Rough outline of part I (list of things not to forget)

1) <u>homotopy type of cats.</u>

1.1. BC C small, homotopy property

BC C essentially small

language: C contractible, $\pi_q(C,X)$ etc.

<u>Examples</u> ~~of contractible categories~~ $C \sim C^0$

~~[illegible]~~ functor with an adjoint ~~is a h.eq.~~ initial, final object,

filtering cats.

<u>Conical contractibility.</u>

1.3. homology of a category

$$H_*(C,L) = L_* \varinjlim L$$

spectral sequence of a functor

Whitehead thm.

1.4. Thm. A. generalized

Subdivision

1.2 fibred cats. and functors

~~[illegible] replacing of [illegible]~~

1.5. Thm. B.

trivial cases [fibres contractible

base contractible

2) S^{-1} construction

2.1) Construction of $\langle S, X\rangle$, $S^{-1}X$

monoidal cat S

action of S on X

construction of $\langle S, X\rangle$ and its functorial properties (define $S, X \to S', X'$).

$$\langle S_1 \times S_2, X\rangle = \langle S_1, \langle S_2, X\rangle\rangle$$

~~[illegible]~~

$S \to X$ map of symm. mon. cats, then $\langle S, X\rangle$ is a symm. mon. cat.

cartesian action of S on X (prefibred) over B
$\Rightarrow \langle S, X\rangle$ (pre)fibred over B fibres $\langle S, X_B\rangle$
(also cofibred).

2.2. Homology of $S^{-1}X$

$S^{-1}X \longrightarrow S^{-1}(pt)$ cofibred ← hence $X \to S^{-1}X$ is a heq when S acts invertibly on X.

spectral sequence

cofinality results: $S' \to S$ cofinal
$\Rightarrow S'^{-1}X \to S^{-1}X$ heq

2.4. Thm: $\exists$ $BGL(A) \longrightarrow BGL(A)^+$, $B\Sigma_\infty$.

2.5: fibration $S^{-1}S \to S^{-1}X \to \langle S, X\rangle$.

so it seems now that we have

Def $\mathcal{S}$ monoid cat

Def $\mathcal{S}$ acts on $\mathcal{X}$

$\langle \mathcal{S}, \mathcal{X} \rangle$ construction of $\mathcal{H}(\mathcal{X}, \mathcal{X}')$

Def $\mathcal{X} \longrightarrow \mathcal{X}'$ map of cats w $\mathcal{S}$ action

const. of $\langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X}' \rangle$ induced functor

~~Prop~~ Def ~~[illegible]~~ fibrewise action of $\mathcal{S}$ on $\mathcal{X}$ over $\mathcal{B}$

Def cartesian action of $\mathcal{S}$ on $\mathcal{X}$ prefibred over $\mathcal{B}$.

Prop: cartesian action $\Rightarrow$ $\langle \mathcal{S}, \mathcal{X} \rangle$ is prefibred over $\mathcal{B}$ with $\langle \mathcal{S}, \mathcal{X} \rangle_B = \langle \mathcal{S}, \mathcal{X}_B \rangle$

Now assume $\mathcal{S}$ groupoid and $\perp S$ faithful. and let $\mathcal{S}^{-1}\mathcal{X} = \langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle$ telescope category for the $\mathcal{S}$-action.

Proposition: $\mathcal{S}^{-1}\mathcal{X} \longrightarrow \mathcal{S}^{-1}(pt)$ is cofibred, etc.

This should be all the tools I need.

DEF. a cat with a nat. assoc. unitary operation.

$\mathcal{S}$ category with internal operation

$$\perp : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$$

denoted $(X, Y) \longmapsto X \perp Y$. We suppose $\perp$ provided with an assoc. iso

$$\varphi : (X \perp Y) \perp Z \xrightarrow{\sim} X \perp (Y \perp Z)$$

satisfying the pentagon condition. We suppose also given an object 0 and isos.

$$\psi_X : 0 \perp X \xrightarrow{\sim} X \qquad \psi'_X : X \perp 0 \xrightarrow{\sim} X$$

such that

a) coincide for $X = 0$

b)

$$\begin{tikzcd}
(0 \perp X) \perp Y \arrow[rr] \arrow[dr] & & 0 \perp (X \perp Y) \arrow[dl] \\
& X \perp Y &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \perp 0) \perp Y \arrow[rr] \arrow[dr] & & X \perp (0 \perp Y) \arrow[dl] \\
& X \perp Y &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \perp Y) \perp 0 \arrow[rr] \arrow[dr] & & X \perp (Y \perp 0) \arrow[dl] \\
& X \perp Y &
\end{tikzcd}$$

commute.

<u>DEF</u> Given $(\mathcal{S}, \perp, \varphi, \psi)$, an <u>action</u> of $\mathcal{S}$ on a cat $\mathcal{X}$ consists of

1) $$\#: \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$$
$$(S, X) \longmapsto S \# X$$

2) nat. transf.
$$(S_1 \perp S_2) \# X \xrightarrow{\sim} S_1 \# (S_2 \# X)$$
$$0 \# X \xrightarrow{\sim} X.$$

such that

a) pentagon:
$$\begin{tikzcd}
((S_1 \perp S_2) \perp S_3) \# X \arrow[r] \arrow[d] & (S_1 \perp S_2) \# (S_3 \# X) \arrow[dd] \\
(S_1 \perp (S_2 \perp S_3)) \# X \arrow[d] & \\
S_1 \# ((S_2 \perp S_3) \# X) \arrow[r] & S_1 \# (S_2 \# (S_3 \# X))
\end{tikzcd}$$

b)
$$\begin{tikzcd}
(0 \perp S) \# X \arrow[r] \arrow[dr] & 0 \# (S \# X) \arrow[d] \\
& S \# X
\end{tikzcd}
\qquad
\begin{tikzcd}
(S \perp 0) \# X \arrow[r] \arrow[dr] & S \# (0 \# X) \arrow[d] \\
& S \# X
\end{tikzcd}$$

commute.

Suppose $\mathcal{S}$ acts on $\mathcal{X}$

The category $\mathcal{H}(X,Y)$, X, Y objects of $\mathcal{X}$:

$$\mathrm{Ob}\,\mathcal{H}(X,Y) = \{\text{pairs } (S,u) \mid S \in \mathcal{S},\ u : S \# X \to Y\}$$

$$\mathrm{Hom}_{\mathcal{H}(X,Y)}\big((S,u),(S',u')\big) = \left\{ v \in \mathrm{Isom}_{\mathcal{S}}(S,S') \;\middle|\; \begin{tikzcd} S\#X \arrow[rr, "v\#id"] \arrow[dr, "u"'] & & S'\#X \arrow[dl, "u'"] \\ & Y & \end{tikzcd} \right\}$$

The pairing

$$\mathcal{H}(X,Y) \times \mathcal{H}(Y,Z) \longrightarrow \mathcal{H}(X,Z)$$

$$(S,u),\ (S',u') \longmapsto (S,u)\cdot(S',u') = (S' \perp S,\ u' * u)$$

where $u' * u$ is the composition

$$(S' \perp S)\# X \xrightarrow{\sim} S'\#(S\#X) \xrightarrow{id\#u} S'\#Y \xrightarrow{u'} Z$$

Lemma: If $X, Y, Z, W \in \mathcal{X}$, then two functors

$$\mathcal{H}(X,Y) \times \mathcal{H}(Y,Z) \times \mathcal{H}(Z,W) \rightrightarrows \mathcal{H}(X,W)$$

$$(S,u),\ (S',u'),\ (S'',u'') \longmapsto (S'' \perp (S' \perp S),\ u'' * (u' * u))$$

$$\longmapsto ((S'' \perp S') \perp S),\ (u'' * u') * u)$$

are canonically isomorphic.

Lemma: If $X \in \mathcal{X}$, let $e_X = (0,\ 0 \perp X \xrightarrow{\psi_X} X) \in \mathcal{H}(X,X)$. Then the functors

$$? \cdot e_X : \mathcal{H}(X,Y) \longrightarrow \mathcal{H}(X,Y),\quad (S,u) \longmapsto (S \perp 0,\ u * \psi_X)$$

$$e_X \cdot ? : \mathcal{H}(Z,X) \longrightarrow \mathcal{H}(Z,X),\quad (S,u) \longmapsto (0 \perp S,\ \psi_X * u)$$

are canonically isomorphic to the identity functors.

__Proof__: Verify that $(S''\perp S')\perp S \longrightarrow S''\perp(S'\perp S)$ constitutes a map from $((S''\perp S')\perp S,\ (u''*u')*u)$ to $(S''\perp(S'\perp S),\ u''*(u'*u))$ in $\mathcal{H}(X,Z)$; this uses the pent. axiom in the def. of action:

$$\begin{tikzcd}
((S''\perp S')\perp S)\# X \arrow[r, "\sim"] \arrow[d, "s"] \arrow[ddddrr, bend right=20, "(u''*u')*u"'] & (S''\perp(S'\perp S))\# X \arrow[r, "\sim"] \arrow[d, phantom, "\text{pentagon}"] & S''\#((S'\perp S)\# X) \arrow[d, "s"] \arrow[dddd, bend left=40, "u''*(u'*u)"] \\
(S''\perp S')\#(S\# X) \arrow[rr, "\sim"] \arrow[d, "id\# u"] & & S''\#(S'\#(S\# X)) \arrow[d] \\
(S''\perp S')\# Y \arrow[rr, "\sim"] & & S''\#(S'\# Y) \arrow[d] \\
 & & S''\# Z \arrow[d, "u''"] \\
 & & W
\end{tikzcd}$$

Verify that $S\perp 0 \xrightarrow{\varphi'_S} S$ constitutes a map from $(S\perp 0,\ u*\varphi_X)$ to (S,u) in $\mathcal{H}(X,Y)$

$$\begin{tikzcd}
(S\perp 0)\# X \arrow[r, "\sim"] \arrow[dr, "\varphi'_S\# id_X"', "\sim"] \arrow[ddr, bend left=60, "u*\varphi_X"] & S\#(0\# X) \arrow[d, "s"', "id\#\varphi_X"] \\
\text{triangle assumed comm.} & S\# X \arrow[d, "u"] \\
 & Y
\end{tikzcd}$$

Also verify that $0 \perp S \xrightarrow{\psi_S} S$ constitutes a map from $(0\perp S, \psi_X * u)$ to (S, u) in $\mathcal{H}(Z, X)$:

$$\begin{tikzcd}
(0\perp S)\# Z \arrow[r] \arrow[d, "\psi_S \# id_Z"'] \arrow[rrdd, bend left, "\psi_X * u"] & 0\#(S\# Z) \arrow[d, "id_0 \# u"] \arrow[dl, "\psi_{S\# Z}"'] \\
S\# Z \arrow[rdd, bend right] & 0\# X \arrow[d, "\psi_X"] \\
& X
\end{tikzcd}$$

$$\begin{tikzcd}
(0\perp S)\# Z \arrow[r, "\sim"] \arrow[rd, "\sim", "\psi_S \#\, id_Z"'] & 0\#(S\# Z) \arrow[r, "id_0 \# u"] \arrow[d, "\wr", "\psi_{S\# Z}"'] \arrow[dl, phantom, "\text{triangle comm. by axiom}"] & 0\# X \arrow[d, "\wr"', "\psi_X"] \arrow[dl, phantom, "\text{nat. of } \psi."] \\
& S\# Z \arrow[r, "u"'] & X
\end{tikzcd}$$

QED

Now set (assuming $\mathcal{S}$ has a set of iso. classes)

$$H(X,Y) = \pi_0 \mathcal{H}(X,Y)$$

and it follows from the lemmas that we ~~have~~ obtain a category with the same objects ~~the~~ as $\mathcal{X}$, but in which ~~a map from X to Y is an class~~ $H(X,Y)$ is the set of morphisms from X to Y.

Notation: $\langle \mathcal{S}, \mathcal{X} \rangle$

Review:

Def. $\mathcal{S}$ monoidal category

$$\perp : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S} \quad , \quad (S_1, S_2) \longmapsto S_1 \perp S_2$$

$$\varphi_{S_1, S_2, S_3} : (S_1 \perp S_2) \perp S_3 \xrightarrow{\sim} S_1 \perp (S_2 \perp S_3)$$

$$\psi_S : 0 \perp S \longrightarrow S$$

$$\psi'_S : S \perp 0 \longrightarrow S$$

Def: Action of $\mathcal{S}$ on $\mathcal{X}$

$$\# : \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X} \qquad (S, X) \longmapsto S \# X$$

$$\varphi_{S_1, S_2; X} : (S_1 \perp S_2) \# X \xrightarrow{\sim} S_1 \# (S_2 \# X)$$

$$\psi_X : 0 \# X \xrightarrow{\sim} X.$$

Def. If $\mathcal{S}$ acts on $\mathcal{X}$ we have $\forall X, X' \in \mathcal{X}$

$$\mathcal{H}(X, X') : \{(S, u : S \perp X \to X') + \text{isos.}\}$$

$$H(X, X') = \pi_0 \mathcal{H}(X, X').$$

$\langle \mathcal{S}, \mathcal{X} \rangle$ has same objects as X but $H(X, X')$ for maps

Def: Map $F : \mathcal{X} \longrightarrow \mathcal{X}'$ of categories with $\mathcal{S}$-action.

The induced functor $F : \langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X}' \rangle$

Def: Morphism $F: \mathcal{X} \to \mathcal{Y}$ of cats. with $\mathcal{S}$-action

Let $\mathcal{X}$ and $\mathcal{Y}$ be two categories on which $\mathcal{S}$ acts. By a morphism $F: \mathcal{X} \to \mathcal{Y}$ of categories with $\mathcal{S}$-action, we mean a functor $\mathcal{X} \to \mathcal{Y}$, $X \mapsto F(X)$, together with a natural isomorphism

$$\eta_{S,X}: \quad S \# F(X) \xrightarrow{\sim} F(S \# X)$$

of functors from $\mathcal{S} \times \mathcal{X}$ to $\mathcal{Y}$ such that η is compatible with the associativity + unity data:

$$\begin{tikzcd}
(S_1 \perp S_2) \# F(X) \arrow[rr, "\eta"] \arrow[d, "\varphi"] & & F((S_1 \perp S_2) \# X) \arrow[d, "F(\varphi)"] \\
S_1 \# (S_2 \# F(X)) \arrow[r, "\eta"] & S_1 \# F(S_2 \# X) \arrow[r, "\eta"] & F(S_1 \# (S_2 \# X))
\end{tikzcd}$$

$$\begin{tikzcd}
0 \# F(X) \arrow[rr, "\eta"] \arrow[dr, "\psi_{F(X)}"'] & & F(0 \# X) \arrow[dl, "F(\psi_X)"] \\
& F(X) &
\end{tikzcd}$$

~~Remark: A morphism $F: \mathcal{X} \to \mathcal{X}'$ of categories with $\mathcal{S}$ action induces a functor $\langle \mathcal{S}, \mathcal{X} \rangle \to \langle \mathcal{S}, \mathcal{X}' \rangle$ sending $X \mapsto F(X)$, $(S,u): X \to$~~

~~Such an F induces functors~~

~~$$\mathcal{H}(X,Y) \to \mathcal{H}(F(X),F(Y)), \quad (S,u) \mapsto (S, F(u)\eta_{S,X})$$~~

~~where ... is the composition~~

~~$$S \# F(X) \xrightarrow{\eta_{S,X}} F(S \# X) \xrightarrow{F(u)} F(Y)$$~~

If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a morphism of categories on which $\mathcal{S}$ operates, then it induces functors

$$(*) \qquad \mathcal{H}(X, X') \longrightarrow \mathcal{H}(FX, FX')$$

sending the couple $(S, u : S \# X \longrightarrow X')$ to $(S, F_* u)$ where $F_* u$ is the composite

$$S \# F(X) \simeq F(S \# X) \xrightarrow{F(u)} F(X').$$

From $(*)$ we obtain maps

$$H(x, x') \longrightarrow H(FX, FX')$$

which we will denote $\gamma \longmapsto F(\gamma)$.

Lemma: $X \longmapsto F(X)$, $\gamma \longmapsto F(\gamma)$ is a functor $\langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{Y} \rangle$.

Proof: $F(\gamma_1 \gamma_2) = F(\gamma_1) F(\gamma_2)$ will follow by showing

$$\begin{tikzcd}
\mathcal{H}(X, X') \times \mathcal{H}(X', X'') \arrow[r, "\cdot"] \arrow[d] & \mathcal{H}(X, X'') \arrow[d] \\
\mathcal{H}(FX, FX') \times \mathcal{H}(FX', FX'') \arrow[r, "\cdot"] & \mathcal{H}(FX, FX'')
\end{tikzcd}$$

commutes. Starting with $(S, u), (S', u')$

$$
\begin{tikzcd}
(S'\perp S)\# F(X) \arrow[r, "\sim"] \arrow[dd, "\wr"] \arrow[ddr, phantom, "\text{axiom}"] & S'\#(S\# F(X)) \arrow[d, "\wr"] \arrow[dr, "F*u"] & & \\
& S'\# F(S\# X) \arrow[r, "id_{S'}\# F(u)"] \arrow[d, "\wr"] \arrow[dr, phantom, "\text{nat of } \eta"] & S'\# F(X') \arrow[d, "\wr"] \arrow[dr, "F*u'"] & \\
F((S'\perp S)\# X) \arrow[r, "\sim"] & F(S'\#(S\# X)) \arrow[r, "F(id_{S'}\# u)"'] & F(S'\# X') \arrow[r, "F(u')"] & F(X'')
\end{tikzcd}
$$

top path is $F(S', u')\cdot F(S, u)$.

bottom path is $F((S', u)\cdot(S, u))$. So it's clear.

Why $F(id_X) = id_{F(X)}$: $id_X = \text{class}(0, 0\# X \xrightarrow{\psi_X} X)$

$$id_{F(X)} = cl\left(0,\ 0\# F(X) \xrightarrow{\psi_{FX}} F(X)\right)$$

$$F(id_X) = cl\left(0,\ 0\# F(X) \xrightarrow{\sim} F(0\# X) \xrightarrow{F(\psi_X)} FX\right)$$

and these are the same by axiom.

$\langle \mathcal{S}, \ \rangle$

~~Telescope~~ construction over a base $\mathcal{B}$:

Suppose $f : \mathcal{X} \longrightarrow \mathcal{B}$ ~~[illegible]~~ functor, and let $\mathcal{S}$ act on $\mathcal{X}$. We say that the action is <u>fibrewise</u> relative to $\mathcal{B}$ if

$$\begin{tikzcd} \mathcal{S} \times \mathcal{X} \arrow[rr, "\#"] \arrow[dr, "f\,pr_2"'] & & \mathcal{X} \arrow[dl, "f"] \\ & \mathcal{B} & \end{tikzcd}$$

commutes and if $\forall S, T \in \mathcal{S}$, $\forall X \in \mathcal{X}$ the maps

$$\varphi_{S,T,X} : (S \perp T) \# X \xrightarrow{\sim} S \# (T \# X)$$

$$\psi_X : \quad 0 \# X \xrightarrow{\sim} X$$

lie over $id_{f(X)}$. In this case ~~$\mathcal{B}$~~ $\mathcal{S}$ acts naturally on the fibres, and in fact on $\mathcal{B}' \times_{\mathcal{B}} \mathcal{X}$ for any $\mathcal{B}'/\mathcal{B}$.

Suppose now that f is fibred. We say the $\mathcal{S}$-action is <u>cartesian</u> relative to $\mathcal{B}$ if it is fibre-wise and if $\forall S \in \mathcal{S}$, the functor $S \# ? : \mathcal{X} \longrightarrow \mathcal{X}$ is a cartesian functor of categories over $\mathcal{B}$. I claim that in this case for any map $u : B' \longrightarrow B$ in $\mathcal{B}$ the base change functor

$$u^* : \mathcal{X}_B \longrightarrow \mathcal{X}_{B'}$$

can be interpreted as a morphism of categories with $\mathcal{S}$-action. To see this let

$$\alpha_X : u^* X \longrightarrow X$$

denote ~~be~~ the ~~[illegible]~~ canonical cartesian map over u with target X. ~~Applying~~ Then by hypothesis

$$S \# u^*X \xrightarrow{S\#\alpha_X} S\#X$$

is cartesian, hence it factors uniquely

$$\begin{tikzcd}
S\# u^*X \arrow[rr, "\eta_{S,X}"] \arrow[dr, "S\#\alpha_X"'] & & u^*(S\#X) \arrow[dl, "\alpha_{S\#X}"] \\
& S\#X &
\end{tikzcd}$$

where $\eta_{S,X}$ is an isomorphism. To check:

$$\begin{tikzcd}
(S_1\perp S_2)\# u^*(X) \arrow[rrrr, "\eta_{S_1\perp S_2, X}"] \arrow[drr, "(S_1\perp S_2)\#\alpha_X"] \arrow[dd, "\varphi"'] & & & & u^*((S_1\perp S_2)\#X) \arrow[dll, "\alpha_{(S_1\perp S_2)\#X}"'] \arrow[dd, "\varphi"] \\
& & (S_1\perp S_2)\#X \arrow[dd, bend right, "\varphi"'] & & \\
S_1\#(S_2\# u^*(X)) \arrow[rr, "S_1\#\eta_{S_2,X}"] \arrow[drr, Rightarrow, "S_1\#(S_2\#\alpha_X)"'] & & S_1\# u^*(S_2\#X) \arrow[rr, "\eta_{S_1,S_2\#X}"] \arrow[d, "S_1\#\alpha_{S_2\#X}"] & & u^*(S_1\#(S_2\#X)) \arrow[dll, "\alpha_{S_1\#(S_2\#X)}"] \\
& & S_1\#(S_2\#X) & &
\end{tikzcd}$$

This diagram proves associativity, as $\alpha_{S_1\#(S_2\#X)}$ is cartesian

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$$\begin{tikzcd}
& u^*X & \\
0\# u^*X \arrow[ur, "\psi_{u^*X}"] \arrow[rr, "\eta_{0,X}"] \arrow[dr, "0\#\alpha_X"'] & & u^*(0\#X) \arrow[ul, "u^*(\psi_X)"'] \arrow[dl, "\alpha_{0\#X}"] \\
& 0\#X &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \# u^*X \arrow[rr, "\eta_{0,X}"] \arrow[dr, "0\#\alpha_X"'] \arrow[rrrr, bend left, "\psi_{u^*X}"] & & u^*(0\# X) \arrow[rr, "u^*(\psi_X)"] \arrow[dl, "\alpha_{0\#X}"] & & u^*(X) \arrow[dl, "\alpha_X"] \\
& 0\# X \arrow[rr, "\psi_X"'] & & X &
\end{tikzcd}$$

shows that η is compatible with the unit data.

Have forgotten to show η is functorial:

$$\begin{tikzcd}
S \# u^*X \arrow[r, "\eta_{S,X}"'] \arrow[rr, bend left, "S\#\alpha_X"] \arrow[d, "\gamma_1 \# u^*(\gamma_2)"] & u^*(S\# X) \arrow[r, "\alpha_{S\#X}"'] \arrow[d, "u^*(\gamma_1\#\gamma_2)"] & S\# X \arrow[d, "\gamma_1\#\gamma_2"] \\
S' \# u^*X' \arrow[r, "\eta_{S',X'}"'] \arrow[rr, bend left, "S'\#\alpha_{X'}"] & u^*(S'\# X') \arrow[r, "\alpha_{S'\#X'}"'] & S'\# X'
\end{tikzcd}$$

Thus if the $\mathcal{S}$- action is cartesian, then $\forall u: B' \to B$

$$u^*: \mathcal{X}_B \to \mathcal{X}_{B'}$$

is compatible with the action. Need only that $\mathcal{X} \to B$ is pre-fibred for this.

η

Lemma: Let $f: \mathcal{X} \to \mathcal{B}$ be prefibred and suppose the action of $\mathcal{S}$ on $\mathcal{X}$ is cartesian relative to f. Then

$$\langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \mathcal{B}$$

is a prefibred category over $\mathcal{B}$ with fibre over B isom. to $\langle \mathcal{S}, f^{-1}(B) \rangle$, and with base change functor over $u: B' \to B$ isomorphic to the functor

$$\langle \mathcal{S}, f^{-1}(B) \rangle \longrightarrow \langle \mathcal{S}, f^{-1}(B') \rangle$$

induced by $u^*: f^{-1}(B) \longrightarrow f^{-1}(B')$ (with its natural structure of morphism of categories with $\mathcal{S}$-action. Moreover if $\mathcal{X}$ is a fibred category over $\mathcal{B}$, then so is $\langle \mathcal{S}, \mathcal{X} \rangle$.

Proposition: Let $\mathcal{S}$ act fibrewise on a category $\mathcal{X}$ over $\mathcal{B}$. If $\mathcal{X}$ is prefibred (resp. fibred) over $\mathcal{B}$, and if the action is cartesian, then $\langle \mathcal{S}, \mathcal{X} \rangle$ is a prefibred (resp. fibred) category over $\mathcal{B}$. In this case the fibre over B is naturally isomorphic to $\langle \mathcal{S}, \mathcal{X}_B \rangle$ and the base change functor over $: B' \to B$ is isom to the functor

$$\langle \mathcal{S}, \mathcal{X}_B \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X}_{B'} \rangle$$

induced by $^*: \mathcal{X}_B \to \mathcal{X}_{B'}$ with its natural structure of morphism of categories with $\mathcal{S}$-action.

<u>Proof.</u> Let $f : \mathcal{X} \to \mathcal{B}$ be the structural map. If $(S, u : S\#X \to X')$ is an object of $\mathcal{H}(X, X')$, then it determines the map $f(u) : fX = f(S\#X) \longrightarrow fX'$ which depends only on the iso class of (S, u), since by hyp. any map $S \to S'$ induces a map $S\#X \to S'\#X$ lying over id_{fX}. Thus we have a well-defined map

$$H(X, X') \longrightarrow Hom(fX, fX')$$
$$cl(S, u) \longmapsto f(u).$$

It is clear that this map is compatible with composition, hence we obtain a functor

$$\bar{f} : \langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \mathcal{B}$$

sending X to fX and $cl(S,u)$ to $f(u)$.

Let $z : B' \to B$ be a map in $\mathcal{B}$, $X \in \mathcal{X}_B$, $X' \in \mathcal{X}_{B'}$ and let $\mathcal{H}(X', X)_z$ be the full subcat of $(S, u : S\#X \to X')$ in $\mathcal{H}(X', X)$ s.t. $f(u) = z$. Then we have a functor

$$(\ ) \qquad \mathcal{H}(X', z^*X)_{id_{B'}} \longrightarrow \mathcal{H}(X', X)_z$$

$$(S, S\#X' \xrightarrow{u} z^*X) \longmapsto (S, S\#X' \xrightarrow{\alpha_z u} X)$$

where $\alpha_z : z^*X \to X$ is the canon. cart. arrow. The functor ( ) is an isomorphism of categories since

$$Hom(S\#X', z^*X)_{id_{B'}} \xrightarrow{\sim} Hom(S\#X', X)_z, \quad u \mapsto \alpha_z u$$

(def. of f being prefibred.) Taking iso. classes in ( )

we get

$$H(X', z^*X)_{id_{B'}} \simeq H(X', X)_z$$

which show that $\langle \mathcal{S}, \mathcal{X} \rangle$ is prefibred over $\mathcal{B}$ with base change functor $X \mapsto z^*X$ assoc. to z.

Somewhere have to ~~show~~ say that $H(X', X)_{id_B}$ is exactly the set of maps from X' to X in $\langle \mathcal{S}, \mathcal{X}_B \rangle$.

So we have the base change functor ~~is~~ for $\langle \mathcal{S}, \mathcal{X} \rangle$ if $\mathcal{X}$ is fibred is $X \mapsto z^*X$ with $\alpha_X : z^*X \to X$. Now, given $B'' \xrightarrow{w} B' \xrightarrow{z} B$ ~~it follows that~~ we have $(zw)^* \Leftarrow w^*z^*$, so it is clear that $\langle \mathcal{S}, \mathcal{X} \rangle$ is fibred over $\mathcal{B}$.

<u>Remark</u>: Preceding prop. holds if we replace fibred by cofibred.

Translation cat $\langle \mathcal{S}, \mathcal{S} \rangle = \mathcal{S}^{-1}(pt)$
telescope cat $\langle \mathcal{S}, \mathcal{S} \times X \rangle = \mathcal{S}^{-1} X$

Translation cat of $\mathcal{S}$: Let $\mathcal{S}$ act on itself in the natural way: $S \# S' = S \perp S'$. Then $\langle \mathcal{S}, \mathcal{S} \rangle$ is the trans. cat.

Lemma: If $\mathcal{S}$ is a groupoid, $\langle \mathcal{S}, \mathcal{S} \rangle$ has 0 for an initial object.

Proof. $\mathcal{H}(0, S)$: $\{(T, T \perp 0 \to S)$ & isos.$\}$ is ~~equivalent~~ isom. to $\{(T, T \to S)$ & isos.$\}$. So when $\mathcal{S}$ is a groupoid, $\mathcal{H}(0, S)$ is isom. to $\{(T, T \simeq S)\}$ which is equivalent to pt. $\Rightarrow$ $H(0, S) = pt$ for all S.

Examples: 1. $\mathcal{S}$ = finite sets & autos. Then $\langle \mathcal{S}, \mathcal{S} \rangle$ is the category of finite sets and injections

2. $\mathcal{S}$ = f.g. proj. A-modules & their autos. Then $\langle \mathcal{S}, \mathcal{S} \rangle$ = the cat of f.g. proj. A-modules in which a map from P to Q is a pair of A-module homos $P \underset{i}{\overset{p}{\leftrightarrows}} Q \ni pi = id$.

__Lemma:__ If every arrow in $\mathcal{X}$ is a monomorphism, and if $\forall X$, $S \longmapsto S \perp X$ from $\mathcal{S}$ to $\mathcal{X}$ is faithful, then $\forall X, X'$, $\mathcal{H}(X, X')$ is equivalent to the discrete cat defined by the set $H(X, X')$.

__Proof:__ suffices to show that two maps $(S, S \perp X \xrightarrow{z} X') \rightrightarrows (S', S' \perp X \xrightarrow{z'} X')$ have to coincide. Let these maps be given by the isos $u, v : S \xrightarrow{\sim} S'$. Then

$$\begin{tikzcd}
S \perp X \arrow[d, shift right, "u \perp X"'] \arrow[d, shift left, "v \perp X"] \arrow[dr, "z"] & \\
S' \perp X \arrow[r, "z'"'] & X'
\end{tikzcd}$$

z' mono $\Longrightarrow$ $u \perp X = v \perp X$ $\Longrightarrow$ $u = v$.

[Assumptions on $\mathcal{S}$: 1) $\mathcal{S}$ is a groupoid
2) $?\perp S : \mathcal{S} \to \mathcal{S}$ is faithful $\forall S$.
2)': $\mathcal{H}(S,S')$ is equivalent to $H(S,S')$, $\forall S, S'$.

Assuming 1), 2) $\Leftrightarrow$ 2'): For $\mathcal{H}(S,S')$ to be $\sim H(S,S')$ means $\exists$ at most one map $T \to T'$ compatible with given $T\perp S \simeq S' \simeq T'\perp S$. $\Leftrightarrow$ $?\perp S$ faithful.

Given $g \in H(S,S')$, if g is rep. by $\alpha : T\perp S \xrightarrow{\sim} S'$, then T is determined up to <u>unique</u> isomorphism. Call T the "cokernel" of g and denote it T_g.

Now let $\mathcal{X}$ be a cat with $\mathcal{S}$-action, and let $\mathcal{S}$ act diag on $\mathcal{S}\times\mathcal{X}$: $T\#(S,X) = (T\perp S, T\# X)$. Then $pr_1 : \mathcal{S}\times\mathcal{X} \longrightarrow \mathcal{S}$ is compatible with $\mathcal{S}$-action in an obvious way, so it induces a functor

$$(1) \quad \pi : \langle \mathcal{S}, \mathcal{S}\times\mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{S} \rangle.$$

rep. by $u : T\perp S \xrightarrow{\sim} S'$.

<u>Lemma</u>: Let (S,X), (S',X') be objects of $\langle \mathcal{S}, \mathcal{S}\times\mathcal{X}\rangle$ and let $g \in H(S,S')$ be a morphism between their images in $\langle \mathcal{S},\mathcal{S}\rangle$. Let $H((S,X),(S',X'))_g$ denote the ~~set of maps~~ ~~subset of $H((S,X),(S',X'))$~~ set of maps $f : (S,X) \longrightarrow (S',X')$ in $\langle \mathcal{S}, \mathcal{S}\times\mathcal{X}\rangle$ lying over g wrt the functor (1). Then there is a bijection

$$\mathrm{Hom}_{\mathcal{X}}(T\# X, X') \xrightarrow{\sim} H((S,X),(S',X'))_g$$

$$v \longmapsto (T,\ u : T\perp S \to S',\ v : T\# X \to X')$$

Proof. A map $f : (S, X) \longrightarrow (S', X)$ is rep by a triple $(T_0,\ u_0 : T_0 \perp S \to S',\ v : T_0 \# X \longrightarrow X')$ such that (T_0, u_0) is isom. to (T, u). This isom. is given by a unique isom of T_0 and T. Thus f has a unique rep. of the form $(T,\ u,\ v : T \# X \longrightarrow X')$, which proves the lemma.

Prop. ① Given $S \in \langle \mathcal{S}, \mathcal{S} \rangle$ define

$$e_S : \mathcal{X} \longrightarrow \langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle$$

by [illegible]

$$e_S(X) = (S, X)$$

$$e_S(v : X \to X') = d\left(0,\ 0 \perp S \xrightarrow{\ell_S} S,\ 0 \# X \xrightarrow{\ell_X} X \xrightarrow{v} X'\right)$$

Then e_S is an isomorphism ~~[illegible]~~ of $\mathcal{X}$ with the fibre category $\pi^{-1}(S)$.

② The functor π is cofibred. If $g : S \to S'$ is the map is $\langle \mathcal{S}, \mathcal{S} \rangle$ represented by $(T,\ u : T \perp S \to S')$, ~~[illegible]~~ then the square of categories

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "e_S"] \arrow[d, "T \# ?"'] & \pi^{-1}(S) \arrow[d, "g_*"] \\
\mathcal{X} \arrow[r, "e_{S'}"] & \pi^{-1}(S')
\end{tikzcd}$$

where g_* is the cobase change functor, commutes up to a canonical isomorphism of functors.

~~Proof. Immediate consequence of the lemmas and definitions. ① id_S is rep by $(0, 0 \perp S \xrightarrow{\psi_S} S)$, and by lemma the map $f: (S,X) \to (S,X')$ over id_S same as maps $0 \# X \to X'$, which are same as maps $X \to X'$. (You can say~~

② The functor π is cofibred. If $g \in H(S,S')$ is represented by $(T, u: T \perp S \xrightarrow{\sim} S')$, then the base-change functor g_* may be taken to be

$$\pi^{-1}(S) \xrightarrow{e_S^{-1}} \mathcal{X} \xrightarrow{T\#?} \mathcal{X} \xrightarrow{e_{S'}} \pi^{-1}(S').$$

Proof. ① ~~$id_S \in H(S,S)$ is represented by~~ Clearly e_S is bijective from objects of $\mathcal{X}$ to objects of $\pi^{-1}(S)$. Now $id_S \in H(S,S)$ is represented by $(0, 0 \perp S \xrightarrow{\psi_S} S)$, so by the lemma there is a bijection between maps $(S,X) \to (S,X')$ over id_S and maps $0 \# X \to X'$ in $\mathcal{X}$. Since $\psi_X: 0 \# X \simeq X$, it is clear e_S is an isomorphism as claimed.

② Since $\mathrm{Hom}_{\mathcal{X}}(T \# X, X) = \mathrm{Hom}_{\pi^{-1}(S')}((S', T\#X), (S', X'))$ by ①, it is clear from the lemma that π is pre-cofibred with $g_* = e_{S'}(T\#?)\,e_S^{-1}$. The fact that it is fibred comes from the assoc. isom

$$T' \# (T \# X) \cong (T' \perp T) \# X.$$

2nd section

Suppose $\mathcal{S}$ acts on $\mathcal{X}$ fibrewise rel. to $f : \mathcal{X} \to \mathcal{B}$. Then f induces maps

$$(*) \qquad \mathrm{Hom}_{\langle \mathcal{S}, \mathcal{X} \rangle}(X, X') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(fX, fX')$$

$$cl(S, u : SX \to X') \longmapsto (f(u) : fX \to fX').$$

Hence f induces a functor

$$\bar{f} : \langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \mathcal{B} \quad , \quad X \longmapsto fX$$

whose effect on morphism is the map $(*)$. Clearly for any B in $\mathcal{B}$ we have

$$\bar{f}^{-1}(B) = \langle \mathcal{S}, f^{-1}(B) \rangle.$$

Suppose now that f is prefibred, let $u : B \to B'$ be a map in $\mathcal{B}$, let $u^* : f^{-1}(B') \to f^{-1}(B)$ be the associated base change functor, let X and X' be objects in the fibres over B and B' respectively. Then denoting by a subscript u the subset of morphisms lying over u, we have

$$\mathrm{Hom}_{\langle \mathcal{S}, \mathcal{X} \rangle}(X, X')_u = \varinjlim_{S} \mathrm{Hom}_{\mathcal{X}}(SX, X')_u$$

$$\xleftarrow{\sim} \varinjlim_{S} \mathrm{Hom}_{f^{-1}(B)}(SX, u^*X')$$

$$= \mathrm{Hom}_{\langle \mathcal{S}, f^{-1}(B) \rangle}(X, u^*X')$$

this isomorphism being induced by image in $\langle \mathcal{S}, \mathcal{X} \rangle$ of the canonical morphism $u^*X' \longrightarrow X'$ in $\mathcal{X}$.

2nd section:

Def: Fibrewise $\mathcal{S}$-action on $\mathcal{X}$ relative to $f: \mathcal{X} \to \mathcal{B}$

$\bar{f}: \langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \mathcal{B}$

fact: $\bar{f}^{-1}(B) \equiv \langle \mathcal{S}, f^{-1}(B) \rangle$

Suppose f prefibred. We will say the $\mathcal{S}$-action on $\mathcal{X}$ is cartesian rel. to f if it is fibrewise and if $X \mapsto SX$ is a cart functor rel. to f for every S in $\mathcal{S}$. In this case let $u: B' \to B$ be a $\mathcal{B}$-map, ~~let $X \in f^{-1}(B)$, $S \in \mathcal{S}$, and let~~ let $X \in f^{-1}(B)$, and let $c_{u,X}: u^*X \longrightarrow X$ denote the canon cart arrow lying over u. If $S \in \mathcal{S}$, then because the action is cartesian, there is a unique isom

(*) $$S(u^*X) \xrightarrow{\sim} u^*(SX)$$

in $f^{-1}(B')$ such that

$$\begin{tikzcd} S(u^*X) \arrow[d] \arrow[dr, "S(c_{u,X})"] & \\ u^*(SX) \arrow[r, "c_{u,SX}"'] & SX \end{tikzcd}$$

commutes.

Prop: Assume f prefibred and that the $\mathcal{S}$-action is cartesian relative to f. Then

a) $\forall u: B' \to B$, the b.c. functor $u^*: f^{-1}(B) \longrightarrow f^{-1}(B')$ together with the isoms (*) is an action-preserving functor.

~~b) The functor $\bar{f}$ is prefibred and the base change functor $u^*: \bar{f}^{-1}(B) \to \bar{f}^{-1}(B')$ is the~~

b) The functor $\bar{f}$ is prefibred. If we identify $\bar{f}^{-1}(B)$ with $\langle S, f^{-1}(B)\rangle$ as above, then the base change functor $\langle S, f^{-1}(B)\rangle \to \langle S, f^{-1}(B')\rangle$ assoc. to $u: B' \to B$ is the functor induced by the action-preserving functor u^*: $f^{-1}(B) \to f^{-1}(B')$ described in a)

c) If f is fibred, then so is $\bar{f}$.

Proof: Part a) involves checking certain diagrams commute and will be left to the reader.

Part b): Let $X \in f^{-1}(B)$, $X' \in f^{-1}(B')$. Denoting by a subscript u the subset of morphisms lying over u, then we have

$$\mathrm{Hom}_{\langle S, X\rangle}(X', X)_u = \varinjlim_{S} \mathrm{Hom}_{\mathcal{X}}(SX', X)_u$$

$$\xleftarrow{\sim} \varinjlim_{S} \mathrm{Hom}_{f^{-1}(B')}(SX', u^*X)$$

$$= \mathrm{Hom}_{\langle S, f^{-1}(B')\rangle}(X', \underset{\text{in } \langle S, X\rangle}{u^*X})$$

where this isom. is induced by the image of the canon. map $c_u : u^*X \to X$. Part b) results directly.

Part c): f fibred $\Rightarrow$ transitivity, $u^*v^* \xrightarrow{\sim} (vu)^*$ for the base change functors assoc. to f $\Rightarrow$ transitivity for ______ to $\bar{f}$ $\Rightarrow$ $\bar{f}$ fibred.

Dually if f is ~~[illegible]~~ pre-cofibred, we say the $\mathcal{S}$-action is cocart. if $X \mapsto SX$ is a cocart functor relative to f, $\forall S \in \mathcal{S}$! In this case there are canon isos

$$u_*(SX) \xrightarrow{\sim} S(u_* X)$$

and we have

<u>Prop.</u> If f is precofibred and the $\mathcal{S}$-action is cocart relative to f, then

a) $\forall u : B \to B'$ in $\mathcal{B}$, the cobase change functor $u_* : f^{-1}(B) \to f^{-1}(B')$ together with $(**)$ is an action-preserving functor

b) $\bar{f}$ is precofibred and the cobase-change functor $\langle \mathcal{S}, f^{-1}(B) \rangle \to \langle \mathcal{S}, f^{-1}(B') \rangle$ ~~is the~~ assoc to $u : B \to B'$ is the functor induced by the action-preserving functor u_* described in a).

c) f cofibred $\Rightarrow$ $\bar{f}$ cofibred.

Proof analogous to preceding one.

<u>Corollary:</u> Let $\mathcal{S}$ act on a category $\mathcal{F}$, and trivially on the category $\mathcal{B}$. Then there is an isomorphism

$$\langle \mathcal{S}, \mathcal{B} \times \mathcal{F} \rangle = \mathcal{B} \times \langle \mathcal{S}, \mathcal{F} \rangle$$

<u>Proof:</u> Apply either one of the preceding props to $f = pr_1 : \mathcal{B} \times \mathcal{F} \longrightarrow \mathcal{B}$.

Cor. Hyp same as preceding, the functor $S^{-1}\mathcal{X} \longrightarrow \mathcal{B}$ is ~~pre~~fibred (resp. fibred) if f is with fibre $S^{-1}(f^{-1}(B))$ over B and basechange induces by u^*.

Proof: Suffices to apply preceding to functor $fpr_1 : S \times \mathcal{X} \longrightarrow \mathcal{B}$ ~~[illegible]~~ which ~~[illegible]~~ is fibred if f is and ~~[illegible]~~ s.t. the action is cartesian if it is so for f.

Cofibred variant.

List of things used

$+ : \mathcal{S}\times\mathcal{S} \longrightarrow \mathcal{S}$ product-preserving

---

$(\mathcal{S}\times\mathcal{T})^{-1}X = \mathcal{T}^{-1}(\mathcal{S}^{-1}X)$

---

canon. map $X \longrightarrow \mathcal{T}^{-1}(X)$ of $\mathcal{T}$ action cats.
and fact it is an h.eq. $\Leftrightarrow$ $\mathcal{T}$ acts inv.

---

notation $q_{\mathbb{Z}} : T_{\mathbb{Z}} + S \xrightarrow{\sim} S$ for

---

$\mathcal{S}^{-1}X \xrightarrow{q} \mathcal{S}^{-1}(pt)$ cofibred with
$q^{-1}(S) = X$, $u^* = T_u \# ? : X \to X$

---

morphism-inverting functors + homotopy invariance of homology

---

$\mathcal{S}^{-1}(pt)$ contractible.

---

spectral sequence for $f : C \to C'$ pre-cofibred

Homology of $S^{-1}\mathcal{X}$

Have can functor

$$\mathcal{X} \longrightarrow S^{-1}\mathcal{X}$$

of cats with S-action, ~~such that~~ hence a map of $(\pi_0 S)$-modules

$$H_*(\mathcal{X}) \longrightarrow H_*(S^{-1}\mathcal{X})$$

since S acts invertibly on $S^{-1}\mathcal{X}$, ~~$\pi_0 S$~~ the monoid $\pi_0 S$ acts invertibly on $H_*(S^{-1}\mathcal{X})$, hence the preceding induces a map

$$(*) \qquad (\pi_0 S)^{-1} H_*(\mathcal{X}) \longrightarrow H_*(S^{-1}\mathcal{X}).$$

<u>Prop</u>: $(*)$ is an isomorphism.

<u>Proof</u>: If M is a $(\pi_0 S)$-module, denote by $\bar{M}$ the functor from $S^{-1}(pt) = \langle S, S \rangle$ to $\mathcal{A}b$ ~~sending each object to M and the~~

$$\bar{M}(S) = M$$

$$\bar{M}(u) = \text{mult. by } cl(T_u)$$

such that $\bar{M}(S) = M$ for every S and such that if u ~~[illegible]~~ is ~~a~~ represented by $T_u + S \xrightarrow{\sim} S'$, then $\bar{M}(u) =$ mult by $cl(T_u) \in \pi_0 S$. ~~If $(\pi_0 S)$ acts invertib~~

Note that if $(\pi_0 S)$ acts invertibly on M, then $\bar{M}$ is a ~~[illegible]~~ morphism-inverting functor (ref), hence

$$(**) \qquad H_n(S^{-1}(pt), \bar{M}) = \begin{cases} M & n = 0 \\ 0 & n > 0 \end{cases}$$

since the homology is a homotopy-invariant (ref) and $S^{-1}(pt)$ is contrac. (ref)

Now $\mathcal{S}^{-1}\mathcal{X}$ is cofibred over $\mathcal{S}^{-1}(pt)$, with fibre over any S = $\mathcal{X}$, and $u^* =$ ~~[illegible]~~ $T_u\#? : \mathcal{X} \to \mathcal{X}$. (ref). Hence have spec seq. (ref.)

$$E^2_{pq} = H_p\left(\mathcal{S}^{-1}(pt), \overline{H_q(\mathcal{X})}\right) \Longrightarrow H_n(\mathcal{S}^{-1}\mathcal{X}).$$

~~and moreover this is a spectral sequence of $(\pi_0\mathcal{S})$~~
Since $\mathcal{S}$ acts on $\mathcal{S}^{-1}\mathcal{X}$ over $\mathcal{S}^{-1}(pt)$, it is a sp. sequence of $(\pi_0\mathcal{S})$-modules. Since localization is exact, we can localize to obtain a spec. seq.

$$E^2_{pq} = H_p\left(\mathcal{S}^{-1}(pt), \overline{(\pi_0\mathcal{S})^{-1}H_q(\mathcal{X})}\right) \Longrightarrow (\pi_0\mathcal{S})^{-1}H_n(\mathcal{S}^{-1}\mathcal{X}) = H_n\mathcal{X}.$$

~~By (**), this spec. seq. is trivial collapses, and from this the proposition follows.~~
which collapses by (**). From this the proposition follows easily.

Cofinality:

Let $f: S \to T$ be a prod.-pres. functors between groupoids with products, and suppose T acts on X. ~~Equipping X with the S-action induce~~ Then there is an induced S-action on X and we have a functor

$$S^{-1}X \longrightarrow T^{-1}X$$

which we will denote $\tilde{f}$.

Prop: If f is cofinal, then $\tilde{f}$ is a heq.

Proof: We let $S \times T$ and $T \times T$ act on X ~~thru~~ the product-preserving functors

$$S \times T \xrightarrow{f \times id} T \times T \xrightarrow{+} T$$

whence we get a comm. diag

$$\begin{tikzcd}
S^{-1}X \arrow[r, "(id_S, 0)^\sim"] \arrow[d, "\tilde{f}"'] & (S \times T)^{-1}X \arrow[d, "(f \times id)^\sim"] & T^{-1}X \arrow[l, "(0, id_T)^\sim"'] \arrow[d, "id"] \\
T^{-1}X \arrow[r, "(id_T, 0)^\sim"] & (T \times T)^{-1}X & T^{-1}X \arrow[l, "(0, id_T)^\sim"']
\end{tikzcd}$$

It suffices to show the horizontal arrows are all heq's.

If we make the identification (ref)

$$(S \times T)^{-1}X = T^{-1}(S^{-1}X)$$

~~$0 \to L \to V \to L' \to 0$~~

~~BK^* but in BGL^+ it comes with a basepoint.~~

~~$GL_n \to GL$.~~

---

the functor $(id_{\mathcal{S}}, 0)^\sim$ ~~in the upper top row~~ becomes identified with the canon inclusion (ref.)

$$\mathcal{S}^{-1}\mathcal{X} \subset \mathcal{T}^{-1}(\mathcal{S}^{-1}\mathcal{X}).$$

Since $\mathcal{S}$ acts invertibly on $\mathcal{S}^{-1}\mathcal{X}$ (ref.), and f is cofinal, it is clear that $\mathcal{T}$ acts invertibly on $\mathcal{S}^{-1}\mathcal{X}$. Thus this inclusion is a heq (ref.), so $(id_{\mathcal{S}}, 0)^\sim$ is a heq. ~~and~~ Similar arg. shows the other horizontal arrows are heqs.

---

first section: of S^{-1} construction

monoidal cat

examples: S, $\mathrm{Hom}(\mathcal{X},\mathcal{X})$, monoids

morphism of monoidal cats.

(left) action of a monoidal cat. on a cat.

~~[illegible]~~ example: S acts on itself

~~action preserving functor (?)~~

Definition of $\langle S, \mathcal{X}\rangle$.

examples: monoid acting on a set

S acting on pt

Prop. 1

~~[illegible]~~ Assume S is a groupoid, and every arrow in $\mathcal{X}$ is a mono, $\forall X$, the functor $S \mapsto SX$ from S to $\mathcal{X}$ is faithful. Then any arrow $x: X \to X'$ in $\langle S, \mathcal{X}\rangle$ is represented by a pair $(S_x, a_x: S_x X \to X')$ which is determined up to unique isomorphism.

Proof: The morphism x is by definition a component of the fibred cat over S consists of pairs $(S, a: SX \to X')$. $\mathrm{Hom}_{\langle S,\mathcal{X}\rangle}(X, X')$ The hypothesis implies $\mathrm{Hom}_{\langle S,\mathcal{X}\rangle}(X,X')$ is a groupoid with at most one arrow between any two objects, whence the lemma.

Example: 1) S = finite sets + autos with $+$ = disj. union.

a map $S \to S'$ in $\langle S, S\rangle$.

2) S = ~~Iso~~ $P(A)$.

Prop. 2: S groupoid $\Rightarrow \langle S, S\rangle$ has initial object O.

~~(Second section (X functorial properties with $\mathcal{S}$ fixed)~~

action-preserving functor $\mathcal{X} \longrightarrow \mathcal{Y}$.

induced map $\langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{Y} \rangle$

Define $\mathcal{S}^{-1}\mathcal{X} = \langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle$ and the functor
$p: \mathcal{S}^{-1}\mathcal{X} \longrightarrow \mathcal{S}^{-1}(pt) = \langle \mathcal{S}, \mathcal{S} \rangle$.

Prop 3. Assume $\mathcal{S}$ is a groupoid and $\forall S$ that $T \mapsto TS$ from $\mathcal{S}$ to $\mathcal{S}$ is faithful. Then $p: \mathcal{S}^{-1}\mathcal{X} \to \langle \mathcal{S}, \mathcal{S} \rangle$ is cofibred with $p^{-1}(S) = \mathcal{X}$ and with $z_*^{\#}: p^{-1}(S) \to p(S')$ equal to the action of S_z on $\mathcal{X}$.

Cor. $\mathcal{S}$ as in preceding prop. Then $i_S: \mathcal{X} \longrightarrow \mathcal{S}^{-1}\mathcal{X}$ is a heq. ($\forall S$) iff $\forall S$, $S\#: \mathcal{X} \longrightarrow \mathcal{X}$ is a heq.

First part of $\mathcal{S}^{-1}$:

$\mathcal{S}$ monodial cat. $+: \mathcal{S}\times\mathcal{S} \longrightarrow \mathcal{S}$ denoted $(S,T) \longmapsto S+T$ or simply ST, together with associativity data and unity data

morphism of monoidal cats. $\mathcal{S} \longrightarrow \mathcal{T}$ consists of a functor $F: \mathcal{S}\longrightarrow\mathcal{T}$ and

$$F(ST) \xrightarrow{\sim} F(S)F(T)$$
$$F(0) \xrightarrow{\sim} 0$$

compatible with assoc. + unity isos.

Action of $\mathcal{S}$ on $\mathcal{X}$ consists of $\#: \mathcal{S}\times\mathcal{X}\longrightarrow\mathcal{X}$ $(S,X)\longmapsto S\#X$, or simply SX, together with assoc. + unity constraints. Same as a morphism

$$\mathcal{S}\longrightarrow \underline{\mathrm{Hom}}(\mathcal{X},\mathcal{X})$$

of monoidal categories. Example: $\mathcal{S}$ acts on itself.

Action-preserving functor: $\mathcal{X}\longrightarrow\mathcal{X}'$ consists of $F:\mathcal{X}\longrightarrow\mathcal{X}'$ together with

$$SF(X) \xrightarrow{\sim} F(SX)$$

compatible with the associativity and unity constraints.

Let $\mathcal{S}$ act on $\mathcal{X}$. We define a new cat $\langle\mathcal{S},\mathcal{X}\rangle$ having the same objects as $\mathcal{X}$ as follows. Given $X,X' \in \mathrm{Ob}\,\mathcal{X}$, let

$$\mathrm{Hom}_{\langle\mathcal{S},\mathcal{X}\rangle}(X,X') = \varinjlim_{S} \mathrm{Hom}_{\mathcal{X}}(SX,X')$$

where the limit is taken category $\mathcal{S}^{\circ}$. Thus an

$\langle \mathcal{S}, \mathcal{X} \rangle$-morphism $X \to X'$ is an equivalence class of pairs (S, u) where $u : SX \to X'$ is a map in $\mathcal{X}$; if we make these pairs into a fibred cat over $\mathcal{S}$ in the evident way, then two pairs are equivalent iff they are in the same component. Define composition, identity maps, and check it works.

An action-preserving functor $F : \mathcal{X} \to \mathcal{X}'$ induces a functor

$$\langle \mathcal{S}, \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X}' \rangle$$

sending $X \mapsto FX$,

$$\mathrm{cl}\,(S, SX \xrightarrow{u} X') \longmapsto (S, SF(X) \simeq F(SX) \xrightarrow{F(u)} FX')$$

<u>Example:</u> 1) $\mathcal{S}$ acting on a point. Then $\langle \mathcal{S}, pt \rangle$ is the monoid $\pi_0 \mathcal{S}$.

2) $\mathcal{S}$ = finite sets and autos. with $+ = \amalg$ then $\langle \mathcal{S}, \mathcal{S} \rangle$ = finite sets and injections.

3) $\mathcal{S} = \mathrm{Iso}(\mathcal{P})$ where $\mathcal{P}$ is an additive category and $+ = \oplus$. Then $\langle \mathcal{S}, \mathcal{S} \rangle$ = objects of $\mathcal{P}$ + complemented injections.

situation:

$\mathcal{S}$ monoid cat acting on $\mathcal{X}$.
Define $\langle \mathcal{S}, \mathcal{X} \rangle$ same objects as $\mathcal{X}$ but

$$\mathrm{Hom}_{\langle \mathcal{S}, \mathcal{X} \rangle}(X, X') = \varinjlim_{\mathcal{S}^{\circ}} \{ S \mapsto \mathrm{Hom}_{\mathcal{X}}(SX, X') \}$$

In other words a map from X to X' is rep. by a couple $(S, u: SX \to X')$, these couples form a fibred cat over $\mathcal{S}$, and two couples represent the same map iff they lie in the same comp. of this fibred cat.

Definition of composition: If $X \to X'$ is rep by $SX \xrightarrow{u} X'$, and if $X' \to X''$ is rep by $S'X \xrightarrow{u'} X''$, then composition is rep by

$$(S'S)X = S'(SX) \xrightarrow{S'(u)} S'X' \xrightarrow{u'} X''$$

One ~~sees~~ verifies easily that comp. is assoc. and that the couple

$$(0, 0X \xrightarrow{=} X)$$

rep. the identity of X. Thus $\langle \mathcal{S}, \mathcal{X} \rangle$ is a well-defined category.

Universal property:

Given $\mathcal{X} \xrightarrow{F} \mathcal{Y}$

and a nat. transf.

$$\theta_{S,X} : F(X) \longrightarrow F(SX)$$

from $F\,pr_2 \longrightarrow F\cdot\# \; : \; \mathcal{S}\times\mathcal{X} \longrightarrow \mathcal{Y}$

satisfying the conditions

a) $\theta_{0,X} : F(X) \longrightarrow F(0X) \longrightarrow F(X)$ is identity

b)
$$\begin{tikzcd}
F(X) \arrow[r, "\theta_S"] \arrow[drr, "\theta_{TS}"'] & F(SX) \arrow[r, "\theta_T"] & F(T(SX)) \arrow[d, equal] \\
 & & F((TS)X)
\end{tikzcd}$$

then $\exists!$ functor

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[dr, "F"'] & \langle \mathcal{S}, \mathcal{X} \rangle \arrow[d, "\exists!"] \\
 & \mathcal{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, mapsto] \\
FX
\end{tikzcd}$$

$$\begin{tikzcd}
(S, & SX \arrow[r, "u"] & X') \\
S' \arrow[u, "\alpha"] & S'X \arrow[u, "\alpha X"] \arrow[ur, "u'"'] &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
(F(X) \arrow[r, "\theta_S"] \arrow[dr, "\theta_{S'}"'] & F(SX) \arrow[r] & F(X')) \\
 & F(S'X) \arrow[u, "F(\alpha X)"'] \arrow[ur] &
\end{tikzcd}$$

nat of θ

Compatible with composition:

Let $\mathcal{S}$ be a monoidal category as defined by Maclane, (true-category AU ~~in~~ the terminology of [Saavedra]). This means $\mathcal{S}$ is provided with a functor

$$\perp : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S} \quad , \quad (S_1, S_2) \longmapsto S_1 \perp S_2$$

together with associativity constraints

$$\varphi_{S_1, S_2, S_3} : (S_1 \perp S_2) \perp S_3 \xrightarrow{\sim} S_1 \perp (S_2 \perp S_3)$$

and unity constraints consisting of 0 and isos.

$$d_S : S \perp 0 \xrightarrow{\sim} S$$

$$g_S : 0 \perp S \xrightarrow{\sim} S .$$

The constraints are required to satisfy certain conditions of compatibility (pentagon + three triangles)

Let $\mathcal{X}$ be a cat. By a (left) action of $\mathcal{S}$ on $\mathcal{X}$ consists of a functor

$$\# : \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X} \qquad (S, X) \longmapsto S \# X$$

together with the following associativity and unity constraints.

Associativity:

$$\varphi'_{S_1, S_2, X} : (S_1 \perp S_2) \# X \xrightarrow{\sim} S_1 \# (S_2 \# X)$$

Unity: $\quad d'_X : 0 \# X \xrightarrow{\sim} X$

subject to the following conditions (pentagon + two unity triangles.)

Example: $\mathcal{S}$ acts on itself: $S \# X = S \perp X$.

Definition of $\langle \mathcal{S}, \mathcal{X} \rangle$: An object of $\langle \mathcal{S}, \mathcal{X} \rangle$ is the same as an object of $\mathcal{X}$. Given X, X' in $\mathcal{X}$ let $\mathcal{H}_{\mathcal{S},\mathcal{X}}(X, X')$ be the groupoid whose objects are pairs (S, u) consisting of an object S of $\mathcal{S}$ and an mor $u : S \# X \to X'$ in $\mathcal{X}$, and in which a morphism $(S,u) \to (S', u')$ is an isomorphism $v : S \to S'$ such that

$$\begin{tikzcd} S\#X \arrow[r, "u"] \arrow[d, "v\#id"'] & X' \arrow[d, equal] \\ S'\#X \arrow[r, "u'"] & X' \end{tikzcd}$$

commutes. Put A morphism from X to X' in $\langle \mathcal{S}, \mathcal{X} \rangle$ is defined to be an iso class of this groupoid:

$$\mathrm{Hom}_{\langle \mathcal{S},\mathcal{X} \rangle}(X, X') = \pi_0 \mathcal{H}_{\mathcal{S},\mathcal{X}}(X, X').$$

and denote by $cl(S,u)$ the iso class of pair (S,u).

and define composition maps

$$\mathrm{Hom}_{\langle \mathcal{S},\mathcal{X} \rangle}(X, X') \times \mathrm{Hom}_{\langle \mathcal{S},\mathcal{X} \rangle}(X', X'') \longrightarrow \mathrm{Hom}_{\langle \mathcal{S},\mathcal{X} \rangle}(X, X'')$$

$$cl(S, u) \,,\; cl(S', u') \longmapsto cl(S \perp S', u' * u)$$

~~where cl(S,u) denotes the iso~~ where $u' * u$ is the composition

$$(S \perp S') \# X \cong S \# (S' \# X) \xrightarrow{id \# u} S \# X' \xrightarrow{u'} X''.$$

The identity morphism of X is $cl(0, g_X : 0 \# X \to X)$.

Verification ~~that composition~~ that the category axioms are satisfied is routine and will be left to the reader.

$$\cancel{S \times X \rightrightarrows X \longrightarrow [\text{redacted}]\ \langle S, X\rangle}$$

$$\cancel{X \xrightarrow{f} Y}$$

$$\cancel{X \longrightarrow S\# X}$$

$$\cancel{X \longrightarrow S\# X}$$

---

associativity: Given

$$X \xrightarrow{d(S,u)} X' \xrightarrow{d(S',u')} X'' \xrightarrow{d(S'',u'')} X'''$$

the assoc. amounts to

$$d\left(S''\perp(S'\perp S),\ u''*(u'*u)\right) = d\,(S \quad \cdots$$

One verifies $(S''\perp S')\perp S \simeq S''\perp(S'\perp S)$ gives the desired iso.

identity: Given

$$X \xrightarrow{d(S,u)} X'$$

$$(S\perp 0,\ u*g_X) \simeq (S,u) \quad \text{furnished by } d_S$$

$$(0\perp S,\ g_X * u) \simeq (S,u) \quad \text{furnished by } g_S.$$

$$\cancel{(S\perp 0)\# X \longrightarrow S\#(0\# X) \xrightarrow{id\# g_X} S\# X \xrightarrow{u} X'}$$

$$\cancel{(S\perp 0)\# X \xrightarrow{d_S\# id} S\# X \xrightarrow{u}}$$

Examples: 1) Let $X = pt$ with the unique S-action. Then $\langle S, pt \rangle$ is the cat. with one object associated to the monoid of isom. classes of S.

2) S = finite sets + their isomorphisms with $\perp$ = disj. union. $X = S$. Then $\mathcal{H}(X, X')$ is the groupoid consisting of (S, u), $u : S \amalg X \xrightarrow{\sim} X'$, and clearly such a couple is det up to isom by the injection $X \to X'$ induced by u. Thus $\langle S, S \rangle$ is isom to the cat of finite sets and injective maps.

3) S = fin. gen. proj. A-modules + their isos with $\perp = \oplus$, $X = S$. A couple (S, u), $u : S \oplus X \cong X'$ ~~determined up to isomorphism by a splitting of X'~~ a decomp of X' as a direct sum of two submodules and an isom of X with the second, or what amounts to the same thing a pair of homom. $X \underset{i}{\overset{p}{\rightleftarrows}} X'$ such that $pi = id_X$. Clearly the pair (p, i) determines (S, u) up to isom. Thus $\langle S, S \rangle$ is equivalent to the cat whose objects are f.g.p A-modules and in which map are such pairs (p, i).

4) $S = Iso(P_A)$, $X = Iso(P_{A'})$ where $A \to A'$ is a homom, action $S \perp X = (A' \otimes_A S) \oplus X$. In this case $\langle S, X \rangle$ has objects $X \in Iso(P_{A'})$ and a map $X \to X'$ is a couple $X \underset{i}{\overset{p}{\rightleftarrows}} X'$ + an iso class of A-reductions of $\mathrm{Ker}(p)$.

Application:

$\mathcal{X}$ cat with prod having a basic object A
$\mathcal{S}$ = finite sets + autos with $+ = \sqcup$.
$\mathcal{S} \to \mathcal{X}$, $n \longmapsto A^n$ cofinal

~~[illegible]~~ Can form $\mathcal{S}^{-1}\mathcal{X}$. Note

$$\pi_0(\mathcal{S}^{-1}\mathcal{X}) = (\pi_0\mathcal{S})^{-1}\pi_0\mathcal{X}$$

$$= \text{group completion of } \pi_0\mathcal{X}$$

$$\overset{\text{denoted}}{=} K_0\mathcal{X}$$

Let $(\mathcal{S}^{-1}\mathcal{X})_0$ = full subcat of $\mathcal{S}^{-1}\mathcal{X}$ cons. of pairs (n,X) $\ni$ $[X] = [A^n]$ in $K_0\mathcal{X}$. = component of $\mathcal{S}^{-1}\mathcal{X}$ assoc to $0 \in K_0\mathcal{X}$

Let $\mathcal{X}(A^n)$ be ~~[illegible]~~ component of $\mathcal{X}$ containing A^n. Have ind. sys

$$\longrightarrow \mathcal{X}(A^n) \longrightarrow \mathcal{X}(A^{n+1}) \longrightarrow \cdots$$

$$X \longmapsto X + A$$

Let $\mathcal{X}(A^\infty) = \varinjlim \mathcal{X}(A^n)$

~~[illegible]~~ Form $\mathrm{Tel}(\mathcal{X}(A^n), n \in \mathbb{N})$ cofibred over the ordered set $\mathbb{N}$. Then have functor

$$\mathrm{Tel}(\mathcal{X}(A^n), n \in \mathbb{N}) \longrightarrow \mathcal{X}(A^\infty)$$

which is a heq (ref.). Have functor

$$(*) \qquad \mathrm{Tel}(\mathcal{X}(A^n), n \in \mathbb{N}) \longrightarrow (\mathcal{S}^{-1}\mathcal{X})_0$$

$$(n, X) \longmapsto (A^n, X).$$

Prop: $(*)$ induces isos. on homology

Proof: Have (ref)

$$(\pi_0 S)^{-1} H_*(\mathcal{X}) \xrightarrow{\sim} H_*(S^{-1}\mathcal{X})$$

Now if $\mathcal{X}(\alpha)$ is the comp of $\mathcal{X}$ belonging to $\alpha \in \pi_0\mathcal{X}$, have grading

$$H_*(\mathcal{X}) = \coprod_{\alpha \in \pi_0\mathcal{X}} H_*(\mathcal{X}(\alpha))$$

of the ring $H_*(\mathcal{X})$ w.r.t. $\pi_0 \mathcal{X}$ and similarly a grading

$$H_*(S^{-1}\mathcal{X}) = \coprod_{\beta \in K_0\mathcal{X}} H_*((S^{-1}\mathcal{X})_\beta).$$

From this we see any element z of $H_*((S^{-1}\mathcal{X})_\beta)$ is rep as a fraction x/ε^n where $\varepsilon = d(A) \in \pi_0\mathcal{X} \in H_0(\mathcal{X}(A))$ and

$$x \in \coprod_{[\beta] = [A^n]} H_*(X_\beta)$$

but mult. num. + denom. but a power of ε, can suppose z rep by x/ε^n $x \in H_*(\mathcal{X}(A^n))$. In other words ~~we have an iso~~ the inclusions $\mathcal{X}(A^n) \longrightarrow (S^{-1}\mathcal{X})_0$, $X \longmapsto (A^n, X)$ lead to an isos

$(*)$
$$\varinjlim_n H_*(\mathcal{X}(A^n)) \xrightarrow{\sim} H_*((S^{-1}\mathcal{X})_0).$$

But we have a comm. diagram

$$
\begin{tikzcd}
 & \varinjlim_{n} H_*(\mathcal{X}(A^n)) \arrow[dl, "\sim"'] \arrow[d] \arrow[dr, "\sim", "(*)"'] & \\
H_*(\mathcal{X}(A^\infty)) & H_*(\mathrm{Tel}(\mathcal{X}(A^n), n \in \mathbb{N})) \arrow[l, "\sim"'] \arrow[r] & H_*((S^{-1}\mathcal{X})_0)
\end{tikzcd}
$$

so done.

Thus have proved:

<u>Theorem</u>: In the homotopy cat. $\exists$ map

$$\mathcal{X}(A^\infty) \longrightarrow (S^{-1}\mathcal{X})_0$$

inducing isos. on homology such that $(S^{-1}\mathcal{X})_0$ is a category with product (hence an infinite loop space by Segal theory).

<u>Examples</u>: 1. finite sets

2. K-theory

$\mathcal{X}$ cat with product having basic object A

$\mathcal{S}$ = finite sets + isos with + = $\amalg$

$\mathcal{S} \longrightarrow \mathcal{X}$ $\qquad\qquad n \longmapsto A^n$

Then have map of cats on which $\mathcal{S}$-acts

$$\mathcal{X} \longrightarrow \mathcal{S}^{-1}\mathcal{X}$$

hence a map of $\pi_0\mathcal{S}$-modules

$$H_*(\mathcal{X}) \longrightarrow H_*(\mathcal{S}^{-1}\mathcal{X})$$

As $\mathcal{S}$ acts invertibly on $\mathcal{S}^{-1}\mathcal{X}$ (ref.) $\pi_0(\mathcal{S})$ acts invertibly on $H_*(\mathcal{S}^{-1}\mathcal{X})$, so get

$$(\pi_0\mathcal{S})^{-1}H_*(\mathcal{X}) \longrightarrow H_*(\mathcal{S}^{-1}\mathcal{X})$$

and have shown this is an isomorphism.

Note

$$\pi_0(\mathcal{S}^{-1}\mathcal{X}) = (\pi_0\mathcal{S})^{-1}\pi_0\mathcal{X}$$

$$= \text{group assoc to } \pi_0\mathcal{X} \qquad = K_0\mathcal{X}$$

Let $(\mathcal{S}^{-1}\mathcal{X})_0$ = comp. of $\mathcal{S}^{-1}\mathcal{X}$ corresp to $0 \in K_0\mathcal{X}$ = full subcat of $\mathcal{S}^{-1}\mathcal{X}$ consis of $(A^n, X) \ni cl(A^n) = cl(X)$.

For each $\alpha \in \pi_0\mathcal{X}$, let $\mathcal{X}_{(\alpha)}$ denote corresp. component of $\mathcal{X}$, and write $\mathcal{X}_{(A^n)}$ for $\alpha = cl(A^n)$. Then we have an ind. system of cats.

$$\longrightarrow \mathcal{X}(A^n) \longrightarrow \mathcal{X}(A^{n+1}) \longrightarrow \cdots\cdots$$

$$X \longmapsto X \oplus A$$

whose inductive limit we will denote $\mathcal{X}(A^\infty)$.

Define $\mathrm{Tel}(\mathcal{X}(A^n))$ to be the cofibred category over the ordered set $\mathbb{N}$ defined by the above inductive system. An object is ~~a couple (n,X) an object X of $\mathcal{X}(A^n)$ for some~~ (n,X) $X \in \mathcal{X}(A^n)$ and a map $(n,X) \longrightarrow (n',X')$ consists of (p,θ) $p \in \mathbb{N}$, $n+p=n'$ and $\theta : X \oplus A^p \longrightarrow X'$ is a map.

Then the canonical functors $i_n : \mathcal{X}(A^n) \longrightarrow \mathcal{X}(A^\infty)$ induce a functor

$$\mathrm{Tel}(\mathcal{X}(A^n)) \longrightarrow \mathcal{X}(A^\infty)$$

$$(n,X) \longmapsto i_n(X)$$

which is a heq by virtue of

<u>Lemma</u>: Let $i \mapsto \mathcal{X}_i$ be an ind. system of categories indexed by a filtering cat I, let $\mathcal{X}_I$ be the corresp cofibred cat over I, and $\mathcal{X} = \varinjlim \mathcal{X}_i$. Then canon functor

$$\mathcal{X}_I \longrightarrow \mathcal{X}$$

is a heq.

~~Correction~~
~~Chevalley reference (§7)~~
~~Segal reference (after Th B)~~
~~Groth ref. §8.~~

~~principal ideal domain~~

~~need Bass's book~~

On the other hand we have a functor

$$\begin{array}{ccc} \mathrm{Tel}(\mathcal{X}(A^n)) & \longrightarrow & (\mathcal{S}^{-1}\mathcal{X})_0 \\ (n, X) & \longmapsto & (A^n, X) \end{array}$$

<u>Lemma:</u> Above functor induces isos. on homology

~~$H_*((\mathcal{S}^{-1}\mathcal{X})_0)$ = degree zero~~

$H_*(\mathcal{X}) = \coprod_{\alpha \in \pi_0 \mathcal{X}} H_*(\mathcal{X}_\alpha)$ graded according to $\pi_0 \mathcal{X}$.

$H_*(\mathcal{S}^{-1}\mathcal{X}) = \coprod_\alpha H_*((\mathcal{S}^{-1}\mathcal{X})_\alpha)$ is graded according to $\alpha \in K_0$.

$H_*((\mathcal{S}^{-1}\mathcal{X})_0)$ consists of fractions x/ε^n where $x \in H_*(\mathcal{X}_\alpha)$ $d(\alpha) = d(A^n)$, and in fact easy to see this is the same as fractions $\frac{x}{\varepsilon^n}$ $x \in H_*(\mathcal{X}(A^n))$, whence we see that

$$H_*(S^{-1}\mathcal{X}_0) = \varinjlim_n H_*(\mathcal{X}(A^n))$$

i.e. $$\mathcal{X}(A^n) \longrightarrow S^{-1}\mathcal{X}_0$$
$$X \longmapsto (A^n, X)$$

induces $$H_*(\mathcal{X}(A^n)) \longrightarrow H_*(S^{-1}\mathcal{X}_0)$$

~~where~~ which in the limit gives

$$\begin{tikzcd}
 & \varinjlim_n H_*(\mathcal{X}(A^n)) \arrow[dl, "\cong"'] \arrow[d] \arrow[dr, "\cong"] & \\
H_*(\mathcal{X}(A^\infty)) & H_*(\operatorname{Tel}_n \mathcal{X}(A^n)) \arrow[l, "\cong"'] \arrow[r] & H_*(S^{-1}\mathcal{X}_0)
\end{tikzcd}$$

Thus have proved

<u>Thm</u>: Let $\mathcal{X}$ be a category with product having a basic element A. Then in the homot. cat $\exists$ a map

$$\mathcal{X}(A^\infty) \longrightarrow (S^{-1}\mathcal{X})_0$$

inducing isos. on homology, where $(S^{-1}\mathcal{X})_0$ is a homotopy commutative associative H-space (in fact an infinite loop space.)