_3) \); then if \( L \in P(W_2 \cap U_3) - P(W_1) \) and \( L \cap L' \in C(U_1, \ldots, U_4) \)

\[ \text{ then } (L \cap L') \cap U_2 = (L \cap L') \cap U_3 = L. \]

Thus \( W_2 \cap U_3 \) is a hyperplane in \( W_2 \cap U_3 \), so for dimensional reasons \( W_2 \cap U_2 = W_2 \cap U_3 \). Thus we
get the picture

Case 4a: \( C(V_1, V_2, V_3, V_4) \subset C(U_1, U_2, U_3, U_4) \)

If one chooses a \( W_2' \) and \( W_1 \) so that

\[
\begin{array}{c}
W_2' \\
V_2 \\
V_1
\end{array} \quad \begin{array}{c}
V_4 \\
V_3 \\
V_1
\end{array} \quad \begin{array}{c}
U_2 \\
U_3 \\
W_1
\end{array}
\]

then we have \( C(V_1, W_2') \subset C(V_1, V_4) \subset C(U_1, U_4) \subset C(W_1, U_4) \), hence \( (V_1, W_2') \leq (W_1, U_4) \). Thus we will get by using all possible choices for \( W_2' \) that \( V_4 \subset U_4 \), and by using all possible choices for \( W_1 \) that \( V_1 \subset U_1 \).

Consider the map \( V_2/V_1 \rightarrow U_4/U_3 \)

Case 4a: This map is zero, i.e. \( V_2 \subset U_3 \).
If \( L_1 \in PV_2 - PV_1 \), \( L_2 \in PV_4 - PV_3 \), then \( L_1 \oplus L_2 \in C(V_1 \cap V_4) < C(U_1 \cap U_4) \), hence \( L_1 \oplus L_2 \rightarrow u_4 / u_3 \Rightarrow (L_1 \oplus L_2) \cap U_3 = L_1 \).

But \((L_1 \oplus L_2) \cap U_3 = (L_1 \oplus L_2) \cap U_2\), so we have

\[
P V_2 - PV_1 \subset PU_2 - PU_1
\]

\[
P V_4 - PV_3 \subset PU_4 - PU_3
\]

\[
\Rightarrow (V_1, V_2) \leq (U_1, U_2) \quad \text{and} \quad (V_3, V_4) \leq (U_3, U_4).
\]

**Case 4b:** \( V_2 / V_1 \rightarrow U_4 / U_3 \). Consider the filtration \( V_1 \cap U_1 \subset V_4 \cap U_2 \subset V_4 \cap U_3 \subset V_4 \). I know \( V_4 \cap U_3 \) is a hyperplane in \( V_4 \) different from \( V_3 \).

Take any \( A \in C(V_1, \ldots, V_4) \). Define \( A \cap V_2 \rightarrow u_4 / u_3 \)

\[
A_2 / U_1 \rightarrow u_2 / U_1
\]

hence we can write \( A \) in the form

\[
A = L_1 \oplus L_2 \quad \text{where} \quad L_1 \in PV_2 - PV_1 \quad \text{and} \quad L_2 \in PV_4 - PV_3
\]

where any \( L_1, L_2 \) occur.

\[
L_2 = (L_1 \oplus L_2) \cap U_3 = (L_1 \oplus L_2) \cap U_2
\]

we see that

\[
P (U_3 \cap V_4) - P (U_3 \cap V_3) \subset PU_2 - PU_1
\]

\[
\Rightarrow (U_3 \cap V_3, U_3 \cap V_4) \leq (U_1, U_2)
\]
Thus $U_3 \cap V_4 < U_2 \implies V_4 \cap U_2 = V_4 \cap U_3$. Since $U_2 < V_4 \cap U_2$ met in $V_4 \cap U_1$, it follows that $(V_4 \cap U_1, V_4 \cap U_2) < (U_1, U_2)$.

So we have $V_4 \cap U_1 < V_4 \cap U_2 = V_4 \cap U_3 < V_4$, and any $A$ in $V_4$ and $A \in C(U_1, \ldots, U_4)$, is in $C(V_4 \cap U_1, V_4)$. Thus we have

$$C(U_1, \ldots, V_4) \subseteq C(V_4 \cap U_1, V_4) \subseteq C(U_1, \ldots, U_4)$$

so what we have found above shows us that we get the picture

Thus in this case we factor the inclusion into

$$C(V_1, \ldots, V_4) \subseteq C(V_4 \cap U_1, V_4) \subseteq C(U_1, V_4 + U_3) \subseteq C(U_1, U_4)$$
Take functor \( j: L_2(V) \rightarrow \mathcal{H}_2(V) \). Given \( C(U_1, U_4) \in \mathcal{H}_2(V) \), we have seen that \( C(W_1, W_2) \leq C(U_1, U_4) \iff (W_1, W_2) \leq (U_1, T) \), where \( T/U_2 \oplus U_3/U_2 = U_4/U_2 \):

\[
\begin{array}{c}
W_2 \\
\downarrow \quad T \\
U_2 \\
\downarrow \quad U_4 \\
W_1 \\
\downarrow \quad U_1 \\
U_3
\end{array}
\]

Therefore \( j/C(U_1, U_4) \) is homotopy equivalent to the set of such \( T \).

Let \( F(U_1, U_4) = \{ T | T/U_2 \text{ comp. to } U_3/U_2 \text{ in } U_4/U_2 \} \)

\[
= \{ T | (U_2, T) \leq (U_3, U_4) \}
\]

Then \( F \) is a covariant functor from \( \mathcal{H}_2(V) \) to sets, and \( L_2(V) \) is homotopy equivalent to the cofibred category \( \mathcal{H}_2(V)_F \).

Similarly, \( C(U_1, U_4) \backslash j \) is homotopy equivalent to the set of complements to \( U_2/U_1 \in U_3/U_1 \).

Take \( j': L_{1,1}(V) \rightarrow \mathcal{H}_2(V) \). Given \( C(V_1, V_4) \leq C(U_1, U_4) \), we have either \( V_2/V_1 \rightarrow U_4/U_3 \) is zero or an isom, thus distinguishing components of \( j'/C(U_1, U_4) \). If this map is zero, there is a unique arrow \( (U_1, U_4) \rightarrow (U_1, U_4) \) in \( L_{1,1}(V) \). If this arrow is \( \neq 0 \), then one
has a unique arrow \( \delta \) in \( L_{1,1}(V) \) to an object \( (U_1, Z, Z, T) \) where \( T/U_2 \) is comp. to \( U_3/U_2 \) in \( U_4/U_2 \), and \( Z/U_1 \) is comp. to \( U_3/U_1 \) in \( T/U_1 \), i.e. \( Z/U_1 \oplus U_3/U_1 = U_4/U_1 \) and \( T = \mathbb{Z} \oplus U_2 \)

Thus we get the functor which assigns to \((U_1, U_2, U_3, U_4)\) the union of a point and the set of lines in \( U_4/U_1 \) complementary to \( U_3/U_1 \), (except when \( U_2 = U_3 \), when we get the set of lines in \( U_4/U_1 \).

Similarly, \( L'_{1,1}(V) \) is homotopy equivalent to the cofibred cat. over \( S^2(V) \) defined by the functor assigning to \((U_1, U_2, U_3, U_4)\) the pairs \((L, T)\) consisting of \( T = T/U_2 \oplus U_3/U_2 = U_4/U_2 \) and \( L \) a line \( L \in T/U_1 \). The map \( L'_{1,1}(V) \to L_2(V) \) forgets \( L \); the map \( L'_{1,1}(V) \to L_{1,1}(V) \) collapses all \((L, T)\) with \( L = U_2/U_1 \) to a point.

Now let me fix \((U_1, \ldots, U_4)\) and compute the map...
\[ \mathbb{Z} \{ (T, L) \} \longrightarrow \mathbb{Z} \{ \frac{N}{N+U_2} \} \otimes \mathbb{Z} \{ T \}. \]

Here \((T, L)\) runs over all pairs: \(T/U_2 \oplus U_3/U_2 = U_4/U_2\), \(L/U_4 = \text{a line in } T/U_4\). The map goes into the first factor. The kernel is \(\text{generated by elements of the form } (T, (U_2/U_4)) - (T_0, U_2/U_4)\), as \(T\) ranges over the complements to \(U_3/U_2\) in \(U_4/U_2\). This hits exactly the augmentation zero part of \(\mathbb{Z}[T]\). Hence we can conclude working with covariant functors.

\[
\begin{array}{ccc}
L_{b'}(V) & \xrightarrow{a} & L_2(V) \\
b & \downarrow & j' \\
L_{b''}(V) & \xrightarrow{j'} & SH_2(V)
\end{array}
\]

That \(L_+ j'! \mathbb{Z} = L_+ j! \mathbb{Z} = L_+(ga)! \mathbb{Z} = 0\) and that

\[
0 \rightarrow (ga)! \mathbb{Z} \rightarrow j_*\mathbb{Z} \otimes j'_* \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0
\]

is exact, whence \((\ast)\) is homotopy-cocartesian.
I now want to generalize the preceding, in order to understand the poset \( \text{Sh}_p(V) \) of Schubert cells in \( G_p(V) \).

What is a Schubert cell? Take a full flag

\[ 0 = V_0 < V_1 < V_2 < \cdots \text{ in } V, \quad \dim V_i = i, \quad \text{and a sequence} \]

\[ 0 < i_1 < i_2 < \cdots < i_p. \]

The corresponding Schubert cell is

\[ \{ A \in G_p(V) \mid \dim (A \cap V_j) = j, \quad j = 1, 2, \ldots, p \}, \]

\[ \dim (A \cap V_0) = 0. \]

The cell is perhaps best described using filtrations

\[ U_1 \subset U_2 \subset \cdots \subset U_{2p} \]

such that \( \dim (U_{2i-1} / U_{2i-2}) = 1 \) for \( 1 \leq i \leq p \). The corresponding Schubert cell is

\[ C(U_1, U_{2p}) = \{ A \in G_p(V) \mid 0 = A \cap U_1 < A \cap U_2 < \cdots < A \cap U_{2p} = \ldots \} \]

Observe that if \( U_{2i} = U_{2i+1} \) for some \( i \) then the cell \( C(U_1, U_{2p}) \) depends only on

\[ U_1, U_{2i-1}, U_{2i+2}, \ldots, U_{2p}. \]

So therefore it would be better to give a filtration

\[ U_1 < U_2 < \cdots < U_{2p} \]

with \( p = \sum \dim U_{2i} / U_{2i-1} \), \( 1 \leq i \leq p \).
and to define \( C(U_1, \ldots, U_9) \) as the set of \( A \) in \( G_p(V) \) such that

\[
A \cap U_{2i} + U_{2i-1} = U_{2i}, \quad 1 \leq i \leq 9
\]

(Note: this implies \( A \cap U_{2i}/A \cap U_{2i-1} = U_{2i}/U_{2i-1} \), hence by dimensional considerations that \( A \cap U_{2i-1} = A \cap U_{2i} \)).

Suppose we have \( a_1 + \cdots + a_9 = p \) with \( a_i > 0 \). Then we define \( L_{a_1, \ldots, a_9}(V) \) to be the subset of \( L_{a_1}(V) \times \cdots \times L_{a_9}(V) \) consisting of

\[
(U_1, U_2), (U_3, U_4), \ldots, (U_{2p-1}, U_{2p})
\]

such that \( U_2 \subset U_3, U_4 \subset U_5, \ldots \). Previous argument should generalize to show that \( L_{a_1, \ldots, a_9}(V) \) is a classifying space for \( BGL_{a_1} \times \cdots \times BGL_{a_9} \).

It might be better to think of \( a_1 + \cdots + a_9 = p \) as a subset of simple roots. (The simple roots for \( SL_p \) are pairs \( (i, i+1) \), \( 1 \leq i \leq p-1 \).) So here the simple roots are \( (a_1, a_1+1), (a_1+a_2, a_1+a_2+1), \ldots, (a_1+\cdots+a_{p-1}, p) \). Thus \( \sigma \) is a subset of \( 1, \ldots, p-1 \). Use the notation \( L_{a}(V) \) for \( L_{a_1, \ldots, a_9}(V) \). \( \sigma \) is allowed to be the empty subset, whence we get \( L_p(V) \).
If now \( \tau \subset \sigma \subset \{1, \ldots, p-1\} \), then we let \( L_{\sigma, \tau}(V) \) be the subset of \( L_{\sigma}(V) \) consisting of \( U_1 \subset \cdots \subset U_{2k} \) such that \( U_{2i} = U_{2i+1} \) for each element of \( \sigma \) not in \( \tau \).

Maybe a better notation would be to label \( \sigma \) as \( 1 \leq i_1 < \cdots < i_8 \leq p \). Then the filtration is

\[
U_{2i-1} < U_{2i} < U_{2i-1} < U_{2i} < \cdots < U_{2i-1} < U_{2i} < \cdots < U_{2i_8}
\]

\( \iota_1 \quad \iota_2 \quad \iota_3 \)

\( \sigma \subset \{1, \ldots, p-1\} \). \( \sigma = \{i_1 < i_2 < \cdots < i_{8-1}\} \). \( L_{\sigma}(V) \) consists of flags

\[
U_i' < U_i'' < U_i' < U_i'' < \cdots < U_i' < U_i'' < U_i' < U_i''
\]
i.e., a succession of layers of dimensions \( i_1, i_2, i_3, \ldots, p-i_8 \).

If \( \tau \subset \sigma \), it is clear what I mean by \( L_{\tau \subset \sigma}(V) \) namely the subset of \( L_{\sigma}(V) \) such that for each minimal interval \( j < j' \) in \( \tau \), the refining \( \sigma \) layers are squeezed together.
Start by trying to understand inclusions:
Given \((U_{1}, \ldots, U_{2p})\), \(\dim U_{2i}/U_{2i-1} = 1\), \(i \leq p\), one first wants to understand an inclusion
\[ C(W_{1}, W_{2}) \subset C(U_{1}, \ldots, U_{2p}) \]
The conjecture is that one has the picture:
\[ \begin{array}{c}
\vdots \\
W_{2} \\
\downarrow \\
W_{1}
\end{array} \]
So we consider the induced filtration \(W_{2} \cap U_{j}\)
\(1 \leq j \leq 2p\). Fixing \(A \in C(W_{1}, W_{2})\), we know
\[ A \cap U_{2i}/A \cap U_{2i-1} \twoheadrightarrow U_{2i}/U_{2i-1} \quad \text{dim} \; 1 \]
\[ \text{hence} \quad W_{2} \cap U_{2i}/W_{2} \cap U_{2i-1} \twoheadrightarrow U_{2i}/U_{2i-1}. \]
\[ \text{Claim:} \quad W_{1} + W_{2} \cap U_{2} > W_{1} \quad \text{is of codim} \; 1, \text{ since} \]
\(\text{any} \; A \in C(W_{1}, W_{2}) \text{ has} \; A \cap U_{2} = L_1 \neq W_1.\)
\[ \text{Claim:} \quad W_{1} + W_{2} \cap U_{2} = W_{1} + W_{2} \cap U_{3}. \; \text{If not,} \; F \]
\[ L_2 \in P(W_2 \cup U_3) - P(W_1 + W_2 \cup U_2). \] Fix \( L_1 \in P(W_2 \cup U_2) - PW_1 \) and extend \( L_1 + L_2 \) to an \( A \in C(W_1, W_2) \). Then

\[ L_1 + L_2 \not\in A \cap U_3 \quad \text{contradiction.} \]

Continuing, one sees that

\[ W_i = W_i + W_2 \cup U_1 < W_i + W_2 \cup W_2 = W_i + W_2 \cup U_3 < \]

etc. Counting dimensions, it follows that \( W_2 \cap U_3 \) has the same codimension in \( W_2 \) as does \( W_i \), thus \( W_i = W_2 \cap U_3 \).

Next consider an inclusion \( C(U_1, \ldots, U_2) \subseteq C(W_1, W_2) \), where we want the picture

Here we have \( U_2 \rightarrow W_2 \cup W_1 \) and so we can as well suppose \( U_2 = W_2 \). What I want to show is that

\[ U_i \subseteq C(W_1) \]

\[ W_i \cap U_{2i-1} = W_i \cap U_{2i} \]

\[ W_i \cap U_{2i+1}/W_i \cap U_{2i} \rightarrow U_{2i+1}/U_{2i} \]

so I will consider the filtration
\[ W_1 \subset U_1 + W_1 \subset U_2 + W_1 \subset \ldots \subset U_{2g} + W_1 = W_2. \]

If \( U_{2g-1} + W_1 = W_2 \), then \( \exists L \in PW_1 - PU_{2g-1} \).

So if \( L_i \in PU_{2i-1} - PU_{2i-1} \), \( i = 1, \ldots, g-1 \), then

\[ A = L_1 + \ldots + L_g \in C(U_1 \ldots U_2) \]

but \( A \notin C(W_1, W_2) \). This contradiction shows \( U_{2g-1} \supset W_1 \).

If \( U_{2g-3} + W_1 = U_{2g-2} + W_1 \), then \( \exists L_{g-1} \in PU_{2g-2} \cap PW_1 - PU_{2g-3} \).

If for some \( i \), \( U_{2i-1} + W_1 = U_{2i} + W_1 \), then

\[ \exists L_i \in PU_{2i} \cap PW_1 - PU_{2i-1}. \]

Then with any other \( L_1, \ldots, L_i \), \( i \), we have

\[ A = L_1 + \ldots + L_g \in C(U_1 \ldots U_2) \]

but \( A \notin C(W_1, W_2) \). Thus conclude

\[ W_1 \subset U_1 + W_1 \subset U_2 + W_1 \subset \ldots \subset U_{2g-1} + W_1 \subset U_{2g} + W_1 = U_2 \]

so dimension-counting shows that \( W_1 = U_1 + W_1 \)

\[ \Rightarrow U_1 \subset W_1 \]

and that \( U_{2i} + W_1 = U_{2i+1} + W_1 \). So we do get the picture

\[
\begin{array}{c}
\text{W} \\
\mid \\
\text{W}_1
\end{array}
\]

that we expected.
Suppose \( C(W_1, W_2) \subset C(U_1, \ldots, U_{2p}) \), where \( \dim(U_i/U_{i-1}) = 1 \) \( i = 1, \ldots, p \). I consider the filtration of \( W_2/W_1 \) induced by \( U_1 \subset \cdots \subset U_{2p} \). Recall

\[
\frac{W_1 + (W_2 \cap U_j)}{W_1 + (W_2 \cap U_{j-1})} = \frac{W_2 \cap U_j}{W_1 U_j} + W_2 \cap U_{j-1}
\]

Choose \( A \in C(W_1, W_2) \), so that

\[
0 = A \cap U_1 < A \cap U_2 = A \cap U_3 < A \cap U_4 = \ldots
\]

Then

\[
\frac{A \cap U_k}{A \cap U_{k-1}} \sim \rightarrow U_{2i}/U_{2i-1}
\]

so \( W_2 \cap U_{2i-1} < W_2 \cap U_{2i} \).
October 20, 1974. Schubert cells

Given a flag of fin. dim. subspaces of $V$

$$U_1 < U_2 < \cdots < U_{2p}$$

with $\dim U_{2i}/U_{2i-1} = 1$ for $1 \leq i \leq p$, we put

$$C(U_1, \ldots, U_{2p}) = \left\{ A \in G_p(V) \mid 0 = A \cap U_1 < A \cap U_2 = A \cap U_3 < \cdots \right\}$$

such a subset of $G_p(V)$ we call a Schubert cell, and we let $S_p(V)$ be the poset of Schubert cells, ordered by inclusion.

Notice that if $U_{2j} = U_{2j+1}$, then $C(U_1, \ldots, U_{2p})$ doesn't depend upon $U_{2j} = U_{2j+1}$, in fact we have

$$C(U_1, \ldots, U_{2p}) = \left\{ A \in G_p(V) \mid \dim (A \cap U_{2i-1}) = \dim (A \cap U_{2i}) = i \right\}$$

for $1 \leq i \leq p$, $i \neq j$.

(The point is that the conditions $\dim A \cap U_{2i-2} = i - 1$
$\dim A \cap U_{2i+2} = i + 1$, $\dim U_{2i+2}/U_{2i-2} = 2$ force
$\dim A \cap U_{2i} = i$. ) When this happens, we write

$$C(U_1, \ldots, U_{2j}, U_{2j+1}, \ldots, U_{2p})$$

for $C(U_1, \ldots, U_{2p})$. In this
way we define $C(W_1, \ldots, W_{2k})$ for any flag $W_1 < \cdots < W_{2k}$
such that $W_{2i-1} < W_{2i}$, $1 \leq i \leq k$, and

$$\sum_{i=1}^{k} \dim (W_{2i}/W_{2i-1}) = p$$
Suppose $C(W_1, W_2) \subset C(U_1, \ldots, U_2p)$. If $W_1 \notin W_2 \cup U_1$,
exists $L_1 \in P(W_2, U_1) - PW_1$, which can be extended to
$A \in C(W_1, W_2)$; but $A \cap U_1 \cap L_1 \neq \emptyset$ so $A \notin C(U_1, \ldots, U_2p)$,
a contradiction. Thus $W_1 \cap W_2 \cup U_1$, i.e. $W_1 = W_1 \cap (W_2 \cup U_1)$.

If $W_1 + W_2 \cap U_{2i-2} \subset W_1 + W_2 \cap U_{2i-1}$, \exists $L_i \in P(W_2 \cap U_{2i-1}) - P(W_1 + W_2 \cap U_{2i-2})$. Choose $A \in C(W_1, W_2)$ so that
$A \cap U_{2i-2}$ has dim $i-1$. Then $L_i + A \cap U_{2i-2}$ can be extended to $A' \in C(W_1, W_2)$; but $A' \cap U_{2i-1} \supset L_i + A \cap U_{2i-2}$
which has dim $i$, a contradiction. Thus
$W_1 + W_2 \cap U_{2i-2} = W_1 + W_2 \cap U_{2i-1}$ for $i = 1, \ldots, p$.

By dimensional considerations, it follows that
$W_1 + W_2 \cap U_{2i-1}$ is a hyperplane in $W_1 + W_2 \cap U_{2i}$, $1 \leq i \leq p$.

If $W_1 \cup U_2 < W_2 \cup U_3$, for some $i < j < p$, then
\[ L_i \in P(W_2 \cap U_{2i+1}) - P(W_2 \cap U_{2i}) \] hence

Can I show $W_2 \cap U_2 = W_2 \cap U_3$?

\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{W_1 \cap U_3}{W_2 \cap U_1} & \rightarrow & \frac{W_2 \cap U_3}{W_1} & \rightarrow & \frac{W_1 + W_2 \cap U_1}{W_1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \sim & & \downarrow \\
& & \frac{W_2 \cap U_2}{W_2 \cap U_1} & & & & & & \\
\end{array}
\]

Thus you see that if $W_2 \cap U_2 < W_2 \cap U_3$, then we can
find a line in $W_2 \cap U_3$ not contained in the hyperplanes $W_2 \cap U_2$, $W_1 \cap U_3$. (Use the fact that projective space has $\geq 3$ elements).

Continue: Assuming we know $W_2 \cap U_{2i} = W_2 \cap U_{2i+1}$ for $i < j$, assume $W_2 \cap U_{2j} < W_2 \cap U_{2j+1}$ for some $j < p$.

Choose $L_j^+ \in \mathbb{P}(W_2 \cap U_{2j+1}) - \mathbb{P}(W_2 \cap U_{2j}) - \mathbb{P}(W_1 + W_2 \cap U_{2j}^-)$

This is possible. Combining this with $A \cap U_{2j-1}$ to get $L_j + A \cap U_{2j-1}$, we can extend this to $A' \in C(W_1, W_2)$. But then $A' \cap U_{2j+1} = L_j + A \cap U_{2j+1}$, which has dimension $\geq j$, hence $A \cap U_{2j+1} = 1 + A \cap U_{2j+1}$. 
But then $A' \cap U_{2j+1}$ contains $j$, which is not.

But then $A' \cap U_{2j+1} > L_j + A_0 U_{2j-1}$ which has dimension $j \Rightarrow A' \cap U_{2j+1} = L_j + A_0 U_{2j-1}$. But $A' \cap U_{2j} \neq A' \cap U_{2j+1}$ as $L_j \notin A' \cap U_{2j}$. Contradiction, so we find $W_2 \cap U_2 = W_2 \cap U_{2i+1}$ for $1 \leq i \leq p-1$.

Now by counting we find that $W_i = W_2 \cap U_1$.

Thus we have proved.

Prop: If $C(W_1, W_2) < C(U_1, \ldots, U_{2p})$, then $(W_1, W_2) \leq (U_1, W_2 + U_1)$, where $W_2 + U_1 / U_1 \in C(U_1 / U_1, \ldots, U_{2p} / U_1)$.

Have picture:

```
   u_2p
  /   /
 W_2 /   /
   /   /
 W_1 /   u_1
```
Suppose now that $C(U_1, \ldots, U_{2p}) \subset C(W_1, W_2)$ and consider the flag

$$W_1 \subset U_1 + W_1 \subset U_2 + W_2 \subset \cdots \subset U_{2p} + W_1 = W_2$$

Suppose $U_{2i-1} + W_1 = U_{2i} + W_1$ for some $i$, $1 \leq i \leq p$. Then $U_{2i-1} + W_1 \cap W_1 = U_{2i}$, hence $E i \in PU_{2i} \cap W_1 - PU_{2i-1}$.

Choosing $L_j \in PU_{2j} - PU_{2j-1}$ for $1 \leq j \leq p$, $j \neq i$, we have $A = L_1 + \cdots + L_p \in C(U_1, \ldots, U_{2p})$, but $A \notin C(W_1, W_2)$, a contradiction. Thus $U_{2i-1} + W_1 \neq U_{2i} + W_1$ for $1 \leq i \leq p$.

Since $W_1$ is of codim $p$ in $W_2$, this forces $W_1 = U_1 + W_1$ (hence $U_1 < W_1$) and $U_{2i} + W_1 = U_{2i+1} + W_1$, $1 \leq i \leq p - 1$, so we get the picture:
Prop. If \( C(U_1, \ldots, U_2) \subset C(W_1, W_2) \), then one has:

\[
\begin{align*}
U_1 &
\subset W_1, \quad U_2 \subset W_2 \\
U_{2i} + W_1 &= U_{2i+1} + W_1, \quad 1 \leq i \leq p-1 \\
U_{2i} \cap W_1 &= U_{2i-1} \cap W_1, \quad 1 \leq i \leq p.
\end{align*}
\]

Next let us consider an inclusion

\( C(V_1, \ldots, V_2) \subset C(U_1, \ldots, U_2) \)

Choose \( C(W'_1, W'_2) \subset C(V_1, \ldots, V_2) \) with \( W'_1 = V_j \)

and \( C(U_1, \ldots, U_2) \subset C(W_1, W_2) \) with \( W_2 = U_{2p} \)

One then has \((W'_1, W'_2) \leq (W_1, W_2)\) showing that \( V_1 \subset U_1, \quad V_{2p} \subset U_{2p} \).

Review the Birkhoff decomposition. Let \( Z \) be a vector space with a full flag \( 0 = Z_0 < Z_1 < \cdots < Z_p = Z \) and let \( 0 = F_0 < F_1 < \cdots < F_p = Z \) be another flag. Recall the Schreier isomorphism.
\[ g^2_i(F_i / F_{i-1}) = \frac{Z_i \cap F_i + F_{i-1}}{Z_{i-1} \cap F_i + F_{i-1}} = \frac{Z_i \cap F_i}{Z_{i-1} \cap F_i + Z_i \cap F_{i-1}} \]

\[ g^1_j(Z_i / Z_{i-1}) = \frac{Z_i \cap F_j + Z_{i-1}}{Z_i \cap F_{j-1} + Z_{i-1}} = \frac{Z_i \cap F_j}{Z_i \cap F_{j-1} + Z_{i-1} \cap F_j} \]

Therefore we get a unique permutation \( \sigma \) of \( \{1, \ldots, p\} \) such that \( g^2_i(F_i / F_{i-1}) \neq 0 \) for \( 1 \leq i \leq p \).

Better, for each \( i \), \( 1 \leq i \leq p \), \( \sigma_i \) is the unique index such that

\[ \frac{F_{\sigma_i} / F_{\sigma_i-1}}{Z_{i-1} \cap F_{\sigma_i} + Z_i \cap F_{\sigma_i-1}} \sim \frac{Z_i}{Z_{i-1}} \]

so now given \( C(V_1, \ldots, V_p) \subset C(U_1, \ldots, U_2p) \), I want to associate a permutation to this inclusion.

Choose \( A \in C(V_1, \ldots, V_p) \) and consider the permutation associated to the two flags

\[ A \cap V_{2i}, \quad A \cap U_{2i} \]

On the other hand we can choose \( C(U_1, \ldots, U_2p) \subset C(W_1, W_2) \), whence we get two flags in \( W_2/W_1 \)

\[ \frac{V_{2i} + W_1}{W_1}, \quad \frac{U_{2i} + W_1}{W_1} \]
Since under the isomorphism \( A \rightarrow W_2/W_1 \), one has \( A \cap V_{2i} \cong V_{2i} + W_1/W_1 \), it follows that the permutations obtained by either choosing \( A \) or \( (W_1/W_1) \) are the same, hence independent of these choices.

Let us consider the two extreme cases.

First suppose the permutation is the identity, i.e. \( A \cap V_{2i} = A \cap U_{2i}, \quad 1 \leq i \leq p \). Since we know \( U_1 \subset V_1 \), \( U_2 = A \cap U_2 \oplus U_1 \), \( V_2 = A \cap V_2 \oplus U_2 \), it follows that \( U_2 \subset V_2 \). Note that if \( A \in C(V_1, \ldots, V_2p) \), then \( A + V_2/V_2 \in C(V_3/V_2, V_4/V_2, \ldots, V_{2p}/V_2) \):

Moreover the resulting map \( C(V_1, \ldots, V_{2p}) \rightarrow C(V_3/V_2, \ldots, V_{2p}/V_2) \) is onto. Claim \( C(V_3/V_2, \ldots, V_{2p}/V_2) \subset C(U_3/V_2, \ldots, U_{2p}/V_2) \). Indeed take \( B \) in the former and lift it to \( A \in C(V_1, \ldots, V_{2p}) \), so \( B = A + V_2/V_2 \).

Then clearly \( A + V_2/V_2 \in C(U_3/V_2, \ldots, U_{2p}/V_2) \).

So we get \( V_3/V_2 \subset U_3/V_2 \), whence \( V_4 = V_3 + V_4 \cap A \subset U_3 + U_4 \cap A = U_4 \) and we can continue.
Prop. If the permutation assoc. to $C(V_{ij}, V_{kp}) \subseteq C(U_{ij}, U_{kp})$ is the identity, then $(V_{2i-1}, V_{2i}) \leq (U_{2i-1}, U_{2i})$ for $1 \leq i \leq p$.

Here's another proof. Let $L_i \in PV_{2i} - PV_{2i-1}$, so that $A = L_1 + \cdots + L_p \in C(V_{ij}, V_{kp}) \subseteq C(U_{ij}, U_{kp})$. We know $\bigcap U_{2i} = \emptyset$. $\bigcap V_{2i} = L_1 + \cdots + L_i$, hence $L_i \in PU_{2i}$. And we know $\bigcap U_{2i+1} = \bigcap U_{2i-2} = L_1 + \cdots + L_i$, so $L_i \in PU_{2i-2}$. Thus $PU_{2i} - PV_{2i-1} \subseteq PU_{2i} - PU_{2i-1}$, so $(V_{2i-1}, V_{2i}) \leq (U_{2i-1}, U_{2i})$.

Next extreme case is where the permutations or reverse the order: $\sigma(i) = p-i+1$. In this case we know that the filtrations $V_{2i} \cap A$ and $U_{2i} \cap A$ are complementary, i.e.

$$V_{2i} \cap A \oplus U_{2(p-i+1)} \cap A = A.$$ 

Or put another way, we have unique lines $L_1, \ldots, L_p$ such that

$$V_{2i} \cap A = L_1 + \cdots + L_i$$

$$U_{2i} \cap A = L_{p-i+1} + \cdots + L_p$$
\[ L_1 \text{ can be arbitrary in } \overline{PV_2 - PV_1} \]

\[ L_2 \quad \frac{(PV_4 - PV_3) \cap PU_{2p-2}}{(PV_6 - PV_5) \cap PU_{2p-4}} \]

\[ (PV_4 - PV_3) \cap PU_{2p-2} = (PV_4 - PV_3) \cap PU_{2p-1} \subset PU_{2p-2} - PU_{2p-3} \]

so \( (V_3 \cap U_{2p-2}, V_4 \cap U_{2p-2}) = (V_3 \cap U_{2p-1}, V_4 \cap U_{2p-1}) \leq (U_{2p-3}, U_{2p-2}) \)

In general \( L_i \in (PV_{2i} - PV_{2i-1}) \cap PU_{2p-i} \)

so \( (V_{2i-1} \cap U_{2p-2i+3}, V_{2i} \cap U_{2p-2i+3}) \leq (U_{2p-2i+1}, U_{2p-2i+2}) \)

First case would be for \( L_p \)

\( (V_{2p-1} \cap U_3, V_{2p} \cap U_3) \leq (U_1, U_2) \)

Write this for \( p = 3 \).

\( (V_5 \cap U_3, V_6 \cap U_3) \leq (U_1, U_2) \)

\( (V_3 \cap U_5, V_4 \cap U_5) \leq (U_3, U_4) \)

\( (V_1 \cap U_7, V_2) \leq (U_5, U_6) \)
The basic question is whether \( C(V_1, \ldots, V_6) \subseteq C(V_1, U_3, V_6) \) and need \( V_6 \cap U_4 = V_6 \cap U_5 \).
Generalize and consider also flag manifolds instead of Grassmannians.

\[ G(\cdots, 1)(V) = \text{set of flags } 0 < A_1 < \cdots < A_p \text{ in } V \]

given in full flag \( V \) with \( \dim A_i/A_{i-1} = 1 \).

Let \( 0 < V_1 < V_2 < \cdots \) be a full flag in \( V \).

For each \( 1 \leq i \leq p \), there is a unique \( \lambda(i) \) such that \( A_i/A_{i-1} = V_{\lambda(i)}/V_{\lambda(i)-1} \) in the Schreier isomorphism.

Thus \( \lambda(i) \) is the unique integer \( \geq 1 \) such that

\[
\frac{A_i \cap V_{\lambda(i)}}{A_i \cap V_{\lambda(i)-1} + A_{i-1}} \neq 0
\]

iff

\[
\frac{A_i}{A_{i-1}} \quad \overset{\text{unique}}{\longrightarrow} \quad \frac{A_i \cap V_{\lambda(i)}}{A_i \cap V_{\lambda(i)-1} + A_{i-1} \cap V_{\lambda(i)}} \quad \overset{\text{unique}}{\longrightarrow} \quad \frac{V_{\lambda(i)}}{V_{\lambda(i)-1}}
\]
Here \( s: \{1, \ldots, p\} \rightarrow \{1, 2, \ldots, \dim V^2\}. \) Question: Fixing \( s, \) does the set of \((0 < s_1 < \ldots < s_p)\) belonging to a form a cell?

Suppose \( V=\{ke_1 + \cdots + ke_i^2\}; \) \( e_1, \ldots, \) basis for \( V. \)

Choosing a basis \( \mathbf{x}_i \) for \( A_p \) with \( A^2_i = ke_1 + \cdots + ke_p \)

we get a matrix \( A \) of size \( p \times \dim (V) \)

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_p
\end{pmatrix}
\]

we are permitted to add multiples of \( x_i \) to \( x_j \)

for \( i \neq j. \) Resulting canonical form for the matrix is

\[
\begin{bmatrix}
  \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star 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\end{bmatrix}
\]

The 1's occur in positions \( s_1, s_2, \ldots, s_p. \) Below any \( = 1 \) are zeroes.

Notice that this cell depends only \( = \) on the layers \((V_0(p-1), V_0(p)) \ldots (V_0(p-1), V_0(p))\). So what seems to happen is that we have a flag \((U_1, \ldots, U_{2p})\) as before together with a permutation \( \sigma \) of \( \{1, \ldots, p\}. \) The
corresponding cell consists of pairs \((A, f)\) where \(A \in C(U_1, \ldots, U_{2p})\), and where \(f\) is a flag in \(A\) bearing the relation \(\sigma\) to the flag \(0 < U_2 \cap A < \cdots < U_{2p} \cap A = A\).

---

Infinite Grassmannian: Let \(V\) contain \(V_0\) such that \(V_0\) and \(V/V_0\) are of infinite dimension. Then we can consider \(A \subset V\) commensurable with \(V_0\), meaning that \(A/A \cap V_0\), \(V_0/A \cap V_0\) are finite dimensional. Call this set \(G(V, V_0)\). Each such \(A\) has an index \(= \dim (A/A \cap V_0) - \dim (V_0/A \cap V_0)\), so

\[
G(V, V_0) = \bigcup_{n} G_n(V, V_0)
\]

where \(G_n(V, V_0)\) consists of those \(A\) of index \(n\).

Clearly

\[
G_p(V, V_0) = \bigcup_{V_1 \subset V_0 \subset V_2} G_{p + \dim (V_2/V_1)}(V_2/V_1)
\]

where \(V_1, V_2\) run over subspaces such that \(V_2/V_0, V_0/V_1\) are finite dimensional. From now on concentrate on index 0.

What is a Schubert cell in \(G_0(V, V_0)\)? Suppose we have a flag which I will suppose to pass through
$V_0 \preceq \cdots \preceq V_i \preceq V_0 \preceq V_1 \preceq \cdots$ If I have $A \in G_0(V,V_0)$, then we get a set of $n$ such that $V_n/V_{n-1}$ appears in $A$, meaning that $V_{n-1} \cap A < V_n \cap A$.

It seems reasonable to suppose $A$ contains $V_N$ for some $N$; this would certainly be the case if I just took $A$ with this property, which would still give me an infinite Grassmannian. Can suppose $A \subset V_N$, whence $[V_{n-1} \cap A < V_n \cap A] \subset [-N,N]$. So it is clear that fixing the flag $\{V_n\}$ and this finite set of integers the Schubert cell I am considering is just the image of a cell in $G(V_n/V_{n-1})$.

These cells can be described as follows: One gives $U_1 \subset \cdots \subset U_{2k}$, $\dim U_i/U_{i-1} = 1$, commensurable with $V_0$ and defines

$$C(U_1,U_2,\ldots,U_{2k}) = \{ A \in G_0(V,V_0) \mid \dim U_{2i} \cap A/U_{2i-1} \cap A = 1 \}$$

$U_1$ will have to have index $-k$. 
Recall how one constructs $BU$ classically. $G_p(C^n) < G_p(C^{n+1}) < \cdots G_p(C^{\infty})$ is a classifying space for $U_p$. Then one realizes $U_p \subset U_{p+1}$ by $G_p(C^n) < G_{p+1}(C^{n+1})$ etc. One obtains in the limit the set of subspaces of $\bigoplus_{n \geq n_0} C_n$ such that $\bigoplus_{n \geq n_0} C_n \subset A$ and such that the codimension of this inclusion is $-n_0$.

What seems to be at stake is that we have a space $V$ with a flag $V = V_1 \subset V_2 \subset \cdots \subset V_n$, $n \in \mathbb{Z}$, and we are taking $\bigcup G_p(V_p/V_{p-1})$.

Variant: We have a Hilbert space $V$ and a splitting $V = V_0 \oplus V_0^\perp$ into two infinite pieces. Then we consider closed subspaces $A$ which essentially "coincide with $A$ mod finite dimensional subspaces." Here are various possible meanings:

i) $\text{codim of } A \cap V_0 \text{ in } A$, $V_0$ is finite.

ii) If $E_A$ and $E_{V_0}$ are the ortho projectors, then $E_A - E_{V_0}$ is compact.

Concentrate on this: In the Caldeire alg, we fix a projector. Then we can consider the not equal to 1.
inverse image of $e$ in $\text{End}(V)$, and inside of this we can consider the space of orthogonal projectors $E$ in $\text{End}(V)$ such that $E \mapsto e$. It is this space of projectors which is the infinite Grassmannian. It should be possible to construct a contractible space over the set of projectors in $C$. Clear.

So the fibration I want is

\[
\begin{array}{c}
\text{inf. Grass.} \\
\text{of } E \mapsto e
\end{array} 
\xrightarrow{\text{orth. proj. } E \text{ such that } \text{Im } E \text{ Im } (1-E) \text{ inf. dim.}} 
\xrightarrow{\text{in } C \mapsto e \neq 0, 1}
\]

\[
\mathbb{R} \times \mathbb{R}/\mathbb{R} \times \mathbb{R}
\]
contractible

Next let us consider Fredholm operators. Consider the fibration

\[
\begin{array}{c}
C^* \\
\xrightarrow{\text{embeddings } C^* \to C \text{ with image } 2} \\
\text{direct summand End} \\
\text{kernel } = C.
\end{array} 
\xrightarrow{\text{projectors } e \in C} 
\text{e \neq 0, 1.}
\]
Lifted version

\[ \left\{ \begin{array}{lcl}
\Theta: V \to V \\
\Theta^* \Theta = 1 \\
\Theta \Theta^* = \Lambda
\end{array} \right\} \rightarrow \left\{ \begin{array}{lcl}
\text{bounded operators} \\
\Theta: V \to V \\
\Theta^* \Theta = 1 \mod K \\
\text{Coker } \Theta \text{ infinite}
\end{array} \right\} \rightarrow \left\{ \begin{array}{lcl}
\text{self-adjoint}
\end{array} \right\}
\]

\[ \lambda \text{ on } V \\
0 \leq \lambda \leq 1 \\
\text{ess. spectrum } 0, 1 \]

Can one relate the Grassmannian + Fredholm operators directly? For example, I can form over the space of E (with proj. to Im E, Im (1-E) inf), the space of pairs consisting of E, \Theta where \Theta : V \to \text{Im} E is a unitary isomorphism. Thus I could consider the space of \Theta : V \to V such that \Theta^* \Theta = \text{id}, which should be contractible.