V vector space of dimension n, V' subspace of dim m

$P(V,V') =$ ~~[illegible]~~ simp. cx. assoc. to ordered set of subspaces $W < V$ which are transv. to V': $W + V' = V$.

Let L be a line in V'. (If $m = 0$, then $P(V,V')$ is ϕ).
and define ~~a~~ the map

$$f: P(V,V') \longrightarrow P(V/L, V'/L) * \mathcal{H}_L \qquad * = \text{join}$$

where $\mathcal{H}_L = \{H \mid H \oplus L = V\}$ as follows:

$$W \longmapsto \begin{cases} W & \text{if } W \in \mathcal{H}_L \\ W+L/L & \text{if } W \notin \mathcal{H}_L \end{cases}$$

A simplex of the join ~~[illegible]~~ $K_1 * K_2$ of two simplicial complexes is a non-empty subset of $K_1 \cup K_2$ whose intersection with K_i is a simplex or empty. So it is clear f is simplicial.

To show f is a homotopy equivalence, let $\mathcal{U}$ be the open subset of $P(V,V')$ consisting of open simplices containing a vertex not in $\mathcal{H}_L$, and let $\mathcal{U}'$ be the open set which is the open star of ~~[illegible]~~ $\mathcal{H}_L$. Then $\mathcal{U}$ retracts to the subcomplex Y complementary to $\mathcal{H}_L$ which contracts to $\{W \mid L < W < V,\ W + V' = V\} \cong P(V/L, V'/L)$ via $W \mapsto L + W$. Thus $f: \mathcal{U} \longrightarrow$ open star of $P(V/L, V'/L)$ in join is a heq. Similar $f: \mathcal{U}' \longrightarrow$ open star of $\mathcal{H}_L$ in join is a heq. But next note

$$\mathcal{U} \cap \mathcal{U}' \longrightarrow P(V/L, V'/L) \times \mathcal{H}_L$$

$$\underbrace{W_0 < \cdots <}_{\notin \mathcal{H}_L} \overset{\in \mathcal{H}_L}{W_q} \longmapsto (W_0 + L/L, \ldots, W_{q-1} + L/L, W_q)$$

is a homomeomorphism, since $(W_i + L) \cap W_q = W_i$ if $L \oplus W_q = V$.

So what we have done is decomposed

$$P(V,V') = U \cup U'$$

and given heq's

$$U \longrightarrow P(V/L, V'/L)$$
$$U' \longrightarrow \mathcal{H}_L$$

such that

$$U \cap U' \longrightarrow P(V/L, V'/L) * \mathcal{H}_L \text{ is a heq.}$$

It follows then that we have a heq

$$P(V,V') \longrightarrow P(V/L, V'/L) * \mathcal{H}_L$$

~~Write this dually: [illegible]~~

~~[illegible]~~

Suppose now that $0 < V_1 < \cdots < V_m = V$ is a flag in V. Then we have heq's.

$$P(V, V_m) \longrightarrow P(V/V_1, V_m/V_1) * \left(\mathbb{P}V^* - \mathbb{P}(V/V_1)^*\right)$$

$$\longrightarrow P(V/V_2, V_m/V_2) * \left(\mathbb{P}(V/V_1)^* - \mathbb{P}(V/V_2)^*\right) * (\qquad)$$

and so we have proved:

<u>Prop:</u> $\exists$ canonical homotopy equivalence

$$P(V,V_m) \longrightarrow \left(\mathbb{P}(V/V_{m-1})^* - \mathbb{P}(V/V_m)^*\right) * \cdots * \left(\mathbb{P}(V/V_0)^* - \mathbb{P}(V/V_1)^*\right)$$

So in particular $P(V,V_m)$ is a bouquet of $(m-1)$-spheres in number $(q^{n-1}-1)\cdots(q^{n-m}-1)$. Also if we identify $P(V,V_{m-1})$ with the simp. cx. of affine subspaces of V_{n-1}, then we have a heq

$$P(V, V_{n-1}) \longrightarrow P(V/V_m, V_{n-1}/V_m) * P(V, V_m)$$

which gives Lusztig's isom

$$\tilde{H}_{n-1}(A(V)) \cong \tilde{H}_{n-m-1}(A(V/W)) \otimes \tilde{H}_{m-1}(P(V,W)).$$

$$n = \dim V, \qquad m = \dim W$$

~~It~~ seems there is virtue in changing notation.

Put $$Q(V,V') = \{W \mid 0<W,\ W \cap V' = 0\}$$

Lemma: Let H be a hyperplane of V containing V'. Then have heq's

$$\theta : Q(V,V') \longrightarrow (\mathbb{P}V - \mathbb{P}H) * Q(H,V')$$

$$W \longmapsto \begin{cases} W & \text{if } W \in \mathbb{P}V - \mathbb{P}H = Q(V,H) \\ W \cap H & \text{if } W \notin \text{———} . \end{cases}$$

Proof: Clearly this map is ~~simplicial~~ order-preserving. On the other hand θ is pre-cofibred: suppose given W and $\theta(W) \xrightarrow{u} Z_0$. ~~To prove $u_*(W)$ exists, I can suppose $u \neq id$. Case 1: $W \in \mathbb{P}V - \mathbb{P}H$ whence $\theta W = W$ and u is $W < Z_0$ where $Z_0 \in Q(H,V')$. Then if $W \subset W_1$, $W_1 \cap H = Z_0$, it is clear that~~ To prove u_* exists:

~~Case 1: $W \notin Q(V,H)$. Given $W \subset W_1$ with $\theta(W_1) = Z_0$ then as $W_1 \notin Q(V,H)$, one has $W_1 \cap H = Z_0$, so if we put $u_*(W) = W + Z_0$ we have $(W+Z_0) \cap H = W \cap H + Z_0 = Z_0$ $\Rightarrow$ $W \subset u_*(W)$ lies over u; also $W + Z_0 \subset W_1$~~

To prove $u_*(W)$ exists, i.e. $\exists$ universal arrow $W < u_*(W)$ over u. Can suppose $u \neq id \Rightarrow Z_0 \in Q(H,V')$. Set

$$u_*(W) = W + Z_0$$

Then $H \cap (W + Z_0) = H \cap W + Z_0 = Z_0$ which implies in particular that $V' \cap (W + Z_0) = 0$; thus $u_*(W)$ is defined and $W \subset u_* W$ sits over u. Next if $W \subset W_1$ sits over $u_1 \Rightarrow Z_0 = H \cap W_1 \Rightarrow W + Z_0 \subset W_1$ which implies universality of the map $W \subset u_*(W)$.

Finally, the fibre of Θ over an L in $PV - PH$ is L itself, and the fibre of Θ over $Z \in Q(H, V')$ consists of $W \in Q(V, H) \ni W \cap H = Z$ and this has a smallest element, namely Z.

This proves the lemma.

<u>Remark</u>: If $f : X \to Y$ is a map of ordered sets which is pre-cofibred, with contractible fibres, then the map $|X| \to |Y|$ of simplicial complexes has the same property. In effect for $y \overset{u}{<} y'$ one has arrows

$$\begin{tikzcd} & X_u \arrow[dl] \arrow[dr] & \\ X_y & & X_{y'} \end{tikzcd}$$

and the precofibred condition $\Rightarrow$ the former has an adjoint $x \mapsto (x \to u_* x)$; so therefore quite generally X_u retracts to its source X_y, etc.

So therefore the proof of the lemmas shows all the fibres of the geometric map $Q(V, V') \longrightarrow Q(V, H) * Q(H, V')$ are contractible. In fact the fibres are

$$\Theta^{-1}\{L, Z_1 < \cdots < Z_q\} = \{L < L + Z_1 < \cdots < L + Z_q\} \qquad \text{even if } q = 0$$

$$\Theta^{-1}\{Z_0 < \cdots < Z_q\} = \{\sigma = W_0 < \cdots < W_k \mid \sigma \cap H = (Z_0 < \cdots < Z_q)\}$$

contracts to simplex $Z_0 < \cdots < Z_q$.

Iterating it follows that we get heq's

$$Q(V, V_m) \longrightarrow Q(V, V_{n-1}) * Q(V_{n-1}, V_m)$$

$$\longrightarrow Q(V, V_{n-1}) * Q(V_{n-1}, V_{n-2}) * \cdots * Q(V_{m+1}, V_m)$$

which can be explicitly described as follows. Namely one takes a subspace $0 < W$, $W \cap V_m = 0$ and starts intersecting it with the flag $W \supset W \cap V_{n-1} \supset \cdots\cdots$.
If j is such that $W \cap V_j \neq 0$, $W \cap V_{j-1} = 0$, then $W \cap V_j \in Q(V_j, V_{j-1}) = \mathbb{P}V_j - \mathbb{P}V_{j-1}$ and this is where the vertex $_1W$ goes.

At the same time one gets heq.

$$Q(V, V_m) \longrightarrow Q(V, V_p) * Q(V_p, V_m) \qquad m \leq p \leq n.$$

Formula:

$$Q(V, V_m) \xrightarrow{heq} (\mathbb{P}V - \mathbb{P}V_{n-1}) * \cdots * (\mathbb{P}V_{m+1} - \mathbb{P}V_m)$$

Conjecture: There is a canonical heq.

$$T(V) \xrightarrow{heq} (\mathbb{P}V / \mathbb{P}V_{n-1}) * \cdots\cdots * (\mathbb{P}V_1 / \mathbb{P}V_0)$$

where for spaces with basepoint one takes the reduced join.

?

More examples.

$T(V) =$ subspaces W, $0 < W < V$.

More generally let $X = \{ W \mid p \leq \dim(W) \leq q \}$

Prop: X is a bouquet of $(q-p)$-spheres.

Proof: Denote the above X by $X^n_{p,q}$, $n = \dim(V)$. Let L be a line in V, and let $S_L = \{ W \mid \dim(W) = q,\ L \not\subset W \}$. $\mathrm{Link}(Z) = \{ W \mid W < Z,\ \dim W \geq p \} \cong X^q_{p,q-1}$ is a bouquet of $(q-p-1)$ spheres by induction. Remove the vertices S_L from $X^n_{p,q}$ and the result collapses by $W \mapsto L+W$ into $\{ W \mid p \leq \dim(W) \leq q,\ L \subset W \} = X^{n-1}_{p-1,q-1}$. by induction this is a bouquet of $(q-p)$-spheres. Use now the basic fact that if one attaches an m-cell to a bouquet of m-spheres one gets a bouquet of m-spheres.

$A(V) =$ affine subspaces $W < V$.

Prop: Let $A_p(V) = \left\{ W \,\middle|\, \begin{array}{l} W < V \\ W \text{ affine subspace} \\ \text{of } V \text{ of } \dim \geq p \end{array} \right\}$. Claim this is a bouquet of $(n-p-1)$-spheres.

Proof: Take point $0 \in V$ and consider map $W \mapsto \overline{W \cup \{0\}} = kW$. In order this this be defined we must remove $\mathcal{H} =$ hyperplanes not passing thru 0.

$$\mathrm{Link}(H) = \left\{ W \,\middle|\, \begin{array}{l} W \text{ affine space of } \dim \geq p \\ W < H \end{array} \right\} = A_p(H)$$

which will be a bouquet of $(n-p-2)$-spheres by ind. Suffices to show that $\left\{ W \,\middle|\, \begin{array}{l} W \text{ affine space of } \dim \geq p \\ 0 \in W \quad W < V \end{array} \right\}$ is a bouquet of $(n-p-1)$-spheres.

But this is $X^n_{p,n-1}$ as above, so it is a bouquet of $(n-p-1)$-spheres. done.

Dec. 8, 1973:

V vector space of dim n, W subspace of dim m.

$X(V,W)$ = simplicial complex whose simplices are non-ϕ subsets $\{v_1,\dots,v_q\}$ of V such that

$$\dim\ kv_1+\cdots+kv_q+W = q+\dim(W)$$

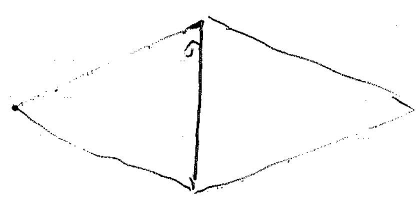
Thus $\dim X(V,W) = n-m-1$. Claim $X(V,W)$ is a bouquet of $(n-m-1)$-spheres. Argue by induction on $n-m$. If $n-m=0$, then $X(V,W)=\phi$ and it's clear. Also if $n-m=1$, then $X(V,W)$ is a non-empty union of points and it is also clear.

So assume $n-m>0$ and let e be a vector in V not in W. Let Z be the link of the vertex e, i.e. the subcomplex of non-ϕ subsets $\{v_1,\dots,v_q\}$ which are independent of $ke+W$. Let C = closed star of e.

~~Now let K be a finite subcomplex of $X(V,W)$ of dimension $< n-m-1$. Let S be the finite set of simple~~

Now let $f: K \longrightarrow X(V,W)$ be a map where K is a finite complex of dim $< n-m-1$. By simplicial approx. we can homotop f to a simplicial map. Let S be the finite set of simplices of $X(V,W)$ which are in $f(K)$ but not in C.

Let $\sigma = (v_1,\dots,v_q) \in S$; then $q < n-m$ and $e \in kv_1+\cdots+kv_q+W$. Consider the link of σ, L so that



locally around σ: Open star$(\sigma) \cong$ Int $\sigma \times$ Cone(L).

First suppose to simplify that $\sigma = (v)$ is a vertex. Then

$$\mathrm{Link}(v) = X(V, kv + W) \subset \mathrm{Link}(e)$$

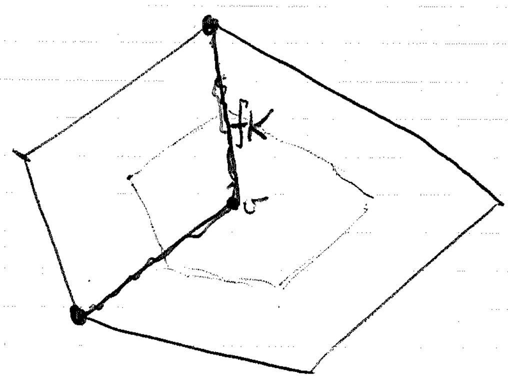
is a bouquet of $(n-m-2)$-spheres. In general if $(v_1, \cdots, v_q) \in X(V,W)$, then

$$\mathrm{Link}(v_1, \cdots, v_q) = X(V, kv_1 + \cdots + kv_q + W)$$

is a bouquet of $(n-m-q-1)$-spheres by induction. And if $ke \subset kv_1 + \cdots + kv_q + W$, then

$$\mathrm{Link}(v_1, \cdots, v_q) \subset \mathrm{Link}(e).$$

But now in the case $\sigma = (v)$ the point is the following. Because $\mathrm{Link}(v)$ begins in dim $(n-m-2)$, and fK is of dim $\leq (n-m-2)$, we should be able to push fK off of v. Precisely,



In a nbd of v, ~~$\mathrm{Link}(fK)$~~ fK is the cone on $\mathrm{Link}(fK)_v$. But $\mathrm{Link}(fK)_v$ contracts to a point in $\mathrm{Link}(X)_v$. Now ~~take the~~ using this homotopy, one can push fK into the link.

<u>Lemma</u>: $Y \subset X$ contractible to a point $\Rightarrow$ $\mathrm{Cone}(Y) \subset \mathrm{Cone}(X)$ homotopic keeping Y fixed to a map of $\mathrm{Cone}(Y) \to X$.

Now for $\sigma = (v_1, \dots, v_q) \in S$, we can use the same argument to push fK onto the boundary of $\sigma * \mathrm{Link}(\sigma)$. Thus we have modified f so that now we have to worry about simplices of the form

$$\{\sigma' \cup \tau\}$$

where $\sigma' < \sigma$ and $\tau \in \mathrm{link}(\sigma)$. ~~Now we note in this process that if we write~~

$$\cancel{e = \sum \lambda_i v_i + w}$$

~~λ_i not unique~~

~~that the rank of the simplex relative to $ke + W$ has increased. NO~~

So now let us consider those simplices σ contained in fK such that if $\sigma = (v_1, \dots, v_q)$ then $e \in kv_1 + \dots + kv_q + W$ and this is not true for any face of σ. Since $v_1, \dots, v_q$ are ind. mod W one has a unique expression

$$e = \sum_i \lambda_i v_i + w$$

so the σ under consideration are those such that $\lambda_i \neq 0$ for all i. Now we will use induction on the number n of this minimal σ. If $n = 0$, then $fK \subset \mathrm{Cone}(e)$ and we are done. Otherwise let σ be one and note that every $\sigma' \cup \tau$ in $\partial(\sigma * \mathrm{Link}(\sigma))$ is in $\mathrm{Cone}(e)$. Thus when we use the preceding pushing construction to push fK over σ, we do not introduce any new minimal simplices.

In fact it would appear that if we let

$\Gamma = \{\sigma \in X(V,W) \mid \sigma \text{ minimal} \ni e \in k\sigma + W\}$,
then each simplex not in Cone(e) contains a unique σ, whence

barycenter of σ

$$\coprod_{\sigma\in\Gamma} C(\mathrm{Link}(\hat{\sigma})) \;\cup_{\coprod_{\sigma\in\Gamma} \mathrm{Link}(\hat{\sigma})}\; \mathrm{Cone}(e) \xrightarrow{\sim} X(V,W)$$

<u>Prop:</u> Let Γ be the set of subsets $v_1,\dots,v_q$ of V which are minimal such that

$$e \in kv_1 + \dots + kv_q + W$$

Then

$$\boxed{X(V,W) = \bigvee_{\sigma\in\Gamma} \Sigma^{|\sigma|} X(V, k\sigma + W)}$$

By induction $X(V, k\sigma+W)$ bouquet of $(n-|\sigma|-m-1)$-spheres so $X(V,W)$ is a bouquet of $(n-m-1)$-spheres. $n=\dim(V)$ $m=\dim(W)$.

V vector space of dim n, W subspace of dim m.

$X(V,W)$ = simplicial complex whose simplices are fin. non-ϕ subsets σ of V $\ni$

$$\dim(k\sigma + W) = \operatorname{card}\sigma + \dim W$$

(one says σ is independent mod W).

Then $X(V,W)$ is of dimension $n-m-1$, and here is how one proves it has the homotopy type of a bouquet of $(n-m-1)$-spheres:

Fix a vector e of V not in W. (If $n=m$, nothing to prove since $X(V,V)=\phi$; otherwise e $\exists$). We try to push $X(V,W)$ into the closed star of e, which is contractible; denote this $\mathrm{Cl\,St}(e)$. Hence we are interested in those σ which ~~Let Γ = set of σ in $X(V,W)$ which are minimal~~ are not in $\mathrm{Cl\,St}(e)$, that is, such that $\{e\} \cup \sigma$ is not independent mod W. If $\sigma = \{v_1, \dots, v_q\}$, then because σ is ind. of W, one has $ke \subset kv_1 + \dots + kv_q + W$, hence

$$(*) \qquad e = \sum_{i=1}^{q} \lambda_i v_i + w$$

for $\lambda_i \in k$, $w \in W$. Moreover the λ_i and w are uniquely determined. Now consider those λ_i which are non-zero; these form a face σ' of σ, and it is clear from the uniqueness of the representation $(*)$, that $\sigma \subset \tau$ for any face τ of σ such that $e \in k\tau + W$.

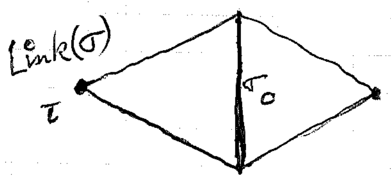
Put $\Gamma = \{\sigma \in X(V,W) \mid \sigma \text{ minimal} \ni e \subset k\sigma + W\}$.

Then we have shown that

$$X(V,W) - \mathrm{Cl\,St}(e) = \coprod_{\sigma \in \Gamma} \mathrm{Op\,St}(\sigma)$$

~~But now [illegible] in a general simplicial complex~~
~~the [illegible] structure around~~

But now consider in an arbitrary simplicial complex X what things look like around an (open) simplex σ_0. A simplex σ containing σ_0 is of the form $\sigma = \sigma_0 \sqcup \tau$ where $\tau \in \mathrm{Link}(\sigma)$.



More carefully: $\mathrm{Link}(\sigma_0)$ is by definition the ~~[illegible]~~ subcomplex of X ~~whose simplices τ are those~~ consisting of those simplices τ of X such that $\sigma_0 \cup \tau$ is a simplex, and $\sigma_0 \cap \tau = \phi$.

$$\mathrm{Op.st.}(\sigma_0) = \{\sigma \mid \sigma \supset \sigma_0\} \qquad \text{open}$$
$$\mathrm{Cl.st}(\sigma_0) = \{\sigma \mid \sigma \cup \sigma_0 \text{ is a simplex}\} \qquad \text{closed}$$
$$\mathrm{Link}(\sigma_0) = \left\{\tau \;\middle|\; \begin{array}{l}\sigma \cup \sigma_0 \text{ is a simplex} \\ \sigma \cap \sigma_0 = \phi\end{array}\right\} \qquad \text{closed.}$$

Thus

$$\mathrm{Cl.st}(\sigma_0) = \text{Closed simp } \sigma_0 \underset{\text{join}}{*} \mathrm{Link}(\sigma_0)$$

($K * L$ = simp comp. whose simp. are non-empty subsets $\sigma \sqcup \tau$ of $K \sqcup L$ such that $\sigma \neq \phi \Rightarrow \sigma$ simplex of K, $\tau \neq \phi \Rightarrow \tau$ simp. of L.)

$$\partial\mathrm{Opst}(\sigma) = \mathrm{Clst}(\sigma) - \mathrm{Opst}(\sigma) = \partial\sigma_0 * \mathrm{Link}(\sigma)$$

But taking b_{σ_0} = barycenter of σ_0, one has a subdivision homeomorphism:

$$\mathrm{Clst}(\sigma) = b_{\sigma_0} * \partial\mathrm{Opst}(\sigma_0)$$

Now going back to $X(V,W)$, note that ~~[illegible]~~

~~then~~ $\mathrm{Link}(\sigma) = X(V, k\sigma + W)$. Thus

$$X(V,W) = \mathrm{Clst}(e) \bigcup_{\coprod_{\sigma\in\Gamma} \partial\, \mathrm{Clst}(\sigma)} \coprod_{\sigma\in\Gamma} \mathrm{Clst}(\sigma)$$

$$= \mathrm{Clst}(e) \bigcup_{\coprod_{\sigma\in\Gamma} \partial\sigma * X(V,k\sigma+W)} \coprod_{\sigma\in\Gamma} \sigma * X(V,k\sigma+W)$$

On using that $\mathrm{Clst}(e)$ is contractible we get first part of:

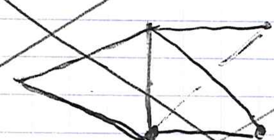
<u>Prop</u>: i) Let $\Gamma = \{\sigma \mid \sigma \text{ minimal } \& \ e \in k\sigma + W\}$. Then

$$X(V,W) \sim \bigvee_{\sigma\in\Gamma} S^{|\sigma|} X(V,k\sigma+W)$$

where $|\sigma| = \mathrm{card}\,\sigma = \dim\sigma + 1$. $\left(S^k Y = \underbrace{S^0 * S^0 * \cdots * S^0}_{k \text{ times}} * Y \right)$

ii) $X(V,W) \sim \bigvee S^{n-m-1}$

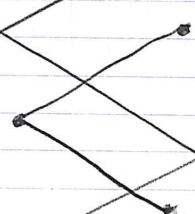
<u>Remarks</u>. Conventions: If $X(V,k\sigma+W) = \phi$, then $\Sigma^{|\sigma|} X(V,k\sigma+W) = \sigma/\partial\sigma = S^{|\sigma|-1}$. If $X(V,k\sigma+W)$ ~~[illegible]~~ a set of points, then $\Sigma^{|\sigma|} X(V,k\sigma+W)$



$\Sigma^{|\sigma|} X$ means $\Sigma(\partial\sigma * X)$

S^0

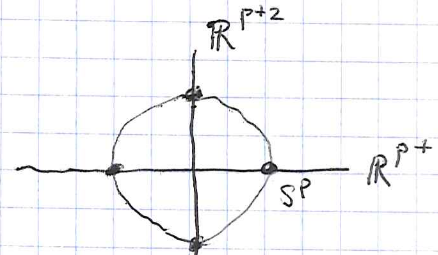
X



here $\Sigma X = S^0 * X$

Proof of ii). Proceed by induction on $\dim(V)-\dim(W)$. Use that suspension carries a bouquet of S^p's into a bouquet of S^{p+1}'s:

$$S^0 * \phi = S^0$$
$$S^0 * S^p = S^{p+1}$$



and

$$S^0 * (A \cup_C B) = (S^0 * A) \cup_{(S^0 * C)} (S^0 * B)$$

and $S^0 * pt = I$.

Generalizations: The proceeding applies in the general context of a set S with a closure relation satisfying the exchange condition: Let $X(S)$ be the simplicial complex of non-empty finite independent sets. Assuming $X(S) \neq \phi$ pick a vertex e, and notice again that if $e \in \bar{\sigma}$, then there is a least face τ of $\sigma \ni e \in \bar{\tau}$. (Proof: Assume τ_1, τ_2 are minimal faces of $\sigma \ni e \in \bar{\tau}_1$, $e \in \bar{\tau}_2$, with $\tau_1 \neq \tau_2$. Let $\tau_1 = (v_1, v_2, \ldots, v_p)$, $v_1 \notin \tau_2$. Since $e \notin \overline{(v_2, \ldots, v_p)}$, $e \in \overline{(v_1, \ldots, v_p)}$, exchange cond. $\Rightarrow v_1 \in \overline{e, v_2, \ldots, v_p} \subset \overline{(v_2 \ldots v_p) \cup \tau_2}$ which contradicts fact that σ is independent.)

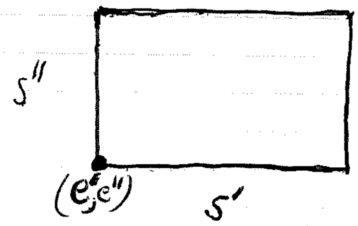
So again we will have

$$X(S) \sim \bigvee_{\sigma \in \Gamma} S^{|\sigma|} \operatorname{Link}(\sigma)$$

where $\Gamma =$ minimal $\sigma \ni e \in \bar{\sigma}$. But now $\operatorname{Link}(\sigma) = X(S_\sigma)$ where $S_\sigma = S$ with the closure relation $cl_\sigma(\tau) = cl(\tau \cup \sigma)$.

Example: Suppose now I try to understand the case of the complex $\Gamma(S',S'')$ = finite $\neq\phi$ subsets σ of $S'\times S''$ such that $pr_1|\sigma: \sigma \to S'$, $pr_2|\sigma: \sigma \to S''$ are injective.

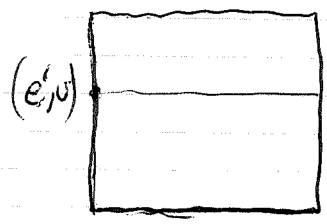
Fix $e = (e',e'') \in S'\times S''$ and let $C = \mathrm{Clst}(e)$ as usual; $\mathrm{Link}(e) = \Gamma(S'-e', S''-e'')$



Now let σ be a simplex not in C. Then σ contains a vertex in $e'\times(S''-e'')$ or $(S'-e')\times e''$. Thus we have to remove these vertices from $\Gamma(S',S'')$ to get C.

Start by removing the vertices $e'\times(S''-e'')$ and call the result C'. The link of (e',v) in $\Gamma(S',S'')$ is clearly

$$\mathrm{Link}\ (e',v) \text{ in } \Gamma(S',S'') = \Gamma(S'-e', S''-v)$$

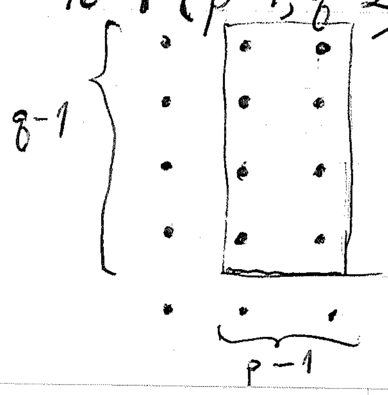


On the other hand once all of these vertices are removed, one has

$$\mathrm{Link\ of\ } (u,e'') \text{ in } C' = \Gamma(S'-e'-u, S''-e'')$$

Set $\Gamma(p,q) = \Gamma(S',S'')$ when $\mathrm{card}\, S' = p$, $\mathrm{card}\, S'' = q$.

Then to construct this starting from $\Gamma(p-1,q-1)$, I first attach cones on subcomplexes $\cong$ to $\Gamma(p-1,q-2)$, and then I attach cones on $\Gamma(p-1,q-1)$.



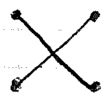
So starting with $\Gamma(1,q) \sim \vee S^0 \qquad q \geq 1$

we should get

$$\Gamma(p,q) \sim \vee S^{p-1} \qquad q \geq 2p-1$$

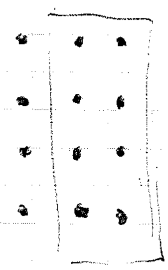
In effect assuming this for $p-1$, suppose $q \geq 2p-1$. The first operation involves collapsing in a contractible space things $\simeq \Gamma(p-1, q-2)$ which $\sim \bigvee S^{p-2}$ since $q-2 \geq 2(p-1)-1$. Thus one has a $\sim \bigvee S^{p-1}$, after that one collapses $\Gamma(p-1,q-1)$ $\sim S^{p-2}$ in a $\bigvee S^{p-1}$, so again one has a $\bigvee S^{p-1}$.

Counterexamples to improving things.

$\Gamma(2,2)$  not $\sim S^1$'s.
$\wr$
S^0

$\Gamma(2,3) =$  $\sim S^1$ so $\Gamma(3,3)$ obtained by attaching cones on three S^0's.

$\therefore \quad \Gamma(3,3) \sim \overset{4}{\bigvee} S^1 \quad \therefore$ not $\sim \bigvee S^2$

$\Gamma(3,4)$  collapse 4 loops in $\Gamma(2,4) \sim \overset{5}{\bigvee} S^1$ (χ comp.)
so $H_1(\Gamma(3,4))$ has rank ≥ 1.
hence $\Gamma(3,4)$ not $\sim \bigvee S^2$.

Dec. 10, 1973:

Lusztig's complex $L(V)$.

V vector space of dimension n, V' subspace of dim. m. Let $|L(V,V')|$ be the simplicial complex assoc to the ordered set $L(V,V')$ consisting of affine subspaces $(\neq \phi)$ W of V such that $0 \notin W$, and $kW \cap V' = 0$. For example, if $V'=0$, then we get Lusztig's complex $L(V)$ of affine subspaces not containing 0. Note that if $\phi \neq W_1 < \cdots < W_q$ is a chain of affine subspaces, then $0 < kW_1 < \cdots < kW_q$ is an increasing sequence, so if $kW_q \cap V' = 0$ on has $q \leq \dim(kW_q) \leq \dim(V/V') = n-m$, and equality is possible. Thus $\dim(L(V,V')) = n-m-1$.

Theorem: $L(V,V') \sim \bigvee S^{n-m-1}$.

Proof: Use induction on $n-m$, the cases $n-m=0$, $n-m=1$ being trivial. Assuming $n-m>0$, let U be a subspace of dim $m+1$ containing V', so that $L(V,U) \subset L(V,V')$. Note that for each W in $L(V,V')$, $(kW \cap U) \cap V' = 0$, so if $W \notin L(V,U)$, then $kW \cap U$ is a line L in U not in the hyperplane V'. Put

$$Z_L = \left\{ W \;\middle|\; \begin{array}{l} 0 \notin W \\ kW \cap U \subset L \end{array} \right\}$$

for each $L \in PU - PV'$. Note that if $L \neq L'$.

$$|Z_L| \cap |Z_{L'}| = |L(V,U)|$$

$$|\bigcup Z_L| = |L(V,V')|$$

In effect if $\sigma: W_1 < \cdots < W_q$ is a chain of affine space not cont. 0, then σ is in $|Z_L|$ iff $W_q \in Z_L$, so $\sigma \in |Z_L| \cap |Z_{L'}| \Leftrightarrow W_q \in Z_L \cap Z_{L'} = L(V,U)$; similarly $\sigma \in |L(V,V')| \Leftrightarrow W_q \in L(V,V') \Leftrightarrow kW_q \cap U \subset$ some L.

Lemma 1: Let a simplicial complex K be the union of subcomplexes L_i all containing A as a subcomplex such that $L_i \cap L_{i'} = A$ for $i \neq i'$. If $L_i \sim VS^q$ for each i, and $A \sim VS^{q-1}$, then $K \sim VS^q$.

Proof: One has a cof. situation

$$\begin{array}{ccc} \coprod_i A & \longrightarrow & A \\ \downarrow & & \downarrow \\ \coprod_i L_i & \longrightarrow & K \end{array}$$

as the hyp. $\Rightarrow K - A = \coprod L_i - A$. The result is clear for $q = 0$ as then $A = \phi$, $L_i \sim$ set so $K \sim$ set. If $q \geq 1$, choose a basepoint in A, whence one has a cocart square

$$(*) \qquad \begin{array}{ccc} \bigvee A & \longrightarrow & A \\ {\scriptstyle i}\downarrow & & \downarrow \\ \bigvee L_i & \longrightarrow & K \end{array}$$

and hence sequences

$$\longrightarrow \tilde{H}_*(A) \longrightarrow \tilde{H}_*(K) \longrightarrow \oplus \tilde{H}_*(L_i/A) \longrightarrow \cdots$$

$$\longrightarrow \tilde{H}_*(A) \longrightarrow \tilde{H}_*(L_i) \longrightarrow \tilde{H}_*(L_i/A) \longrightarrow \cdots$$

which show $\tilde{H}_*(K)$ is free abelian in dim q, zero elsewhere. For $q \geq 2$, $\pi_1(K) = 0$ by applying van Kampen to $(*)$, so OK. For $q = 1$, can suppose up to a heq that each L_i is a connected graph and A is a bunch of pts. Then K is a connected graph + done.

<u>Lemma 2</u>: $|Z_L| \sim \bigvee S^{n-m-1}$. $\quad \forall L \in PU - PV'$

Assuming this and the induction hyp. $L(V,U) \sim \bigvee S^{n-m-2}$ we have $L(V,V') \sim \bigvee S^{n-m-1}$ by lemma 1, and the theorem is proved.

<u>Proof of lemma 2</u>: Recall $Z_L = \left\{ W \,\middle|\, \begin{matrix} W \in L(V) \\ kW \cap U \subset L \end{matrix} \right\}$.

Note if $W \in Z_L - L(V,U)$, then $kW \cap U = L$. Let $L = ke$ and suppose $w_0, \dots, w_\delta \in W$ is a ~~base for W~~ basis for kW. ~~a unique expression~~ Then we have

$$e = \sum_{i=0}^{\delta} \lambda_i w_i$$

for unique λ_i. If $\sum \lambda_i \neq 0$, then

$$\frac{1}{\sum \lambda_i} e = \sum \left(\frac{\lambda_i}{\sum \lambda_i} \right) w \in W$$

And so $W \cap L =$ a point of $L^* = L - \{0\}$; (the intersection cannot be L or else $0 \in W$). On the other hand if $\sum \lambda_i = 0$, then because any w has an expression $\sum t_i w_i$ with $\sum t_i = 1$, it is clear that $ke + W = W$, i.e. $L + W = W$.

Now put
$$C(e) = \{ W \mid \overline{e \cup W} \in Z_L \} \qquad e \in L - \{0\}$$
i.e. $(ke + W) \cap U \subset L$. Then $C(e) \subset Z_L$. If $W \in C(e)$, and $e \notin W$, then $0 \notin \overline{e \cup W}$, so $0, e, W$ are independent $\Rightarrow e \notin kW \Rightarrow kW \cap U = kW \cap U \cap L = 0 \Rightarrow W \in L(V, U)$. Thus
$$C(e) = \{ W \in Z_L \mid e \in W \} \cup L(V,U).$$

Put
$$C(L) = \{ W \mid L + W \in Z_L \}.$$
Then $C(L) \subset Z_L$ and if $W \in C(L)$ and $W < L + W$, then $kW < L + kW \Rightarrow kW \cap U \subset kW \cap U \cap L = 0 \Rightarrow W \in L(V,U)$.
So
$$C(L) = \{ W \in Z_L \mid L + W = W \} \cup L(V,U).$$
So we get that
$$Z_L = \bigcup_{e \in L - \{0\}} C(e) \cup C(L)$$
where the subcomplexes $C(e)$, $C(L)$ pairwise intersect in $L(V,U)$.

Because $C(e)$ is contractible, Z_L is therefore obtained by starting from $C(L)$ and attaching a cone on $L(V,U)$ for each $e \in L - \{0\}$. Now $C(L)$ contracts via $W \mapsto L + W$ to
$$\{ W \in Z_L \mid L + W = W \}$$

and the mapping

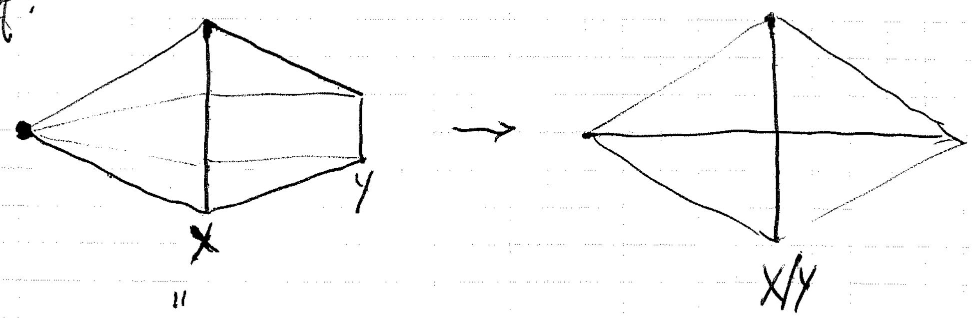
$kW \cap U = L$

$$L(V,U) \longrightarrow \{W \in Z_L \mid L + W = W\}$$

has a section, namely, take a complement H to L and send W to $H \cap W$. Therefore in virtue of

<u>Lemma:</u> If Y is a retract of X, then the mapping cone of the retract map $r: X \longrightarrow Y$ is the suspension of X/Y.

<u>Proof:</u>



$\shortparallel$
cone on r $\qquad$ = susp of X/Y.

What this means is that

$$C(e) \cup_{L(V,U)} C(L) \sim S\big(L(V,U)/L(H, U \cap H)\big)$$

By induction $L(V,U)$ is a bouquet of S^{n-m-2}'s, so $S(L(V,U)/L(H, U\cap H)$ will be a bouquet of S^{n-m-1}'s. Then as we attach further cones on $L(V,U)$, this will remain the case, and so we get a proof of lemma 3.

December 11, 1973

We now have many examples of spherical simplicial complexes, and it is now necessary to understand what implications result for the homology.

~~First let us consider the Tits building $T(V)$~~

First let S be a partially ordered set with a dimension function $d : S \longrightarrow N$ such that $d(s_2) = d(s_1) + 1$ whenever s_2 immediately follows s_1. Then filter S by putting

$$F_p S = \{ s \mid d(s) \le p \}$$

$$\phi = F_{-1}S \subset F_0 S \subset F_1 S \subset \cdots\cdots$$

Put $S_p = \{ s \mid d(s) = p \}$, and let $\overline{\{s\}} = \{ s' \le s \}$. ~~[illegible]~~ Now given $F : S \longrightarrow \mathcal{Ab}$ one has the chain complex

$$C_q(S, F) = \coprod_{s_0 < \cdots < s_q} F(s_0)$$

for computing $L_q \varinjlim_S F = H_q(S, F)$, and if $S' \subset S$, one puts ~~$C_q(S,S';F) = C_q(S,F)/C_q(S',F)$~~ $C_q(S,S';F) = \dfrac{C_q(S,F)}{C_q(S',F)}$

a short exact sequence of complexes

$$0 \longrightarrow C_q(S',F) \longrightarrow C_q(S,F) \longrightarrow C_q(S,S';F) \longrightarrow 0$$

hence a long exact sequence in homology.

Thus from the filtration $\{F_p S\}$ we will get a spectral sequence

$$E^1_{pq} = H_{p+q}(F_p S, F_{p-1} S; F) \Longrightarrow H_{p+q}(S; F)$$

But ~~because any element of F_pS~~ because the predecessors of an elt $s \in F_p$ lie in F_{p-1}

$$C_q(F_pS, F_{p-1}S; F) = \coprod_{\substack{s_0 < \cdots < s_q \\ s_q \in F_pS - F_{p-1}S}} F(s_0)$$

~~by the sort that~~ and this breaks up into a ~~complex~~ sum over the elements of S_p:

$$C_*(F_pS, F_{p-1}S; F) = \coprod_{s \in S_p} C_*(\overline{\{s\}}, \overline{\{s\}} - \{s\}; F)$$

and so our spectral sequence takes the form

$$E^1_{pq} = \bigoplus_{s \in S_p} H_{p+q}(\overline{\{s\}}, \overline{\{s\}} - \{s\}; F) \Longrightarrow H_{p+q}(S; F)$$

Further it clear that from

$$\cdots \longrightarrow H_*(\overline{\{s\}} - \{s\}, F) \longrightarrow H_*(\overline{\{s\}}, F) \longrightarrow H_*(\overline{\{s\}}, \overline{\{s\}} - \{s\}; F) \longrightarrow \cdots$$

$$H_*(\overline{\{s\}}, F) = \begin{cases} 0 & * \neq 0 \\ F(s) & * = 0 \end{cases}$$

To simplify notation put $\overline{\{s\}} = \bar{s}$ and $\overline{\{s\}} - \{s\} = \bar{s} - s$.

<u>Cor:</u> <u>Assume</u> $(\bar{s}, \bar{s} - s)$ <u>is</u> F-<u>spherical</u> <u>of</u> <u>dim</u> $d(s)$, <u>i.e.</u>

$$H_i(\bar{s}, \bar{s} - s; F) = 0 \qquad i \neq d(s)$$

<u>And that</u> S <u>is</u> F-<u>spherical</u> of dim $= \max d(s) = n$.

Then one gets an exact sequence

$$0 \longrightarrow H_n(S; F) \longrightarrow \bigoplus_{s \in S_n} H_n(\bar{s}, \bar{s} - s; F) \longrightarrow \cdots \longrightarrow \bigoplus_{s \in S_0} F(s) \longrightarrow H_0(S, F) \to 0$$

Better statement is that if

$$H_i(\bar{s}, \bar{s}-s; F) = 0 \qquad i \neq d(s)$$

then the complex which is the E^1_{*0} – term:

$$\cdots \longrightarrow \bigoplus_{s\in S_1} H_1(\bar{s}, \bar{s}-s; F) \longrightarrow \bigoplus_{s\in S_0} F(s) \longrightarrow 0$$

can be used to compute $H_*(S; F)$.

Example: Let S be the ordered set of simplices in a simplicial complex K, ~~[illegible]~~ $d(\sigma) = \dim(\sigma)$ so that

$$H_i(\bar{\sigma}, \bar{\sigma}-\sigma; \mathbb{Z}) = \begin{cases} 0 & i \neq d(\sigma) \\ \cong \mathbb{Z} & i = d(\sigma) \end{cases}$$

Then one gets that one can compute $H_*(K, \mathbb{Z})$ using the complex of simplicial chains on K.

Example 2: Let V be a vector space of dim $n > 0$. and $S =$ proper subspaces of V. Put

$$I(V) = H_{n-1}(\bar{V}, \bar{V}-V; \mathbb{Z}) \left(= \tilde{H}_{n-2}(T(V)) \text{ if } n \geq 2\right)$$

where $\bar{V} = \{W \leq V \mid 0 < W\}$, $\bar{V}-V = T(V)$. Then one has from the above, exact sequences

$$0 \to I(V^n) \longrightarrow \bigoplus_{V^{n-1}\subset V} I(V^{n-1}) \longrightarrow \bigoplus_{V^{n-2}\subset V} I(V^{n-2}) \to \cdots \to \bigoplus_{L\subset V} \mathbb{Z} \longrightarrow \mathbb{Z} \to 0$$

Now let $GL(V)$ act on the above exact sequence, tensored with M where M is a $GL(V)$-module. One gets in dimension p

$$\bigoplus_{V^p \subset V} I(V^p) \otimes M$$

which is a module induced from the stabilizer of V_p; here choose a "base" flag $0 = V_0 < V_1 < \cdots < V_n = V$, and let $GL(V, V_p)$ be the stabilizer of V_p. Then one has

$$\bigoplus_{V^p \subset V} I(V^p) \otimes M = \mathbb{Z}[GL(V)] \otimes_{\mathbb{Z}[GL(V,V_p)]} M$$

so by Shapiro:

$$H_*\Big(GL(V), \bigoplus_{V^p \subset V} I(V^p) \otimes M\Big) = H_*\big(GL(V,V_p), I(V_p) \otimes M\big).$$

So we get a spectral sequence

$$E^1_{pq} = H_q\big(GL(V,V_p), I(V_p) \otimes M\big) \Rightarrow \overset{0}{\cancel{H_{p+q}(GL(V), M)}}.$$
$$0 \le p \le n$$

Here M could be more generally a complex of $GL(V)$-modules. I have in mind the case where V is a subspace of Z, in which case we have

$$1 \to GL^0(Z, V) \longrightarrow GL(Z, V) \longrightarrow GL(V) \longrightarrow 1$$

so one has

$$H_*(GL(Z,V), \mathbb{Z}) = $$
$$= H_*\big(C.(P_{GL(Z,V)}, \mathbb{Z})_{GL(Z,V)}\big)$$
$$= H_*\big(GL(V), C.(P_{GL(Z,V)}, \mathbb{Z})_{GL^0(Z,V)}\big).$$

Recall the structure of $I(V)$:

$$T(V) = \bigvee_{H\in\mathcal{H}_L} S\,T(H)$$

where H runs over the hyperplanes complementary to the line L. Thus we have

$$I(V) = \tilde{H}_{n-1}(ST(V)) = \bigoplus_{H\in\mathcal{H}_L} \tilde{H}_{n-2}(ST(H))$$

or

$$I(V) = \bigoplus_{H\in\mathcal{H}_L} I(H)$$

which determines the structure of $I(V)$ as a module over $GL(V,L) = GL(H\oplus L, L) = [GL(H)\times GL(L)] \tilde{\times} \operatorname{Hom}(H,L)$ namely

$$\boxed{I(V) = \mathbb{Z}[GL(V,L)] \otimes_{\mathbb{Z}[GL(H)\times GL(L)]} I(H)}$$

(In general it is probably true that if $V^n \supset W^m$, then as a $GL(V,W)$-module

$$I(V) = \mathbb{Z}[GL(V,W)] \otimes_{\mathbb{Z}[GL(A)\times GL(W)]} I(A)\otimes I(W)$$

where A is a complement for W in V.)

From now on the ground field is finite with $q = p^d$ elements

Lemma: If M is a $\mathbb{Z}_{(p)}$-module, then

$$H_+(GL(V); I(V)\otimes M) = 0$$

Proof: The subgroup $GL(V,L)$ is of index $= \operatorname{card}(\mathbb{P}V)$ $= q^{n-1}+\cdots+1 \equiv 1 \pmod p$, hence from transfer theory one knows that $H_+(GL(V,L), I(V)\otimes M)$ maps onto the homology in question. From (*) and Shapiro, this latter is isom. to

$$H_*(GL(H)\times GL(L),\ I(H)\otimes M)$$

which as $GL(L)$ is prime to p, is a quotient of

$$H_*(GL(H),\ I(H)\otimes M)$$

which is zero by induction. (Check lemma trivial for $n=0,1$.)

<u>Lemma</u>: If M is a $\mathbb{Z}_{(p)}$-module on which $GL(V)$ acts trivially, then $H_0(GL(V), I(V)\otimes M)=0$ for $n\geq 2$.

<u>Proof</u>: Enough by preceding argument to check for $n=2$. Here one has

$$0\longrightarrow I(V_2)\longrightarrow \bigoplus_{L\subset V_2}\mathbb{Z}\xrightarrow{\ \Sigma\ }\mathbb{Z}\longrightarrow 0$$

the diagonal $\Delta:\mathbb{Z}\longrightarrow\bigoplus_{L\subset V_2}\mathbb{Z}$ composed with Σ is multiplication by $\mathrm{card}(\mathbb{P}V_2)=q+1$ which is invertible in $\mathbb{Z}_{(p)}$. Thus over $\mathbb{Z}_{(p)}$ the above sequence splits equivariantly so for any $GL(V_2)$-module M

$$0\to H_0(GL(V_2),I(V_2)\otimes M)\longrightarrow H_0(GL_2,\mathbb{Z}[GL(V_2)]\otimes_{\mathbb{Z}[B]}M)\longrightarrow H_0(GL(V_2),M)\to 0$$

is exact.

$$H_0(GL_2,\mathbb{Z}[GL(V_2)]\otimes_{\mathbb{Z}[B]}M)=H_0(B,M)$$

where $B=$ stabilizer of a line. Since $G=GL(V_2)$ acts trivially on M, one has $H_0(B,M)\xrightarrow{\sim}H_0(G,M)$, and so one wins.

<u>Lemma</u>: The exact sequence at bottom of page 3 splits as an exact sequence of $GL(V)$-modules after tensoring with $\mathbb{Z}_{(p)}$. seems to be false already for $n=3$.

<u>Proof:</u> I will only prove this at the bottom. I already know that $\bigoplus_{L<V} \mathbb{Z} \to \mathbb{Z}$ has an equivariant splitting over $\mathbb{Z}_{(p)}$ since $\operatorname{card}(\mathbb{P}V) = q^{n-1} + \cdots + 1 \equiv 1 \ (p)$. Now consider

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{H_2<V} I(H) \arrow[r] \arrow[dr, bend left, "d_1"'] & \bigoplus\limits_{L<H_2<V} \mathbb{Z} \arrow[r] \arrow[d, "\alpha"] & \bigoplus\limits_{H_2<V} \mathbb{Z} \arrow[r] \arrow[d, "\beta"] & 0 \\
& & \bigoplus\limits_{L<V} \mathbb{Z} \arrow[r, "d_1"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The exact sequence splits canonically and α and β have canonical sections (β because $\operatorname{card}\{H_2<V\} = \dfrac{GL(V)}{GL(V,H_2)}$ is prime to p). Thus by diagram chasing? one can construct canonical contracting homotopies for the d_1 sequence?

To get a section of α one takes a function $L \mapsto f(L) \in \mathbb{Z}$ and extends it to $\bar{f}(L<H) = f(L)$. Then $(\alpha\bar{f})(L) = \sum_{L<H} f(L) = [\mathbb{P}(V/L)]\cdot f(L)$

This is only reasonable section of α.

$$\begin{tikzcd}
& & \bigoplus I(L) \arrow[rr] \arrow[d] & & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus\limits_{H} I(H) \arrow[r] \arrow[d] & \bigoplus\limits_{L<H} \mathbb{Z} \arrow[rr] \arrow[d] & & \bigoplus\limits_{H} \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & \bigoplus\limits_{L} \mathbb{Z} \arrow[rr] & & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
& & & & 0 &
\end{tikzcd}$$

December 13, 1973

Theorem: Let G_S be the group of perm. of an infinite set S. Then $\tilde{H}_*(G_S) = 0$.

Lemma 1: Let $S = S' \amalg S''$ where S', S'' are of same card as S, and let $G_{S'}$ be the subgroup of G_S fixing the elements of S''. Then the inclusion of G' in G_S induced the zero map on reduced homology.

Proof: Consider the permutative category $\mathcal{C}$ consisting of sets having the same card as S together with ϕ, where the operation is $\amalg$. Then it is clear that K-theory of $\mathcal{C}$ is trivial, for one has countable direct sums (standard "flaskness" argument). Thus $\varinjlim_{[S]} H_*(G_S) = 0$ where $[S]$ runs over the translation cat of the iso. classes of $\mathcal{C}$. But there are two iso classes $\{1, e\}$, $e^2 = e$. Thus mult. by e is zero on $\tilde{H}_*(G_S)$ which proves the lemma.

Next consider the partially ordered set J whose elements are subsets S' of S $\ni$ S' & $S - S'$ are of same card as S. On puts $S' < S''$ if $S' \subset S''$ and $S'' - S' \simeq S$. Clearly G acts on J.

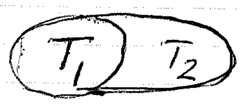
Lemma 2: J is contractible

Proof: It suffices to show any finite subset K of J with the induced order contracts to a point in J. Let $S_0 = \bigcap_{S' \in K'} S'$ where K' is a maximal subset of K such that S_0 has the same card as S. Then

~~given any $S_1 \in K$ either $S_1 \in K'$ and $S_0 \subset S_1$ or $S_1 \notin K'$ and so $S_0 \cap S_1$ is of card < card(S). Now split $S_0 = S_{01} \amalg S_{02}$ where card(S_{01}) = card(S_{02}) = card(S). Then for $S_1 \in K'$ we have $S_{01} < S_1$ and for $S_1 \in K - K'$, we have $S_1 \cup S_{01} \in J$ and $S_1 < S_1 \cup S_{01} > S_{01}$. (Check this last: I know card$(S_1 \cap S_0)$ < card(S) = card(S_1), hence card$(S_1 \cap S_{01})$ < card(S_1) and so $S_1 \cup S_{01} - S_1 = S_{01} - S_1 \cap S_{01}$ has card = card(S); also $S_1 \cup S_{01} - S_{01} = S_1 - S_1 \cap S_{01}$ also has card(S).). Therefore for all $S_1 \in K'$ we have $S_1 \leq S_1 \cup S_{01} \geq S_{01}$ in J~~

given $T \in K$ either $T \in K'$ and $S_0 \subset T$ or $T \notin K'$ and so card$(T \cap S_0)$ < card(S). Now split $S_0 = T_0 \amalg U_0$ where $T_0 \approx U_0 \approx S$. Then given $T \in K'$ we have $T = T \cup T_0 > T_0$ (as $T - T_0 \supset S_0 - T_0 = U_0$). If $T \in K - K'$, then $T \cup T_0 \in J$ because its complement contains $U_0 - T \cap U_0$ and $U_0 \cap T \subset S_0 \cap T$ is negligible. Moreover $T < T \cup T_0$ for same reason, and $T \cup T_0 > T_0$ also. So thus we have $T \leq T \cup T_0 > T_0$ for all $T \in K$. Now I have only to check that $T_1 < T_2 \Rightarrow T_1 \cup T_0 \leq T_2 \cup T_0$.

~~No problem if $T_1, T_2 \in K'$ for then $T_1 \cup T_0 = T_1$, $T_2 \cup T_0 = T_2$. Also no problem if $T_1 \in K'$ and $T_2 \in K - K'$ for we have seen that $T_1 \cup T_0 = T_0 < T_2 \cup T_0$. If $T_1, T_2 \in K - K'$, then $T_1 \cup T_0$ $T_2 \cup T_0$~~ However the only way this could fail would for $T_2 \cup T_0 - T_1 \cup T_0 = T_2 - (T_1 \cup T_2 \cap T_0)$ to be negligible. However this means that we have to make T_0 so small that it doesn't eat up the differences $T_2 - T_1$. (This should be clear anyway, for



no problem if $S_0 \subset T_1$, or if $S_0 \cap T_2$ is negligible, or if $S_0 \cap T_1$ is negligible and $S_0 \subset T_2$.) So done.

Now this gives us a resolution of $\mathbb{Z}$:

$$\cdots \longrightarrow \coprod_{T_0 < T_1} \mathbb{Z} \longrightarrow \coprod_{T_0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

(Chains on the associated simplicial complex). since G acts transitively on the simplices of a given dimension and the stabilizer of $T_0 < T_1 < \cdots < T_q$ is G^{q+2} one gets a spectral sequence

$$E^1_{pq} = H_q(G^{p+2}) \Longrightarrow H_{p+q}(G).$$

and clearly $E^2_{*0} = \mathbb{Z}$ in degree 0. Now if one has $H_q(G) = 0$ $q < r$, then $\tilde{H}_q(G^s) = 0$ $q < r$ by Kunneth. Thus from the spec. seq. we get

$$\begin{array}{c} \tilde{H}_r(G^2) \longrightarrow\!\!\!\!\rightarrow \tilde{H}_r(G) \\ \wr\!\wr \\ H_r(G) \oplus H_r(G) \end{array}$$

zero map by lemma 1.

Suppose E, F are projective A-modules. Define $\theta : E \longrightarrow F$ to be "compact" if

$$\theta(e) = \sum_{i=1}^{n} \lambda_i(e) f_i$$

for $\lambda_i \in \mathrm{Hom}_A(E, A)$, $f_i \in F$. Thus the compact maps are those in the image of the canonical maps

$$\mathrm{Hom}_A(E, A) \otimes_A F \longrightarrow \mathrm{Hom}_A(E, F)$$

(Is this map always injective? Seems so, for $F = A^{(S)}$ hence in general)

<u>Example:</u> Take $E = A^{(I)}$ $F = A^{(J)}$. Then for θ to be compact means that if we write its matrix

$$\theta(\{x_i\}) = \{\textstyle\sum a_{ji} x_i\}$$

Then only finitely many of the rows of the matrix are non-zero.

Another way of stating that θ is compact is to say that θ factors

$$E \longrightarrow P \longrightarrow F$$

where P is a f.t. projective A-module.

<u>Question:</u> Assume $\theta \in \mathrm{End}(E)$ is compact and $1+\theta$ is an automorphism. Does it follow that $(1+\theta)^{-1} = 1 + K$ with K compact?

Yes, because

$$(1+\theta)(1+K) = 1 \implies \theta + K + \theta K = 0$$
$$\implies K = -\theta - \theta K$$

which is compact as the compact operators form an ideal.

Question: Take $E = A^{(\infty)} = Ae_1 \oplus Ae_2 \oplus \cdots\cdots$ so take we have $GL(A) \subset Aut_{1+c}(A^{(\infty)})$. Does this inclusion induce an isomorphism on homology?

Here's how we might prove this when A is a field. I observe that given a decomposition $E = P \oplus F$ with $P \in \mathcal{P}_A$, then there is an embedding

$$GL(P) \subset Aut_{1+c}(E)$$

obtained by taking the direct sum of an auto of P with the identity of F. Assume now that F' is another complement for F, say $F' = \{(h(f), f) \in P \oplus F\}$ where $h: F \to P$. Then if we denote by $i_*: GL(P) \to Aut_{1+c}(E)$ the homo defined by the complement F and i' the one defined by F' we have

$$i'_*(\theta)(1+h)f = (1+h)f$$

$$i'_*(\theta)\, p = \theta(p) = i_*(\theta)p$$

$$\therefore \quad i'_*(\theta) = (1+h)\, i_*(\theta)(1+h)^{-1}$$

where $1+h: p+f \longmapsto p+h(f)+f$ translates F to F'. Thus the homos i_*, i'_* are conjugate, and so we obtain a well-defined homomorphism

$$i_{P\subset E}: H_*(GL(P)) \longrightarrow H_*(Aut_{1+c}(E))$$

depending only on the direct summand P of E. Now if P, Q are direct summands of E with $P \subset Q$, then P is a direct summand of Q, so we can choose $P \oplus C = Q$, $Q \oplus F = E$, and take the complement $C \oplus F$ for

P. Then it is clear that one gets a comm. diag.

$$\begin{tikzcd} H_*(GL(P)) \arrow[dr, "i_{PCE}"] \arrow[d, "i_{PCQ}"'] & \\ H_*(GL(Q)) \arrow[r, "i_{QCE}"'] & H_*(Aut_{1+c}(E)) \end{tikzcd}$$

(This is the familiar fact that modulo inner autos. one has a functor $P \mapsto GL(P)$ with respect to injections onto direct summands. Question: I get in this way a lien of some sort - any interesting gerbes around?)

Thus putting all this together I get a map

$$(*) \qquad \varinjlim_P H_*(GL(P)) \longrightarrow H_*(Aut_{1+c}(E))$$

where P runs over the fin. type. direct summands of E.

Thus the question under discussion comes down to whether the above map is an isomorphism.

So let G be a finitely gen. subgroup of $Aut_{1+c}(E)$. It would probably be enough to know that for any such G one can find a decomposition $E = P \oplus F$ stable under G such that G acts trivially on F. Certainly this implies the map (*) is onto. But also any element α in the kernel of i_{PCE} would comes from an α' in the homology of a f.t. subgp G of $GL(P)$, and for this to go to zero in $H_*(Aut_{1+c}(E))$ would mean α' dies in $H_*(G')$ for some finite type $G' \subset Aut_{1+c}(E)$. Thus

$$\begin{tikzcd} G \arrow[d, phantom, "\cap"] & \subset & GL(P)^F \arrow[r, hook] & Aut_{1+c}(E) \\ G' & \subset & GL(P')^{F'} \arrow[ur] & \end{tikzcd}$$

if we could find $E = P' \oplus F'$ G' stable such that G' acts trivially on F' we would win.

So suppose $g_1, \ldots, g_n$ are a finite number of elements of $\mathrm{Aut}_{1+c}(E)$, and put $\theta_i = 1-g_i$. These are compact, hence in particular $\mathrm{Im}(\theta_i)$ is finitely generated. Put $M = \sum \mathrm{Im}(\theta_i)$ and note that the set of $u \in \mathrm{End}(E)$ $\ni$ $\mathrm{Im}(u) \subset M$ is a right ideal, in particular closed under sum and product. Thus if G is the group generated by the g_i, then $\mathrm{Im}(g-1) \subset M$ for any $g \in G$. (To make this more intelligible observe that $g \mapsto g-1$ transforms product to $x+y+xy$ which shows that the set of $g \in G$ $\ni$ $\mathrm{Im}(g-1) \subset M$ is closed under product. Also if $(1+x)^{-1} = 1+y$, then $y = -x-xy$ has its image contained in M).

Next put $K = \bigcap \mathrm{Ker}(\theta_i)$ and note that $g = 1$ on K, for all $g \in G$. Thus we have identified G with a subgroup of the group of quasi-invertible elements in the ring (without unit) of endos. θ $\ni$ $K \subset \mathrm{Ker}(\theta)$, $\mathrm{Im}(\theta) \subset M$.

$$\mathrm{Hom}(E/K, M) \subset \mathrm{Hom}(E,E)$$

Now I would like to find a decomposition $E = P \oplus F$ such that $P \supset M$, $F \subset K$, whence each $g \in G$ would preserve this decomposition and act trivially on F. Now if this could be done, then $E/K \twoheadleftarrow E/F \cong P$ would be finitely generated. Since

$$0 \longrightarrow K \longrightarrow E \xrightarrow{(\theta_i)} F^n$$

there is no reason for this to be true in general. Example:

take $\lambda : A^{(\infty)} \longrightarrow A$, $\lambda(x_i) = \sum a_i x_i$. In order that $\mathrm{Ker}(\lambda)$ be cofinitely generated, it is necessary that the $\mathrm{Im}(\lambda)$ = ideal gen. by the a_i be finitely generated. Thus we must assume A is noetherian.

If A is noetherian, then

$$E/K \subset \prod (\mathrm{Im}\,\theta_i)$$

so E/K is finitely generated. At least when $E = A^{(\infty)}$ this means we can find a direct summand P of E with $P \in \mathcal{P}_A$ such that $P \hookrightarrow E \longrightarrow E/K$ is onto. Then

$$\begin{tikzcd} & K \arrow[d] & \\ P \arrow[r] \arrow[dr, two heads] & E \arrow[r, two heads] \arrow[d] & E/P \\ & E/K & \end{tikzcd} \quad \Rightarrow \quad K \twoheadrightarrow E/P$$

~~[illegible]~~ and since E/P is projective we can find $F \subset K$, $F \oplus P = E$. Now if we started with a P which not only maps onto E/K but also contains M, then we have $P \supset M$, $F \subset K$ as desired. This proves:

<u>Lemma</u>: Let G be a fin. type subgroup of $\mathrm{Aut}_{f+c}(E)$, E = free infinite type A-module, A noetherian. Then $\exists$ $E = P \oplus F$ with P free fin. type such that G preserves this decomposition and such that G acts trivially on F.

Counterexample when A is not noetherian: Choose $Aa_2 < Aa_2 + Aa_3 < \cdots$ and define $\theta(x_i) = x_1 + \sum_{i \geq 2} a_i x_i$. Then $\mathrm{Ker}(\theta - 1)$ does not contain an F because $\mathrm{Im}(\theta - 1) = Aa_1 + \cdots$ is not of finite type.

Ideas: To what extent do the decompositions of the form $E = P \oplus F$ partially ordered by requiring $(P,F) \leq (P',F')$ if $P \subset P'$, $F \supset F'$ form a directed set. In the noetherian case this is so, because given $(P_1,F_1),(P_2,F_2)$, $E/F_1 \cap F_2 \subset E/F_1 \times E/F_2$ so $E/F_1 \cap F_2$ is of finite type. Thus as above choosing P a fin. type dir. summand of E sufficiently big so as to include $P_1 + P_2$, and map onto $E/F_1 \cap F_2$, then we have $F_1 \cap F_2 \twoheadrightarrow E/P$ so we can find a complement F to P contained inside $F_1 \cap F_2$.

Therefore in the noetherian case with $E = A^{(I)}$ one really obtains $\mathrm{Aut}_{1+c}(E)$ as a filtered direct limit of $GL(P)$

Last time I was intrigued by the fact that given a finite type subgroup G of matrices of the form

$$\begin{array}{c} n \\ \infty \end{array} \begin{pmatrix} * & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

I have managed to conjugate it into a subgroup of the form

$$\begin{pmatrix} * & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

but where $*$ is very big.

Suppose A is a field to simplify. I can consider then the category of ~~[illegible]~~ vector spaces over A of countable dimension ~~[illegible]~~ under addition. This is an abelian category in which exact sequences split, and its K-theory is trivial by the usual countable sum argument. Thus if $E = A^{(\infty)}$ we find as in the case of countable sets that the idempotent endomorphism of $H_*(GL(E))$ obtained from $E \oplus E \simeq E$ is zero. To put this another way, if we decompose E into $E' \oplus E''$ where E' and E'' are both countable, then the ~~[illegible]~~ induced homomorphism $\tilde{H}_*(GL(E')) \longrightarrow \tilde{H}_*(GL(E))$ is trivial.

So as before we consider the ordered simplicial complex whose q-simplices are decompositions

$$E = F_0 \oplus F_1 \oplus \cdots \oplus F_{q+1}$$

with each F_i of infinite dimension. Precisely a vertex is a decomposition $E = F_0 \oplus F_1$ and one says this decomposition is $<$ another $E = F_0' \oplus F_1'$ if

$$F_0 \subset F_0' \quad \text{and} \quad F_1 \supset F_1' \quad \text{and} \quad \underset{\simeq F_1 \cap F_0'}{F_0'/F_0} \text{ is inf-diml.}$$

In this case we get a 2-simplex $E = F_0 \oplus F_1 \cap F_0' \oplus F_1'$.

Another description of a decomposition $E = F_0 \oplus F_1$ is to given the projection on F_0. In this way a q-simplex appears as a decomp $1 = e_0 + \cdots + e_{q+1}$ into orthogonal idempotents such that e_i is a projection onto an infinite dimension subspace.

Now one wants to show this simplicial complex is contractible. So suppose we are given a finite set $\mathcal{F}$ of idempotents e in $\mathrm{End}(E)$ $\ni$ $\mathrm{Im}(e)$ & $\mathrm{Im}(1-e)$ are of infinite dimension. Suppose that we have two

$$E = F_0 \oplus F_1 = F_0' \oplus F_1'$$

such that $F_0 \cap F_0'$ has inf. dim. Then it might happen that $F_1 + F_1' = E$. (e.g. even if $F_0 = F_0'$) so our previous argument for sets will not work.

Start by trying to understand if this simplicial complex is connected. Thus I suppose given two decompositions $E = A_1 \oplus B_1 = A_2 \oplus B_2$ which I want to connect. Consider the map $A_1 \hookrightarrow E \twoheadrightarrow B_2$ whose kernel is $A_1 \cap A_2$.

If $A_1 \cap A_2$ is infinite, then by subdividing it into two infinite pieces if necessary I then reach the case where $A_1 = A$

December 15, 1973

Let A be a field, let E be a vector space over k (not nec. fin. dim.) and let $\mathrm{Aut}_c(E)$ denote the group of Autos of E of the form $1+\theta$ where θ is of finite rank. I know that any finite type subgroup G of $\mathrm{Aut}_c(E)$ stabilizes a splitting $E = P \oplus F$ with P fin. dim. and G acts trivially on F. (The point is that such splittings form a filtered set under the ordering $(P,F) \leqslant (P',F')$ if $P \subset P'$ and $F \supset F'$; for given (P_i, F_i) $i=1,2$, then these are dominated by (P,F) where P is chosen containing $P_1 + P_2$ such that $P + (F_1 \cap F_2) = E$, and F is a complement of $P \cap F_1 \cap F_2$ in $F_1 \cap F_2$). Thus E determines an ind. object in the category of complemented injections of finite dimensional vector spaces, and $\mathrm{Aut}_c(E)$ is just the limit of $\mathrm{Aut}(P)$ as (P,F) runs over this ind. object.

Let V be a $\mathbb{C}$-vector space of dimension N, and $G_p(V)$ the Grassmannian of p planes in V. Let $e_1, \dots, e_N$ be a basis for V, $V_i = ke_1 + \dots + e_i$, $0 \le i \le n$. One knows $G_p(V)$ has a cell decomposition given by the Shubert cells as follows. I take $p=2$ to simplify.

Given a 2 plane A in V after performing row operations on the $2 \times N$ matrix given by a basis for A, one gets a canonical form for A of the form:

$$\begin{pmatrix} \overbrace{* \cdots *}^{} & 1 & 0 & \cdots & & & \cdots \\ \underbrace{\underbrace{* \cdots *}_{r_1} & 0 & * \cdots * & 1}_{r_2+1 \text{ entries (the 1 counts the 0)}} & 0 & \cdots \end{pmatrix}$$

where the $*$'s are arbitrary complex numbers. Call C_{r_1,r_2} the set of A with canonical form of the above type. It is a cell of cx. dimension r_1+r_2. One clearly gets a decomposition of $G_2(V)$ into cells C_{r_1,r_2} for each $0 \le r_1 \le r_2 \le N-2$; as these cells are even-dimensional each cell C_{r_1,r_2} gives rise to a homology class $[C_{r_1,r_2}]$ in $H_{2(r_1+r_2)}(G_2(V))$, and in this way we get a basis for $H_*(G_2(V))$.

Clearly if $V_i = ke_1 + \dots + ke_i$ then

$$C_{r_1,r_2} = \{A \mid 0 = \dots = A \cap V_{r_1} < A \cap V_{r_1+1} = \dots = A \cap V_{r_2+1} < A \cap V_{r_2+2} = \dots = A\}$$

which shows these cells depend only on the flag $\{V_i\}$.

Another way of seeing this is to note that the cell $C_{r_1 r_2}$ is simply the orbit under the action of the group

$$N = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix} = \text{stability group of the flag}$$

of the matrix:

$$\begin{array}{c} \begin{array}{cc} r_1+1 & \end{array} \\ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ \begin{array}{cc} & r_2+2 \end{array} \end{array}.$$

Thus these cells are the N-orbits (or also the B-orbits, B = upper Δ matrices) on $G_2(V)$.

One has

$$\overline{C}_{r_1 r_2} = \bigcup_{\substack{(a \le b) \\ a \le r_1,\ b \le r_2}} C_{ab}$$

$$= \left\{ A_2 \,\middle|\, \dim(A \cap V_{r_1+1}) \geq 1,\ \dim(A \cap V_{r_2+2}) \geq 2 \right\}$$

where the closure is the same for both $\mathbb{C}$-topology and the Zariski topology. $\overline{C}_{r_1 r_2}$ is a cycle in the alg. variety $G_2(V)$ which can be desingularized as follows.

Let $G_{11}(V)$ be the flag bundle of the subbundle on $G_2(V)$, i.e. the space of pairs, (ℓ, A) where ℓ is a line in the 2-plane A. Put

$$\widetilde{C}_{r_1 r_2} = \left\{ (\ell, A) \,\middle|\, \ell \subset V_{r_1+1},\ A \subset V_{r_2+2} \right\}$$

This is non-singular because it is the projective bundle over $\mathbb{P}V_{r_1+1}$ associated to the vector bundle $\ell \mapsto V_{r_2+2}/\ell$.

It maps onto $\overline{C}_{r_1 r_2}$ bijectively over $C_{r_1 r_2}$.

Now enlarging N doesn't change the cells C_{r_1, r_2}, it only adds some more. Thus in the limit as $G_2(V) \longrightarrow BU_2$ we have a basis for $H_*(BU_2)$ given by $[C_{r_1, r_2}]$ with $0 \leq r_1 \leq r_2$. On the other hand one knows that

$$\left[H_*(BU_1) \otimes H_*(BU_1)\right]_{\Sigma_2} \xrightarrow{\sim} H_*(BU_2)$$

whence if $b_i \in H_{2i}(BU_1)$ is the class of $\mathbb{P}^i$, we get the basis $b_i b_j$ $i \leq j$ for $H_*(BU_2)$. The problem now is to relate these bases.

First the map $BU_1 \times BU_1 \longrightarrow BU_2$ can be viewed as the flag bundle of the canonical bundle. Let $G_{1,1}(V)$ be the flag bundle of the canonical 2-plane bundle on $G_2(V)$. Then $G_{1,1}(V) = \{(\ell \subset A_2)\}$ which can also be identified with the manifold of pairs (ℓ_1, ℓ_2) where ℓ_1, ℓ_2 are orthogonal lines in V. This gives us maps

$$\mathbb{P}V \times \mathbb{P}V \xleftarrow{\ i\ } G_{11}(V) \longrightarrow G_2(V)$$

where i is an equivalence in a range increasing with $\dim V$. This gives us the map $BU_1 \times BU_1 \longrightarrow BU_2$.

Denote by L_1, L_2 the line bundles on $G_{11}(V)$ whose fibres at (ℓ_1, ℓ_2) are ℓ_1, ℓ_2 respectively. Then $G_{11}(V) =$ projective bundle of the quotient bundle $V/\mathcal{O}(-1)$ on $\mathbb{P}V$. Recall:

<u>Lemma 1</u>: Let E be an n-dim. vector bundle over X, $f : \mathbb{P}E \longrightarrow X$ the assoc. projective bundle, $t = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}E)$. Then

$$H^*(\mathbb{P}E) = H^*(X)[t] \big/ \left(t^n + c_1(E)t^{n-1} + \dots + c_n(E)\right)$$

and if $a(T) \in H^*(X)[T]$, then

$$(*) \qquad f_*(a(t)) = \operatorname{res}\left(\frac{a(T)\,dT}{T^n + c_1(E)T^{n-1} + \dots + c_n(E)}\right)$$

Proof of $(*)$. Embed E in a trivial bundle V of rank N so that one has

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}V \arrow[d, "g"] \\ & X \end{tikzcd}$$

One has that $\mathbb{P}E$ is where the homo

$$\mathcal{O}_{\mathbb{P}V}(-1) \subset g^*V \longrightarrow\!\!\!\!\rightarrow g^*(V/E)$$

vanishes, whence
$$\begin{aligned} i_* 1 &= e\left(\mathcal{O}(1) \otimes g^*(V/E)\right) \\ &= t^q + g^*c_1(V/E)t^{q-1} + \dots + g^*c_q(V/E) \end{aligned}$$

where $q + n = N$, and $t = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}V)$. Then

$$\begin{aligned} f_* a(t) &= g_* i_* i^* a(t) = g_*\left(i_* 1 \cdot a(t)\right) \\ &= \text{coeff of } T^{N-1} \text{ in } \left(T^q + \dots + c_q(V/E)\right) a(T) \\ &= \operatorname{res} \frac{\left(T^q + \dots + c_q(V/E)\right) a(T)\,dT}{T^N} \end{aligned}$$

since V is trivial. But $T^N = [T^q + \dots + c_q(V/E)][T^n + \dots + c_n(E)]$ so cancelling we get $(*)$.

Now apply this to $G_{11}(V)$:

$$\begin{tikzcd} G_{11}(V) \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}V \times \mathbb{P}V \arrow[d, "pr_1"] \\ & \mathbb{P}V \end{tikzcd}$$

$E = V/L_1 \subset V$

$f(\ell_1, \ell_2) = \ell_1$

and one finds $H^*(G_{11}(V))$ has a base over $\mathbb{Z}$ given by the monomials $t_1^a t_2^b$ $\quad 0 \le a < N, \; 0 \le b < N-1$ and

$$f_*(a(t_1,t_2)) = \operatorname{res} \frac{a(t_1,T_2)\,dT_2}{T_2^{N-1} + c_1(V/L_1)T_2^{N-2} + \dots}$$

$$= \operatorname{res} \frac{[T_2 + c_1(L_1)]\, a(t_1,T_2)\,dT_2}{T_2^N}$$

$$= \operatorname{res} \, [T_2 - t_1]\, a(t_1,T_2) \frac{dT_2}{T_2^N}$$

Thus

$$\boxed{\int_{G_{11}(V)} a(t_1,t_2) = \text{coeff of } (T_1T_2)^{N-1} \text{ in } (T_2 - T_1)\, a(T_1,T_2)}$$

$$= a_{N-1,N-2} - a_{N-2,N-1} \quad \text{if } a = \sum a_{ij} T_1^i T_2^j$$

Thus we get

<u>Lemma 2</u>: The homology class of $G_{11}(V)$ regarded as the submanifold of $\mathbb{P}V \times \mathbb{P}V$ of orthogonal line pairs is $b_{N-1} \otimes b_{N-2} - b_{N-2} \otimes b_{N-1}$.

But now

$$\widetilde{C}_{r_1 r_2} = \{(\ell_1, \ell_2) \mid \ell_1 \subset V_{r_1+1},\ \ell_2 \subset V_{r_2+2},\ \ell_1 \perp \ell_2\}$$
$$= G_{11}(V) \cap (\mathbb{P}V_{r_1+1} \times \mathbb{P}V) \cap (\mathbb{P}V \times \mathbb{P}V_{r_2+2})$$

and this intersection is proper because it gives something of the right codimension. Thus the class of $\widetilde{C}_{r_1 r_2}$ in $H_*(\mathbb{P}V^2)$ is

$$(b_{N-1} \otimes b_{N-2} - b_{N-2} \otimes b_{N-1}) \cap b_{r_1} \otimes b_{r_2+1}$$
$$= b_{r_1} \otimes b_{r_2} - b_{r_1-1} \otimes b_{r_2+1}$$

Thus we see that

$$\begin{tikzcd}
 & cl(\widetilde{C}_{r_1 r_2}) \arrow[r, mapsto] \arrow[ddl, bend right, mapsto] & cl(C_{r_1 r_2}) \arrow[d, phantom, "\cap"] \\
H_*(\mathbb{P}V^2) \arrow[d, hook] & H_*(G_{11}(V)) \arrow[l] \arrow[r] & H_*(G_2(V)) \arrow[d, phantom, "\cap"] \\
H_*(BU_1 \times BU_1) \arrow[rr] & & H_*(BU_2)
\end{tikzcd}$$

$$b_{r_1} \otimes b_{r_2} \otimes b_{r_1-1} \otimes b_{r_2+1}$$

and so we obtain

<u>Proposition</u>: In $H_*(BU_2)$ one has

$$cl(C_{r_1 r_2}) = \begin{vmatrix} b_{r_1} & b_{r_2+1} \\ b_{r_1-1} & b_{r_2} \end{vmatrix}.$$

Generalize to $G_3(V)$.

$$G_{111}V \xrightarrow{f} G_{11}V \longrightarrow \mathbb{P}V \longrightarrow pt$$

$$f_* \, a(t_1,t_2,t_3) = \operatorname{res} \frac{a(t_1,t_2,T_3)\,dT_3}{T_3^{N-2} + c_1(V/L_1+L_2)T_3^{N-3} + \cdots}$$

$$= \operatorname{res} \frac{(T_3+c_1(L_1))(T_3+c_1(L_2))\,a(t_1,t_2,T_3)\,dT_3}{T_3^N}$$

$$= \operatorname{res} \frac{(T_3-t_1)(T_3-t_2)\,a(t_1,t_2,T_3)\,dT_3}{T_3^N}$$

$$\int_{G_{111}(V)} a(t_1,t_2,t_3) = \text{coeff of } (T_1T_2T_3)^{N-1} \text{ in } (T_3-T_1)(T_3-T_2)(T_2-T_1)\,a(T_1,T_2,T_3)$$

Lemma: The cohomology class of $G_{111}(V) \subset \mathbb{P}V^3$ is

$$\prod_{i>j}(t_i-t_j) = \begin{vmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{vmatrix}$$

Now in general if you use the fact that

$$\prod_{i>j}(t_i-t_j) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_p \\ & & & \\ t_1^{p-1} & t_2^{p-1} & \cdots & t_p^{p-1} \end{vmatrix}$$

then one will find that the coh. class of $G_{\underbrace{11\cdots 1}_{p}}(V) \subset \mathbb{P}V^p$ is this Vandermonde determinant.

Now to find the class of $\tilde{C}_{r_1\cdots r_p}$ in $(\mathbb{P}V)^p$ you take cup product of the ~~classes~~ coh. class corresponding to $\mathbb{P}V_{r_1+1} \times \cdots \times \mathbb{P}V_{r_p+p}$ which is $t_1^{n-r_1}\, t_2^{n-r_2-1} \cdots t_p^{n-r_p-p}$ and you get the class:

$$\begin{vmatrix} t_1^{n-r_1} & t_2^{n-r_2-1} & \cdots & t_p^{n-r_p-p+1} \\ t_1^{n-r_1+1} & t_2^{n-r_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ t_1^{n-r_1+p-1} & \cdots & \cdots & t_p^{n-r_p} \end{vmatrix}$$

which then capped with the fundamental cycle of $\mathbb{P}V^p$ which is $b_n \otimes \cdots \otimes b_n$ gives the same determinant, but where t_i^{n-j} is replaced by $b_j^{(i)}$, $b_j^{(i)} = 1 \otimes \cdots \otimes b_j \otimes \cdots \otimes 1$ i-th place. Thus its clear that we get

<u>Proposition:</u> In $H_*(BU_p)$, the class associated to the Schubert cycle

$$\bar{C}_{r_1 \cdots r_p} = \{ A_p \mid \dim(A_p \cap V_{r_i+i}) \geq i , \quad i = 1, \dots, p \}$$

of dim $r_1 + \cdots + r_p$ is the determinant

$$\begin{vmatrix} b_{r_1} & b_{r_2+1} & \cdots & b_{r_p+p-1} \\ b_{r_1-1} & b_{r_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{r_1-p+1} & \cdots & \cdots & b_{r_p} \end{vmatrix}$$

<u>Cohomology side</u>: Here I use the canonical form

$$C^{r_1,r_2} = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 1 & * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & * & \cdots & * \end{pmatrix} \\ \qquad\quad \uparrow \qquad\qquad \uparrow \\ \qquad\quad r_1+1 \qquad r_2+2 \end{array}$$

which will be a cell of codim $= r_1 + r_2$. One has

$$C^{r_1 r_2} = C_{N-2-r_2,\, N-2-r_1}$$

$$= \left\{ A \,\middle|\, \dim(A \cap V_{N-1-r_2}) \geq 1,\ \dim(A \cap V_{N-r_1}) \geq 2 \right\}$$

and

$$\tilde{C}^{r_1 r_2} = \left\{ (l_1, l_2) \,\middle|\, \begin{array}{l} l_1 \perp l_2 \\ l_1 \subset V_{N-1-r_2},\ l_2 \subset V_{N-r_1} \end{array} \right\}$$

$$= \left\{ (l_1, l_2) \in G_{11}(V) \,\middle|\, l_1 \subset \mathbb{P}V_{N-1-r_2},\ l_2 \subset \mathbb{P}V_{N-r_1} \right\}$$

has the cohomology class

$$t_1^{r_2+1}\, t_2^{r_1}$$

But now if $f: G_{11}(V) \longrightarrow G_2(V)$ is the projection we know that

$$f_* \, cl(\tilde{C}^{r_1 r_2}) = cl(C^{r_1 r_2})$$

and that $\exists!\ \alpha, \beta \in H^*(G_2(V))$ with

$$t_1^{r_2+1} t_2^{r_1} = cl(\tilde{C}^{r_1 r_2}) = f^*(\alpha) + f^*(\beta)\, t_1$$

~~Now work universally where we know that~~

$$\cancel{f^*: H^*(BU_2) \xrightarrow{\sim} H^*(\mathbb{P}V^2)^{\Sigma_2}}$$

~~Then Applying the interchange we have~~

where $t_1 = c_1(L_1^*)$ is the generator for $H^*(G_{11}(V))$ over $H^*(G_2(V))$.
We also know that $f_*(t_1) = 1$, whence

$$\beta = f_* \, cl(\tilde{C}^{r_1,r_2}) = cl(C^{r_1,r_2}).$$

But applying the interchange of L_1, L_2 which is a symmetry of $G_{11}(V)$ over $G_2(V)$, one gets the equations

$$t_1^{r_2+1} t_2^{r_1} = f^*(\alpha) + f^*(\beta)\, t_1$$

$$t_2^{r_2+1} t_1^{r_1} = f^*(\alpha) + f^*(\beta)\, t_2$$

so solving

$$f^*(\beta) = \frac{\begin{vmatrix} 1 & t_1^{r_2+1} t_2^{r_1} \\ 1 & t_2^{r_2+1} t_1^{r_1} \end{vmatrix}}{\begin{vmatrix} 1 & t_1 \\ 1 & t_2 \end{vmatrix}} = \frac{\begin{vmatrix} t_1^{r_1} & t_2^{r_1} \\ t_1^{r_2+1} & t_2^{r_2+1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ t_1 & t_2 \end{vmatrix}}$$

So we have proved:

Proposition: ~~Lemma~~ In $H^*(BU_2) \simeq H^*(BU_1 \times BU_1)^{\mathbb{Z}/2}$ we have the following formula for the class assoc. to the Schubert cycles

$$cl(C^{r_1,r_2}) = \frac{\begin{vmatrix} t_1^{r_1} & t_2^{r_1} \\ t_1^{r_2+1} & t_2^{r_2+1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ t_1 & t_2 \end{vmatrix}} = t_1^{r_1} t_2^{r_2} + t_1^{r_1+1} t_2^{r_2-1} + \dots + t_1^{r_2} t_2^{r_1}$$