Let $L$ be a line in $V'$. (If $m=0$, then $P(V,V')$ is $\emptyset$.

and define $f$ the map

$$f: P(V,V') \rightarrow P(V/L, V'/L) \times \mathcal{H}_L$$

where $\mathcal{H}_L = \{H | H \cap L = V\}$ as follows:

$$W \mapsto \begin{cases} W & \text{if } W \in \mathcal{H}_L \\ W + L/L & \text{if } W \notin \mathcal{H}_L \end{cases}$$

A simplex of the join $K_1 \times K_2$ of two simplicial complexes is a non-empty subset of $K_1 \cup K_2$ whose intersection with $K_2$ is a simplex or empty.

So it is clear $f$ is simplicial.

To show $f$ is a homotopy equivalence, let $U$ be the open subset of $P(V,V')$ consisting of open simplices containing a vertex not in $\mathcal{H}_L$, and let $U'$ be the open set which is the open star of $\mathcal{H}_L$. Then $U$ retracts to the subcomplex $\mathcal{Y}$ complementary to $\mathcal{H}_L$ which contracts to $\{W | L \subseteq W < V, \; W + L = V\} = P(V/L, V'/L)$ via $W \mapsto L + W$. Thus $f: U \rightarrow \text{open star of } P(V/L, V'/L)$ in join is a homtopy. Similar $f: U' \rightarrow \text{open star of } \mathcal{H}_L$ in join is a homtopy. But note next:

$$U \cap U' \rightarrow P(V/L, V'/L) \times \mathcal{H}_L$$

is a homeomorphism, since $(W_0/L) \cap W_0 = W_0$ if $L \cap W_0 = V$. 

$$W_0 \mapsto \begin{cases} W_0 & \text{if } W_0 \in \mathcal{H}_L \\ (W_0 + L/L, \ldots, W_0 + L/L, W_0) & \text{otherwise} \end{cases}$$
So what we have done is decomposed
\[ P(V, V') = U \cup U' \]
and given flags
\[ U \rightarrow P(V/L, V'/L) \]
\[ U' \rightarrow \phi_L \]
such that
\[ U \cup U' \rightarrow P(V/L, V'/L) \times \phi_L \]

It follows then that we have a flag
\[ P(V, V') \hookrightarrow P(V/L, V'/L) \times \phi_L \]

Write this down...

\[ \text{Proposition 2.1: } \quad \phi_L \text{ is a flag.} \]

Suppose now that \( 0 < V_1 < \cdots < V_n = V \) is a flag in \( V \). Then we have flags
\[ P(V, V_m) \rightarrow P(V/V_1, V_m/V_1) \times (P(V/V_1)^* - P(V/V_1)^*) \]
\[ \rightarrow P(V/V_2, V_m/V_2) \times (P(V/V_2)^* - P(V/V_2)^*) \]
and so we have proved:

\[ \text{Prop. 2.2; canonical homotopy equivalence} \]
\[ P(V, V_m) \hookrightarrow (P(V/V_{m-1})^* - P(V/V_m)^*) \times \cdots \times (P(V/V_0)^* - P(V/V_1)^*) \]

So in particular \( P(V, V_m) \) is a bouquet of \((m-1)\)-spheres in number \((n-1)\) \(-\cdots\) \((n-m-1)\). Also \( \Rightarrow \) if we identify \( P(V/V_{m-1}) \) with the simp. ex. of affine subspaces of \( V_{m-1} \), then we have a flag
\[ P(V, V_{n-1}) \hookrightarrow P(V/V_m, V_{n-1}/V_m) \times P(V, V_m) \]
which gives Lusztigs win
\[ \tilde{\mathcal{H}}_{n-1}(A(V)) \cong \tilde{\mathcal{H}}_{n-m-1}(A(V/W)) \otimes \tilde{\mathcal{H}}_{m-1}(P(V,W)) \]
\[ n = \dim V, \quad m = \dim W \]

\[ \tilde{\mathcal{H}} \text{ seems there is virtue in changing notation.} \]

Put \[ Q(V,V') = \{ W \mid 0 \leq W, W \cap V' = 0 \} \]

**Lemma:** Let \( H \) be a hyperplane of \( V \) containing \( V' \). Then there have heq's
\[ \Theta : Q(V,V') \to (PV - PH) \times Q(H,V') \]
\[ \begin{aligned}
W & \mapsto \\
& \begin{cases} 
W & \text{if } W \in PV - PH = Q(V,H) \\
W \cap H & \text{if } W \notin 
\end{cases}
\end{aligned} \]

**Proof:** Clearly, this map is order-preserving. On the other hand \( \Theta \) is cofibrational. Suppose given \( W \) and \( G(W) \to Z_0 \). To prove \( u_x(W) \) exists. \( \text{For suppose } u_x(W) \text{ exists. } \)

\[ \begin{aligned}
\text{To prove } u_x(W) \text{ exists, i.e. there is a universal arrow } W \leq u_x(W) \text{ over } u. \text{ Can suppose } u_x(id) = Z_0 \in Q(H,V'). \text{ Set } \\
u_x(W) = W + Z_0
\end{aligned} \]
Then \( H \cap (W + Z_0) = H \cap W + Z_0 = Z_0 \) which implies in particular that \( V \cap (W + Z_0) = 0 \); thus \( u_*(w) \) is defined and \( W \cap u_* W \) sits over \( u_* \). Next if \( W \subseteq W_1 \) sits over \( u_* \), \( Z_0 = H \cap W_1 \Rightarrow W + Z_0 \subseteq W_1 \), which implies universality of the map \( W \subseteq u_*(W) \).

Finally, the fibre of \( \Theta \) over an \( L \) in \( PV-PH \) is \( L \) itself, and the fibre of \( \Theta \) over \( Z \in Q(V, V') \) consists of \( W \subseteq Q(V, H) \Rightarrow W \cap H = Z \) and this has a smallest element, namely \( Z_0 \).

This proves the lemma.

**Remark:** If \( f : X \to Y \) is a map of ordered sets which is pre-fibred, with contractible fibres, then the map \( |X| \to |Y| \) of simplicial complexes has the same property. In effect for \( y \leq y' \) one has arrows:

\[
X_y \leftarrow X_u \rightarrow X_{y'}
\]

and the pre-fibred condition \( \Rightarrow \) the former has an adjoint \( \times \mapsto (\times \mapsto u_*\times) \); so therefore quite generally \( X_u \) retracts to its source \( X_y \), etc.

So therefore the proof of the lemmas shows all the fibres of the geometric map \( Q(V, V') \to Q(V, H) \times Q(H, V') \) are contractible. In fact the fibres are

\[
\Theta^{-1} \left\{ L, Z_1 < \ldots < Z_8 \right\} = \left\{ L < L + Z_1 < \ldots < L + Z_8 \right\}
\]

even if \( q = 0 \)

\[
\Theta^{-1} \left\{ Z_0 < \ldots < Z_8 \right\} = \left\{ w_0 < \ldots < w_k \mid \text{in } H = (Z_0 < \ldots < Z_8) \right\}
\]

contracts to simplex \( Z_0 < \ldots < Z_8 \).
Iterating it follows that we get heq's
\[ Q(V, V_m) \rightarrow Q(V, V_{n-1}) \ast Q(V_{n-1}, V_m) \]
\[ \rightarrow Q(V, V_{n-1}) \ast Q(V_{n-1}, V_{n-2}) \ast \cdots \ast Q(V_{n+1}, V_m) \]

which can be explicitly described as follows. Namely one takes a subspace \( 0 < W, \ W \cap V_m = 0 \) and starts intersecting it with the flag \( W \supset W \cap V_{n-1} > \cdots \). If \( j \) is such that \( W \cap V_j \neq 0, \ W \cap V_{j-1} = 0 \), then \( W \cap V_j \in Q(V_j, V_{j-1}) = PV_j - PV_{j-1} \) and this is where the vertex \( W \) goes.

At the same time one gets heq's
\[ Q(V, V_m) \rightarrow Q(V, V_p) \ast Q(V_p, V_m) \]

\[ \text{Formula:} \]
\[ Q(V, V_m) \xrightarrow{\text{heq}} (PV - PV_{n-1}) \ast \cdots \ast (PV_{m+1} - PV_m) \]

Conjecture: There is a canonical heq
\[ T(V) \xrightarrow{\text{heq}} (PV/PV_{n-1}) \ast \cdots \ast (PV_1/PV) \]

where for spaces with basepoint one takes the reduced join.
More examples:

\[ T(V) = \text{subspaces } W, \ 0 < W < V. \]

More generally let \( X = \{ W \mid p \leq \dim(W) \leq q \} \)

**Prop:** \( X \) is a bouquet of \((q-p)\)-spheres.

**Proof:** Denote the above \( X \) by \( X_{p,q}^n \), \( n = \dim(V) \).

Let \( L \) be a line in \( V \) and let \( S_L = \{ W \mid \dim(W) = q, \ L \not\subset W \} \).

\( \text{Link}(Z) = \{ W \mid W < Z, \ \dim W \geq p \} \cong X_{p,q-1}^p \) is a bouquet of \((q-p-1)\)-spheres by induction. Remove the vertices \( S_L \) from \( X_{p,q}^n \), and the result collapses by \( W \mapsto L + W \) into \( \{ W \mid p \leq \dim(W) \leq q, \ L \subset W \} = X_{p-1,q-1}^{n-1} \).

By induction this is a bouquet of \((q-p)\)-spheres. Use now the basic fact that if one attaches an \( m \)-cell to a bouquet of \( m \)-spheres one gets a bouquet of \( m \)-spheres.

\[ A(V) = \text{affine subspaces } W < V. \]

**Prop:** Let \( A_p(V) = \{ W \mid W \text{ affine subspace of } V, \ \dim W \geq p \} \). Claim this is a bouquet of \((n-p-1)\)-spheres.

**Proof:** Take point \( O \in V \) and consider map \( W \mapsto W \{O\} = kW. \) In order this be defined let \( kW \) remove \( H = \text{hyperplanes not passing through } O. \)

\( \text{Link}(H) = \{ W \mid W \text{ affine space of } \dim \geq p \} = A_p(H) \)

which will be a bouquet of \((n-p-2)\)-spheres by ind. Sufﬁces to show that \( \{ W \mid W \text{ affine space of } \dim \geq p \} \) is a bouquet of \((n-p-1)\)-spheres.
But this is $X_{p,n-1}$ as above, so it is a bouquet of $(n-p-1)$-spheres. Done.
vector space of dim n, W subspace of dim m.

\[ \dim X(V, W) = \text{simplicial complex whose simplices are non-}\phi \text{ subsets } \{v_1, \ldots, v_k\} \text{ of } V \text{ such that} \]
\[ \dim k\nu_1 + \ldots + k\nu_k + W = \phi + \dim(W) \]

Theo \( \dim X(V, W) = n-m-1 \). Claim \( X(V, W) \) is a bouquet of \((n-m-1)\)-spheres. Argue by induction on \( n-m \). If \( n-m = 0 \), then \( X(V, W) = \emptyset \) and it's clear. Also if \( n-m = 1 \), then \( X(V, W) \) is a non-empty union of points and it is also clear.

So assume \( n-m > 0 \) and let \( e \) be a vector in \( V \) not in \( W \). Let \( \mathcal{Z} \) be the link of the vertex \( e \), i.e. the subcomplex of \( \{v_1, \ldots, v_k\} \) which are independent of \( ke + W \). Let \( C = \text{closed star of } e \).

Now let \( f : K \to X(V, W) \) be a map where \( K \) is a finite complex of dim \( < n-m-1 \). By simplicial approx. we can homotop \( f \) to a simplicial map. Let \( S \) be the finite set of simplices of \( X(V, W) \) which are in \( f(K) \) but not in \( C \).

Let \( \sigma = (v_1, \ldots, v_k) \in S \); \( \phi \) then \( \sigma \in (n-m) \) and \( e \in k\nu_1 + \ldots + k\nu_k + W \). Consider the link of \( \sigma \), \( L \) so that

locally around \( \sigma \) : \( \text{Open star } (\sigma) \cong \emptyset \text{ Int } \sigma \times \text{ Cane } (L) \)
First suppose to simplify that \( \sigma = (v_2) \) is a vertex. Then
\[
\text{Link}(v) = X(V, kv + W) \subset \text{Link}(e)
\]
is a bouquet of \((n-m-2)\)-spheres. In general if \((v_1, \ldots, v_6) \in X(V, W)\), then
\[
\text{Link}(v_1, \ldots, v_6) = X(V, kv_1 + \ldots + kv_6 + W)
\]
is a bouquet of \((n-m-g-1)\)-spheres by induction. And if \(ke < kv_1 + \ldots + kv_6 + W\), then
\[
\text{Link}(v_1, \ldots, v_6) \subset \text{Link}(e).
\]

But now in the case \( \sigma = (v) \) the point is the following: Because \( \text{Link}(v) \) begins in \( \text{dim (n-m-2)} \), and \( \text{FK} \) is of \( \text{dim (n-m-2)} \), we should be able to push \( \text{FK} \) off of \( v \). Precisely,

In a nbhd of \( v \), \( \text{FK} \) is the cone on \( \text{Link(FK)} \). But \( \text{Link(FK)} \) contracts to a point in \( \text{Link(X)} \). Now using this homotopy, one can push \( \text{FK} \) into the link.

Lemma: \( Y \subset X \) contractible to a point \( \Rightarrow \)
\( \text{Cone}(Y) \subset \text{Cone}(X) \) homotopic keeping \( Y \) fixed to a map of
\( \text{Cone}(Y) \rightarrow X \).
Now for $\sigma = (v_1, \ldots, v_k) \in S$, we can use the same argument to push $fK$ onto the boundary of $\sigma \star \text{link}(\sigma)$. Thus we have modified $f$ so that we now have $e_1 + \cdots + e_k$, where $e_1, \ldots, e_k$ are the simplices of the form $\tau_1, \ldots, \tau_k$ where $\tau < \sigma$ and $\tau \in \text{link}(\sigma)$. Now we note in this process that if we write

$$e = \sum l_i v_i + w$$

then the rank of the simplex $e$ has increased.

So now let us consider those simplices $\tau$ contained in $fK$ such that if $\tau = (v_1, \ldots, v_k)$ then $e \in kv_1 + \cdots + kv_k + W$ and this is not true for any face of $\sigma$. Since $v_1, \ldots, v_k$ are ind. mod $W$ one has a unique expression

$$e = \sum l_i v_i + w$$

So the $\tau$ under consideration are those such that $l_i \neq 0$ for all $i$. Now we will use induction on the number of this minimal $\sigma$. If $n = 0$, then $fK \subseteq \text{cone}(e)$ and we are done. Otherwise let $\tau$ be one and note that every $\tau' \cup \tau$ in $\sigma(\sigma \star \text{link}(\sigma))$ is in $\text{cone}(e)$. Thus when we use the preceding pushing construction to push $fK$ over $\tau$, we do not introduce any new minimal simplices.

In fact it would appear that if we let $
\[ \Gamma = \{ \sigma \in \pi(V, W) \mid \sigma \text{ minimal } \implies \sigma \in \text{Cone}(e) \}, \]

then each simplex not in \text{Cone}(e) contains a unique \( \sigma \), whence

\[ \bigsqcup_{\sigma \in \Gamma} \text{Cone}(e) \cup \bigsqcup_{\sigma \in \Gamma} \text{Link}(e) \rightarrow X(V, W) \]

**Prop:** Let \( \Gamma \) be the set of subsets \( \{ V_1, \ldots, V_6 \} \) of \( V \) which are minimal such that \( e \in kV_1 + \cdots + kV_6 + W \)

Then

\[ X(V, W) = \bigvee_{\sigma \in \Gamma} \bigvee_{k=1}^{10} X(V, k\sigma + W) \]

By induction \( X(V, k\sigma + W) \) is a bouquet of \((n-101-m-1)\)-spheres, so \( X(V, W) \) is a bouquet of \((n-m-1)\)-spheres. \( n = \dim(V) \), \( m = \dim(W) \).
$V$ vector space of dim $n$, $W$ subspace of dim $m$.

$X(V,W) =$ simplicial complex whose simplices are fin. un +

$\tau \in \mathfrak{S} \subset V$ s.t.

$$\dim (k\sigma + W) = \dim \sigma + \dim W$$

(one says $\sigma$ is independent $W$).

Then $X(V,W)$ is of dimension $n-m-1$, and here is how

one proves it has the homotopy type of a bouquet

of $(n-m-1)$-spheres.

Fix a $\sigma$ vector $e \in V$ not in $W$. (If $n=m$, nothing
to prove since $X(V,W)=\emptyset$; otherwise $e \not\in W$).

We try to push $X(V,W)$ into the closed star of $e$, which is contractible; denote

this $\mathfrak{S}(e)$. Hence we are interested in those $\tau$ which

are not in $\mathfrak{S}(e)$, that is, such that $\{e\} \cup \tau$ is not

independent mod $W$. If $\tau = \{v_1, \ldots, v_k\}$, then because $\sigma$ is

ind. of $W$, one has $ke \subset k\tau + \cdots + k\tau + W$; hence

$$e = \sum_{i=1}^k \lambda_i v_i + w$$

for $\lambda_i \in \mathbb{K}$, $w \in W$. Moreover the $\lambda_i$ and $w$ are uniquely
determined. Now consider those $\lambda_i$ which are non-zero;

these form a face $\tau'$ of $\tau$, and it is clear from the uniqueness

of the representation $(*)$, that $\tau' \subset \tau$ for any face $\tau'$ of

$\tau$ such that $e \in k\tau + W$.

But $\Gamma = \{\sigma \in X(V,W) \mid \sigma$ minimal $e \in k\sigma + W\}$.

Then we have shown that

$$X(V,W) - \mathfrak{S}(e) = \bigsqcup_{\sigma \in \Gamma} \mathfrak{S}(\sigma)$$
But now consider in an arbitrary simplicial complex $X$ what things look like around an (open) simplex $\sigma_0$. A simplex $\sigma$ containing $\sigma_0$ is of the form $\sigma = \sigma_0 \cup \tau$ where $\tau \in \text{Link}(\sigma)$.

More carefully: $\text{Link}(\sigma)$ is by definition the subcomplex of $X$ consisting of those simplices $\tau \in X$ such that $\sigma_0 \cup \tau$ is a simplex, and $\sigma_0 \cap \tau = \emptyset$.

\[ \text{Ope. st. } (\sigma_0) = \{ \sigma \mid \sigma \supseteq \sigma_0 \} \quad \text{open} \]

\[ \text{Cl. st. } (\sigma_0) = \{ \sigma \mid \sigma \cup \sigma_0 \text{ is a simplex} \} \quad \text{closed} \]

\[ \text{Link } (\sigma_0) = \{ \tau \mid \sigma_0 \cup \tau \text{ is a simplex} \} \quad \text{closed} \]

Thus,

\[ \text{Cl. st. } (\sigma_0) = \text{Closed simp. } \sigma_0 \star \text{Link } (\sigma_0) \]

$(K \star L = \text{ simp. comp. where simp. are subsets } \sigma \in \text{K} \star \text{L} \text{ such that } \sigma \neq \emptyset \Rightarrow \sigma \text{ simplex of } K \text{, } \tau \neq \emptyset \Rightarrow \tau \text{ simplex of } L.)$

\[ \partial \text{Ope. } (\sigma) = \text{Cl. st. } (\sigma) - \text{Ope. } (\sigma) = \partial \sigma_0 \star \text{Link } (\sigma) \]

But taking $b_{\sigma_0} = \text{ barycenter of } \sigma_0$, one has a submersion homeomorphism:

\[ \text{Cl. st. } (\sigma) = b_{\sigma_0} \star \partial \text{Ope. } (\sigma_0) \]
Now going back to \( X(V, W) \), note that

\[
\text{Link}(\sigma) = X(V, k\sigma + W). \quad \text{Thus}
\]

\[
X(V, W) = \text{Clst}(e_0) \cup \bigcup_{\sigma \not\in \Gamma} \text{Clst}(\sigma)
\]

\[
= \text{Clst}(e_0) \cup \bigcup_{\sigma \not\in \Gamma} \sigma \times X(V, k\sigma + W)
\]

\[
\bigcup_{\sigma \not\in \Gamma} \partial\sigma \times X(V, k\sigma + W)
\]

In using that \( \text{Clst}(e) \) is contractible we get first part of:

**Prop.** Let \( \Gamma = \{ \sigma \mid \sigma \text{ minimal } \varepsilon \in k\sigma + W \}. \) Then

\[
X(V, W) \sim \bigvee_{\sigma \in \Gamma} S^{\lvert \sigma \rvert} X(V, k\sigma + W)
\]

where \( \lvert \sigma \rvert = \text{card } \sigma = \dim \sigma + 1 \)

\[ (S^{\ast}Y = S^0 \ast S^0 \ast \ldots \ast S^0 \ast Y) \]

\[ k \text{ times} \]

\[ \text{Where } \lvert \sigma \rvert = \text{card } \sigma = \dim \sigma + 1 \]

\[ (S^{\ast}Y = S^0 \ast S^0 \ast \ldots \ast S^0 \ast Y) \]

\[ k \text{ times} \]

**Remarks & Conventions**: If \( X(V, k\sigma + W) = \emptyset \), then

\[
\Sigma^{1\lvert \sigma \rvert} X(V, k\sigma + W) = \sigma/\partial\sigma = S^{\lvert \sigma \rvert - 1}.
\]

If \( X(V, k\sigma + W) \) a set of points, then

\[
\Sigma^{1\lvert \sigma \rvert} X(V, k\sigma + W)
\]

The meaning is:

\[
\Sigma (\partial\sigma \ast X)
\]

\[ S^0 \ast X \]

Here \( \Sigma X = S^0 \ast X \)
Proof of ii). Proceed by induction on \( \dim(V) - \dim(W) \).

Use that suspension carries a bouquet of \( S^p \)'s into a bouquet of \( S^{p+1} \)'s:

\[
S^0 \times \emptyset = S^0 \\
S^0 \times S^p = S^{p+1}
\]

and

\[
S^0 \times (A \cup C B) = (S^0 \times A) \cup (S^0 \times C) (S^0 \times B)
\]

and \( S^0 \times pt = \emptyset \).

Generalizations: The proceeding applies in the general context of a set with a closure relation satisfying the exchange condition. Let \( X(S) \) be the simplicial complex of non-empty finite independent sets. Assuming \( X(S) \neq \emptyset \) pick a vertex \( e \), and notice again that if \( e \in \bar{\tau} \), then there is a least face \( \tau \) of \( \sigma \ni e \in \bar{\tau} \). (Proof: Assume \( \tau_1, \tau_2 \) are minimal faces of \( \sigma \ni e \in \bar{\tau}_1, e \in \bar{\tau}_2 \), with \( \tau_1 \neq \tau_2 \). Let \( \tau_1 = (v_1, v_2, \ldots, v_p) \), \( v_1 \notin \tau_2 \). Since \( e \notin (v_2, \ldots, v_p) \), \( e \in (v_1, \ldots, v_p) \), exchange cond. \( \Rightarrow v_1 \in \bar{e}, v_2, \ldots, v_p \in (v_2, \ldots, v_p) \cup \tau_2 \) which contradicts fact that \( \sigma \) is independent.)

So again we will have

\[
X(S) \sim \bigvee_{e \in \bar{\tau}^0} \text{Link}(e) 
\]

where \( \Gamma = \text{minimal } \sigma \ni e \in \bar{\tau} \). But now \( \text{Link}(e) = X(S_0) \)

where \( S_0 = S \) with the closure relation \( \overline{e \cup \sigma} \).

where \( \Gamma = \text{minimal } \sigma \ni e \in \bar{\tau} \). But now \( \text{Link}(e) = X(S_0) \)

where \( S_0 = S \) with the closure relation \( \overline{e \cup \sigma} \).
Example: Suppose now I try to understand the case of the complex $\Gamma(S', S''') = \Gamma(S' - \sigma, S'' - \sigma)$, such that $pr_1|\sigma: \sigma \to S'$, $pr_2|\sigma: \sigma \to S''$ are injective.

Fix $e=(e',e'') \in S' \times S''$ and let $C=\text{Clst}(e)$ as usual; $\text{Link}(e) = \Gamma(S'-e', S''-e')$

Now let $\sigma$ be a simplex not in $C$. Then $\sigma$ contains a vertex $e' \in S' - e''$ in $e' \times (S'' - e'')$ or $(S'-e') \times e''$. Thus we have to remove these vertices from $\Gamma(S', S'')$ to get $C$.

Start by removing the vertices $e' \times (S'' - e'')$ and call the result $C'$. The link of $(e', e'')$ in $\Gamma(S', S'')$ is clearly

$$\text{Link}((e', e'') \text{ in } \Gamma(S', S'') = \Gamma(S'-e', S''-e'')$$

On the other hand once all of these vertices are removed, one has

$$\text{Link of } \emptyset \text{ in } C' = \Gamma(S'-e'-u, S''-e'')$$

Let $\Gamma(p,q) = \Gamma(S', S'')$ when $\text{card } S' = p$, $\text{card } S'' = q$.

Then to reconstruct this starting from $\Gamma(p-1, q-1)$, I first attach cones on the subcomplexes to $\Gamma(p-1, q-2)$, and then I attach cones on $\Gamma(p-1, q-1)$. So starting with $\Gamma(1, q) \sim \nu^{S_{p-1}}$, $q \geq 1$ we should get

$$\Gamma(p,q) \sim \nu^{S_{p-1}}$$

$\nu^{S_{p-1}}$, $q \geq 2p-1$
In effect assuming this for \( p-1 \), suppose \( g \geq 2p-1 \). The first operation involves collapsing in a contractible space things \( \Gamma(p-1, q-2) \) which \( \sim V S^{p-2} \) since \( q-2 \geq 2(p-1)-1 \). Thus one has a \( \sim V S^{p-1} \), after that one collapses \( \Gamma(p-1, q-1) \) \( \sim S^{p-2} \) in a \( V S^{p-1} \), so again one has a \( \sim V S^{p-1} \).

Counterexamples to improving things:

\[
\Gamma(2, 2) \quad \times \quad \text{not} \sim S^{1}/S.
\]

\[
\Gamma(2, 2) = \quad \times \quad \sim S^{1} \quad \text{so} \quad \Gamma(2, 3) \quad \text{obtained by}
\]

attaching cones on three \( S^{2} \)’s.

\[
\therefore \quad \Gamma(2, 3) \sim V S^{1} \quad \text{not} \sim V S^{2}
\]

\[
\Gamma(3, 4) \quad \text{collapse} \quad 4 \quad \text{loops in} \quad \Gamma(2, 4) \sim V S^{1} \quad \text{so} \quad H_{1}(\Gamma(3, 4)) \quad \text{has rank} \geq 1.
\]

hence \( \Gamma(3, 4) \) not \( \sim V S^{2} \).
Lusztig's complex $L(V)$

$V$ vector space of dimension $n$, $V'$ subspace of dim. $m$. Let $L(V, V')_1$ be the simplicial complex associated to the ordered set $\mathcal{W}$ consisting of affine subspaces $(\neq) W$ of $V$ such that $0 \notin W$, and $kW \cap V' = 0$. For example, if $V' = 0$, then we get Lusztig's complex $L(V)$ of affine subspaces not containing $0$. Note that if $\emptyset \neq W_1 < \cdots < W_k$ is a chain of affine subspaces, then $0 < kW_1 < \cdots < kW_k$ is an increasing sequence, so if $kW_k \cap V' = 0$ on has $q \leq \dim(kW_k) \leq \dim(V/V') = n - m$, and equality is possible. Thus $\dim(L(V, V')) = n - m - 1$.

Theorem: $L(V, V') \sim \vee S^{n-m-1}$.

Proof: Use induction on $n - m$, the case $n - m = 0$, $n - m = 1$ being trivial. Assuming $n - m > 0$, let $U$ be a subspace of dim $m + 1$ containing $V'$, so $L(V, U) \subset L(V, V')$. Note that for each $W$ in $L(V, V')$, $(kW \cap U) \cap V' = 0$, so if $W \notin L(V, U)$, then $kW \cap U$ is a line $L$ in $U$ not in the hyperplane $V'$. Put

$$Z_L = \{ W \mid 0 \notin W \land kW \cap U \subset L \}$$

for each $L \in PU - PV'$. Note that if $L \neq L'$,

$$|Z_L| \cap |Z_{L'}| = |L(V, U)|$$

$$|U \cap Z_L| = |L(V', V')|$$
In effect if $W_1, \ldots, W_i$ is a chain of affine spaces not cont. to $0$, then $\sigma$ is in $\mathbb{Z}_L$ iff $W_i \in \mathbb{Z}_L$, so $\sigma \in \mathbb{Z}_L \cap \mathbb{Z}_L'$ iff $W_i \in \mathbb{Z}_L \cap \mathbb{Z}_L' = L(V_j)$, similarly $\sigma \in L(V_j')$ iff $W_i \in L(V_j') \Rightarrow \sigma \in \mathbb{Z}_L \cap \mathbb{Z}_L'$.  

Lemma: Let a simplicial complex $K$ be the union of subcomplexes $L_i$ all containing $A$ as a subcomplex such that $L_i \cap L_i' = A$ for $i \neq i'$. If $L_i \sim V S_b$ for each $i$, and $A \sim V S_b^{-1}$, then $K \sim V S_b$. 

Proof: One has a cof. situation 

\[
\begin{array}{ccc}
\coprod A & \longrightarrow & A \\
\downarrow & \circlearrowleft & \downarrow \\
\coprod L_i & \longrightarrow & K
\end{array}
\]

as the hyp. $\Rightarrow K - A = \coprod L_i - A$. The result is clear for $q = 0$ as then $A = \emptyset$, $L_i \sim$ set so $K \sim$ set. If $q \geq 1$, choose a basepoint in $A$, whence one has a cocart square 

\[
\begin{array}{ccc}
\bigvee A & \longrightarrow & A \\
\downarrow & & \downarrow \\
\bigvee L_i & \longrightarrow & K
\end{array}
\]

and hence sequences 

\[
\begin{array}{cc}
\tilde{H}_*(A) \longrightarrow \tilde{H}_*(K) & \longrightarrow \tilde{H}_*(L_i/A) \\
\longrightarrow \tilde{H}_*(A) & \longrightarrow \tilde{H}_*(L_i) & \longrightarrow \tilde{H}_*(L_i/A)
\end{array}
\]

which show $\tilde{H}_*(K)$ is free abelian in dim $q$, zero elsewhere. For $q \geq 2$, $\pi_1(K) = 0$ by applying van Kampen to $(\ast)$, so OK. For $q = 1$, can suppose up to a hyp. that $L_i$ is a
connected graph and \( A \) is a bunch of pts. Then \( K \) is a connected graph + done.

**Lemma 2.** \( |Z_L| \sim V S^{n-m-1} \). \( \forall L \in PU-PV' \)

Assuming this and the induction hyp. \( L(V, u) \sim V S^{n-k-2} \)
we have \( L(V, V') \sim V S^{n-m-1} \) by lemma 1, and the theorem is proved.

**Proof of lemma 2.** Recall \( Z_L = \{ W | kw(u) \} \).

Note if \( W \in Z_L-L(V, u) \), then \( kw(u) = L \).

Let \( L = L' \) and suppose \( w_0, \ldots, w_d \in W \). Then \( e \) is a basis for \( W \).

\( e = \sum_{i=0}^{d} \lambda_i w_i \)

for unique \( \lambda_i \). If \( \sum \lambda_i \neq 0 \), then

\[
\frac{1}{\sum \lambda_i} e = \sum_{i=0}^{d} \left( \frac{\lambda_i}{\sum \lambda_i} \right) w_i \in W
\]

and so \( W \cap L = \{ L^* \} \) a point of \( L^* = L-\{ u \} \) (the intersection cannot be \( L \) or else \( 0 \in W \)). On the other hand if \( \sum \lambda_i = 0 \), then because any \( w \) has an expression \( \sum t_i w_i \) with \( \sum t_i = 1 \), it is clear that \( kw(u) = W \), i.e. \( L+W = W \).
Now put
\[ C(e) = \{ W \mid e \in W \in Z_L \} \quad e \in L - \{0\} \]

i.e. \((k+e) \cap U = L\). Then \( C(e) \subset Z_L \) and if \( wc(e) \subset V \)

is a \((k+e) \cap U = L\), then by embedding if \( wc(e) \subset V \)

and \( e \in W \), then \( 0 \in e \in W \), so \( 0, e, W \) are

independent \(\Rightarrow e \in k \Leftrightarrow k \cap W = k \cap W \cap U = 0 \)

\(\Rightarrow W \in L(V, U) \). Thus

\[ C(e) = \{ W \mid e \in W \} \cup L(V, U) \]

Put
\[ C(L) = \{ W \mid L+W \in Z_L \} \]

Then \( C(L) \subset Z_L \) and if \( wc(L) \) and \( W \subset L+W \), then

\[ k \cap W < L \Leftrightarrow k \cap W \cap U = 0 \Rightarrow W \in L(V, U) \]

As
\[ C(L) = \{ W \mid L+W = W \} \cup L(V, U) \]

so we get that
\[ Z_L = \bigcup_{e \in L - \{0\}} C(e) \cup C(L) \]

where the subcomplexes \( C(e), C(L) \) pairwise intersect in \( L(V, U) \).

Because \( C(e) \) is contractible, \( Z_L \) is therefore obtained by starting from \( C(L) \) and attaching a cone on \( L(V, U) \) for each \( e \in L - \{0\} \). Now-

\( C(L) \) contracts via \( W \mapsto L+W \) to

\[ \{ W \in Z_L \mid L+W = W \} \text{.} \]
and the mapping

\[ L(V,u) \rightarrow \{ w \in \mathbb{Z}_L^I \mid L + w = w \} \]

has a section, namely, take a complement \( H \) to \( L \) and send \( W \) to \( H \cap W \). Therefore in virtue of

**Lemma:** If \( Y \) is a retract of \( X \), then the mapping cone of the retract map \( r: X \rightarrow Y \)

is the suspension of \( X/Y \).

**Proof:**

\[ \text{cone on } r \]

\[ \rightarrow \]

\[ \text{susp of } X/Y. \]

What this means is that

\[ C(e)_{L(V,u)} \cong C(L) \sim S(L(V,u)/L(H, u \cap H)) \]

By induction \( L(V,u) \) is a bouquet of \( S^{m-2} \)'s, so

\[ S(L(V,u)/L(H, u \cap H)) \]

will be a bouquet of \( S^{m-1} \)'s. Then as we attach further cones on \( L(V,u) \), this will remain the case, and so we get a proof of Lemma 3.
We now have many examples of spherical simplicial complexes, and it is now necessary to understand what implications result for the homology.

First let us consider the chain building \( T(S) \).

First let \( S \) be a partially ordered set with a dimension function \( d: S \to \mathbb{N} \) such that \( d(a_2) = d(a_1) + 1 \) whenever \( a_2 \) immediately follows \( a_1 \). Then filter \( S \) by putting

\[
F_p S = \{ s \mid d(s) \leq p \}
\]

\( \varnothing = F_0 S \subset F_1 S \subset \cdots \)

Put \( S_p = \{ s \mid d(s) = p \} \), and let \( \mathcal{F} = \{ S_p \} \).

Now given \( F: S \to \mathbb{A}^{\mathbb{N}} \) one has the chain complex

\[
C_p(S, F) = \bigoplus_{s_0 < \cdots < s_p} F(s_0)
\]

for computing \( \lim_{\rightarrow s} F = H_p(S, F) \), and if \( S' \subset S \), one puts

\[
C_p(S', F) = \frac{C_p(S, F)}{C_p(S, F)}
\]

a short exact sequence of complexes

\[
0 \longrightarrow C_p(S', F) \longrightarrow C_p(S, F) \longrightarrow C_p(S, S'; F) \longrightarrow 0
\]

hence a long exact sequence in homology.

Thus from the filtration \( \{ F_p S \} \) we will get a spectral sequence

\[
E^{1}_{pq} = H_{p+q}(F_p S, F_{p+1} S; F) \Rightarrow H_{p+q}(S; F)
\]
because the predecessors of an $s \in F_p$ lie in $F_{p-1}$

$$C_x(S_p ; s_j) = \bigcap_{\alpha} \mathcal{F}(\alpha)$$

and this breaks up into a sum over the elements of $S_p$:

$$C_x(S_p ; s_j) = \bigcap_{\alpha} C_x(S_{p-1} ; s_j)$$

and so our spectral sequence takes the form

$$E_{pq}^1 = \bigoplus_{s \in S_p} H_p(S_{p-1} ; s_j) \Rightarrow H_{pq}(S_j ; F)$$

Further it clear that from

$$\cdots \longrightarrow H_x(\overline{s_j} \setminus s_j ; F) \longrightarrow H_x(\overline{s_j} ; F) \longrightarrow H_x(\overline{s_j} - s_j ; F) \longrightarrow \cdots$$

To simplify notation put $\overline{s_j} = \overline{s}$ and $\overline{s_j} - s_j = \overline{s} - s$.

Cor. Assume $(\overline{s}, \overline{s} - s)$ is $F$-spherical of dim $d(s)$, i.e.

$$H_i(\overline{s}, \overline{s} - s ; F) = 0 \quad i \neq d(s)$$

and that $S$ is $F$-spherical of dim $d = \max d(s) = n$.

Then one gets an exact sequence

$$0 \longrightarrow H_n(S_j ; F) \longrightarrow \bigoplus_{s \in S_n} H_n(\overline{s_j} ; s_j ; F) \longrightarrow \cdots \longrightarrow \bigoplus_{s \in S_0} H_n(\overline{s_j} ; s_j ; F) \longrightarrow 0$$
Better statement is that if
\[ H_2(\tilde{S}; F) = 0 \quad i \neq d(\sigma) \]
then the complex which is the \( E_{i0}^1 \)-term:
\[ \cdots \rightarrow \bigoplus_{\sigma \in S_1} H_1(\tilde{S}, \tilde{S} - \sigma; F) \rightarrow \bigoplus_{\sigma \in S_0} F(\sigma) \rightarrow 0 \]
can be used to compute \( H_*(S; F) \).

Example 1: Let \( S \) be the ordered set of simplices in a simplicial complex \( K \), \( d(\sigma) = \dim(\sigma) \) so that
\[ H_2(\tilde{S}, \tilde{S} - \sigma; \mathbb{Z}) = \begin{cases} 0 & i \neq d(\sigma) \\ \mathbb{Z} & i = d(\sigma) \end{cases} \]
Then one gets that one can compute \( H_*(K, \mathbb{Z}) \) using the complex of simplicial chains on \( K \).

Example 2: Let \( V \) be a vector space of \( \dim n > 0 \) and \( \mathcal{S} = \) proper subspaces of \( V \). Put
\[ I(V) = H_{n-1}(\tilde{V}, \tilde{V} - V; \mathbb{Z}) = \tilde{H}_{n-2}(T(V)) \quad \text{if} \ n > 2 \]
where \( \tilde{V} = \{ W \leq V \mid 0 < W \} \), \( \tilde{V} - V = T(V) \). Then one has from the above, exact sequence:
\[ 0 \rightarrow I(V) \rightarrow \bigoplus I(V^{n-1}) \rightarrow \bigoplus I(V^{n-2}) \rightarrow \bigoplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \]
Now let $GL(V)$ act on the above exact sequences tensored with $\mathbb{M}$ where $\mathbb{M}$ is a $GL(V)$-module. One gets in dimension $p$

$$\bigoplus_{V_p \in \mathbb{M}} I(V_p) \otimes \mathbb{M}$$

which is a module induced from the stabilizer of $V_p$; here choose a "base" flag $0 = V_0 < V_1 < \cdots < V_n = V$, and let $GL(V, V_p)$ be the stabilizer of $V_p$. Then one has

$$\bigoplus_{V_p \in \mathbb{M}} I(V_p) \otimes \mathbb{M} = Z[GL(V)] \otimes_{Z[GL(V, V_p)]} \mathbb{M}$$

so by Shapiro:

$$H_\ast(GL(V), \bigoplus_{V_p \in \mathbb{M}} I(V_p) \otimes \mathbb{M}) = H_\ast(GL(V, V_p), I(V_p) \otimes \mathbb{M})$$

so we get a spectral sequence

$$E_{p+q}^1 = H_q(GL(V, V_p), I(V_p) \otimes \mathbb{M}) \Rightarrow H_{p+q}(GL(V), \mathbb{M})$$

Here $\mathbb{M}$ could be more generally a complex of $GL(V)$-modules. I have in mind the case where $V$ is a subspace of $\mathbb{Z}^n$ in which case we have

$$1 \rightarrow GL(\mathbb{Z}, V) \rightarrow GL(\mathbb{Z}, V) \rightarrow GL(V) \rightarrow 0$$

so one has

$$H_\ast(GL(\mathbb{Z}, V), \mathbb{Z}) = H_\ast(C_\ast(GL(\mathbb{Z}, V), \mathbb{Z})) = H_\ast(GL(V), \mathbb{Z})$$

$$C_\ast(GL(\mathbb{Z}, V), \mathbb{Z}) \otimes Z[GL(\mathbb{Z}, V), \mathbb{Z}]$$
Recall the structure of $I(V)$:

$$I(V) = \bigvee_{H \in \mathcal{H}_L} S I(H)$$

where $H$ runs over the hyperplanes complementary to the line $L$. Thus we have

$$I(V) = \tilde{H}_{n-1}(ST(V)) = \bigoplus_{H \in \mathcal{H}_L} \tilde{H}_{n-2}(ST(H))$$

or

$$I(V) = \bigoplus_{H \in \mathcal{H}_L} I(H)$$

which determines the structure of $I(V)$ as a module over $GL(V, L) = GL(H \oplus L, L) = [GL(H) \times GL(L)] \rtimes \text{Hom}(H, L)$ namely

$$I(V) = \mathbb{Z}[GL(V, L)] \otimes \mathbb{Z}[GL(H) \times GL(L)] I(H)$$

(In general it is probably true that if $V \rightarrow W$, then as a $GL(V, W)$-module

$$I(V) = \mathbb{Z}[GL(V, W)] \otimes \mathbb{Z}[GL(A) \times GL(W)] I(A) \otimes I(W)$$

where $A$ is a complement for $W$ in $V$.)

From now on the ground field is finite with $q = p^d$ elements.

Lemma: If $M$ is a $\mathbb{Z}_p$-module, then

$$H_+(GL(V), I(V) \otimes M) = 0$$

Proof: The subgroup $GL(V, L)$ is of index $\text{card}(p V) = q^{n-1} + \cdots + 1 \equiv 1 \pmod{p}$, hence from transfer theory one knows that $H_+(GL(V, L), I(V) \otimes M)$ maps onto the homology
in question. From (*) and Shapiro, this latter is isom. to
\[ H_1(G\ell(H) \times G\ell(L), I(H) \otimes M) \]
which as \( G\ell(L) \) is prime to \( p \), is a quotient of
\[ H_1(G\ell(H), I(H) \otimes M) \]
which is zero by induction. (Check lemma trivial
for \( n=0,1 \).)

**Lemma:** If \( M \) is a \( \mathbb{Z}(p) \)-module on which \( G\ell(V) \)
acts trivially, then \( H_0(G\ell(V), I(V) \otimes M) = 0 \) for \( n \geq 2 \).

**Proof:** Enough by preceding argument to check for \( n=2 \). Here one has
\[ 0 \to I(V) \to \bigoplus_{L \in V_2} \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z} \to 0 \]

where \( \Sigma \) means the diagonal \( \Delta : \mathbb{Z} \to \bigoplus_{L \in V_2} \mathbb{Z} \)
composed with \( \Sigma \) is multiplication by \( \text{card}(V_2) = s+1 \)
which is invertible in \( \mathbb{Z}(p) \). Thus over \( \mathbb{Z}(p) \) the above
sequence splits equivariantly so for any \( G\ell(V_2) \)-module \( M \)
\[ 0 \to H_0(G\ell(V_2), I(V) \otimes M) \to H_0(G\ell_2, \mathbb{Z}[G\ell(V_2)] \otimes \mathbb{Z}[B] \otimes M) \to H_0(G\ell, M) \to 0 \]
is exact.

\[ H_0(B, M) \]
where \( B \) = stabilizer of a line. Since \( G = G\ell(V) \) acts trivially
on \( M \), one has \( H_0(B, M) \cong H_0(G, M) \), and so one wins.
Lemma: The exact sequence at bottom of page 3 splits as an exact sequence of $GL(V)$-modules after tensoring with $\mathbb{Z}(p)$. Seems to be false already for $n=3$.

Proof: I will only prove this at the bottom. I already know that $\bigoplus_{L \in GL(V)} \mathbb{Z} \to \mathbb{Z}$ has an equivalent splitting since $\text{card}(PV) = q^n - 1 \equiv 1 \text{ (p)}$. Now consider:

$$
0 \to \bigoplus \mathbb{Z} \to \mathbb{Z} \to \bigoplus \mathbb{Z} \to 0
$$

The exact sequence splits canonically and $\alpha$ and $\beta$ have canonical sections ($\beta$ because $\text{card}\{H_{2cV}\} \equiv 0 \text{ (GL(V), H_{2cV})}$ is prime to $p$). Thus by diagram chasing, one can construct canonical contracting homotopies for the $d_1$ sequence.

To get a section of $\alpha$, one takes a function $\sum f(L) \in \mathbb{Z}$ and extends it to $\bar{f}(L_{cH}) \to f(c)$. Then $(\alpha f)(L) = \sum f(L) = [\mathcal{R}(4) - f]$. This is only reasonable section $j: \bigoplus \mathbb{Z} \to \mathbb{Z}$.

$$
0 \to \bigoplus \mathbb{Z} \to \mathbb{Z} \to \bigoplus \mathbb{Z} \to 0
$$

$$
0 \to \bigoplus \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0
$$
Theorem. Let $G_5$ be the group of perm. of an infinite set $S$. Then $\tilde{H}_*(G_5) = 0$.

**Lemma 1:**
Let $S = S' \cup S''$ where $S', S''$ are of same card as $S$ and $S'_A$, and let $G'_S$ be the subgroup of $G_S$ fixing the elements of $S''$. Then the inclusion of $G'$ in $G_S$ induces the zero map on homology.

**Proof.** Consider the permutative category $\mathcal{C}$ consisting of sets having the same card as $S$ together with $\phi$, where the operation is $\cdot$. Then it is clear that $K$-theory of $\mathcal{C}$ is trivial, for one has countable direct sums (standard "flaskeness" argument). Thus $\tilde{H}_*(G_S) = 0$ where $[S]$ runs over the translation cat of the iso. classes of $\mathcal{C}$.

But there are two iso classes $[\{ e \}, e^2 = e]$. Thus mult. by $e$ is zero on $\tilde{H}_*(G_S)$ which proves the lemma.

Next consider the partially ordered set whose elements are subsets $S'$ of $S$ where $S', S''$ are of same card as $S$. On put $S'_0 < S''_0$ if $S' < S''$ and $S'' - S' \subset S$. Clearly $G$ acts on $\mathcal{T}$.

**Lemma 2:** $\mathcal{T}$ is contractible.

**Proof.** It suffices to show any finite subset $K$ of $\mathcal{T}$ with the induced order contracts to a point in $\mathcal{T}$. Let $S_0 = \bigcap S'$ where $K'$ is a maximal subset of $K$ such that $S_0$ has the same card as $S$. Then
Given any $S \in \mathcal{K}$ either $S \in \mathcal{K}'$ and so $S_0 < S_1$ or $S_1 \in \mathcal{K}'$ and so $S_0 S_1 < S_1$ of card $< \text{card}(S)$. Now split $S_0 = S_{01} \cup S_{02}$ where $\text{card}(S_{01}) < \text{card}(S_{02}) = \text{card}(S)$. Then for $S_1 \in \mathcal{K}'$ we have $S_0 S_1 < S_1$ and for $S_1 \in \mathcal{K} - \mathcal{K}'$, we have $S_1 \cup S_{01} \notin J$ and $S_1 < S_{1} S_{01} < S_{01}$.

(Proof in text: I know $\text{card}(S_1 \cap S_0) < \text{card}(S) = \text{card}(S)$, hence $\text{card}(S_1 \cap S_{01}) < \text{card}(S_1)$ and so $S_{1} S_{01} - S_{1} S_{01} - S_{1} = S_{1} S_{01} - S_{1} S_{01} \leq \text{card}(S)$. Therefore, for all $S_1 \in \mathcal{K}'$ we have $S_1 \leq S_{1} S_{01} \leq S_{01}$ in J.)

given $T \in \mathcal{K}$ either $T \in \mathcal{K}'$ and so $\text{card}(T \cap S_0) < \text{card}(S)$. Now split $S_0 = T_0 \cup U_0$ where $T_0 \cup U_0 \neq S$. Then given $T \in \mathcal{K}'$ we have $T = T_0 \cup T_0 > T_0$ (as $T - T_0 > S_0 - T_0 = U_0$). If $T \in \mathcal{K} - \mathcal{K}'$, then $T_0 U_0 \in \mathcal{J}$ because its complement contains $U_0 - T_0 U_0$ and $U_0 T_0 \neq S$, so $\mathcal{J}$ is negligible. Moreover $T < T_0$ for some reason, and thus we have $T < T_0$ for all $T \in \mathcal{K}$. Now I have only to check that $T_1 < T_2 \Rightarrow T_1 U_0 < T_2 U_0$.

The problem of $T_1 T_2 \in \mathcal{K}'$ for $T_1 \neq T_2$ is solved by proving $T_1 U_0 < T_2 U_0$. Also the problem $T_1 T_2 \in \mathcal{K}'$ and $T_2 \in \mathcal{K}'$ for $T_2 \neq T_0$ we have seen that if $T_0 = T_0 < T_0 U_0$ then $T_1 U_0 < T_2 U_0$. However the only way this could fail would for $T_2 U_0 - T_1 U_0 = T_2 (T_1 U_0 T_2 U_0)$ to be
negligible. However, this means that we have to make $T_0$ so small that it doesn't eat up the difference $T_2 - T_1$. (This should be clear anyway, for $T_0$."

no problem if $S_0 \subseteq T_1$, or if $S_0 \cap T_2$ is negligible, or if $S_0 \cap T_1$ is negligible and $S_0 \subseteq T_2$.) So done.

Now this gives us a resolution of $\mathbb{Z}$:

$$\cdots \rightarrow \frac{\mathbb{Z}}{T_0 < T_1} \rightarrow \frac{\mathbb{Z}}{T_0} \rightarrow \mathbb{Z} \rightarrow 0$$

(Chains on the associated simplicial complex).

since $G$ acts transitively on the simplices of a given dimension and the stabilizer of $T_0 < T_1 < \cdots T_n$ in $G^{+2}$ one gets a spectral sequence

$$E_1^{pq} = H_q^G(G^{p+2}) \Rightarrow H_{p+q}(G).$$

and clearly $E_2^{20} = \mathbb{Z}$ in degree 0. Now if one has $H_q^G(G) = 0$ for $q < n$, then $H_q^G(G^s) = 0$ for $q < n$ by Kunneth. Thus from the spec. seq. we get

$$\tilde{H}_q(G^2) \rightarrow \tilde{H}_q(G)$$

and

$$H_q(G) \oplus H_q^s(G) \rightarrow \text{zero map by lemma 1}.$$
Suppose \( E, F \) are projective \( A \)-modules. Define \( \Theta: E \to F \) to be "compact" if

\[
\Theta(e) = \sum_{i=1}^{n} \lambda_i(e) f_i
\]

for \( \lambda_i \in \text{Hom}_A(E, A) \), \( f_i \in F \). Thus the compact maps are those in the image of the canonical maps

\[
\text{Hom}_A(E, A) \otimes_A F \to \text{Hom}_A(E, F)
\]

(Do this map always injective? Seems so, for \( F = A^{(1)} \) hence in general)

**Example:** Take \( E = A^{(5)} \), \( F = A^{(3)} \). Then for \( \Theta \) to be compact means that if we write its matrix

\[
\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5
\end{bmatrix}
\]

Then only finitely many of the rows of the matrix are non-zero.

Another way of stating that \( \Theta \) is compact is to say that \( \Theta \) factors

\[
E \to P \to F
\]

where \( P \) is a f.p. projective \( A \)-module.

**Question:** Assume \( \Theta \in \text{End}(E) \) is compact and \( 1 + \Theta \) is an automorphism. Does it follow that \( (1 + \Theta)^{-1} = 1 + K \) with \( K \) compact?

Yes, because

\[
(1 + \Theta)(1 + K) = 1 \implies \Theta + K + \Theta K = 0
\]

\[
\implies K = -\Theta - \Theta K
\]

which is compact as the compact operators form an ideal.
Question: Take $E = A^{(\infty)} = Ae_1 \oplus A e_2 \oplus \ldots$, so take we have $GL(A) \subset Aut_{1+}(A^{(\infty)})$. Does this inclusion induce an isomorphism on homology?

Here's how we might prove this when $A$ is a field. I observe that given a decomposition $E = P \oplus F$ with $P \in P_A$, then there is an embedding $GL(P) \subset Aut_{1+}(E)$ obtained by taking the direct sum of an auto of $P$ with the identity of $F$. Assume now that $F'$ is another complement for $F$, say $F' = \{ \phi(h(f), f) \in P \oplus F \}$ where $h: F \to P$. Then if we denote by $\lambda_x: GL(P) \to Aut_{1+}(E)$ the homomorphism defined by the complement $F$ and $\lambda'$ the one defined by $F'$, we have

$$\lambda_x'(\phi)(1+h)f = (1+h)f$$

$$\lambda_x'(\phi)P = \phi(P) = \lambda_x(\phi)P$$

$$\lambda_x'(\phi) = (1+h)\lambda_x(\phi)(1+h)^{-1}$$

where $1+h: p+f \mapsto p+h(f) + f$ translates $F$ to $F'$. Thus the homomorphisms $\lambda_x, \lambda_x'$ are conjugate, and so we obtain a well-defined homomorphism

$$\lambda_{PC}: H_x(GL(P)) \to H_{x}(Aut_{1+}(E))$$

depending only on the direct summand $P$ of $E$. Now if $P, Q$ are direct summands of $E$ with $P \lt P Q$, then $P$ is a direct summand of $Q$, so we can choose $P \oplus C = Q$, $Q \oplus F = E$, and take the complement $C \oplus F$ for
P. Then it is clear that one gets a comm. diag.

\[
\begin{array}{c}
H^*_x(\text{GL}(P)) \xrightarrow{\cdot p_{\text{ce}}} \\
\downarrow p_{\text{ce}} \quad \downarrow \\
H^*_x(\text{GL}(Q)) \xrightarrow{\cdot q_{\text{ce}}}
\end{array}
\]

(This is the familiar fact that modulo inner auto,
one has a functor \( P \rightarrow \text{GL}(P) \)
with respect to injections onto direct summands. \( \text{Ques.} \) I get in this way a 
\[ \text{Ques.} \) I get in this way a \
\text{biene of some sort? any 

\text{interesting gerbes around?} \)

Thus putting all this together I get a map

\[
(*) \quad \lim_{P} H^*_x(\text{GL}(P)) \longrightarrow H^*_x(\text{Aut}_{\text{\text{ht}}}(E))
\]

where \( P \) runs over the fin. type. direct summands of \( E \).

Thus the question under discussion comes down to
whether the above map is an isomorphism.

So let \( G \) be a finitely gen. subgroup of \( \text{Aut}_{\text{\text{ht}}}(E) \).

It would probably be enough to know that \( \text{co} \) for
any such \( G \) one can find a decomposition \( E = P \oplus F \)
stable under \( G \) such that \( G \) acts trivially on \( F \).

Certainly this implies the map \( (*) \) is onto. But also
\[ \text{any element} \ x \ \text{in the kernel of} \ \text{co} \ \text{would}
\]
\[ \text{come from the homology of a f.t. suby of} \ \text{GL}(P), \text{and}
\]
\[ \text{for this to go to zero in} \ H^*_x(\text{Aut}_{\text{\text{ht}}}(E)) \text{\ would mean} \ x \text{\ does in}
\]
\[ H^*_x(G') \text{\ for some finite type} \ G' \subset \text{Aut}_{\text{\text{ht}}}(E). \] Thus
\[ \text{if we could find}
\]
\[ G \subset \text{GL}(P)^\text{c} \xrightarrow{\cdot} \text{Aut}_{\text{\text{ht}}}(E) \]
\[ \downarrow \]
\[ G' \subset \text{GL}(P')^\text{c'} \]

\[ \text{E} = P \oplus F \text{\ stable such that} \ G' \text{\ acts trivially on} \ F' \text{\ we would win.} \]
To suppose \( g_1, \ldots, g_n \) are a finite number of elements of \( \text{Aut}(E) \), and put \( E_i = 1 - g_i \). These are compact, hence in particular \( \text{Im}(\Theta_i) \) is finitely generated. Put \( M = \sum \text{Im}(\Theta_i) \) and note that the set of \( \mu \in \text{End}(E) \) \( \cap \text{Im}(\Theta) \subset M \) is a right ideal, in particular closed under sum and product. Thus if \( G \) is the group generated by the \( g_i \), then \( \text{Im}(g^{-1}) \subset M \) for any \( g \in G \).

(To make this more intelligible, observe that \( g \mapsto g^{-1} \) transforms product to \( x + y + xy \) which shows that the set of \( g \in G : \text{Im}(g^{-1}) \subset M \) is closed under product.

Also if \( (1+x)^{-1} = 1 + y \), then \( y = -x - gxy \) has its image contained in \( M \).

Next put \( K = \bigcap \text{Ker}(\Theta_i) \) and note that \( g = 1 \) on \( K \), for all \( g \in G \). Thus we have identified \( G \) with a subgroup of the group of quasi-invertible elements in the ring (without unit) of endomorphism \( \Theta \) of \( K \subset \text{Ker}(\Theta) \), \( \text{Im}(\Theta) \subset M \).

\[ \text{Hom}(E/K, M) \subset \text{Hom}(E, E) \]

Now I would like to find a decomposition \( E = P \oplus F \) such that \( P \supset M \), \( F \subset K \), whence each \( g \in G \) would preserve this decomposition and act trivially on \( F \). Now, if this could be done, then \( E/K \approx E/P \) would be finitely generated. Hence

\[
0 \to K \to E \xrightarrow{(g)} F^n
\]

there is no reason for this to be true in general. Example:
take $\lambda : A^{(\infty)} \rightarrow A$, $\lambda : \bigoplus_i A_i \rightarrow A$. In order that
$\ker (\lambda)$ be cofinitely generated, it is necessary that the
$\text{Im}(\lambda)$ = ideal gen. by the $a_i$ be finitely generated. Thus
we must assume $A$ is noetherian.

If $A$ is noetherian, then

$$E/K \subset \text{Im}(\lambda)$$

so $E/K$ is finitely generated. At least when $E = A^{(\infty)}$
this means we can find a direct summand $P$ of $E$
with $P \leq PA$ such that $P \rightarrow E \rightarrow E/K$ is onto. Then

$$P \rightarrow E \rightarrow E/P \rightarrow K \rightarrow E/P$$

for $E/P$ is projective. And since $E/P$ is projective we can find $F \leq K$, $F \oplus P = E$. Now
if we started with a $P$ which not only maps onto $E/K$
but also contains $M$, then we have $P \oplus M \subset F \oplus K$
as desired. This proves:

**Lemma:** Let $G$ be a finitetype subgroup of
$\text{Aut}(A^\infty)$, $E =$ free infinite type $A$-module, $A$ noetherian.
Then $E = P \oplus F$ with $P$ free finitetype such that
$G$ preserves this decomposition and such that $G$ acts
trivially on $F$.

Counterexample when $A$ is not noetherian: Choose
$A_1 < A_2 < A_3 < \cdots$ and define $G(x_i) = x_i + \sum_{i \geq 2} a_i x_i$. Then
$\ker (G - 1)$ does not contain an $F$ because $\text{Im}(G - 1) = A_1 + \cdots$
is not of finitetype.
Ideas: To what extent do the decompositions of the form $E = P \oplus F$ partially ordered by requiring $(P, F) \preceq (P', F')$ if $P \supseteq P'$, $F \supseteq F'$ form a directed set. In the noetherian case this is $\emptyset$ because given $(P_1, F_1), (P_2, F_2)$, $E/F_1 \cap F_2 \subseteq E/F_1 \times E/F_2$ so $E/F_1 \cap F_2$ is of finite type. Thus, choosing $P$ a finite type direct summand of $E$ sufficiently big so as to include $P_1 + P_2$, and map onto $E/F_1 \cap F_2$, then we have $F_1 \cap F_2 \hookrightarrow E/P$ so we can find a complement $F$ to $P$ contained inside $F_1 \cap F_2$.

Therefore in the noetherian case with $E = A^{(I)}$ one really obtains $\text{Aut}_{\text{nic}}(E)$ as a filtered direct limit of $\text{GL}(P)$.

Last time I was intrigued by the fact that given a finite type subgroup $G$ of matrices of the form

$$
\begin{pmatrix}
\star & \star \\
\circ & \text{Id}
\end{pmatrix}
$$

I have managed to conjugate it into a subgroup of the form

$$
\begin{pmatrix}
\star & 0 \\
0 & \text{Id}
\end{pmatrix}
$$

but where $\star$ is very big.
Suppose $A$ is a field to simplify, I can consider then the category of vector spaces over $A$ of countable dimension under addition. This is an abelian category in which exact sequences split, and its $K$-theory is trivial by the usual countable sum argument. Thus if $E = A^{(\infty)}$ we find as in the case of countable sets that the idempotent endomorphism of $H_\ast(\text{GL}(E))$ obtained from $E \oplus E \cong E$ is zero. To put this another way, if we decompose $E$ into $E' \oplus E''$ where $E'$ and $E''$ are both countable, then the induced homomorphism $H_\ast(\text{GL}(E')) \to H_\ast(\text{GL}(E''))$ is trivial.

So as before we consider the simplicial complex whose $g$-simplices are decompositions

$$E = F_0 \oplus F_1 \oplus \cdots \oplus F_{g+1}$$

with each $F_i$ of infinite dimension. Precisely a vertex is a decomposition $E = F_0 \oplus F_1$ and one says this decomposition is $\prec$ another $E = F'_0 \oplus F'_1$ if $F'_0 \subset F_0$ and $F'_1 \supset F_1$ and $F'_0/F_0$ is indistinct. In this case we get a 2-simplex $E = F_0 \oplus F_1 \oplus F' \oplus F_1'$.

Another description of a decomposition $E = F_0 \oplus F_1$ is to given the projection $E$ on $F_0$. In this way a $g$-simplex appears as a decomposition $1 = e_0 + \cdots + e_{g+1}$ into orthogonal idempotents such that $e_i$ is a projection onto an infinite dimension subspace.
Now one wants to show this simplicial complex is contractible. So suppose we are given a finite set $F$ of idempotents $e$ in $\text{End}(E) \cap \text{Im}(e) \cap \text{Im}(1-e)$ all of infinite dimension. Suppose that we have two

$$E = F_0 \oplus F_1 = F'_0 \oplus F'_1$$

such that $F_0 \cap F'_0$ has infinite dim. Then it might happen that $F_1 + F'_1 = E$. (e.g. even if $F_0 = F'_0$) so our previous argument for sets will not work.

Start by trying to understand if this simplicial complex is connected. Thus I suppose given two decompositions $E = A_1 \oplus B_1 = A_2 \oplus B_2$ which I want to connect. Consider the map $A_1 \to E \to B_2$ whose kernel is $A_1 \cap A_2$.

If $A_1 \cap A_2$ is infinite, then by subdividing it into two infinite pieces if necessary I then reach the case where $A_1 = A$.  

December 15, 1973

Let $A$ be a field, let $E$ be a vector space over $A$ (not nec. fin. dim.) and let $\text{Aut}_c(E)$ denote the group of automorphisms of $E$ of the form $1 + \Theta$ where $\Theta$ is of finite rank. I knew that any finite type subgroup $G$ of $\text{Aut}_c(E)$ stabilizes a splitting $E = P \oplus F$ with $P$ fin. dim. and $G$ acts trivially on $F$. (The point is that such splittings form a filtered set under the ordering $(P, F) \leq (P', F')$ if $P \subseteq P'$ and $F \supseteq F'$, for given $(P_i, F_i)_{i=1,2}$, then these are dominated by $(P, F)$ where $P$ is chosen containing $P_i + P_2$ such that $P + (F_i \cap F_2) = E$, and $F$ is a complement of $P \cap F \cap F_2$ in $F \cap F_2$. Thus $E$ determines an ind. object in the category of complemented injections of finite dimensional vector spaces, and $\text{Aut}_c(E)$ is just the limit of $\text{Aut}(P)$ as $(P, F)$ runs over this ind. object.)
Let $V$ be a $C$-vector space of dimension $N$, and $G_p(V)$ the Grassmannian of $p$ planes in $V$. Let $e_1, \ldots, e_n$ be a basis for $V$, $V_i = ke_1 + \cdots + e_i$, $0 \leq i \leq n$.

One knows $G_p(V)$ has a cell decomposition given by the Schubert cells as follows. I take $p = 2$ to simplify.

Given a 2-plane $A$ in $V$ after performing row-operations on the $2 \times N$ matrix given by a basis for $A$, one gets a canonical form for $A$ of the form:

$$
\begin{pmatrix}
*x & \cdots & * & 1 & 0 & \cdots \\
*x & \cdots & 0 & * & 1 & \cdots
\end{pmatrix}
$$

with $n_2 + 1$ entries (the 1 counts the 0)

where the $*$'s are arbitrary complex numbers. Call $C_{n_1, n_2}$ the set of $A$ with canonical form of the above type. It is a cell of ex. dimension $r_1 + r_2$. One clearly gets a decomposition of $G_2(V)$ into cells $C_{n_1, n_2}$ for each $0 \leq r_1 \leq r_2 \leq N - 2$; as these cells are even-dimensional, each cell $C_{n_1, n_2}$ gives rise to a homology class $[C_{n_1, n_2}]$ in $H_{2(r_1 + r_2)}(G_2(V))$, and in this way we get a basis for $H_*(G_2(V))$.

Clearly if $V_i = ke_1 + \cdots + ke_i$ then

$$
C_{n_1, n_2} = \{ A | 0 = \cdots = A_n V_{n_1} < A_n V_{n_1+1} = \cdots = A_n V_{n_2+1} < A_n V_{n_2+2} = \cdots = A \} 
$$

which shows these cells depend only on the flag $\{ V_i \}$. 

December 18, 1973  Grassmannian (groggly again)
Another way of seeing this is to note that the cell $G_{n_1r_2}$ is simply the orbit under the action of the group

$$N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{r_1} = \text{stability group of the flag of the matrix:} \begin{pmatrix} 1 \\ 1 \\ r_2+2 \end{pmatrix}$$

Thus these cells are the $N$-orbits (or also the $B$-orbits, $B = \text{upper } \Delta \text{ matrices}$) on $G_2(V)$.

One has

$$\overline{C_{n_1r_2}} = \bigcup_{a \leq b} \overline{C_{ab}}$$

$$= \left\{ A_2 \mid \dim(AnV_{n_1+1}) \geq 1, \dim(AnV_{r_2+2}) \geq 2 \right\}$$

where the closure is the same for both $C$-topology and the Zariski topology. $\overline{C_{n_1r_2}}$ is a cycle in the alg. variety $G_2(V)$ which can be desingularized as follows.

Let $G_{11}(V)$ be the flag bundle of the subbundle on $G_2(V)$, i.e. the space of pairs, $(l, A)$ where $l$ is a line in the 2-plane $A$. Put

$$\overline{C_{n_1r_2}} = \left\{ (l, A) \mid l \subset V_{n_1+1}, A \subset V_{r_2+2} \right\}$$

This is non-singular because it is the projective bundle over $PV_{n_1+1}$ associated to the vector bundle $l \mapsto V_{r_2+2}/l$. 
It maps \( L \) onto \( C_{r_1r_2} \) bijectively over \( C_{r_1r_2} \).

Now enlarging \( N \) doesn't change the cells \( C_{r_1r_2} \), it only adds some more. Thus in the limit as \( G_2(V) \to BU_2 \), we have a basis for \( H_*(BU_2) \) given by \([C_{r_1r_2}]\) with \( 0 \leq r_1 \leq r_2 \). On the other hand one knows that

\[
[H_*(BU_1) \otimes H_*(BU_1)]_{r_2} \xrightarrow{\sim} H_*(BU_2)
\]

whence if \( b_i \in H^i(BU_1) \) is the class of \( P^i \), we get the basis \( b_i \cdot \tilde{b}_i \) for \( H_*(BU_2) \). The problem now is to relate these bases.

First the map \( BU_1 \times BU_1 \to BU_2 \) can be viewed as the flag bundle of the canonical bundle. Let \( G_{11}(V) \) be the flag bundle of the canonical 2-plane bundle in \( G_2(V) \). Then \( G_{11}(V) = \{ (l_1, l_2) \} \), which can also be identified with the manifold of pairs \( (l_1, l_2) \) where \( l_1, l_2 \) are orthogonal lines in \( V \). This gives us maps

\[
P^V \times P^V \xleftarrow{i} G_{11}(V) \longrightarrow G_2(V)
\]

where \( i \) is an equivalence in a range increasing with \( \dim V \). This gives us the map \( BU_1 \times BU_1 \to BU_2 \).

Denote by \( L_1, L_2 \) the line bundles on \( G_{11}(V) \) whose fibres at \( (l_1, l_2) \) are \( l_1, l_2 \) respectively. Then \( G_{11}(V) = \) projective bundle of the quotient bundles \( V/O(-1) \) on \( P^V \). Recall:
Lemma 1. Let $E$ be an $n$-dim vector bundle over $X$, $f : PE \to X$ the associated projective bundle, $t = c_1(O(1)) \in H^2(PE)$. Then

$$H^*(PE) = H^*(X)[t] / (t^n + c_{1}(E)t^{n-1} + \cdots + c_{n}(E))$$

and if $a(T) \in H^*(X)[T]$, then

$$(x) \quad f_{*}(a(t)) = \text{res} \left( \frac{a(T)dT}{T^n + c_{1}(E)T^{n-1} + \cdots + c_{n}(E)} \right)$$

Proof of (x). Embed $E$ in a trivial bundle $V$ of rank $N$ so that one has

$$\begin{array}{cc}
PE & \xrightarrow{f} \xrightarrow{g} \xrightarrow{a} X \\
\xrightarrow{\alpha} & \xrightarrow{\beta} & \xrightarrow{\gamma} \\
\xrightarrow{\delta} & \xrightarrow{\epsilon} & \xrightarrow{\zeta} \\
\end{array}$$

One has that $PE$ is where the homomorphism

$$O_{PV}(-1) \subset g^*V \to g^*(V/E)$$

vanishes, whence

$$i_{\alpha}^{*}(t) = e(C(1)) \otimes g^*(V/E)$$

= $T^q + g_{c_{1}(V/E)}T^{q-1} + \cdots + g_{c_{n}(V/E)}$ where $q + n = N$, and $t = c_{1}(O(1)) \in H^2(PV)$. Then

$$f_{*}(a(t)) = g_{*}i_{\alpha}^{*}a(t) = g_{*}(i_{\alpha}^{*}T) = \text{coeff of } T^{N-1} \text{ in } (T^{q} + \cdots + c_{n}(V/E))a(t)$$

= $\text{res} \left( T^{N-1} + c_{n}(V/E) \right) a(t) dT$

since $V$ is trivial. But $\prod [T^{N}] = [T^{q} + \cdots + c_{n}(V/E)][T^{n} + \cdots + c_{n}(E)]$ so cancelling we get (x).
Now apply this to $G_n(V)$:

$$G_n(V) \xrightarrow{f} PV \times PV \xrightarrow{p_{L_1}} PV$$

and one finds $H^*(G_n(V))$ has a base over $\mathbb{Z}$ given by the monomials $t_1^a t_2^b$ $0 \leq a < N$, $0 \leq b < N-1$ and

$$f_*(a(t_1, t_2)) = \int a(t_1, t_2) \frac{a(t_1, t_2) dT_2}{T_2^{-1} + c_1(V/L) T_2^{-2} + \ldots} = \int \frac{[T_2 + c_1(L_1)] a(t_1, t_2) dT_2}{T_2^{N-1}}$$

$$= \int \frac{[T_2 - t_1] a(t_1, T_2) dT_2}{T_2^{N}}$$

Thus,

$$\int a(t_1, t_2) = \text{coeff of } (T_1 T_2)^{N-1} \text{ in } (T_2 - T_1) a(t_1, T_2)$$

$$G_n(V)$$

$$= \sum a_{N,N-2} a_{N-1,N-1} \text{ if } a = \sum_{i,j} a_{i,j} T_1^i T_2^j$$

Thus we get

**Lemma 2.** The homology class of $G_n(V)$ regarded as the submanifold of $PV \times PV$ of orthogonal line pairs is $b_{N-1} \otimes b_{N-2} - b_{N-2} \otimes b_{N-1}$. 
But now
\[ \overline{C}_{l_1 l_2} = \{(l_1, l_2) | l_1 \in V_{l_{1+1}}, l_2 \in V_{l_{2+2}}, l_1 \neq l_2\} \]
\[ = G_{II}(V) \cap (PV_{l_{1+1}} \times PV) \cap (PV \times PV_{l_{2+2}}) \]
and this intersection is proper because it gives something of the right codimension. Thus the class of \( \overline{C}_{l_1 l_2} \) in \( H_\ast (P^2 V) \) is

\[ (b_{N-1} \otimes b_{N-2} - b_{N-2} \otimes b_{N-1}) \cap b_{l_1} \otimes b_{l_2+1} \]
\[ = b_{l_1} \otimes b_{l_2} - b_{l_1-1} \otimes b_{l_2+1} \]

Thus we see that

\[ \begin{array}{c}
H_\ast (P^2 V) \xrightarrow{\pi} H_\ast (G_{II}(V)) \xrightarrow{\cap} H_\ast (G_2(V)) \\
\downarrow \hspace{2cm} \downarrow \hspace{2cm} \downarrow \\
H_\ast (B U_1 \times B U_1) \xrightarrow{\cap} H_\ast (B U_2) \\
\end{array} \]

and so we obtain

**Proposition:** In \( H_\ast (B U_2) \) one has

\[ \text{cl} (\overline{C}_{l_1 l_2}) = \begin{bmatrix}
  b_{l_1} & b_{l_2} \\
  b_{l_2} & b_{l_1} \\
  b_{l_1-1} & b_{l_2+1} \\
  b_{l_1+1} & b_{l_2-1} \\
\end{bmatrix} \]
Generalize to $G_3(V)$.

\[ G_{III} V \xrightarrow{f} G_{II} V \rightarrow PV \rightarrow \mathfrak{p} \mathfrak{t} \]

\[ f_* \alpha(t_1,t_2,t_3) = \alpha_0 \frac{a(t_1,t_2,T_3) \, dT_3}{T_3^{N-2} + c_1(V/L_1 + L_2) T_3^{N-3} + \ldots} \]

\[ = \alpha_0 \frac{(T_3 + c_1(L_1))(T_3 + c_1(L_2)) \, a(t_1,t_2,T_3) \, dT_3}{T_3^N} \]

\[ = \alpha_0 \frac{(T_3 - t_1)(T_3 - t_2) \, a(t_1,t_2,T_3) \, dT_3}{T_3^N} \]

\[ \int a(t_1,t_2,t_3) = \text{coeff of } (T_1 T_2 T_3)^{N-1} \text{ in } (T_3 - T_1)(T_3 - T_2)(T_3 - T_1) \alpha(t_1,t_2,t_3) \]

**Lemma:** The cohomology class of $G_{III}(V) \subset PV^3$ is

\[ \Pi_{i>j} (t_i - t_j) = \begin{vmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_3 & t_2 & t_1 \end{vmatrix} \]

Now in general if you use the fact that

\[ \Pi_{i>j} (t_i - t_j) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_p \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \ldots & t_p \end{vmatrix} \]

one will find that the coh. class of $G_{III}(V) \subset PV^p$ is this Vandermonde determinant.

Now to find the class of $G_{n_1 \ldots n_p}$ in $(PV)^p$ you take cup product of the cohomology class corresponding to $PV_{n_1+1} \times \cdots \times PV_{n_p+1}$ which is $t_1 \cdots t_{n_1-1} \cdots t_{n_p-1}$ and you get the class.
which then capped with the fundamental cycle of $\mu^p$ which is $b_1 \otimes \cdots \otimes b_n$ gives the same determinant, but when $t_{n-i}$ is replaced by $b_i^{(j)}$, $b_i^{(j)} = 1 \otimes \cdots \otimes b_j \otimes \cdots \otimes 1$, $i$-th place. Thus it's clear that we get

**Proposition:** In $H_\ast(B\mu_p)$, the class associated to the Schubert cycle

$$\mathcal{C}_{n_1,\ldots,n_p} = \left\{ A_p \mid \dim (A_p \cap V_{n_1+i}) \geq i \quad i=1,\ldots,p \right\}$$

of $\dim \ n_1+\cdots+n_p$ is the determinant

$$\begin{vmatrix}
:b_1 & b_{n+1} & \ldots & b_{p+1} \\
:b_2 & b_{n+2} & \ddots & \vdots \\
& b_n & \ddots & \ddots \\
& & b_{n+1} & \ddots & b_{p+1}
\end{vmatrix}$$
Cohomology side. Here I use the canonical form

\[
C^{n_1, n_2} = \begin{pmatrix}
0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{pmatrix}
\]

which will be a cell of codim = \(n_1 + n_2\). One has

\[
C^{n_1, n_2} = C_{N-2-n_2, N-2-n_1}
\]

\[
= \{ A \mid \dim(A \cap V_{N-1-n_2}) \geq 1, \dim(A \cap V_{N-n_1}) \geq 2 \}
\]

and

\[
\tilde{C}^{n_1, n_2} = \{(l_1, l_2) \mid l_1 \subseteq V_{N-1-n_2}, l_2 \subseteq V_{N-n_1}\}
\]

\[
= \{(l_1, l_2) \in G_{11}(V) \mid l_1 \subseteq P V_{N-1-n_2}, l_2 \subseteq P V_{N-n_1}\}
\]

has the cohomology class

\[
t_1^{n_2+1} t_2^{n_1}
\]

But now if \(f : G_{11}(V) \to G_2(V)\) is the projection we know that

\[
f_* \text{cl}(\tilde{C}^{n_1, n_2}) = \text{cl}(C^{n_1, n_2})
\]

and that \(\exists! \alpha, \beta \in H^*(G_2(V))\) with

\[
t_1^{n_2+1} t_2^{n_1} = \text{cl}(\tilde{C}^{n_1, n_2}) = f^*(\alpha) + f^*(\beta) t_1
\]

This is the end of the page, where we have finished with the discussion on the cohomology classes and their properties. Applying the interchange, we have

\[
f^* : H^*(G_2(V)) \to H^*(P V_2)\]
where \( t_1 = c_1(L_1^+) \) is the generator for \( H^*(G_1(V)) \) over \( H^*(G_2(V)) \). We also know that \( f_*(t_1) = 1 \), whence

\[
\beta = f_*(c(G^1, \alpha)) = c(G^1, \alpha) = \beta.
\]

But applying the interchange of \( L_1, L_2 \) which is a symmetry of \( G_1(V) \) over \( G_2(V) \), one gets the equations

\[
\begin{align*}
_t^{r_2+1} & = f(a) + f(b) t_1 \\
_t^{r_2+1} & = f(a) + f(b) t_1
\end{align*}
\]

so solving

\[
\begin{bmatrix}
1 & t_1^{r_1+1} & t_1^{r_1} \\
1 & t_1^{r_1+1} & t_1^{r_1} \\
1 & t_1 \\
1 & t_2 \\
1 & t_2
\end{bmatrix}
\begin{bmatrix}
\beta \\
\beta \\
1 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
_t^{r_2+1} \\
_t^{r_2+1} \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
\]

so we have proved:

**Proposition:** In \( H^*(BU) \cong H^*(BU_1 \times BU_2) \), we have the following formula for the class assoc. to the Schubert cycles:

\[
\text{cl}(C^1 \cup C^2) = t_1^{r_1} t_2^{r_2} + t_1^{r_1} t_2^{r_2} t_1^{r_1} t_2^{r_2} + \cdots + t_1^{r_1} t_2^{r_2}
\]