The Stability thesis for $GL(\mathbb{R})$, $k = \mathbb{F}$

Let $V$ be a subspace of $\mathbb{R}^n$ such that

$$W \subset P(\mathbb{R}^n)$$
$$V \subset P(\mathbb{R}^n)$$
$$W + V = V$$

Then $P(\mathbb{R}^n) = P(W) \times P(V)$. Define $\Theta : P(\mathbb{R}^n) \rightarrow P(W) \times P(V)$

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The condition of the stability.
Proof: Take $2$ to be a line $L$ in $V$ where $P(V; L)$ is the set of hyperplanes complementary to $L$. Then $|P(V; V_0)|$ is the join of $|P(V/L; V_0/L)|$ and a set to one more by induction.

Put $J(V; V_0) = \bigwedge_{i=1}^n (P(V; V_0))$. $d = \dim(V_0)$. It is a free abelian group. One gets from the lemma a rank isomorphism $J(V; V_0) \cong J(V/L; V_0/L) \otimes J(V; Z)$. When $k = F_q$, the number of spheres in $P(V; V_0)$ is $(q^{n^2} - 1) \cdots (q^{n^2} - 1)$, $n = \dim(V)$, $s = \dim(V_0)$. $|W \in P(V; V_0)| = d(W) = \dim(W) - \dim(W_0) = \dim(W \cap V_0)$. Hence for $Z \in P(V; V_0)$ one has

\[ \{ W \in \mathbb{Z} \mid W \cap V_0 = \emptyset \} = \{ W \in \mathbb{Z} \mid W + (Z \cap V_0) = Z \} \] \[ P(Z, Z \cap V_0). \]

which is a bouquet of spheres of dimension $\dim(Z \cap V_0) = d(Z)$. Thus if we set $F_\ast \{ P(V; V_0) \} = \{ W \mid \dim(W \cap V_0) \leq p \}$ we have that $H_\ast(F_\ast P/F_{\ast-1} P)$ will be concentrated in degree $p$. 


In fact

\[ H_{p+q}(F, F/F_{p-1}F) = \bigoplus_{d(z) = p} H_{p+q}(Wz, Wz + Wz + \ldots + Wz) \]

\[ = \begin{cases} 0 & \text{if } q \neq 0 \\ \bigoplus J(Z, ZV_0) & \text{if } d(z) = p \end{cases} \]

Since \( F \) is contractible, the spec. sequence

\[ \mathbb{E}^1 = H_{p+q}(F, F/F_{p-1}F) \Rightarrow H_{p+q}(F) \]

degenerates into an exact sequence

\[ \ldots \to \mathbb{E}^1_{p-1} \to \mathbb{E}^1_p \to \mathbb{E}^1_{p+1} \to \ldots \to \mathbb{E}^1_{\infty} \to Z \to 0 \]

i.e. we get an exact sequence naturally assoc. to \((V, V_0)\):

\[ \ldots \to \bigoplus J(Z, ZV_0) \to \bigoplus \mathbb{Z} \to \mathbb{Z} \to 0 \]

\[ \dim(\mathbb{Z}V_0) = 1 \]

Now we use the above sequence to get a spectral sequence. Let \( GL^*(V, V_0) = \{ x \in GL(V) | x(V_0) = V_0 \} \) and \( x = id \) on \( V/V_0 \). Let \( F \) be the element of \( GL^*(V, V_0) \).

Then \( GL^*(V, V_0) \) acts transitively on the element of \( F \) of the same height. In effect if \( W + V_0 = V \), \( \dim(WV_0) = p \), then we can choose a complement to \( V_0 \) in \( V \); \( W = V \setminus V_0 \), \( W = C \setminus W \cap V_0 \); since \( GL^*(V_0) = GL(V_0) \times Hom(C, V_0) \), any two complements of \( V_0 \) in \( V \) are \( GL^*(V, V_0) \)-conjugate and the stabilizer of any of them.
which is isom. to $GL(V_0)$ acts transitively on the subspaces of dimension $p$.

Hence

$$V = \mathbb{R}^q + \mathbb{R}^q = (\mathbb{R}e_1 + \cdots + \mathbb{R}e_q) + (\mathbb{R}e_{q+1} + \cdots + \mathbb{R}e_{q+p})$$

$$V_0 = \mathbb{C} + \mathbb{R}^q$$

so that

$$GL''(V/V_0) = \begin{pmatrix} 1 \alpha & 0 \\ \mathbb{C}^p & GL_\mathbb{C} \end{pmatrix}$$

Now take $Z = \mathbb{R}^q + \mathbb{R}^p = \mathbb{R}e_1 + \cdots + \mathbb{R}e_{q+p}$. The stabilizer of $Z$ in $GL''(V/V_0)$ is

$$\begin{pmatrix} 1 \alpha & 0 \\ 0 & GL_\mathbb{R} \end{pmatrix} \cap \begin{pmatrix} 1 \alpha & 0 \\ \mathbb{C}^p & GL_\mathbb{C} \end{pmatrix} = \begin{pmatrix} 1 \alpha & 0 \\ 0 & GL_\mathbb{R} \end{pmatrix}$$

Thus I get a spectral sequence

$$E^1_{pq} = H_q \left( \begin{pmatrix} 1 & 0 & 0 \\ \mathbb{C}^p & \mathbb{C}^p & \mathbb{C}^p \\ 0 & 0 & GL_\mathbb{C} \end{pmatrix} \right) \to H_p \left( 1 \mathbb{C}^q \mathbb{C}^p \mathbb{C}^p \right)$$

whose edge homomorphism $E^1_{pq} \to H_p$ is the map

$$H_q \left( 1 \mathbb{C}^p GL_\mathbb{C} \right) \to H_p \left( 1 \mathbb{C}^q \mathbb{C}^p GL_\mathbb{C} \right)$$

which is always injective as there is an evident retraction.
Let \( \mathbf{V} \) be a vector space.

The exact sequence is

\[
0 \to J(V, L) \to \mathbb{F} \to \mathbb{F} \to 0
\]

and the spectral sequence in question is the long exact sequence

\[
\cdots \to H_1(\mathbb{F}, t^2) \to H_0(\mathbb{F}, t^1) \to H_0(\mathbb{F}, t^0) \to 0
\]

Proof: If \( \text{dim } (\mathbb{F}) > 2 \), then \( H_0(\mathbb{F}, t^0) \to H_0(\mathbb{F}, t^0) \to 0 \)

Proof: We have

\[
H_1(\mathbb{F}, t^0) \to H_1(\mathbb{F}, t^0) \to H_0(\mathbb{F}, t^0) \to 0
\]

Now from the extension

\[
1 \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \to 1
\]

One gets

\[
H_1(\mathbb{F}, t^0) = H_0(\mathbb{F}, t^0) \oplus H_1(\mathbb{F}, t^0)
\]

But

\[
H_0(\mathbb{F}, t^0) = \mathbb{R}^+ / \langle \lambda - 1, \frac{1}{2} \rangle \quad \text{if } \mathbb{F} = \mathbb{R}^2, \mathbb{C} \in \mathbb{R}^2
\]

\[
= \begin{cases} \mathbb{R}^2 & \text{if } \mathbb{F} = \mathbb{F}_2 \\ 0 & \text{if } \mathbb{F} \neq \mathbb{F}_2 \end{cases}
\]
**Prop.** If \( \text{card}(k) > 2 \), then \( H_\ast \left( \frac{1}{k}, 0, \frac{1}{k^0} \right), J_k(k^0, 0 + k^0) = 0, \forall \ast \geq 1 \).

**Proof.** Use induction on the isomorphism

\[
J(V, V_0) = J(V/L, V_0/L) \otimes J(V/L, V_0/L)
\]

Then \( H_\ast (GL^\ast (V, V_0), J(V, V_0)) \) is a quotient of \( H_\ast (G, J(V, V_0)) \) for any subgroup \( G \subset GL^\ast (V, V_0) \). So take \( G = GL^\ast (V, L) \subset GL^\ast (V, V_0) \):

\[
\begin{pmatrix}
1 & 0 \\
0 & GL_4
\end{pmatrix} \Rightarrow \begin{pmatrix}
1_{x+1-1} & 0 \\
x & 1
\end{pmatrix}
\]

Then \( GL(V, L) \) acts trivially on \( J(V/L, V_0/L) \), so

\[
H_\ast (GL^\ast (V, L), J(V, V_0)) = J(V/L, V_0/L) \otimes H_\ast (GL^\ast (V, L), J(V, L))
\]

and \( H_\ast (GL^\ast (V, L), J(V, L)) \) is zero as we have already shown.

Suppose now that we assume \( k \) is such that

\[
H_\ast \left( \frac{1}{k}, 0, GL_1 \right) = H_\ast \left( \frac{1}{k}, 0, GL_1 \right)
\]

for example \( \text{char}(k) = 0 \) or \( \text{char}(k) = p > 0 \) and

i) ignore \( p \)-torsion

ii) \( k \) contains arbitrarily large finite subfield.

Then I want to show that

**Thm.** \( H_\ast \left( \frac{1}{k}, 0, GL_0 \right), J_k(k^0, 0 + k^0) = 0 \) for \( \ast \geq 1 \).
Proof: Proceed by induction on $s$. For $s = 1$ we have the long exact sequence on page 6 which gives the result. Now note that since we know for any $k$-vector space $V$ that

$$H^\ast (k^\ast \times V) \to H^\ast (k^\ast)$$

it follows that

$$H^\ast \left( \frac{k^\ast}{\beta} \right) = H^\ast \left( \frac{k^\ast}{0} \right)$$

for any $\alpha, \beta$. So when I consider the spectral sequence

$$E_{pq}^1 = H^p \left( \frac{k^\ast}{\beta} \right) \chi \left( k^\ast + k^\ast, \alpha + k^\ast \right) \Rightarrow H_{pq} \left( \frac{k^\ast}{\beta} \right)$$

we can write the $E_1$ term as

$$H^p \left( \frac{k^\ast}{\beta} \right) \chi \left( k^\ast + k^\ast, \alpha + k^\ast \right)$$

so arguing by induction this will be zero for $1 < p < s$. Now take $n = s$ and then

$$E_{0s}^1 = H^s \left( \frac{k^\ast}{\beta} \right) \chi \left( k^\ast + k^\ast, \alpha \right)$$

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and the other columns are zero. Since $E_{0s}^1 \Rightarrow$ abut done.
Now observe that we have a map
\[ P(V, V_0) \to P(V \oplus k, V_0 \oplus k) \]
\[ W \to W \]
compatible with filtration:
\[ W \cap (V_0 \oplus k) = W \cap V_0 \quad \text{if} \quad W \cap V_0 \]
hence this map will induce a map of exact sequences
\[ \cdots \to \bigoplus J(W, W \cap V_0) \to \cdots \]
\[ W + V_0 = V \]
\[ \dim(W \cap V_0) = p \]
\[ \bigoplus J(W, W \cap (V_0 \oplus k)) \to \cdots \]
\[ W + (V_0 \oplus k) = V \oplus k \]
\[ \dim(W \cap (V_0 \oplus k)) = p. \]

And this is evidently compatible with the homomorphism
\[ GL^n(V, V_0) \to GL^n(V \oplus k, V_0 \oplus k). \]

If \[ V = \mathbb{k}^n + k^n, \quad V_0 = 0 + k^n \]
\[ GL^n(V, V_0) = \begin{pmatrix} 1 & * \\ * & GL_n \end{pmatrix} \]
\[ GL^n(V \oplus k, V_0 \oplus k) = \begin{pmatrix} 1 & * \\ * & GL_{n+1} \end{pmatrix}. \]

Representative for \[ W \] of height \[ p \] is
\[ \mathbb{k}^n + \mathbb{k}^p = \mathbb{k}e_1 + \cdots + \mathbb{k}e_{n+p} \]

works for both the \( n, n+1 \) cases. Thus \( \mathcal{E} \) will have a relative spectral sequence.
\[ E_{n+1}(u) = \begin{cases} 
H_t\left( \begin{bmatrix} 1 & 0 \\ \ast & GL_{n+1} \end{bmatrix} ; J(x+1, 0) \right) \\
GL_{n+1} & GL_n \\
\end{cases} \]

for \( 0 \leq d \leq n \)

\[ H_t\left( \begin{bmatrix} 1 & 0 \\ \ast & GL_{n+1} \end{bmatrix} ; J(x+n+1, n+1) \right) \quad d = n+1 \]

\[ 0 \quad d > n+1. \]

And this converges to \( H_t\left( \begin{bmatrix} 1 & 0 \\ \ast & GL_{n+1} \end{bmatrix} ; \ast GL_n \right) \)

\[ \text{Statements to be proved by induction:} \]

\[ I_a : \forall x, H_t\left( \begin{bmatrix} 1 & 0 \\ 0 & GL_{n+1} \end{bmatrix} ; \begin{bmatrix} 1 & 0 \\ 0 & GL_n \end{bmatrix} \right) \rightarrow H_t\left( \begin{bmatrix} 1 & 0 \\ \ast & GL_{n+1} \end{bmatrix} ; \ast GL_n \right) \]

\[ \text{for} \quad n+1 \geq i \]

\[ II_a : \forall x, H_t\left( \begin{bmatrix} 1 & 0 \\ \ast & GL_{n+1} \end{bmatrix} ; \ast GL_n \right) = 0 \quad n > i \]

Clearly \( II_0 \) is true, and \( I_0 \) provided that when \( n=0 \) we interpret the empty group in the obvious way.
Now let me check $I_2$. Using the spectral sequence, I have to show that

$$E_1 = H_t(\mathbb{Z} \otimes GL_{n+1}, GL_n)$$

maps onto the abelianization in degree 1:

$$H_1(GL_{n+1}, GL_n)$$

for $n > 0$. It suffices to show that $E_1^{1,0}$ is zero. One has

$$E_1^{1,0} = H_0(GL_{n+1}, GL_n)$$

for $n > 1$. To show 0, one examines

$$H_i(GL_{n+1}, GL_n) \otimes H_j(GL_{n+1}, GL_n)$$

for $i + j = 0$. For $j = 0$, the term is zero.

For $n = 0$,

$$E_1^{1,0} = H_0(GL_{n+1}, GL_n)$$

and we've seen this is zero. Thus we have proved $I_2$.

Now I want to show that $I_{i,j} = I_{i+1,j}$ for $i < 0 \Rightarrow I_{i,j}$.

In the spectral sequence if $n+1 \geq 8$ I have to show $E_1^{1,0} = H_t(GL_{n+1}, GL_n)$ goes onto abut $H_0(GL_{n+1}, GL_n)$.

It's enough to show $E_t^{1,0} = 0$ for $s+t = 8$, $t \geq 1$.

$$E_t^{1,0} = H_t(GL_{n+1}, GL_n)$$

for $1 \leq 0 \leq n$. If $s+t = 8 < n+1$, then $0 \leq n$ unless $n = 7$

$t = 0$ in which case $E_1^{1,0} = H_0(GL_{n+1}, GL_n)$ and we've seen this is zero for $n > 0$. So can forget this case.
To show $E_{oc}^1(x) = 0$ for $a + t = \frac{g}{5} n + 1$, we use

$H_i(\mathbb{1}_x \oplus \mathbb{G}_{n} ; J(x+c, 2) \otimes H_j\left(\frac{\mathbb{1}_x \oplus \mathbb{1}_x}{\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}}\right))$

is zero for $i + j = t$.

If $i > 0$, then $j < t \leq n+1 - a$, and also $j < t \leq g$

so $I_g, I_j$ give

$H_i\left(\frac{\mathbb{1}_x \oplus \mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}}{\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}}\right) = H_j\left(\frac{\mathbb{1}_x \oplus \mathbb{g}_n \oplus \mathbb{g}_n}{\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}}\right) = 0$

(Here I use the fact that $\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}$ is an $i$-sum of

$\begin{pmatrix}
\mathbb{1}_x \\
\times \mathbb{G}_{n+1} \\
\times \mathbb{G}_{n-1}
\end{pmatrix} 
\begin{pmatrix}
\mathbb{1}_x \\
\times \mathbb{G}_{n+1} \\
\times \mathbb{G}_{n-1}
\end{pmatrix}
\begin{pmatrix}
\mathbb{1}_x \\
\times \mathbb{G}_{n+1} \\
\times \mathbb{G}_{n-1}
\end{pmatrix} 
\begin{pmatrix}
\mathbb{1}_x \\
\times \mathbb{G}_{n+1} \\
\times \mathbb{G}_{n-1}
\end{pmatrix}$)

$H_j i = 0$, then $j = t = n+1 - a$, and $j < t \leq g$

(as $x, c$). Thus $I_g$ gives

$H_j\left(\frac{\mathbb{1}_x \oplus \mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}}{\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}}\right) = H_j\left(\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}\right)$

and so we see that

$\mathbb{G}_{n+1} \oplus \mathbb{G}_{n-1}$ acts trivially on this. So now done because

$H_\beta\left(\frac{\mathbb{1}_x \oplus \mathbb{G}_{n}}{\mathbb{G}_{n}}\right) ; J(a+c) \otimes A) = H_\beta\left(\frac{\mathbb{1}_x \oplus \mathbb{G}_{n}}{\mathbb{G}_{n}}\right) J(a+c, a) \otimes A$

$= 0$,

if $A$ has trivial action.
Next I want to show

\[ I_i \text{ for } i \leq g, \quad \text{II}_i \text{ for } i < g \implies \text{II}_g \]

Here use spec. sequence

\[
E^1_{0t} = H^t_c\left( \frac{G_{n-1} \ast \cdots \ast G_{n-1}}{G_{n+1} \ast \cdots \ast G_{n+1}} \right) \quad 0 \leq s \leq n
\]

\[
= H^t_c\left( G_{n+1} \ast I(n+1) \right) \quad s = n+1
\]

\[
= 0 \quad s > n+1
\]

which converges to zero. I will first show that \( E^1_{1g} \cong E^1_{0g} \) assuming that \( n \geq g + 1 \). As the spectral seq. converges to zero, it suffices to show that \( E^1_{0t} = 0 \) for \( s + t = g + 2 \), \( t \leq g \). As \( 0 \leq g + 2 \leq n + 1 \), we can have \( s = n + 1 \) only if \( t = 0 \), and

\[ 0 = g + 2 = n + 1 \]

in which case \( E^1_{0n+1} = H^0_c(G_{n+1}, I(n+1)) = 0 \) (as \( n^2 = g + 2 \geq 3 \)).

So we can assume \( 2s \leq n \). By spec. seq. have to show

\[
H^t_i\left( G_{n-1} \ast \cdots \ast G_{n-1} \right) \ast H^s_j\left( \frac{G_{n+1} \ast \cdots \ast G_{n+1}}{G_{n+1} \ast \cdots \ast G_{n+1}} \right) = 0
\]

for \( i + j = t \). If \( i > 0 \), then \( j = t - i < t = g + 2 - s \leq g \), as \( s \geq 2 \), and \( j < g + 2 - s \leq n + 1 - s \).

Thus \( j < n \), \( j \leq n - 1 \) so

\[
H^t_i\left( \frac{G_{n+1} \ast \cdots \ast G_{n+1}}{G_{n+1} \ast \cdots \ast G_{n+1}} \right) \ast \frac{G_{n+1} \ast \cdots \ast G_{n+1}}{G_{n+1} \ast \cdots \ast G_{n+1}} = 0
\]

If \( i = 0 \), then \( j \leq g \), \( j \leq n + 1 \), so \( H^s_j = 0 \).
and $G_{n}$ acts trivially so again get $0$, as

$$H_0(GL_n, GL_n) = 0 \quad n \geq 2.$$  

Now we have proved:

$$E_i^j \rightarrow E_0^j$$  

$$H_0(GL_n, GL_n) \rightarrow H_0(GL_n, GL_n)$$  

From the spec. seq.

$$E_{ij} = H_i(GL_n) \rightarrow H_i(GL_n) \rightarrow H_i(GL_n) \rightarrow 0$$  

$\quad j \leq n-1$  

The "filter" is zero in degrees $< g$, so

$$H_0(GL_n) \rightarrow H_0(GL_n) \rightarrow H_0(GL_n) \rightarrow 0$$  

Thus we find that

$$H_0(GL_n, GL_n) \rightarrow H_0(GL_n, GL_n)$$  

for $g \leq n-1$.  

Thus Conclude

$$H_0(GL_n, GL_n) \rightarrow H_0(GL_n, GL_n)$$  

where the embedding is given by $A \mapsto 1 \otimes A$.

Now one uses the hypothesis...
Lemma: \( H_j(\text{GL}_n, \text{GL}_{n+1}) \to H_j(\text{GL}_{n+2}, \text{GL}_{n+1}) \) is zero, the map being induced by \( A \mapsto I_2 \oplus A \).

Proof:

\[
\begin{align*}
H_j(\text{GL}_n, \text{GL}_{n+1}) & \xrightarrow{\partial} H_{j-1}(\text{GL}_{n+1}, \text{GL}_{n+2}) & \xrightarrow{\partial} H_{j-2}(\text{GL}_{n+2}) \\
\end{align*}
\]

Since \( A \mapsto A \oplus I \) is conjugate to \( A \mapsto I \oplus A \), these induce the same map on \( H_j \). The dotted arrow exists, and going one more step one gets zero.

Thus, we get \( H_j(\text{GL}_n, \text{GL}_{n+1}) = 0 \) for \( j \leq n-1 \), which is \( H_j \) and we have proved.

Then: If field \( \mathbb{F}_2 \) then

\[
egin{align*}
H_j(\text{GL}_j) & \to H_j(\text{GL}_{j+1}) \to H_j(\text{GL}_{j+2}) \to \\
\end{align*}
\]
Stability problem. It is now necessary to write out proofs for those results which you think you can prove.

1). A finite commutative ring $\Rightarrow K_i(A)$ finite

2). A finite $R$-algebra where $R$ is a finitely generated comm. alg. over an infinite field $K$, then $H_i(GL_n(A), K)$ stabilizes as a function of $n$.

3). Let $A$ be a Dedekind domain, and let $p$ be a prime number which is a unit in $A$. Then $H_i(GL_n(A), \mathbb{F}_p)$ stabilizes.
January 3, 1974

Stability.

What is the stability problem in alg. K-theory?

The K-theory of a ring \( A \) is constructed out of the category \( \mathcal{P}(A) \) of fin. gen. proj. \( A \)-modules, their isomorphisms, and their extensions. The classifying space for the theory, namely \( \text{Ker} \mathcal{A} \times \text{BGL}(\mathcal{A})^+ \), is built up from an infinite number of objects. The stability problem in general terms consists in showing that restricting the K-theory to spaces of dim \( \leq m \), one needs only use projective modules of rank \( \leq \varphi(m) \).

Example of \( \text{Ker} \mathcal{A} \):

(i) \( \{ [P] \mid \text{rg}(P) \leq N \} \) is a set of generators for \( \text{Ker} \mathcal{A} \).

(ii) Every element of \( \text{Ker} \mathcal{A} = \text{Ker} \{ \text{Ker} A \to H(A,A,Z) \} \) is of the form \( [P] - [A^n] \) with \( \text{rg}(P) = 0, n \leq N \).

Formulation (i) is perhaps natural from the point of view of the \( \mathbb{Q} \)-category and schemes. However it is (ii) that I should concentrate on now, for the following reasons:

(i) Ultimately I think you want to construct
the $F$-filtration of the $K$-theory, and especially
the theory of a fixed weight. Therefore as a start
you must be able to construct the part of filtration
$>1$ which means we must kill the rank. This
seems to mean that the objects of interest are
things of the form $[P]-[A^r]$, $rg(P)=n$.

2) The proof of the Lefschetz theorem tends to show
that the basic geometric part of stability is
splitting off a trivial line bundle. For example:
if $I_n$ is the set of iso classes of $P$ of rank $n$, then
one has an inductive system

$$I_n \rightarrow I_{n+1} \rightarrow \cdots$$

with limit $\tilde{K}_0(A)$, and Lefschetz thm. says $I_n \rightarrow \tilde{K}_0(A)$
is onto, while Bass's thm. says $I_n \rightarrow \tilde{K}_0(A)$ is injective
for $n$ large.

Let us do this discussion a bit more
carefully. Suppose that $I$ let $S =$ set of iso.
classes of $P(A)$. Assume $A$ connected (commutative)
so that one has then homs. of monoids

$$N \rightarrow S \rightarrow N$$

$[A^n]$

with composition = the identity. If $S_n = \eta^{-1}(n)$, then
we have the inductive system

$$\cdots \rightarrow S_n \rightarrow S_{n+1} \rightarrow \cdots$$

whose limit is $\tilde{K}_0(A)$. It is clear that

$S_d \rightarrow S_{d+1} \rightarrow \cdots \rightarrow \tilde{K}_0(A)$
iff one has\[ \text{'} \quad \mathrm{rk}(P) > d \Rightarrow P \cong A \oplus P' \quad \text{and that one has} \]
\[ S_{d+1} \cong \]

iff one has Cancellation: \[ A \oplus P' \cong A \oplus P'' \quad \mathrm{rk}(P) > d \]
\[ \Rightarrow P' \cong P'' . \] In practice one has these results with \[ d = \dim (\operatorname{Max}(A)) . \] For example:\[
\begin{aligned}
\text{A local: } & S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow \cdots \quad \Rightarrow \text{ pt } \\
\text{A Dedekind: } & S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow \cdots \quad \Rightarrow \text{ pt } \ 	ext{R}_u(A) \\
\end{aligned}
\]

(This actually tends to be better than one would expect because in the Dedekind situation one has \( S_1 \longrightarrow S_2 \) instead of just \( S_1 \longrightarrow S_2 \). But perhaps this is exceptional, and there are problems with cancellation for general dimension rings. No -- because of the determinant which says \[ S_1 \longrightarrow S_k \text{ given by } " \).

Now cancellation is exactly the injectivity of the maps \[ S_n \longrightarrow S_{n+1} \longrightarrow \cdots \] But in topology, when you prove something is injective you often prove that some relative group is zero, and this gives some extra information. Thus any really good proof of cancellation should also prove a surjectivity result in dimension \( 1 \).
Now let's discuss carefully the stability situation for manifolds. Let $E$ be a vector bundle over a manifold $X$ which is connected of dimension $d$. To split off a trivial bundle, one chooses a section $s$ of $E$ transversal to the zero section. Recall the proof that this can be done: Let $V$ be a space of sections generating $E$.

One chooses $v_0 \in V$ to be a regular value of the map $S \to V$, where $S = \ker \{x \mapsto E_x \}$. The corresponding section $s_v$ of $E$ is then transversal to the zero section and conversely.

When $\text{rg}(E) > d$, then the section $s$ is nowhere-vanishing, hence gives an exact sequence

$$0 \to L \to E \to E/s \to 0$$

The uniqueness version of this proof goes as follows. One has two regular values $v_1, v_2 \in V$ for the map $S \to V$, one joins them by a path and moves the path transversal to $S \to V$, keeping the endpoints fixed. Then one obtains a family $s_t$ of non-vanishing sections, and one argues that the bundles $E/s_t$ are homotopic, hence isomorphic.
The preceding argument makes sense for affine varieties over a field of char. zero. Precisely, suppose I write $E$ as a quotient of a trivial bundle generated by a $k$-vector space $V$. Then in the case where $\text{rg}(E) > d$, $S$ is of $\dim < \dim(V)$ and so $\exists$ non-vanishing sections. Notice that $S \to V$ is homogeneous of degree one, hence the image of $S$ is a cone in $V$.

\[
\dim(S) = \dim \mathcal{X} + \text{rg}(S) \\
= \dim(\mathcal{X}) + \dim(V) - \text{rg}(E) \\
= \dim(V) - [\text{rg}(E) - \dim(\mathcal{X})]
\]

Assuming now that $\text{rg}(E) \geq d + 2$, then I ought to be able to argue that the complement of $S$ in $V$ is connected by lines. Replaces the image of $S$ by its closure $\overline{S}$. Then $\overline{S}$ gives a subvariety $Z$ of codim $\geq 2$ in $PV$. Now because $Z$ has codim $2$, most lines do not intersect $Z$. Thus if I have a point $l \in PV - Z$, then the set of $l' \neq l$ such that the line $(l, l')$ doesn't meet $Z$ is open and dense. Thus we see that given $l_1, l_2 \in PV - Z$, we can find an $l_3$ such that the lines $(l_1, l_3)$, $(l_2, l_3)$ do not meet $Z$. In terms of our original vector bundle, this means that given two non-vanishing sections
$s_1, s_2$ we can find a third $s_3$ independent of both $s_1$ and $s_2$. (This argument is slightly incomplete in that $S$ is not closed. One will eventually have to beef it up by showing that any non-vanishing section $s$ can be made part of a generating family $V$ such that $s$ is not the limit of somewhere-vanishing sections.)

Philosophy: All this discussion involving transversality, even if we can make it work for varieties over finite fields, it does not help us over a finite field. What I want to do is to somehow get at the higher connectivity structure even over a point.
Let $V$ be a vector space over a field $k$, let $W$ be a subspace. Denote by $\gamma_2(V, W)$ the simplicial complex whose simplices are finite non-empty subsets $\tau = \{L_1, \ldots, L_q\}$ of 2-planes $L_1 + \cdots + L_q + W$ such that the sum $L_1 + \cdots + L_q + W$ is direct. If $\text{cod}(W) = 2m + \{1\}$, then $\gamma_2(V, W)$ is of dimension $m - 1$. I want to prove it is a bouquet of $(m - 1)$-spheres by induction on the codim of $W$.

Let $e$ be a vector not in $W$. Then $\gamma_2(V, ke + W)$ is a subcomplex of $\gamma_2(V, W)$. Let $\tau = \{L_1, \ldots, L_q\}$ be a simplex of $\gamma_2(V, W)$ which is not in $\gamma_2(V, ke + W)$, that is, $L_1 + \cdots + L_q + ke + W$ is not a direct sum, although $L_1 + \cdots + L_q + W$ is. Hence $e \in L_1 \oplus \cdots \oplus L_q \oplus W$. Writing $e = \sum e_i + w$ with $e_i \in L_i$, one sees that a face $\tau' = \{L_1, \ldots, L_p\}$ of $\tau$ has the property that $e \in L_1 \oplus \cdots \oplus L_p \oplus W$ if and only if $i \neq 0 \Rightarrow j = i$. Thus there is at least a face $\gamma$ of $\tau$ such that $e \in \ker + W$ if and only if $\sigma' \supset \sigma$; $\sigma'$ is the face consisting of the $L_i$ in $\tau$ such that $e_i \neq 0$. Denote by $\Gamma$ the set of simplices $\tau$ of $\gamma_2(V, W)$ such that $e \in \ker + W$ and such that no proper face of $\tau$ has this property. Then we have shown

$$\gamma_2(V, W) - \gamma_2(V, ke + W) = \bigsqcup_{\tau \in \Gamma} \text{Openstar}(\tau)$$

and so we know that $\gamma_2(V, W)$ is obtained from $\gamma_2(V, ke + W)$ by attaching a cone on $\partial \star \text{link}(\tau)$ for every $\tau$ in $\Gamma$. Better:
\[ Y_2(V, W) = \bigcup_{\tau \in \Gamma} \left[ Y_2(V, ke+W) \cup \tau \ast \text{Link}(\tau) \right] \]

where \( Z_\tau \ast Z_{\tau'} = Y_2(V, ke+W) \) if \( \tau, \tau' \) are distinct elements of \( \Gamma \). Now

\[ \text{Link}(\tau) = Y_2(V, ke+W) \]

so we have control over this by induction.

Suppose codimension of \( W = n \), and assume known that \( Y_2(V, W) \) is quasi-spherical of dimension \( \left\lceil \frac{n}{2} \right\rceil - 1 \) for lower codimension. Put \( m = \left\lceil \frac{n}{2} \right\rceil \), \( n = 2m+e \).

If \( \tau \in \Gamma \) and card(\( \tau \)) = 8, then \( \text{Link}(\tau) = Y_2(V, ke+W) \) is quasi-spherical of dimension \( \left\lceil \frac{n-2e}{2} \right\rceil - 1 = m - e - 1 \), so \( \tau \ast \text{Link}(\tau) \) is quasi-spherical of dimension \( (m - e) + (e - 2) + 1 \) which is \( m - 2 \). If \( e = 1 \), then \( Y_2(V, ke+W) \) is a bouquet of \((m-1)\)-spheres, so \( Y_2(V, W) \) is obtained by attaching cones on bouquets of \((m-2)\)-spheres in a bouquet of \((m-1)\)-spheres, hence \( Y_2(V, W) \) is a bouquet of \((m-1)\)-spheres.

But if \( e = 0 \), i.e. cod(\( W \)) is even, then \( Y_2(V, ke+W) \) is a bouquet of \((m-2)\)-spheres, and we must prove that

\[ Y_2(V, ke+W) \longrightarrow Y_2(V, W) \] is null-homotopic when \( \dim(V/W) \) is even.

Let \( e' \) be ind. of \( ke+W \), and let \( \Gamma' \) be the set of simplices in \( Y_2(V, ke+W) \) such that \( e' \in L_{e'} \ast L_{e+ke+W} \) and such that the component \( e' \) of \( e' \) in \( L_{e'} \) is non-zero. Then I know from the above that...
\[
Y_2(V, ke' + ke + w) = \bigcup_{r \in \mathcal{R}'} \left( Y_2(V, ke' + ke + w) \cup \tau \times Y_2(V, ke' + ke + w) \right) \cup 2 \tau \times Y_2(V, ke' + ke + w)
\]

On the other hand, the subcomplex \( Y_2(V, ke' + ke + W) \) in \( Y_2(V, W) \) is contained in the link of the vertex \( ke' + ke \), hence it contracts to a point.

**Lemma**: Let \( Z = \bigcup_{i \in I} Z_i \) (i.e., \( \bigcup_{i \in I} Z_i \) is a subspace of \( X \). Assume that \( A \) contracts to a point in \( X \), and that this contraction extends to \( Z_i \). Then \( Z \) contracts to a point in \( X \).

This is obvious. We will apply it to

\[
Z_\tau = Y_2(V, ke' + ke + W) \cup \tau \times Y_2(V, ke' + ke + W) \cup 2 \tau \times Y_2(V, ke' + ke + W)
\]

\( A = Y_2(V, ke' + ke + W) = \text{Link}(ke' + ke) \)

and \( X = Y_2(V, W) \), and the conical contraction of \( A \) to \( ke' + ke \).

**Lemma**: Let \( Z = A \cup B \subset X \) and suppose given contractions of \( A \) to a point and \( B \) to a point in \( X \). Assume the two retractions restricted to \( A \cap B \) are homotopic. Then the given contraction of \( A \) to a point extends to a contraction of \( B \) to a point.
Now I apply this lemma to $Z = Z_{\tau}, A = Y_2(V, ke + ke + W)$ and with $B = \tau \ast Y_2(V, k\tau + ke + W)$ which contracts to the barycenter of $\tau$. Now I have to show these two contractions on $A \cap B = \partial \tau \ast Y_2(V, k\tau + ke + W)$ are consistent.

By assumption, \( \dim(V/W) \) is even, hence $k\tau + ke + W$ contains $Y_2(V, M \oplus W)$. Thus I should be able to find a 2-plane $\Pi$ independent of $ke \oplus ke \oplus W$, and also of $k\tau \oplus W$. This is clear—given two hyperplanes $H, H'$ in a vector space one can always find a vector outside of both. (Note—this fails for (8+1)-hyperplanes and $\dim(W) = 8$.)

But now it is clear that $\text{link}(M) = Y_2(V, M \oplus W)$ contains $\partial \tau \ast Y_2(V, k\tau + ke + W)$.

I want to find a 2-plane $\Pi$ of $Y_2(V, W)$ such that $\Pi$ is independent of $k\tau \oplus W = L_1 \oplus \cdots \oplus L_8 \oplus W \Rightarrow \tau \in \text{link}(M)$ and $(ke + ke) \oplus k\tau \oplus W = ke \oplus ke \oplus L_1 \oplus \cdots \oplus L_8 \Rightarrow (ke + ke) \oplus \tau \in \text{link}(M)$.

Total of a (8+1)-hyperplane in a space of dimension $2g + 1 + \dim(W)$ namely $ke \oplus ke \oplus \tau \oplus W$.\]
Now here's the idea. What's happening is that I have \( \frac{1}{2} (V, k\gamma + ke + W) \) which can be joined to each of the hyperplanes

\[
k\gamma: \quad L_1 \oplus \cdots \oplus L_7
\]

\[
ke + ke + k_2: \quad L_1 \oplus L_5 \oplus L_6 \oplus L_7 \oplus ke + ke
\]

in \( k\gamma + ke \). These hyperplanes, actually, they are \((g+1)-\)
\(g\)-simplices which form the boundary of a \(g\)-simplex, which would be the simplex with vertices \( L_1, \ldots, L_7, ke + ke \)
if this existed. (Everything would be trivial if this simplex existed, for then it would be clear that the barycenter of \(\gamma\) agrees with the contraction to \(p = ke + ke\).

Picture if \( g = 1 \):

The imaginary simplex would fill in \( P \times \partial\gamma \) and would show the consistency of the contractions of

\( \partial\gamma \times \frac{1}{2} (V, k\gamma + ke + W) \) to \( P \) and to the barycenter of \(\gamma\).
The problem is therefore summarized by the picture:

One has 2 contractions of $\partial \times Y_2(V, k\tau + k\varepsilon + W)$ to a point—one to the barycenter of $\tau$ and the other to $P$. The problem is to show the consistency of these two contractions, i.e., that the inclusion

$$(P \times \partial \tau \cup \tau) \times Y_2(V, k\tau + k\varepsilon + W) \subset Y_2(V, W)$$

is null-homotopic.

Here's the hope. Suppose $\tau$ is so large that $k\tau + k\varepsilon + W$ is a hyperplane in $V$, whence $Y_2(V, k\tau + k\varepsilon + W) = \emptyset$, hence what we have to show is that $p \times \partial \tau \cup \tau$ is null-homotopic in $Y_2(V, W)$. I will do this by producing a 2-plane $M$ in $Y_2(V, W)$ independent of the faces of $p \times \partial \tau \cup \tau$.

$$\dim (V/W) = 2g + 2$$
$$\dim (k\tau + e) = 2g + 1$$

One vector of $M$ we find outside of $k\tau + k\varepsilon + W$. For the other we need a vector of $k\tau + k\varepsilon$ not contained in any of the faces of $p \times \partial \tau \cup \tau$. To write

$$k\tau + k\varepsilon = \underbrace{k\varepsilon + k\varepsilon_3 + \cdots + k\varepsilon_{2g-1} + k\varepsilon_{2g}}_{L_1} + \underbrace{k\varepsilon_{2g+1}}_{L_2}$$
where \( e' = e_1 + e_3 + \ldots + e_{2q-1} + \lambda e + w \)

so put \( e'' = e_2 + \ldots + e_{2q} + e \). Then \( e'' \) is not in

\[
\begin{cases}
L_1 \oplus \ldots \oplus L_q \oplus W \\
L_1 + \ldots + L_q + L_0 + ke' + ke + W
\end{cases}
\]

and so it works. (So this in \( L_1 \oplus \ldots \oplus L_q \oplus ke + W \).)

Problem: If \( W \leq W' \leq V \), then have an inclusion

\[
\gamma_2(W, W) \times \gamma_2(V, W') \longrightarrow \gamma_2(V, W)
\]

In effect given \( L_1, \ldots, L_q \subseteq W \) independent of \( W \), and \( M_1, \ldots, M_p \subseteq V \) independent of \( W' \), then \( L_1, \ldots, L_q, M_1, \ldots, M_p \) is independent of \( W \). The problem is to show this is still homotopic when \( \dim(V/W) \) is even and \( \dim(W/W) \) is odd.

So what we have shown is this. I can find a vector \( e'' \in kT + ke \) which is independent of the planes

\[
\begin{cases}
kT + W \\
k_2T + P + W
\end{cases}
\]

belonging to the faces of \( P \times T \cup T \). This means that if \( e'' \) is outside of \( kT + ke + W \), then

\[
(P \times T \cup T) \times \gamma_2(V, ke'' + kT + W)
\]
contracts to the plane $ke^{n} + ke^{m}$. 

But now take apart
\[
\frac{\gamma(V, k\xi + ke + W)}{\gamma(V, k\xi + ke + W)}
\]

into pieces indexed by those simplices $\xi \subseteq \gamma(V, k\xi + ke + W)$ which are minimal such that
\[
ke^{n} \in k\xi + k\tau + ke + W
\]
The piece I must worry about is then
\[
\delta \xi \ast \gamma(V, k\xi + k\tau + ke + W)
\]
which I will contract to the barycenter of $\delta \xi$ and to the plane $Q = ke^{n} + ke^{m}$. Thus I must worry about
\[
(Q \ast \delta \xi + \xi) \ast \gamma(V, k\xi + k\tau + ke + W)
\]
I think that everything should now follow from the fact that
\[
(P \ast \delta \tau + \tau) \ast (Q \ast \delta \xi + \xi) \ast \gamma(V, k\xi + k\tau + ke + W)
\]
is a well-defined subcomplex of $\gamma(V, W)$, plus some sort of explicit contraction of the first two factors, or maybe induction.
January 6, 1974 (More stability)

In the following \( A \) will be a semi-local (not necessarily commutative) ring. This means \( A/\text{rad}(A) \) is semi-simple (finite product of matrix rings over skew-fields; equivalently, the category of modules is semi-simple), where \( \text{rad}(A) = \text{jacobson radical} = \) an ideal \( \in A \) such that \( \text{mod rad}(A) \Rightarrow A \) a unit.

Let \( X(A^n) \) be the unimodular complex of \( A^n \). It is a simplicial complex of dimension \( n-1 \).

**Hypothesis 1:** \( X(A^n) \) is a bouquet of \((n-1)\)-spheres up to homotopy for every \( n \).

It follows from this that we have an exact sequence

\[
0 \rightarrow H_{n-1}(X(A^n)) \rightarrow C_{n-1}(X(A^n)) \rightarrow \cdots \rightarrow C_0(X(A^n)) \rightarrow \mathbb{Z} \rightarrow 0
\]

Put \( X_p(A^n) = \) set of \( p \)-frames of \( A^n \). Then

\[
C_{p-1}(X(A^n)) = \mathbb{Z}[X_p(A^n)] \otimes \mathbb{Z}^{|\Sigma_n|}
\]

**Lemma 1:** A semi-local \( \Rightarrow \text{Gl}_n(A) \) acts transitively on \( X_p(A^n) \), \( p=1, \ldots, n \).

**Proof:** Suppose we have \( u: A^p \rightarrow A^n \)

and \( v: A^p \rightarrow A^n \) injections onto direct summand. Then

\[
A^n \cong u(A^p) \oplus \text{Cok}(u) \cong v(A^p) \oplus \text{Cok}(v).
\]

Cancellation \( \Rightarrow \) \( \text{Cok}(u) \cong \text{Cok}(v) \), hence we get an isom. of \( A^n \).
transforming $u$ to $v$.

Thus

\[ X_p(A^n) = \frac{GL_n(A)}{\text{stabilizer of } e_1, \ldots, e_p} \]

\[ = \frac{GL_p(A)}{ \begin{pmatrix} 1 & X \\ 0 & GL_{n-p}(A) \end{pmatrix} } \]

For later purposes

Put $p = C_{p-1}(X(A^n))$ \hspace{1cm} $J_n = H_{n-1}(X(A^n))$

$0 \rightarrow J_n \rightarrow p_n \rightarrow p_{n-1} \rightarrow \cdots \rightarrow p_o \rightarrow 0$

so we get a spectral sequence

\[ E^1_{pq} = H_q(GL_n, p_p) \implies H_{p+q}(GL_n, J_n[n]) = \begin{cases} 0 & \text{if } p+q \neq n \\ ? & \text{otherwise} \end{cases} \]

Now

\[ H_*(GL_n, \mathbb{Z}[X_p] \otimes_{\mathbb{Z}[\Sigma_p]} \mathbb{Z}[g^n]) = H_*(GL_n, \mathbb{Z}[GL_n] \otimes_{\mathbb{Z}[GL_{n-p}] \mathbb{Z}[g^n]} \mathbb{Z}[g^n]) \]

\[ = H_* \left( \begin{bmatrix} \Sigma_p & * \\ 0 & GL_{n-p} \end{bmatrix}, \mathbb{Z}[g^n] \right) \]

by the Shapiro Lemma.
Let $G$ be the affine group of a vector space $V$; $G = \text{GL}(V) \ltimes \bar{V}$, where $\bar{V}$ is interpreted as translations.

Let $Y(V)$ be the affine building of $V$, that is, the poset of affine subspaces $W < V$. One knows (Lusztig) that if $\dim(V) = n$, then $Y(V)$ is $q$-spherical of dim. $n-1$, and that it is a bouquet of $(q^{n-1}) \cdots (q-1)$ spheres of dim. $(n-1)$. What sort of spectral sequence do we get from these buildings?

Let $J(V) = \tilde{H}_{n-1}(Y(V), \mathbb{Z})$. Then we get an exact complex

$$0 \to J(V) \to \bigoplus_{\text{odd}(n+1)} J(H) \to \cdots \to \bigoplus_{\text{even} V} \mathbb{Z} \to \mathbb{Z} \to 0$$

the stabilizer of an affine subspace $W$.

Now suppose $V = \sum_{i=0}^{n} e_i$. Then

$$G = \begin{pmatrix} GL_n & \ast \\
0 & 1 \end{pmatrix}$$

Now if $W = \sum_{i=0}^{m} k e_i + e_{m+1}$, then the stabilizer $G_W$ is

$$G_W = \begin{pmatrix} GL_p & \ast & \ast \\
0 & GL_{n-p} & 0 \\
0 & 0 & 1 \end{pmatrix}$$

It is clear that a better notation would be this: $V = \sum_{i=0}^{n} e_i$; $W = \sum_{i=0}^{p} k e_i$. Then

$$G = \begin{pmatrix} 1 & \ast \\
\ast & GL_n \end{pmatrix} \quad G_W = \begin{pmatrix} 1 & \ast & \ast \\
\ast & GL_p & \ast \\
0 & 0 & GL_{n-p} \end{pmatrix}$$

\[ V \text{ vector space, } V' \text{ subspace, } G = \{ g \in GL(V) | g(V) \subseteq V' \} \]
\[ T(V,V') = \text{subspaces } W < V \implies V' + W = V. \]

If \( m = \dim(V/V') \), then \( T(V,V') \) has dimension \( m - 1 \), and one knows from Lusztig that it is a bouquet of \((q^n - 1) \cdots (q^m - 1)\) spheres.

Example to keep in mind: \( V' \) a hyperplane - here \( G \) is the affine group of \( V'. \) If \( V = k e_0 + k e_1 + \cdots + k e_m \) and \( V' = k e_1 + \cdots + k e_m \), then
\[ G = \left\{ \begin{pmatrix} 1 & 0 \\ \star & \text{GL}_m \end{pmatrix} \right\} \]

Now if \( W_0 \in T(V,V') \), then its "boundary" \( \{ W \in T(V,V') | W < W_0 \} \) is equal to \( T(W_0, W_0 \cap V') \). In effect
\[ W + V' = V \iff \exists W_0 \in T(V,V'), W_0 \cap W + W_0 \cap V' = W_0 \cap (W + V') = W_0 \]

Let \( J(V,V') = \tilde{H}_{m-1}(T(V,V')). \)

Then we have the following exact sequence
\[ 0 \to J(V,V') \to \cdots \to \bigoplus_{W \in T(V,V')} J(W,W \cap V') \to \bigoplus_{W \in T(V,V')} \mathbb{Z} \to \mathbb{Z} \to 0 \]

by Lusztig's theory.
On the other hand, the group $G$ acts transitively on $T_0(V, V') = \{ W \in T(V, V') \mid \dim(W) = n + p \}$. Put $V = \bigoplus_{i} \mathbb{Q} \oplus k_{e_i} \oplus \cdots \oplus k_{e_n}$ and compute the stabilizer $G_W$ where

$W = \mathbb{Q} \oplus k_{e_1} \oplus \cdots \oplus k_{e_p}$.

\[
G = \begin{pmatrix}
\text{id}_{\mathbb{Q}} & 0 \\
\ast & \mathcal{G}_n
\end{pmatrix}
\]

\[
G_W = \begin{pmatrix}
\text{id}_{\mathbb{Q}} & 0 & 0 \\
\ast & \mathcal{G}_p & \ast \\
0 & 0 & \mathcal{G}_{n-p}
\end{pmatrix}
\]

Notation: $G(V, \text{id}_{W^nV})$ for $G$. Then given $W$ its stabilizer $G_W$ is

$G_W = G(W, \text{id}_{W^nW'}) \times G(V', W^nV') \big/ G(W^nV')$

and so we have an exact sequence

$1 \rightarrow G(V', \text{id}_{W^nV'}) \rightarrow G_W \rightarrow G(W, \text{id}_{W^nW'}) \rightarrow 0$

In the case where $V'$ is a hyperplane, $W$ corresponds to an affine subspace of $e + V'$, so this corresponds to the exact sequence which one obtains by restricting an
affine transformation preserving $e_0$ to $e_n$, one maps into the affine group of $W_n V$ and the kernel is those linear transformations of $V'$ which induce the identity on $W_n V'$.

Thus one ends with a spectral sequence:

$$E_1^{pq} = H_p \left( \begin{array}{c} \text{id}_m \\ \times \text{GL}_p \\ \times \text{GL}_{n-p} \end{array} \right), \quad J(\otimes^{k+1} + \otimes^p, \otimes^1 + \otimes^p)$$

converging to the homology of

$$G = \begin{pmatrix} \text{id}_m & 0 \\ \times & \text{GL}_n \end{pmatrix}$$

I want to use this to prove a stability theorem for the mod p homology.

For this it would appear necessary to know something about the groups

$$H_* \left( \begin{array}{c} \text{id}_m \\ \times \text{GL}_p \end{array} \right), \quad J(\otimes^{k+1} + \otimes, \otimes^1 + \otimes^p)$$

For example take $p=1$. Here $T(k^{m+1} + k^1)$ is the set of hyperplanes transversal to the line $k^1$, i.e. the splittings of

$$0 \rightarrow k^1 \rightarrow k^{m+1} \rightarrow k^m \rightarrow 0$$

and $G = \text{those autos. of this sequence inducing identity on the quotient } k^m$. 

$$\therefore G = k^* \times \text{Hom}(k^m, k) \text{ and } T(k^{m+1}, k) = G/k^*, \text{ so}$$
\[
\begin{array}{c}
0 \rightarrow J(m+1, 1) \rightarrow \mathbb{Z}[G/\ell^*] \rightarrow \mathbb{Z} \rightarrow 0
\end{array}
\]

and so

\[
H_*(G, J(m+1, 1)) \rightarrow H_*(\ell^*, \mathbb{Z}) \rightarrow H_*(G, \mathbb{Z})
\]

which shows that $H_*(G, J(m+1, 1))$ is essentially the same mod $p$ as

\[
H_*(\begin{pmatrix} \text{id}_m & 0 \\ \ell^* & -\ell^* \end{pmatrix})
\]

the sort of group I get rid of with my splitting theorem when $k$ is infinite.

Idea: let $n \rightarrow \infty$ in the above spectral sequence. One gets

\[
E^1_{p*} = H_*(\begin{pmatrix} \text{id}_m & 0 \\ \ell^* & \ell^P \end{pmatrix} \otimes \mathbb{F}_p) \otimes H_*(GL_{\infty})
\]

converging to $H_*(GL_{\infty})$. This suggests that there should be a very reasonable way to get a contractible complex exhibiting the groups $H_*(\begin{pmatrix} \text{id}_m & 0 \\ \ell^* & \ell^P \end{pmatrix} \otimes \mathbb{F}_p)$.
Theorem: Let \( k = F_p, \, s = p^d \). Then for each \( i > 0 \) one has \( H^i(\text{GL}_n(F_q), F_p) = 0 \) for \( n \) sufficiently large.

We will prove that for any \( m \) and \( i < q \) one has

\[
\tilde{H}^i \left( \begin{array}{c|c} \text{id}_m & 0 \\ \hline 0 & \text{GL}_n \end{array} \right) = 0
\]

(mod \( p \) coefficients) for \( n \) sufficiently large. Assume this has been established for \( i < q \), and we want to get it for \( q \).

Lemma 1: It is possible to fit the spectral sequences

\[
E^1_{ab}(n) = H^a \left( \begin{array}{c|c} \text{id}_m & 0 \\ \hline 0 & \text{GL}_n \end{array} \right), \quad J(k^{m+n}, k^d) \Rightarrow H^a(k^{m+n}, \text{GL}_n)
\]

into an inductive system with the obvious effect on \( E_1 \) and \( E_{ab} \).

Assume this for the moment. By induction we know that in degrees \( i < q \) the group \( \left( \begin{array}{c|c} \text{id}_m & 0 \\ \hline 0 & \text{GL}_n \end{array} \right) \) has trivial homology if \( n \) is large. Hence

\[
E^1_{ab}(n) \to E^1_{ab}(\infty)
\]

for \( n \) large.

\[\text{for } n \geq s, \text{ the spectral sequence converges as expected.}\]
Lemma: Let \( E^\bullet_{st}(f) \): \( E^s_{st} \rightarrow E^r_{st} \) be a map of spectral sequences (first quadrant homology type) such that \( E^1_{st}(f) \) is an isomorphism for \( s+t \leq g, \ t < g \) and is surjective for \( s+t = g+1, \ t < g \). Then \( E^s_{st}(f) \) is an isomorphism for \( s+t \leq g, \ t < g \) and all \( r \), and it is surjective for \( s+t = g+1, \ t \leq g-r \).

Picture:

```
\begin{center}
\begin{tikzpicture}
\draw[help lines] (0,0) grid (10,10);
\draw[->] (0,0) -- (10,0);
\draw[->] (0,0) -- (0,10);
\node at (5,5) {critical line \( \text{deg } g \)};
\end{tikzpicture}
\end{center}
```

Proof by induction on \( r \), the case \( r=1 \) being hypothesis. Assume true for \( r \) suppose \( s+t \leq g, \ t < g-1 \).

\[
\begin{align*}
E^s_{s+t-m} \xrightarrow{d_r} E^s_{s+t} \xrightarrow{d_r} E^s_{s-r, t+1} & \quad \text{total degree } s+1 \leq g+1 \quad \Rightarrow \quad \text{because degree } < g \\
t-r+1 \leq g-1 \Rightarrow t+1 = g-1 - t+1 = g-2 \quad \text{hence this is also.}
\end{align*}
\]

Conclude in this case that some cycle \& boundary groups \( \Rightarrow E^{s+1}_{s+t}(f) \) isom for \( s+t \leq g, \ t < g-1 \).

On the other hand, suppose \( \exists \ s+t = g+1, \ t \leq g-r-1 \).
\[ E_{\ast t}^r(f) \xrightarrow{d_h} E_{s+r, t+r+\alpha}^r(f) \]

into \( s \geq 0 \), \( t < q \)  

\[ \cong (\text{total degree} = s, \ t + r - 1 \leq q - 2) \]

so one gets \( E_{\ast t}^{r+1}(f) \) is onto.

(Actually the proof seems to give surjectivity in the range \( a + t = q + 1, \ t < q - r + 1 \) and \( r \geq 2 \))

Now returning to the situation on page 8, I know in the case of a finite field of char \( p \) and mod \( p \) homology that \( E_{\ast t}^0(\infty) = 0 \), \( (s, t) \neq (0, 0) \). Therefore for \( n \) large one has \( E_{\ast t}^0(n) = 0 \) for \( a + t = p \), \( t < q \). So the only term of degree \( q \) is \( E_{q}^0(n) \) which is a quotient of \( E_{q}^0(n) = H_0(\mathrm{id}_n \otimes G_n) \).

Thus I get

\[ H_0^q(GL_n) \to H_0^q(\mathrm{id}_n \otimes GL_n) \]

for \( n \) large. On the other hand, it is injective obviously.

By duality one has

\[ H_0^q(GL_n) \to H_0^q(\mathrm{id}_n \otimes GL_n) \]

for \( n \) large.

Now having obtained this result I can go to the spectral sequence obtained using the Tits building:
\[ E^1_{st} = H^s_b (I_{s-t} \otimes \text{GL}_n) \Rightarrow H_{s+t}(\text{GL}_n) \]

For \( t \leq q \) now we know that \( \text{GL}_n \) acts trivially on the homology of \( (I_s \otimes \text{GL}_{n-r}) \) for \( s \) large.

And since I have seen that \( H^s_b (\text{GL}_n, M \otimes I_s) = 0 \) for a trivial \( M \) of characteristic \( p \), we have \( E^1_{st} = 0 \) for \( s \) odd, \( t \leq q \), \( s \geq 2 \). This gives then the isomorphism

\[ H^s_b (I_s \otimes \text{GL}_{n-t}) \rightarrow H^s_b (\text{GL}_n) \]

so one wins.

Let $k$ be a finite field with $q$ elements, let $W(k)$ be the ring of Witt vectors, and let $V$ be a vector space over $k$ of dim $n$. Lusztig has constructed a basic representation $D(V)$ of $GL(V)$ over $W(k)$ which I really should understand at least for $n = 2$.

It has rank $(q-1) \cdots (q^{n-1}-1)$, hence rank $q-1$ when $n = 2$. When $n = 2$ one has an exact sequence

$$0 \to D(V) \otimes_W k \to \sum_{L \subseteq V} D(L) \otimes_W k \to V \to 0$$

Here is Lusztig's definition when $n = 2$. He considers the simplicial complex of affine subspaces $A$ of $V$ not containing zero, and looks at the 1-cycles, that is, functions $f(A \circ A_1)$ with values in $W$ which are cycles. This gives an $H_1$ of rank $(q^2-1)(q-1)$ which he cuts down first by putting a homogeneity condition, then he takes an eigenspace of some operator.
I would like to understand Steifel-Whitney homology classes of Euler spaces. Perhaps there is some way to get at them using the diagonal embedding $X \to X \times X$ and the action of the cyclic group of order 2 on $X \times X$.

So let $G = \mathbb{Z}/2\mathbb{Z}$ and let $X$ be a $G$-polyhedron and work out what relations you can.

First suppose $X$ is a manifold with boundary $\partial X$; then $X^G$ is a submanifold with boundary, let $d$ be its codimension.

\[
\begin{align*}
H_G^*(X, X^G) &\twoheadrightarrow H_G^*(X) \twoheadrightarrow H_G^*(X^G) \xrightarrow{\delta} \\
\cong H^*_G(X/G, X^G)
\end{align*}
\]

The localization then says

\[
H_G^*(X) \otimes e^{-d} \cong H_G^*[e^{-d}] \otimes H_G^*(X^G)
\]

One knows also that

\[
\chi(X) = 2 \chi(X^G) - \chi(X^G)
\]

In particular, $\chi(X) \equiv \chi(X^G) \mod 2$. So far we've only used homotopy results, and so if we have used that $X$ is a manifold. But now—
consider

\[ \longrightarrow H^*_G(X, X - X^G) \longrightarrow H^*_G(X) \longrightarrow H^*_G(X - X^G) \stackrel{\delta}{\longrightarrow} \]

Again

\[ H^*_G(X^G) \]

and the composition

\[ H^*_G(X^G) \longrightarrow H^*_G(X) \longrightarrow H^*_G(X) \]

is multiplication by the Euler class of the normal bundle \( \nu(X^G, X) \). As a \( G \)-bundle over the trivial \( G \)-space \( X^G \), it is the tensor product of the non-trivial rep. of \( G \) with the normal bundle, i.e.

\[ \nu(X^G, X) = \eta \otimes \nu(X^G, X) \], so its Euler class is

\[ \omega_d(\eta \otimes \nu) = e^d + \omega_d(\nu) e^{d-1} + \ldots + \omega_d(\nu) \]

where \( \omega_d(\nu) \in H^d(X^G) \) are the Stiefel-Whitney classes of the normal bundle of \( X^G \) in \( X \).

Since

\[ H^*_G(X^G) = H^*_G \otimes H^*(X) \]

and the classes \( \omega_d(\nu) \) are nilpotent, one sees that \( \omega_d(\eta \otimes \nu) \) is a non-zero-divisor in \( H^*_G(X^G) \). In particular one gets short exact sequences

\[ 0 \longrightarrow H^*_G(X, X - X^G) \longrightarrow H^*_G(X) \longrightarrow H^*_G(X - X^G) \longrightarrow 0 \]

One should note that this sequence is not homotopy-invariant because if we multiply \( X \) by \( R \) with
-1 action, then d increases by one.

Suppose now that X is a G-polyhedron, again we have the exact sequence

\[ H^*_G(X, X-G) \to H^*_G(X) \to H^*_G(X-G) \to \]

but now we don’t have a Gysin homomorphism.

Example: \( X = \text{Cone}(Z) \) where G acts freely on Z. Then \( (X, X-G) \sim (X, Z) \) so we get

\[ H^*_G(X, Z) \to H^*_G(Y) \to H^*_G(Z) \to \]

\[ H^*(BG) \to H^*(Z/G) \to \]

This shows that the long exact sequence needn’t split.

Observation: Let \( p \) be an odd prime, \( X \) a smooth manifold on which \( G = \mathbb{Z}/p\mathbb{Z} \) acts, and assume \( H^*(X, F_p) \) finite dimensional. From the exact sequence

\[ 0 \to H^*_G(X, X-G) \to H^*_G(X) \to H^*_G(X/G-X^G) \to 0 \]

defined above

\[ H^*_G(X^G) \]

one sees that if \( \beta \in H^1_G \) is the generator, then on taking the homology of \( H^*_G(X) \) with respect to \( \beta \) (recall one has \( \beta^2 = 0 \)), one gets that \( \chi(X/G-X^G) \) can be computed from \( H^*_G(X) \). (This perhaps is not surprising since by duality \( \chi(X/G-X^G) = \chi(X/G) - \chi(X^G). \))
January 10, 1977

Stability for a field.

Let \( k \) be a field. If \( k \) is infinite, I think I can prove the stability of \( H^i(\text{GL}_n(k), \mathbb{Z}) \) starting with \( n > i \), i.e., \( H^i(\text{GL}_n(k)) \rightarrow H^i(\text{GL}_{n+1}(k)) \rightarrow \cdots \), except possibly for 2-torsion.

Recall the proof \( \pmod{\text{torsion}} \). One lets \( \text{GL}_n(k) \) act on the unimodular complex of \( k^n \) which gives a resolution

\[
0 \rightarrow H_{n-1}(X(k^n)) \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow 0
\]

where

\[
L_0 = C_{n-1}(X(k^n)) = \mathbb{Z}[X'_n(k^n)] \otimes \mathbb{Z}^{\mathbb{Q}^n} \mathbb{Z}[S_n]
\]

where \( X'_n(k^n) \) is the set of \( k^n \)-frames in \( k^n \). Now

\[
H^i(\text{GL}_n, L_0) = H^i\left( \frac{\Sigma_n \times X'_n}{\text{GL}_{n-1}} \right, \mathbb{Z}^{\mathbb{Q}^n})
\]

Because \( k \) is infinite I know that

\[
H^i\left( \begin{array}{c} 1_n \\ \times \\ \text{GL}_{n-1} \end{array} \right) \leftarrow H^i(\text{GL}_{n-1})
\]

by using (Casimir operator) style arguments. Hence I know that \( \Sigma_n \) acts trivially on \( H^i\left( \begin{array}{c} 1_n \\ \times \\ \text{GL}_{n-1} \end{array} \right) \). Now over the rationals \( \Sigma_n \) is cocompact so I get

\[
H^i(\text{GL}_n, L_0) = H^i(\Sigma_n \times H^i\left( \begin{array}{c} 1_n \\ \times \\ \text{GL}_{n-1} \end{array} \right, \mathbb{Z}^{\mathbb{Q}^n}) \mod \text{torsion}
\]

\[
= \begin{cases} 0 & n \geq 2 \\ H^i\left( \begin{array}{c} 1_n \\ \times \\ \text{GL}_{n-1} \end{array} \right) & n = 0, 1 \end{cases}
\]
Thus in the spectral sequence assoc. to the above complex one has three non-zero lines.

\[ H^s(GL_n, H^1_m X(k^n)) \Rightarrow H^s(GL_{n+1}) \Rightarrow H^s(GL_n) \]

\[ s = 0 \quad 0 = n \]

\[ a = n + 1 \]

\[ \begin{array}{c}
\text{n-1} \\
\text{n}
\end{array} \]

\[ \begin{array}{c}
h+1 \\
1 \\
o
\end{array} \]

hence one gets

\[ H^1(GL_{n+1}) \simeq H^1(GL_n) \quad \text{for } i \leq n-2 \]

\[ H_0(GL_n, H^1_{n-1} X(k^n)) \Rightarrow H_1(GL_n) \Rightarrow H_0(GL_{n+1}) \Rightarrow 0 \]

\[ H_1(GL_n, H^1_{n-1} X(k^n)) \Rightarrow H_0(GL_{n+1}) \Rightarrow H_0(GL_n) \]

Thus one has what I claimed -

\[ H_n(GL_n) \Rightarrow H_n(GL_{n+1}) \simeq H_n(GL_{n+2}) \Rightarrow \ldots \]

at least modulo torsion. As an example one has

\[ H_1(GL_1) \Rightarrow H_1(GL_2) \simeq H_1(GL_3) \Rightarrow \ldots \]

modulo torsion.
Now ask whether the torsion really matters. The situation involves calculating $d_1$ it seems. So let us set up the situation carefully.

We are considering the complex of chains on the unimodular simplicial complex of $k^n$

$$0 \to H_{n-1}(X) \to L_n \to \cdots \to L_0 \to 0$$

where $L_s = C_s(X)$  \text{ s=0, \ldots , n}

$$= \mathbb{Z} \quad \text{ s=0}$$

Because $GL_n$ acts transitively on $X_s = s$-frames in $k^n$, I know that $\mathcal{L}_s$ is an induced module from the stabilizer of $(e_1, \ldots, e_s)$ which is $(\Sigma_s \ast 0 \ast GL_{n-s}) \subset GL_n$ and so therefore I get the Shapiro isomorphism

$$(\ast) \quad H_\ast (GL_n, \mathcal{L}_s) \cong H_\ast (0 \ast GL_{n-s}, \mathbb{Z}^{\Sigma_s})$$

Now in terms of this isomorphism I wish to compute the map induced by $d : L_s \to L_{s-1}$.

**Lemma**: With respect to the Shapiro iso, $(\ast)\ast$ The homom. $H_\ast (GL_n, \mathcal{L}_s) \to H_\ast (GL_n, \mathcal{L}_{s-1})$ induced by $d$ is the composition of the transfer for

$$\left( \Sigma_{s-1} \ast \right) \cong \left( \Sigma_0 \ast \right)$$

with the inclusion

$$\left( \Sigma_{s-1} \ast \right) \cong \left( \Sigma_0 \ast \right)$$
I will assume this for now and see what stability range results. If I work over a field coefficients

$$E^1_{0*} = H^*(\Sigma_{\alpha}, I_0) = H^*(\Sigma_{\alpha}, I_0) \otimes H^*(\text{GL}_{\alpha})$$

so $$d_1: E^1_{0*} \to E^1_{0-1,0*}$$ is the product of transfer

$$H^*(\Sigma_{\alpha}, I_0) \to H^*(\Sigma_{\alpha-1}, I_0)$$

with ind. map $$H^*(\text{GL}_{\alpha}) \to H^*(\text{GL}_{\alpha-1})$$

Lemma 2: Over $$\mathbb{F}_p$$ one has $$H^*(\Sigma_{\alpha}, I_0) = 0$$ except if $$\alpha \equiv 0, 1 \pmod{2}$$, and the transfer hemo

$$H^*(\Sigma_{\alpha}, I_0) \to H^*(\Sigma_{\alpha-1}, I_{\alpha-1})$$

for $$\alpha \equiv 1 \pmod{2}$$ is an isomorphism.

For $$p = 2$$, one has $$I_0$$ is trivial

$$H^*(\Sigma_{\alpha}) \xrightarrow{\text{tr}} H^*(\Sigma_{\alpha-2}) \xrightarrow{\text{known by Ded}} H^*(\Sigma_{\alpha})$$

mull. by index = $$\alpha$$

Thus transfer $$H^*(\Sigma_{\alpha}) \to H^*(\Sigma_{\alpha-1})$$ is an eine for $$\alpha$$ odd $$(\alpha \equiv 1 \pmod{2})$$ and zero for $$\alpha$$ even. OKAY.

Thus $$H^*(\Sigma_0, I_0) = 0$$ unless $$\alpha \equiv 0, 1 \pmod{2}$$. If $$\alpha \equiv 1 \pmod{2}$$, then transferring down to $$\Sigma_{\alpha-1}$$ and restriction...
back multiples by $s \neq 0$, hence $\text{tr}: H^*_k(\Sigma_0, I_0) \to H^*_k(\Sigma_0, I_0)$ is injective. Finally the other composition

$$H_k(\Sigma_0, I_0) \to H_k(\Sigma_0, I_0) \to H_k(\Sigma_{d-1}, I_{d-1})$$

is given by a double coset formula. Now $\Sigma_0/\Sigma_{d-1} = \{0\}$, and $\Sigma_{d-1}$ has two orbits, one fixed and another with stabilizer $\Sigma_{d-2}$. However we know that $H_k(\Sigma_{d-2}, I_{d-2}) = 0$ if $d \equiv 1 \mod p$. Thus above composition is the identity, qed.

Suppose now I try to see what follows when $p$ is odd. Fix an integer $g$ and assume that

$$H_k^*(GL_g) \to H_k^*(GL_{g+1})$$

for each $i < g$. Now go to spectral sequence.

To prove that $H^*_k(GL_g) \to H^*_k(GL_{g+1})$

I take $n = g+1$, and I want to show that

$$E^2_{0, g+1} = 0$$

for $n \geq 2$.

If we take $A = 0$ so that $\text{tr}: H^*_k(\Sigma_0, I_0) \to H^*_k(\Sigma_0, I_0)$ then and such that the transfer $H^*_k(\Sigma_0, I_0) \to H^*_k(\Sigma_{d-1}, I_{d-1})$ is zero, then from

$$E^1_{0, g+1} = H_k^*(\Sigma_0, I_0) \otimes H_k^*(GL_{g+1})$$

etc. one sees that

$$E^2_{0, g+1} = H_k^*(\Sigma_0, I_0) \otimes \text{Coker}\{H_k^*(GL_{g+1}) \to H_k^*(GL_{g+1})\}$$

Because $p \text{ odd} \Rightarrow H^*_k(\Sigma_0, I_0)$ begins in degree $1 + g+1 - d = g+2$
and so \( E^{2}_{0,2} = 0 \). Now for \( s+1 = 1 \) (p)

\[
E^{2}_{0+1, x} = H_{x}^{*}(\Sigma_{0}, I_{0}) \otimes \operatorname{Ker}\left\{ H_{x}^{*}(\text{GL}_{s+1-0}) \rightarrow H_{x}^{*}(\text{GL}_{s+1-0}) \right\}
\]

which begins in degree \( 1 + q+1-s-1 = q-s+1 \), in particular \( E^{2}_{0+1, q-s+1} = 0 \).

Thus we see all of the \( E^{2} \) terms of degree \( q+1 \) are zero, except for \( E^{2}_{q, q} \), and so we deduce that \( H_{q}^{*}(\text{GL}_{q}) \rightarrow H_{q}^{*}(\text{GL}_{q+1}) \). Analogously, we will get that \( H_{q}^{*}(\text{GL}_{q+1}) \rightarrow H_{q}^{*}(\text{GL}_{q+2}) \rightarrow \ldots \).

This argument doesn't work for the prime 2, because we now have \( H_{0}^{*}(\Sigma_{0}, I_{0}) \neq 0 \). What is the range you get? Again to get \( H_{q}^{*}(\text{GL}_{q}) \rightarrow H_{q}^{*}(\text{GL}_{q+1}) \) I want \( E^{2}_{q,s} = 0 \) for \( s \geq 2 \). Now one has

- For \( q = 1 \) get \( H_{1}^{*}(\text{GL}_{1}) \rightarrow H_{1}^{*}(\text{GL}_{2}) \rightarrow \ldots \)
- For \( q = 2 \) get \( E_{2,1}^{2} \) begins in degree 1 if \( n = 2 \)

\( E_{3,0}^{2} = 0 \)

so for surj. \( H_{2}^{*}(\text{GL}_{1}) \rightarrow H_{2}^{*}(\text{GL}_{2}) \rightarrow \)

To get surjectivity I need \( E^{2}_{q,s} = 0 \) \( s \geq 3 \).
\[ E_{3,0} = H_*(\Sigma_2, I_2) \otimes \text{Ker} \{ H_*(\text{GL}_{n-2}) \to H_*(\text{GL}_{n-1}) \} \]

Thus we get

\[ H_2(\text{GL}_3) \to H_2(\text{GL}_4) \to H_2(\text{GL}_5) \to H_2(\text{GL}_6) \to \cdots \]

\(q=3: \) For surjectivity I need \( E_{2,2} = 0 \) \( \Rightarrow \) \( n < 6 \)

\[ E_{2,2} = H_*(\Sigma_2, I_2) \otimes \text{Ker} \{ H_*(\text{GL}_{n-2}) \to H_*(\text{GL}_{n-1}) \} \]

\[ = 0 \quad \text{if} \quad n > 5 \]

\[ E_{3,1} = \text{Ker} \quad H_1(\text{GL}_{n-3}) \]

\[ = 0 \quad \text{if} \quad n > 5 \]

So \( H_3(\text{GL}_5) \to H_3(\text{GL}_6) \to \cdots \)

For injectivity I need \( E_{2,1} = 0 \) \( \Rightarrow \) \( n > 6 \)

\[ E_{2,1} = 0 \quad \text{if} \quad n > 6 \]

Thus I get \( H_3(\text{GL}_5) \to H_3(\text{GL}_6) \to H_3(\text{GL}_7) \to \cdots \)

So it appears that this method proves

\[ H_0(\text{GL}_{2q-1}) \to H_0(\text{GL}_{2q}) \to \cdots \]

One should notice the for the symmetric groups this result is best possible. In effect one knows \( \otimes H_*(B\Sigma_n) \) is a polynomial ring, and \( I \) generator in we \( H_1(\Sigma_2) \) whose powers whose \( H_*(B\Sigma_{2n}) \) is not in the image of \( H_*(B\Sigma_{2n-1}) \).

ERROR.

The error was caused by fact that to get \( H_0(\text{GL}_n) \to H_0(\text{GL}_{n+1}) \) we forgot to consider \( E_{2,0}^2 \).
$E_{n,0} = H_0^*(\Sigma_n; I_n) \otimes H_0^*(GL_0)$ and if \( n \) is even (\( p=2 \) here), there is no $E_{n+1,0}$ term to cancel it. Thus one has to start with \( n=2 \) to get $H_1^*(GL_2) \rightarrow H_1^*(GL_{n+3})$, in which case we get

$H_1^*(GL_2) \rightarrow H_1^*(GL_3) \rightarrow H_1^*(GL_4) \rightarrow \ldots$.

Then we get $H_2^*(GL_4) \rightarrow H_2^*(GL_5) \rightarrow \ldots$ and similarly $H_0^*(GL_2^0) \rightarrow H_0^*(GL_{2^0+1}) \rightarrow \ldots$ in general. And this is best possible for the symmetric groups.

One should recall that for \( p \) odd, $H_0^*(\Sigma_p)$ begins in dimension $2(p-1)-1 = 2p-3$. Thus for the symmetric groups one has a basic element in $H_3^*(\Sigma_3)$. 

I want to know if the 2-torsion peculiarities, which I have obtained via the symmetric groups, actually is a K-phenomenon. Thus consider the ring \( \mathbb{Z} \), let \( \text{GL}_n(\mathbb{Z}) \) act on the Tate building of \( \mathbb{Q}^n \), and consider the groups \( H_* (\text{GL}_n(\mathbb{Z}), I_n) \) where \( I_n = H_{n-2}(T(\mathbb{Q}^n)) \). These are analogous to the groups \( H_* (\Sigma_n, \mathbb{Z}^{\text{sgn}}) \), and yesterday's work showed that the 2-torsion peculiarities in stability are a consequence of the fact that

\[
H_0 (\Sigma_n, \mathbb{Z}^{\text{sgn}}) = \mathbb{Z}/2 \quad n \geq 2
\]

So I now will compute \( H_0 (\text{GL}_2(\mathbb{Z}), I_2) \). Now

\[
0 \rightarrow I_2 \rightarrow \mathbb{Z} [f, g] \rightarrow \mathbb{Z} \rightarrow 0
\]

\[
\mathbb{Z} [f, g]
\]

\[
\mathbb{Z} [\text{GL}_2(\mathbb{Z})/(\mathbb{Z}/2)]
\]

since any f.t. torsion free \( \mathbb{Z} \)-module is free. (all this would also go through \( \mathbb{Z} \) for a PID).

So

\[
H_4 (\mathbb{Z}/2) \rightarrow H_4 (\text{GL}_2(\mathbb{Z})) \rightarrow H_0 (\text{GL}_2(\mathbb{Z}), I_2) \rightarrow \mathbb{Z} \rightarrow 0
\]

One has

\[
0 \rightarrow \{ \pm 1 \} \rightarrow \text{SL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z}) \rightarrow 1
\]

\( \mathbb{Z}/2 \times \mathbb{Z}/3 \)

exact extension so

\[
H_2 (\text{PSL}_2(\mathbb{Z})) \rightarrow \mathbb{Z}/2 \rightarrow H_1 (\text{SL}_2(\mathbb{Z})) \rightarrow H_1 (\text{PSL}_2(\mathbb{Z})) \rightarrow 0
\]

\( \mathbb{Z}/2 \oplus \mathbb{Z}/3 \)
\[ \text{But in fact } \text{SL}_2 \mathbb{Z} = \mathbb{Z}/4 \times \mathbb{Z}/2 \mathbb{Z}/6 \overset{10}{\to} H_1(\text{SL}_2 \mathbb{Z}) = \mathbb{Z}/12 \text{ generated by the matrices corresponding to mult. by } i \text{ in } \mathbb{Z}[i] \]

\[ i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ in } \mathbb{Z}[i] \]

To see that \( \omega \) generates a cyclic subgroup of order 3 in \( H_1(\text{SL}_2 \mathbb{Z}) \), one uses the map

\[ \text{SL}_2 \mathbb{Z} \rightarrow \text{SL}_2 \left( \mathbb{Z}/3 \mathbb{Z} \right) \]

isomorphic to \( \mathbb{Z}/3 \mathbb{Z} \times \text{quat. grp order 8} \)

And if one uses

\[ \text{SL}_2 \mathbb{Z} \rightarrow \text{GL}_2 \left( \mathbb{Z}/2 \mathbb{Z} \right) = \mathbb{Z}/3 \]

\[ \text{SL}_2 \mathbb{Z}/\left\{ \pm 1 \right\} \]

which shows \( \omega \) is non-trivial in \( H_1(\text{PSL}_2 \mathbb{Z}) \) hence \( \omega \) is non-trivial of order 4 in \( H_1(\text{SL}_2 \mathbb{Z}) \).

Now from the exact sequence

\[ 1 \rightarrow \text{SL}_2 \mathbb{Z} \rightarrow \text{GL}_2 \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 1 \]

one has

\[ H_1(\text{GL}_2 \mathbb{Z}) = H_0(\mathbb{Z}/2, H_1(\text{SL}_2 \mathbb{Z})) \oplus \mathbb{Z}/2 \]

so take

\[ \omega^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \]
Thus on the $\mathbb{Z}/3$ part the quotient $\mathbb{Z}/2$ acts as the inverse, and so there is no $3$-torsion in $H_1(\text{GL}_2 \mathbb{Z})$. But
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
so one sees that $i$ gives rise to a element of order 2 in $H_0(\mathbb{Z}/2, H_1(\text{GL}_2 \mathbb{Z}))$. Thus
\[
H_1(\text{GL}_2 \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}/2 \quad \text{generators } \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]
Note in fact the one of these is detected by the determinant and the other by
\[
H_1(\text{GL}_2 \mathbb{Z}) \longrightarrow H_1(\text{GL}_2(\mathbb{Z}/2)) = \mathbb{Z}/2 \Sigma_3.
\]
Now one can compute the map
\[
H_1(\mathbb{Z}/2) \longrightarrow H_1(\text{GL}_2 \mathbb{Z})
\]
and one finds that $\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}$ goes to the generator $\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}$ while $\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}$ goes to an element of order 2 of $\text{GL}_2(\mathbb{Z}/2)$. Conclude
\[
H_1(\mathbb{Z}/2) \longrightarrow H_1(\text{GL}_2 \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

Summary:

$H_1(\text{GL}_2 \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$H_0(\text{GL}_2 \mathbb{Z}, I(\mathbb{R}^2)) = 0$
I see now that the preceding calculation has been stupid. In effect, by the Euclidean algorithm any $4 \times 4$ matrix in $GL_2(\mathbb{Z})$ is the product of elementary matrices and \((\begin{smallmatrix}1 & 1 \\ 0 & 1 \end{smallmatrix})\). Thus the kernel of $H_1(\mathbb{Z}^2; \mathbb{Z}^2) \to H_1(GL_2\mathbb{Z})$ is clearly zero.

**Prop:** $H_0(GL_n(\mathbb{Z}), I_n(\mathbb{Q})) = 0$ for $n \geq 2$.

**Proof:** If 
\[ 0 \to V' \to V \to V'' \to 0 \]

is an exact sequence of vector spaces, and if we choose a splitting, so that we get a homo.

$GL(V') \times GL(V'') \to GL(V, V'')$

then I know that as $GL(V, V'')$-modules

\[ I(V) = \mathbb{Z}[GL(V'] \otimes \mathbb{Z}[GL(V') \times GL(V'')] I(V) \otimes I(V''). \]

Thus

\[ H_0(GL(V’, V’’), I(V)) = H_0(GL(V’’) \times GL(V’’), I(V’) \otimes I(V’’)) \]

\[ = H_0(GL(V’), I(V’)) \otimes H_0(GL(V’’), I(V’’)) \]

so this will be zero if say dim $(V’') \geq 2$. But $H_0(GL(V), I(V))$ is a quotient of $H_0(GL(V’, V’’), I(V))$, so one wins.

**Question:** Is $H_0(GL_n(\mathbb{Z}), I_n(\mathbb{Q})) = 0$ for $n \geq 2$?
Assume now that we are over a field \( k \), and let us consider the spectral sequence associated to \( \text{GL}_n \) acting on the Tits building \( T(k^n) \).

\[
E_1^\infty(\mathbb{A}^\infty) = H_* \left( \left( \frac{\text{GL}_n \times}{\text{GL}_{n-d}} \right), I_\bullet \right) \Rightarrow 0
\]

I recall this is obtained by taking the Lusztig sequence

\[
L(V) : 0 \rightarrow I(V) \rightarrow \bigoplus I(V') \rightarrow \cdots \rightarrow \bigoplus I(V) \rightarrow \mathbb{Z} \rightarrow 0
\]

If now \( V \) is a subspace of \( W \), then \( L(V) \) is a subcomplex of \( L(W) \) and it is stable under the action of \( \text{GL}(W,V) \). Thus one has

\[
\begin{align*}
L(V) & \rightarrow L(W) \\
\text{GL}(V) & \rightarrow \text{GL}(W)
\end{align*}
\]

so that if we choose a complement for \( V \) in \( W \), so that we get a hom. \( \text{GL}(V) \rightarrow \text{GL}(W) \), then \( L(V) \rightarrow L(W) \) is equivariant for \( \text{GL}(V) \rightarrow \text{GL}(W) \), and so we obtain an induced map of spectral sequences.

So now I have then a map of spectral sequences

\[
E_1^\infty(\mathbb{A}^\infty) = H_* \left( \left( \frac{\text{GL}_n \times}{\text{GL}_{n-d}} \right), I_\bullet \right) \Rightarrow 0
\]

\[
E_1^\infty(\mathbb{A}^\infty) = H_* \left( \left( \frac{\text{GL}_n \times}{\text{GL}_{n-d}} \right), I_\bullet \right) \Rightarrow 0
\]
Now then suppose we try to use this to prove a stability theorem. First try for $H_1$.

line $g = 0$ is $O \to O \to \mathbb{Z} \oplus \mathbb{Z}$
line $g = 1$ is $H_1(GL_n, I) \to H_1(GL_n, I)$

The point is similar — one has in the limit.

It would seem that to use the comparison theorem for spectral sequences is the wrong approach. I would prefer to understand what is happening.

Question 1: I seem to be finding that there is a spectral sequence

$$E^1_{pq} = H_t(GL_n, I_p) \Rightarrow H_{p+q}(Q)$$

Recall that by filtering the $Q$-category I got a spectral sequence. If $F_n(Q) = \text{module of rank } \leq n$, then

$$\cdots \to H_\ast(F_{n-1}Q) \to H_\ast(F_nQ) \to H_{n-\ast}(GL_n, I_n) \to H_{\ast+1}(F_{n-1}Q) \to \cdots$$

Can check this as follows. $F_1 = \Sigma(B\mathfrak{t})^\vee$, $F_0 = \mathfrak{t}$
Consider the \( \mathbb{Q} \)-category of a field \( k \). Forgetting unipotent subgroups, it is the simplicial space which is the bar construction of the monoid

\[
\mathcal{M} = \coprod_{n \geq 0} BGL_n
\]

and it looks like this:

\[
\text{BM: } M \times M \xrightarrow{\mu} M \xrightarrow{n \mu} \cdots \\
\text{with faces determined by the rules:}
\]

\[
\begin{align*}
\alpha_0 (m_1, \ldots, m_p) &= (m_2, \ldots, m_p) \\
\alpha_i (\ldots, m_i, \ldots) &= (\ldots, m_i, m_{i+1}, \ldots) \\
\alpha_p (\ldots, m_i, \ldots) &= (m_1, \ldots, m_{p-1})
\end{align*}
\]

One filters BM by

\[
F_n(BM) = \left\{ F_n(BM) = \coprod_{a_1 + a_2 \leq n} BGL_{a_1} \times \cdots \times BGL_{a_p} \right\}
\]

whence

\[
F_n(BM)/F_{n-1}(BM) = \left\{ p \arrow{2} \text{pt} \coprod_{a_1 + a_2 = n} BGL_{a_1} \times \cdots \times BGL_{a_p} \right\}
\]
Algebraically what is happening: Put
\[ R = \bigoplus_{n \geq 0} H^*_*(B\mathbb{G}_m) \]
coefficients in a field F. Then the simplicial space BM gives a spectral sequence
\[ E^1_{s,t} = H^*_*(M^s) \implies H^*_*(BM) \]
where \( d^1 \) is given by the formula
\[ d^1_*(h_1 \otimes \cdots \otimes h_s) = e(h_1) h_2 \otimes \cdots \otimes h_s - \sum_{i=1}^{s-1} (-1)^i h_1 \otimes \cdots \otimes \widehat{h_i} \otimes \cdots \otimes h_s + (-1)^s h_1 \otimes \cdots \otimes h_{s-1} \cdot e(h_s). \]
I should recall that \( e : R \to k \) is the map sending \( H_*(B\mathbb{G}_m) \to H_*(pt) \), hence it is the augmentation
\[ R \to \bigoplus H_0(B\mathbb{G}_m) = k[T] \to k. \]
Now I know that \( (E^1_{s,t}) \) is just the bar construction of the augmented algebra \( R \), hence
\[ E^2_{s,t} = \operatorname{Tor}^R_s(k, k) \]
\( k \) being regarded as an algebra via the augmentation \( e \).
(The proof is to exhibit the semi-simplicial resolution
\[ \implies R \otimes R \implies R \to k \]
with
\[ d^1(\cdot \otimes \cdots \otimes \cdot) = \cdots \otimes \widehat{r_{t+1}} \otimes \cdots \]
\[ d_0(\cdot) = \cdot \otimes \cdots \otimes r_0 \]
which is contractible because \( s_1 = (r_0 \otimes r_1) = r_0 \otimes r_1 \).
Because \( k \) is a field, this is a free resolution of \( k \) regarded as an \( R \)-module via \( e \).

Now we have a natural increasing filtration of the ring \( R \):

\[
F_n R = \bigoplus_{a \leq n} H(BG_{la})
\]

and what I am doing is to consider the induced filtration of the bar construction. By passing from \( R \) to \( gr(R) \) the ring doesn't change, but rather the augmentation.

Thus as a space

\[
F_n(BM)/F_{n-1}BM = \left\{ \frac{p}{\sum a_i \in \mathbb{N}} \right\}^{\oplus a_i, \ldots, a_p = n}
\]

whence we have a spectral sequence

\[
E_r^{i,n} = \bigoplus_{a_1, \ldots, a_p = n} H^*_k(Gl_{a_1}) \otimes \cdots \otimes H^*_k(Gl_{a_p}) \Rightarrow \tilde{H}^*_k(FB/M/F_{r-1}BM)
\]

where here:

\[
d^0_0(e_{a_1} \otimes \cdots \otimes e_{a_p}) = 0 \quad \text{if} \quad a_p = \deg(e_{a_p}) > 0
\]

\[
d^0_p(e_{a_1} \otimes \cdots \otimes e_{a_p}) = 0 \quad \text{if} \quad a_p = \deg(e_{a_p}) > 0.
\]

Thus here

\[
E_2^{i,n} = T_{R_0}(k^{(e_{la_1})} \otimes \cdots \otimes k^{(e_{la_p})})
\]

where here \( k^{(e)} \) denotes \( k \) regarded as an \( R \)-module via the augmentation sending \( \mu_k(Gl_n) \rightarrow 0 \) for \( n > 0 \).

Now suppose we compute this for the algebraic closure of \( \overline{F_p} \) with coeffs in \( F_p \). Then one has
\[ R = \oplus H^*_x(\text{GL}_n) = \bigoplus \mathbb{R} \left[ \eta_0, \eta_1, \ldots \right] \]

where \( \eta_i \in H^*_{2i}(\text{GL}_1) \) is dual to \( u^2 \in H^{2i}(\text{GL}_1) \), \( u \) being a generator of \( H^2(\text{GL}_1) \). Thus in the bigrading \( (i, j) \) of \( R \), \( \eta_i \) has degree \( (2i, i) \). Now because \( R \) is a polynomial ring,

\[ \text{Tor}_R^*(k, k) = \bigwedge^d \text{Tor}_{2i}^*(k, k) \]

where \( \text{Tor}_1^R(k, k) = \mathbb{I} / \mathbb{I}^2 \)

I denoting the augmentation ideal. Thus in the case at hand, \( \text{Tor}_1^R(k, k) \) is concentrated in the dimension degree 1, which means that \( E^2_{st}(n) = 0 \) for \( s \neq n \). So the spectral sequence degenerates and we find that

\[ \bigoplus_n \overline{H}_*(F_n \text{BM}/F_{n-1} \text{BM}) \]

is an exterior algebra on \( \overline{H}_*(F_n \text{BM}/F_{n-1} \text{BM}) = \overline{H}_*(\Sigma S \text{GL}_1) \)

\[ = \bigoplus \mathbb{F}_2 \eta_i \text{ where } \eta_i \in H_{2i+1} \]

Basic fact is that for \( \mathbb{F}_p \)

\[ \bigoplus_n H^*_x(\text{GL}_n, \mathbb{F}_p) = \bigwedge \left[ \eta_1, \eta_2, \ldots \right] \text{ odds. odd } \mathbb{F}_2 \]

where \( \eta_i \) is a basis for \( H^*_x(\text{GL}_1, \mathbb{F}_p) = H^2(\text{GL}) = \mathbb{F}_2 \).
January 14, 1974

Stability proof in the "direct sum" situation.

Consider the following example: Let $M = \amalg B_{\Sigma}$
action $M' = \amalg B_{\Sigma} \times B_{\Lambda}$ in the obvious way and form
the $\infty$-category which plays the role of $M'/M$. It
equals the simplicial space

$$M^2 \times M' \xrightarrow{M \times M'} M'$$

which looks like

$$\amalg_{a_0} B_{\Sigma} \times B_{\Lambda} \xrightarrow{\amalg_{a_0} B_{\Sigma} \times B_{\Lambda}} \amalg_{a_0} B_{\Sigma} \times B_{\Lambda}$$

Filter this in the obvious way and then

$$F_n / F_{n-1} \xrightarrow{\amalg B_{\Sigma} \times B_{\Lambda}} \amalg B_{\Sigma} \times B_{\Lambda}$$

I want to identify this gadget.

Maybe it would be clearer if I introduced
the $A$-category. It has as objects those of
$M$, in this case, vector spaces over $k$, and a morphism
from $V$ to $V'$ is an
isomorphism $V \oplus k S \to V'$ where $S$ is a set.

Now filter this category according to its dimension.

$F_n = \text{full subcat. consisting of } V$ of rank $\leq n$. Then have inclusion

$$F_{n-1} \subset F_n$$
and the spectral sequence

\[ E_{pq}^2 = \lim_{i \to \infty} (V \mapsto H_p(i/V)) \Rightarrow H_{p+q}(F_{-i}) \]

Now \( i/V \) has a final object if \( r_g(V) < n \), and if \( V = k^i \), then \( i/V \) is equivalent to the ordered set \( Z(V) \) consisting of splittings

\[ V = W \oplus kS \quad \text{with} \quad W \preceq V \]

in which

\[ (W, S) \preceq (W', S') \iff W \preceq W', S = S' \]

Now one has the functor

\[ \text{GL}_n \to F_n \]

such that if \( M \) is a \( \text{GL}_n \) module, then one has

\[ (j! M)(V) = \lim_{i \to \infty} M = \begin{cases} 0 & \text{if } r_g(V) < n \\ \pi M & \text{if } r_g(V) = n. \end{cases} \]

Thus the Leray spectral sequence for \( j \) degenerates yielding

\[ H_q(F_n, j_! M) = H_q(\text{GL}_n, M). \]

So we have

\[ E_{pq}^2 = H_p(F_n, V \mapsto H_q(i/V)) \Rightarrow H_{p+q}(F_{-i}) \]

\[ H_0(i/V) = \begin{cases} H_0(j^* V) & r_g(V) < n \\ H_0(j^* i/V) & r_g(V) = n \end{cases} \]

Thus for \( q > 0 \) one has

\[ H_0(i/V) = j_! H_0(j(V)) \]

and we have:
Thus one has an exact sequence of complexes

\[ 0 \rightarrow i_1^* \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_{F_n} \rightarrow 0 \]

and we get a cofibration situation:

\[ (\mathrm{J}(V), \mathrm{GL}(V)) \subset (\mathrm{Gr} \mathrm{J}(V), \mathrm{GL}(V)) \]

\[ F_{n-1} \subset F_n \]

Now semi-simplicial, one constructs the core by shifting one dimension. Precisely the non-degenerate simplices \( \sigma \) of the cone are of the form \( \sigma \) or \( \sigma \mathrm{J} \), where \( \sigma \) is a non-degenerate simplex of \( X \). Thus if we realize \((\mathrm{J}(V), \mathrm{GL}(V))\) as the simplicial space without degeneracies,

\[ \therefore \coprod \Sigma a_0 \times \Sigma a_1 \times \Sigma a_2 \rightarrow \coprod \Sigma a_0 \times \Sigma a_1 \]

then \( \mathrm{Core} \mathrm{J}(V), \mathrm{GL}(V) \) is the simplicial space without degeneracies.

\[ \therefore \coprod \Sigma a_0 \times \Sigma a_1 \times \Sigma a_2 \rightarrow \coprod \Sigma a_0 \times \Sigma a_1 \]
and the cofibre is the simplicial space

\[ \text{cofibre} = \text{cofibre of } \bigwedge_{a_1 + \cdots + a_n = n} (B\Sigma_{a_1} \times B\Sigma_{a_2} \times \cdots \times B\Sigma_{a_n}) \]

which we obtained before.

Suppose now we put in the fact that \( J(V) \), which is an ordered set of dimension \( n-1 \), is spherical of this dimension, so that we get the following exact sequence:

\[ \cdots \rightarrow H_*(F_{n-1}) \rightarrow H_*(F_n) \rightarrow H_{*+n}(GL_n) \rightarrow \tilde{H}_{*+n}(J^{(n)}) \rightarrow \cdots \]

How can we deduce from this information the stability of \( H_*(GL_n) \)?

Idea: We consider the result of right multiplying by the basepoint of \( BG_1 \), call this \( \exists \). This gives us a map of simplicial spaces:

\[ M^2 \times M' \Rightarrow M \times M' \Rightarrow M' \]

which is a cofibration in each dimension. Now my idea was to work with the simplicial pair of spaces \( \text{pair of spaces} \Rightarrow (M', M') \), and to filter it as before:

\[ \rho \mapsto \bigwedge_{a_1 + \cdots + a_n = n} (B\Sigma_{a_1} \times B\Sigma_{a_2} \times \cdots \times B\Sigma_{a_n}) \]

The point now is that we know, by the connectivity of \( F_n \), that \( F_n/\tilde{F}_{n-1} \Rightarrow F_{n+1}/F_n \) is a fibration in a range, and this gives us a spectral sequence.
\[ E^{-2}_{pq}(n) = \text{Tor}_p^R(k, \bigoplus_n H_\bullet(G_n, G_{n-1})) \to 0 \text{ in a range increasing with } n. \]

Now we compute the Tor in a pleasant manner. Suppose therefore I have a minimal resolution of \( k \) over \( R = \bigoplus H_\bullet(\Sigma_n) \)

\[ \to R \otimes V_{2x} \to R \otimes V_{1x} \to R \overset{=} \to k \]

where \( V_{px} = \text{Tor}_p^R(k, k) \). This will give an \( \text{E}' \) term for computing the Tor:

\[ \to V_{1x} \otimes \bigoplus_n H_\bullet(G_n, G_{n-1}) \to \bigoplus_n H_\bullet(G_n, G_{n-1}) \]

which hopefully yields the desired stability result. For example, if \( k = \mathbb{Q} \), then \( \bigoplus H_\bullet(\Sigma_n) = \mathbb{Q}[\Sigma] \), so one has a Tor_0 = \( \bigoplus H_\bullet(G_n, G_{n-1}) \) and a Tor_1, hence all the differentials are zero.
I now seem to understand better what is going on in the stability proof for $H_*(GL_n)$ using the unimodular complex.

First of all I have succeeded in relating the two different stability proofs that I had before, one based on the unimodular complex and which involved $H_*(Z, \text{sign})$, the other using the $Q$-category which involved $H_*(GL_n, \text{steinberg})$. And also I have made the proof less dependent on special features, i.e. the complex of simplicial chains.

Consider the unimodular complex of $F^n$, denote it $X(F^n)$. It is a simplicial complex, but for purposes of generalization, it is perhaps desirable to view it as a poset, whose elements are finite non-empty unimodular subsets of $F^n$, under inclusion. Now then...

Since $X(F^n)$ is spherical of dim $n-1$, one gets a complex of $GL_n$-modules:

$$0 \rightarrow l_n \rightarrow \cdots \rightarrow l_0 \rightarrow 0$$

which is acyclic except in degree $n$, where

$$l_n = \prod_{1 \leq i < j \leq n} \mathbb{Z}$$
Using transitivity of $G_{ln}$ on unimodular subsets and splitting of exact sequences one has

$$H^* (GL_n, L_0) = \bigoplus H^* (\Sigma_{n} \times \Sigma_{n-1} \times GL_{\Sigma_{n-1}})$$

which I can recognize in the following way. Assume over a field $k$ of coefficients, so that we have Kunneth

$$H^* (\Sigma_{n} \times \Sigma_{n-1} \times GL_{\Sigma_{n-1}}) = H^* (\Sigma_{n}) \otimes \cdots \otimes H^* (GL_{\Sigma_{1}}).$$

Then if I put

$$R = \bigoplus H^* (\Sigma_{n}) \quad R' = \bigoplus H^* (GL_{\Sigma_{n}})$$

and grade these by the degree $a$ to be called from now on the "size", one finds that the complex

$$a \mapsto H^* (GL_n, L_0)$$

is the size $n$ part of the standard bar complex

$$\cdots \Rightarrow R \otimes R \otimes R' \Rightarrow R \otimes R' \Rightarrow R'.$$

which are known complexes

$$\text{Tor}^R (k, R'), \quad \frac{R}{k} = R/R' \otimes H^* (\Sigma_{n}).$$

**Question:** Is the expression just derived:

$$H^* (\Sigma_{n} \mapsto H^* (GL_n, L_0)) = \text{Tor}^R (k, R') (n)$$

independent of the choice of chain complex used for
Computing the homology of $X(F^n)$? In other words, if I had used another complex $L_*$, would I have gotten the same result?

Answer - yes in some sense, because any two such complexes, if obtained geometrically, will be equivariantly homotopy equivalent, so the two complexes of the form $\pi_0 H_2(G_{L_2}, L_2)$ will be also homotopic.

This is the first part of the argument, namely the use of the unimodular complexes to tell us something about $Tor^R(H, R)$. This part of the argument is quite general and I can easily replace symmetric groups by the general linear groups of a subfield.

Why this is related to the $Q$-construction: suppose I let the symmetric groups act by direct sum on the general linear groups and form the quotient, i.e., the simplicial space

$$
\coprod \Sigma_2 \times \mathbb{S}_{a, 2} \rightarrow \coprod \Sigma_2 \times \mathbb{S}_{a, 1} \rightarrow \coprod \mathbb{S}_{a, 0}
$$

which is related to $Q$-of the symmetric groups.

Now filtering this according to total rank leads to quotients

$$
\frac{F_{n,l}}{F_{n-1}} \rightarrow \coprod \mathbb{S}_{a, 0,x} \times \mathbb{S}_{a, l}
$$
which is essentially the unimodular complex of $F^n$ divided out by the action of $GL_n$. 
January 18, 1974

Idea: Make a detailed study of categories obtained by the following gluing procedure

\[ \begin{array}{ccc}
X & \overset{G}{\longrightarrow} & (CX, G) \\
\downarrow & & \downarrow \\
C' & \overset{f}{\longrightarrow} & C
\end{array} \]

where \( X \) is a "G-space", in practice, a poset.

Example: Let \( F \) be a field and let \( k \) be a subfield. Let \( C \) be the category whose objects are \( F \)-vector spaces in which a map from \( V \) to \( V' \) is an isomorphism \( V \oplus F \otimes_k W \cong V' \), \( W \) an \( k \)-subspace of \( V' \).

Then if \( C_n \) is the subcategory of \( V \) of rank \( \leq n \), then one should have a cocartesian square

\[ \begin{array}{ccc}
(f)^n, Gl, F & \overset{f}{\longrightarrow} & (f)^n, Gl, F \\
\downarrow & & \downarrow \\
C_{n-1} & \overset{f}{\longrightarrow} & C_n
\end{array} \]

So here \( X = f/F \) is the ordered set consisting of decompositions

\[ F^n = V \oplus F \otimes_k W \]

with \( V < F^n \).

Question: Given a field \( F \) and a subfield \( k \), consider the poset consisting of non-zero \( k \)-subspaces \( W \) where \( F \otimes_k W \hookrightarrow V \). Is this spherical?