October 1973

Construction of a spectrum assoc. to an exact category.

Review of Segal's construction of a spectrum associated to a category-with-product.

Let $\mathcal{E}$ be a category-with-product. If S is a finite set let $\mathcal{E}(S)$ be the following cat. An object of $\mathcal{E}(S)$ consists of i) an object E_T of $\mathcal{E}$ for each $T \subset S$ ii) an isomorphism $E_T \perp E_{T'} \xrightarrow{\sim} E_{T \cup T'}$ for each pair $T, T' \subset S$ such that $T \cap T' = \emptyset$; these data are subject to unity conditions:

$$E_\emptyset = 0 \qquad 0 = \text{the given unit object of } \mathcal{E}$$

$$\begin{tikzcd} E_\emptyset \perp E_T \arrow[r, "\sim"] \arrow[d, equal] & E_T \\ 0 \perp E_T \arrow[ur, "\sim"'] & \end{tikzcd}$$

commutes

and commutativity + assoc. conditions which I won't write. A morphism $\{E_T\} \longrightarrow \{E'_T\}$ in $\mathcal{E}(S)$ is a family of morphisms $E_T \longrightarrow E'_T$ compatible with the isos of the form ii).

I will assume known that the functor

$$(1) \qquad \begin{aligned} \mathcal{E}(S) &\longrightarrow \mathcal{E}^S \\ \{E_T\} &\longmapsto \{s \mapsto E_{\{s\}}\} \end{aligned}$$

is an equivalence of categories. This ought to be a standard consequence of coherence theory.

Now let Γ denote the category having as objects finite sets, and in which an arrow $S \to S'$ is a map $P(S) \to P(S')$ preserving unions and intersections and empty sets. Equivalently an arrow

$S \to S'$ is a basepoint-preserving map $S' \cup \{\infty\} \to S \cup \{\infty\}$.

Then it is clear that to the arrow $S \to S'$ one has an induced map $\mathcal{E}(S') \longrightarrow \mathcal{E}(S)$ sending $\{E_{T'}, T' \subset S'\}$ to $\{E_{f(T)}, T \subset S\}$. Moreover $S \mapsto \mathcal{E}(S)$ is a contravariant functor from Γ to categories such that for each S, we have the equivalence (1).

~~Next let Ord denote the usual simplicial category~~

Next I want to define a functor

(2)
$$\underbrace{\mathrm{Ord} \times \cdots \times \mathrm{Ord}}_{k \text{ times}} \longrightarrow \Gamma$$

First we have the functor

$$\begin{aligned} \mathrm{Ord} &\longrightarrow \Gamma \\ p &\longmapsto \{1, \ldots, p\} \end{aligned}$$

where if $\theta : p \to q$ is monotone, then we get the Γ-map $\{1, \ldots, p\} \longrightarrow \{1, \ldots, q\}$ sending $i \longmapsto \{\theta(i-1) < j \leq \theta(i)\}$. On the other hand we have an evident functor.

$$\begin{aligned} \Gamma \times \Gamma &\longrightarrow \Gamma \\ S, T &\longmapsto S \times T \end{aligned}$$

So that the functor (2) I want

$$\mathrm{Ord}^k \longrightarrow \Gamma$$

sends $(p_1, \ldots, p_k) \longmapsto \{1, \ldots, p_1\} \times \cdots \times \{1, \ldots, p_k\}$

and it sends the arrow $(\theta_a) : (p_a) \longrightarrow (q_a)$ into the Γ-map

$$(i_1, \ldots, i_k) \longmapsto \{(j_1, \ldots, j_k) \mid \theta(i_a - 1) < j_a \leq \theta(i_a)\}.$$

Therefore from (2) I get a k-fold simplicial category

$$(p_1,\dots,p_k) \longmapsto \mathcal{E}(\{1,\dots,p_1\}\times\cdots\times\{1,\dots,p_k\})$$

whose realization I denote $B_k(\mathcal{E})$.

[Consequence of (1):

(1') $$\mathcal{E}(S \amalg T) \longrightarrow \mathcal{E}(S)\times\mathcal{E}(T)$$

is an equivalence of categories.]

Introduce notation $\underline{n}$ for the set $\{1,\dots,n\}$ as an object of Γ.

We are going to get at $B_k(\mathcal{E})$ by

$$B_k(\mathcal{E}) = \left| p_k \longmapsto \left|(p_1,\dots,p_{k-1}) \longmapsto \mathcal{E}(\underline{p}_1\times\cdots\times\underline{p}_k)\right|\right|$$

But if p_k is fixed we have from (1') an equivalence of categories

$$\mathcal{E}(p_1\times\cdots\times p_k) \longrightarrow \mathcal{E}(p_1\times\cdots\times p_{k-1})^{p_k}$$

functorial in $p_1,\dots,p_{k-1}$; hence we get a h.eq.

$$|(p_1,\dots,p_{k-1}) \longmapsto \mathcal{E}(p_1\times\cdots\times p_k)| \longrightarrow B_{k-1}(\mathcal{E})$$

induced by the face operators of style $(0,1) \longmapsto (i,i+1)$ on the last coordinate. It follows that $B_k(\mathcal{E})$ is the realization of the special simplicial space

$$p_k \longmapsto |(p_1,\dots,p_{k-1}) \longmapsto \mathcal{E}(p_1\times\cdots\times p_k)|$$

which is isomorphic to $B_{k-1}\mathcal{E}$ in dimension 1. But if $k \geq 2$, $B_{k-1}\mathcal{E}$ is connected, so we have by Segal's theory a homotopy equivalence

$$\Omega B_k(\mathcal{E}) \longleftarrow B_{k-1}(\mathcal{E}).$$

The K-spectrum of an exact category.

Definition of $\mathcal{M}^{(p)}$: $\mathcal{M}$ is an exact category. An object of $\mathcal{M}^{(p)}$ is a functor $(i,j)\mapsto M_{ij}$ from the ordered set $\tilde{p}$ of pairs (i,j), $0\le i\le j\le p$ with the product ordering, to $\mathcal{M}$ such that $\forall$ $i\le j\le k$

$$0\to M_{ij}\to M_{ik}\to M_{jk}\to 0$$

is an exact sequence in $\mathcal{M}$. Claim such functors form an exact cat. in an obvious way. <u>Proof</u>: Let $\mathcal{M}$ be a full subcat of an abelian cat $\mathcal{A}$ closed under extensions, let $\mathcal{A}'$ be the cat of functors $(i,j)\mapsto A_{ij}$ from the ordered set above $\tilde{p}$ to $\mathcal{A}$ such that $A_{ii}=0$ for $0\le i\le p$. Then $\mathcal{A}'$ is abelian and $\mathcal{M}^{(p)}$ is a full subcat. of $\mathcal{A}'$ closed under extensions.

Claim $\mathcal{M}^{(p)}$ is equivalent to the category of objects of $\mathcal{M}$ equipped with an admissible filtration of length p. The equivalence is given by sending $\{M_{ij}\}$ to the object M_{0p} with the admiss. filtration

$$0 = M_{00}\to M_{01}\to\cdots\to M_{0p}$$

If $\theta: p\to q$ is a monotone map, it induces a functor $\tilde{p}\to\tilde{q}$, + hence a functor $\theta^*:\mathcal{M}^{(q)}\to\mathcal{M}^{(p)}$ which is an exact functor. Thus $p\mapsto\mathcal{M}^{(p)}$ is a simplicial exact category.

<u>Example for the intuition</u>: Let $\partial_i: p-1\to p$ be the i-th face map. If we identify an obj of $\mathcal{M}^{(p)}$ with an object ~~$\{M_i,\ 0\le i\le p,\ M_0=0\}$, then~~ equipped with an

admissible filtration $0 \subset F_1 M \subset \cdots \subset F_p M$ ~~[illegible]~~ of length p, then

$$d_i(0 \subset F_1 M \subset \cdots \subset F_p M = M) = \begin{cases} (0 \subset F_1 M \subset \cdots \subset F_{p-1} M) & i = p \\ (0 \subset F_1 M \subset \cdots \subset \widehat{F_i M} \subset \cdots \subset F_p M) & 0 < i < p \\ (0 \subset F_2 M / F_1 M \subset \cdots \subset F_p M / F_1 M) & i = 0 \end{cases}$$

$$s_i(0 \subset F_1 M \subset \cdots \subset F_p M) = (0 \subset \cdots \subset F_i M \subset F_i M \subset F_{i+1} M \subset \cdots \subset F_p M)$$

Next we consider the simplicial groupoid

$$p \longmapsto \mathrm{Iso}\, \mathcal{M}^{(p)}$$

and form the ~~[illegible]~~ associated fibred category over ~~[illegible]~~ Ord. Tentative notation: ~~[illegible]~~ $\mathcal{S}^*(\mathcal{M})$. An object is a pair (p, M) ~~[illegible]~~ with $M \in \mathcal{M}^{(p)}$. and a map $(p, M) \longrightarrow (p', M')$ is given by a map $\theta : p \to p'$ in Δ plus an isom ~~[illegible]~~ $M \simeq \theta^*(M')$.

Claim $\mathcal{S}^*(\mathcal{M})$ is heq to $Q(\mathcal{M})$: Define a functor

$$f : \mathcal{S}^*(\mathcal{M}) \longrightarrow Q(\mathcal{M})$$

as follows. On objects: ~~[illegible]~~

$$f(p, M) = M_{0p}$$

On maps: Given $(p, M) \to (p', M')$ sep. by $\theta : p \to p'$ and $\alpha : M \simeq \theta^* M'$. Thus ~~[illegible]~~ α consists of compatible iso.

$$\alpha_{ij} : M_{ij} \simeq M'_{\theta(i)\theta(j)}$$

In particular we have

$$\alpha_{0p} : M_{0p} \simeq M'_{\theta(0)\theta(p)}$$

and the latter is an admissible subquotient of $M'_{0p'}$:

$$\begin{tikzcd} M_{oj} \arrow[r, tail] \arrow[d, two heads] & M_{op'} \\ M'_{ij} & \end{tikzcd}$$

Thus we have assoc. to the map $(p, M) \to (p', M')$ a map from $f(p, M) = M_{op}$ to $f(p', M') = M'_{op'}$ in $Q(\mathcal{M})$. This defines the functor f.

To show f is a heq we will prove the category f/N is contractible for every N in $Q(\mathcal{M})$.

Claim f/N is equivalent to the fibred cat over Δ assoc. to the simplicial set which is the nerve of the ordered set of admissible subobjects of N. Precisely an object of f/N consists of an object (p, M) of $\mathcal{S}^*(\mathcal{M})$ together with an map $u: M_{op} \to N$ in $Q(\mathcal{M})$. ~~We can associate to this object the map u is~~ Thus we get a chain of admiss. subobjects

$$(*) \qquad 0 \subseteq N_0 \subset N_1 \subset \cdots \subset N_p \subset N$$

where u is given by an isom $M_{op} \simeq N_p/N_0$ and N_i/N_0 is the image of M_{oi} under this iso. One sees without difficulty that by assoc. to each object of f/N the above described chain in N that we get then a heq of f/N with the fibred cat over Δ defined by the simplicial set whose simplices are chains of admissible subobjects of N.

Since the category of admissible suboby of N has an initial object, it follows f/N is contractible.

DIGRESSION 7

To make preceding more clear suppose we have a simp. cat. $p \mapsto \mathcal{C}_p$ and let $\mathcal{C}$ be the assoc. fibred cat. over Δrd. Let $f: \mathcal{C} \to \mathcal{C}'$ be a functor whence we get functors $f_p : \mathcal{C}_p \to \mathcal{C}'$ $\forall p \in Ob(Ord)$ and nat. transf.

$$\begin{tikzcd} \mathcal{C}_p \arrow[d, "\theta^*"'] \arrow[dr, "f_p"] \arrow[dr, phantom, shift right=3ex, "\theta_f \Uparrow"] & \\ \mathcal{C}_q \arrow[r, "f_q"'] & \mathcal{C}' \end{tikzcd}$$

$\theta^* Y \xrightarrow{can} Y$

$f_q(\theta^* Y) \to f_p Y$

for every θ: $q \to p$ in Ord.

And these natural transf θ_f have to be transitive. Now form the standard fact. of f denote it $E(f)$ whose objects are $(X, Y, fX \to Y)$. Then $E(f)$ is co~~fibred~~ scinded over $\mathcal{C}'$ with fibres f/Y and ~~fibred~~ scinded over $\mathcal{C}$ with fibred $fX\backslash\mathcal{C}'$. On the other hand I can form the simplicial category $p \mapsto E(f_p)$ which is a simp. object of the cat of cofibred cats over $\mathcal{C}'$ and cocart functors. I claim that $E(f)$ is the ~~fibred~~ scinded cat over Δ assoc. to $p \mapsto E(f_p)$. First of all $E(f)$ is fibred over $\mathcal{C}$, & ~~base~~ $\mathcal{C}$ fibres over Ord, so $E(f)$ fibres over Ord. And the fibre over p has objects $(X, Y, f_p(X) \to Y)$ where $X \in \mathcal{C}_p$, so its clear that $E(f)_p = E(f_p)$. Now given $p \xrightarrow{\theta} q$ one has $\theta^* : \mathcal{C}_q \to \mathcal{C}_p$ and $\theta_f : f_p \theta^* \to f_q$, hence one has $\theta^* : E(f_q) \to E(f_p)$ given by

$$(X, Y, f_q X \to Y) \longmapsto (\theta^* X, Y, f_p \theta^* X \xrightarrow{\theta_f} f_q X \to Y)$$

and I have only to check this is the base change functor $E(f)_q \to E(f)_p$. But if we have $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ fibred functors $gf(A) = C$ and $u : C' \to C$, then the base change $u^*(A)$ is computed by first taking canonical map $u' : u^*(fA) \to fA$ and then taking $u'^*(A)$. So this means that if $A = (X, Y, f_q X \xrightarrow{v} Y) \in E(f)$, then ~~...~~ $fA = X$, and $u^*(X) = \theta^*(X)$, $u' : \theta^*(X) \to X$ being the can. map in $\mathcal{C}$; and $u^*(A)$ is

$$(\theta^*(X), Y, f\theta^*(X) \xrightarrow{f(\mathrm{can})} fX \xrightarrow{v} Y)$$

so it all works since $\theta_f : f_p \theta^* X \to f_q X$ is the result of applying f to can map. What we have:

<u>Lemma</u>: Given $\mathcal{B} \xleftarrow{g} \mathcal{C} \xrightarrow{f} \mathcal{C}'$ where g is fibred. Then $E(f)$ is fibred over $\mathcal{B}$ with $E(f)_B = E(f|\mathcal{C}_B)$. Let $u : B' \to B$ be a map in $\mathcal{B}$, and $u^* : \mathcal{C}_B \to \mathcal{C}_{B'}$ is the base change ~~...~~, let $f_B = f|\mathcal{C}_B$ and let $u_f : f_{B'} u^* \to f_B$ denote the image of the canonical cart. arrow $u^*(X) \to X$ $X \in \mathcal{C}_B$. Then the base change $E(f)_B \to E(f)_{B'}$ wrt u is given by

$$\underset{\hat{\mathcal{C}}_B}{(X, Y, f_B X \xrightarrow{v} Y)} \longmapsto (u^*(X), Y, f_{B'} u^*(X) \xrightarrow{\theta_f} f_B X \xrightarrow{v} Y)$$

So in my example we know that $\mathrm{Iso}(\mathcal{M}^{(p)})/N$ is equivalent to the set of chains $0 < N_0 < N_1 < \cdots < N_p < N$ of admissible subobjects, and the base change functors

$$\theta^* : \mathrm{Iso}(\mathcal{M}^{(q)})/N \longrightarrow \mathrm{Iso}(\mathcal{M}^{(p)}/N)$$

assoc. to a monotone map $\theta : p \to q$ may be identified with

$$\theta^*(N_1 < \cdots < N_q) \longmapsto N_{\theta(1)} < \cdots < N_{\theta(p)}$$

So everything works.

Review: Have now established a heq between the fibred cat. over Δ assoc. to $p \mapsto \mathrm{Iso}\, M^{(p)}$ and the cat. $Q(M)$.

~~Segal's idea? one realizes the bisimplicial space~~

Segal's idea was to form the simplicial space $p \mapsto B(\mathrm{Iso}\, M^{(p)})$ and to realize it:

$$B_1(M) = |p \longmapsto B(\mathrm{Iso}\, M^{(p)})|.$$

Now I want to show this has the same homotopy type as $BQ(M)$. In view of preceding where we have established a heq. between $Q(M)$ and the fibred cat over Ord associated to the s. cat $p \mapsto \mathrm{Iso}\, M^{(p)}$ it suffices to prove:

Lemma: Let C be the fibred cat over Ord associated to a simp. cat $p \mapsto C_p$. Then BC is naturally homotopy equivalent to $|p \mapsto BC_p|$.

Proof: Consider the bisimplicial set

$$(p,q) \longmapsto \{(p \to fX_0,\ X_0 \to \cdots \to X_q)\} = S_{pq}$$

where $f: C \to \mathrm{Ord}$ is the canonical functor. Realizing with respect to p we get

$$|p \mapsto S_{pq}| = \coprod_{X_0 \to X_1 \to \cdots \to X_q} \Delta_{fX_0}$$

and since Δ_n is contractible it follows we get a heq

$$|(p,q) \mapsto S_{pq}| \longrightarrow \left| \coprod_{X_0 \to \cdots \to X_q} pt \right| = BC.$$

On the other hand realizing wrt q

$$|q \mapsto S_{pq}| = B(p\backslash f)$$

and because f is fibred we have a heq $p\backslash f \longrightarrow \mathcal{C}_p$ sending $(X, p \xrightarrow{\theta} fX)$ to $\theta^*(X)$. Thus we get a heq

$$|(p,q) \mapsto S_{pq}| = |p \mapsto B(p\backslash f)| \longrightarrow |p \mapsto B\mathcal{C}_p|$$

concluding the proof.

Next it is necessary to discuss Segal's methods for constructing a spectrum for a category with product. Let $\mathcal{E}$ be a cat with product, and let p be an integer ≥ 0. I want to define another category with product $\mathcal{E}^{(p)}$ which will be equivalent to $\mathcal{E}^p$.

An object of $\mathcal{E}^{(p)}$ will consist of a family of objects E_{ij} of $\mathcal{E}$ for each $0 \leq i \leq j \leq p$ together with isos. $0 \xrightarrow{\sim} E_{ii}$ $\quad 0 \leq i \leq p$ and isos.

$$E_{ij} \perp E_{jk} \xrightarrow{\sim} E_{ik} \qquad 0 \leq i \leq j \leq k \leq p$$

such that the following conditions hold:

α) unit

$$\begin{tikzcd} E_{jj} \perp E_{jk} \arrow[r, "\sim"] \arrow[d, "\wr"'] & E_{jk} \\ 0 \perp E_{jk} \arrow[ur, "\sim"'] & \end{tikzcd} \qquad \begin{tikzcd} E_{ij} \perp E_{jj} \arrow[r, "\sim"] \arrow[d, "\wr"'] & E_{ij} \\ E_{ij} \perp 0 \arrow[ur, "\wr"'] & \end{tikzcd}$$

commute

β) assoc.

$$\begin{tikzcd}
E_{ij} \perp (E_{jk} \perp E_{kl}) \arrow[rr, "\sim"] \arrow[d, "\wr"] & & E_{ij} \perp E_{jl} \arrow[dd, "\wr"] \\
(E_{ij} \perp E_{jk}) \perp E_{kl} \arrow[d, "\wr"] & & \\
E_{ik} \perp E_{kl} \arrow[rr, "\sim"] & & E_{il}
\end{tikzcd}$$

commutes

~~this construction makes sense for a monoidal~~

With the obvious notion of morphisms one gets a category $\mathcal{E}^{(p)}$. Clearly $p \mapsto \mathcal{E}^{(p)}$ is a simp. category.

Claim
$$\mathcal{E}^{(p)} \longrightarrow \mathcal{E}^{p}$$
$$(E_{ij}, ..) \longmapsto (E_{01}, E_{12}, \ldots, E_{p-1,p})$$

is an equivalence of categories. (This requires coherence).
(We could define $\mathcal{E}^{(p)}$ so that $E_{ii} = 0$ if we wanted.)

~~The next point is less~~

Wait: This does not seem to be the best method.
What is essential in Segal's construction is that he can construct k-simplicial categories
$$p_1, \ldots, p_k \longmapsto \mathcal{E}(p_1, \ldots, p_k)$$
where
$$\mathcal{E}(p_1, \ldots, p_k) \longrightarrow \prod_{p_1} \prod_{p_2} \cdots \prod_{p_k} \mathcal{E}(1, \ldots, 1)$$
is an equivalence and $\mathcal{E}(1, \ldots, 1)$ is equivalent to $\mathcal{E}$.

So perhaps it would be better starting with a category with product $\mathcal{E}$ to construct for any set S the category $\mathcal{E}(S)$ of chains $\sum_{s \in S} E_s$ on S with coefficients in $\mathcal{E}$.

Given then a map $f: S \cup \{\infty\} \longrightarrow S' \cup \{\infty\}$ continuous one has

$$\mathcal{E}(S) \longrightarrow \mathcal{E}(S')$$

induced map. In fact what one should really want is a "correspondence" between S and S'

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ S & & S' \end{tikzcd}$$

such that Z is proper over S. ~~To simplify will suppose always that S is finite.~~

Define $\mathcal{E}(S)$ to be the category of product-preserving functors from finite sets over S (& autos) to $\mathcal{E}$. Thus for each $T \longrightarrow S$ have E_T and for each $\begin{tikzcd}[row sep=small, column sep=small] T \arrow[r, "\sim"] \arrow[dr] & T' \arrow[d] \\ & S \end{tikzcd}$ have $E_T \xrightarrow{\sim} E_{T'}$ and have also

$$E_\phi \cong 0 \qquad E_{T \cup T'} \cong E_T \times E_{T'}$$

compatible with the assoc. & unity isos. Now one needs coherence to show that

$$\mathcal{E}(S) \longrightarrow \mathcal{E}^S$$

is an equivalence. Sometime you will have to make a proof.

Now for Segal's purpose one needs only certain types of correspondences, namely, where Z is a subset of S. The point is somehow that a monotone map $p \to q$ will be interpreted as a corresp. $\{1,\dots,p\} \Longleftarrow \{1,\dots,q\}$. And similarly a map

$$(p_1,\dots,p_k) \longrightarrow (q_1,\dots,q_k)$$

will be viewed as a correspondence

$$\{1,\dots,p_1\}\times\dots\times\{1,\dots,p_k\} \Longleftarrow \{1,\dots,q_1\}\times\dots\times\{1,\dots,q_k\}$$

For example the face

$$1 \longrightarrow p \qquad \begin{array}{l}\partial_i(0)=i\\ \partial_i(1)=i+1\end{array} \qquad 1,\dots,i,i+1,\dots,p$$

will be viewed as the correspondence relating i to 1

In general given $p \xrightarrow{\theta} q$ monotone one gets $0 \le \theta(0) \le \theta(1) \le \dots \le \theta(p) \le q$ which are views as relating

$$\{\theta(0) < j \le \theta(1)\} \text{ to } 1$$
$$\{\theta(1) < j \le \theta(2)\} \text{ to } 2$$
etc.

For example $\partial_i : p-1 \longrightarrow p$ omitting i gives rise to the correspondence

$$\begin{array}{ccc} 1 & \longleftarrow & 1 \\ i-1 & \longleftarrow & i \\ i+1 & & i+1 \\ p-1 & \longleftarrow & p \end{array}$$

NO

while $s_i : p \longrightarrow p-1$ collapsing $i, i+1$ gives rise to the correspondence

$$\begin{array}{ccc} 1 & \longleftarrow & 1 \\ i & \longleftarrow & i \\ i+2 & \longleftarrow & i+1 \\ p & \longleftarrow & p-1 \end{array}$$

From my point of view what's natural is to find something that fits well with exact sequences. A heuristic principle to organize things might be that we have the same basic symmetries around but also orderings (i.e. (W, B)).

So what in fact happens is this: The basic objects for me are multifiltered objects. Thus before with the cats with product essentially an object is a collection $\{E_s, s\in S\}$ indexed by a finite set.

Guess is that we now go from a finite Boolean algebra to a finite distributive lattice.

In any case what seems to be the situation in dim 1 is that we are concerned with linearly ordered finite sets

~~What is an object of $M^{(p_1)(p_2)\cdots(p_k)}$? The idea is that it should be an object M of M together with a k multifiltration~~

$$\cancel{F_{j_1\cdots j_k}M \qquad 0\le j_1\le p_1,\dots,0\le j_k\le p_k}$$

~~i.e. a map of~~

$$\cancel{p_1\times\cdots\times p_k \longrightarrow \text{Admiss. subobjects of } M}$$

Idea: Consider $\{1,2,\dots,p\}$ as a top space in which the closed sets are $\{1,2,\dots,i\}$ $\quad i=0,\dots,p$. Then an ~~object of $M^{(p)}$ is a functor from closed~~

~~subsets of {1,…,p}~~ an admissible filtration of length p of M in $\mathcal{M}$ is a map from closed subsets of $\{1,\ldots,p\}$ to admissible subobjects of M which is compatible with joins + meets.

Now perhaps an object of $\mathcal{M}^{(p_1)\cdots(p_k)}$ is an object M of $\mathcal{M}$ together with a join + meet compatible functor from closed subsets of $\{1,\ldots,p_1\}\times\cdots\times\{1,\ldots,p_k\}$ to admissible subobjects of M. Assume this for the moment.

In any case instead of a finite set S maybe we want to consider a finite partially ordered set.

<u>Conjecture</u>: Let X be a finite poset and L the distributive lattice of closed subsets of X $(y \le x \in F \Rightarrow y \in F)$. Let $\mathcal{M}(L)$ be the category of functors from L into admissible monos of L strictly compatible with meets and joins. This should be the same as functors $(F',F'') \longmapsto M(F',F'')$ from pairs of elements of L $\ni$ $F' \le F''$ to $\mathcal{M}$ such that $M(F',F')=0$ and $\forall$ $F' \le F'' \le F'''$

$$0 \to M(F',F'') \to M(F',F''') \to M(F'',F''') \to 0$$

is exact. Then in addition $\mathcal{M}^{(p_1)\cdots(p_k)} = \mathcal{M}(L)$ where $X = \{1<\cdots<p_1\}\times\cdots\times\{1<\cdots<p_k\}$.

The only thing worth examining is whether if these conjectures are true, then how does one interpret maps. ~~[illegible]~~

Good idea: Let X be a poset and let $x \longmapsto M_x$ be a functor from X into the ~~[illegible]~~ ordered set of admissible subobjects of M. Then say the family $x \mapsto M_x$ intersects cleanly if one can extend it:

$$x_1 \cup \cdots \cup x_n \longmapsto M_{x_1} + \cdots + M_{x_n}$$

to a functor on the distributive lattice generated by X. which is compatible with meets and joins.

Now given $f : X' \longrightarrow X$ a map of posets one gets a lattice map

$$L(X') \longleftarrow L(X)$$

and hence a functor

$$m(L(X')) \longrightarrow m(L(X))$$

Very confusing. The point ultimately is that ~~[illegible]~~ one has $m(L)$, where L is a distributive lattice, consisting of systems M_{F_1, F_2} indexed by the layers in L, compatible with the lattice operations. And one has a map $m(L) \longrightarrow m(L')$ whenever one has $L' \rightarrow L$ monotone compatible with meet and join.

The point is that this is the usual simplicial nonsense except we enlarge Δ to include more general ordered sets.

Go on. We were trying to describe Segal's construction of a spectrum starting with a category with product $\mathcal{E}$. First given a finite set S he constructs $\mathcal{E}(S)$ which is contravariant in S as S ranges over Γ. (A map from S to S' in Γ is a map $S' \cup \{\infty\} \longrightarrow S \cup \{\infty\}$ of sets with basepoint, i.e. a map from subsets of S to subsets of S' compatible with unions and intersections

$$S \supset A \longmapsto f_A \subset S'$$

and sending ϕ to ϕ.) Then he uses the functor

$$\begin{aligned} \mathrm{Ord} \times \cdots \times \mathrm{Ord} &\longrightarrow \Gamma \\ p_1, \ldots, p_k &\longmapsto \{1,\ldots,p_1\} \times \cdots \times \{1,\ldots,p_k\} \end{aligned}$$

to obtain from the Γ space $S \mapsto S(\mathcal{E})$ a k-simp. space for every k.

It might be better to set up his construction iteratively.

Thus put $\mathcal{E}^{(p)} = \mathcal{E}(\{1,\ldots,p\})$; this is a simplicial category with product. Now iterate and you get $\mathcal{E}^{(p_1)(p_2)\cdots(p_k)}$ which is a k-simplicial category with product whose realization will be denoted $B_k(\mathcal{E})$. Now the point is that if we fix $p_1, \ldots, p_{k-1}$, then

$$\mathcal{E}^{(p_1)\cdots(p_{k-1})(p_k)} \longrightarrow \left(\mathcal{E}^{(p_1)\cdots(p_{k-1})}\right)^{p_k}$$

is an equivalence, so

$$\left| p_1 \cdots p_{k-1} \longmapsto B\mathcal{E}^{(p_1)\cdots(p_{k-1})(p_k)} \right| \longrightarrow \left(B_{k-1}\mathcal{E}\right)^{p_k}$$

is a h.eq. for each p_k. Thus $p_k \mapsto |p_1 \cdots p_{k-1} \mapsto B\mathcal{E}^{(p_1)\cdots(p_{k-1})(p_k)}|$
is a special simplicial space and so by Segal's results, we know its realization $B_k\mathcal{E}$ has loop space $B_{k-1}\mathcal{E}$. (for $k \geq 2$ so that $B_1\mathcal{E}$ is connected.)
Thus get a spectrum.

Now suppose we have our exact category $\mathcal{M}$. Then we have a simplicial exact category

$$p_1, \cdots, p_k \longmapsto \mathcal{M}^{(p_1)(p_2)\cdots(p_k)}$$

October 8, 1973.

Notes on Wagoner's paper.

Let G be a group and let $\mathcal{H}$ be a family of subgroups of G. Make $\mathcal{H}$ into an ordered set using inclusion.

~~Consider the~~ Let $G\mathcal{H} = \{gH \mid g \in G, H \in \mathcal{H}\}$ be the family of left cosets of the subgroups in $\mathcal{H}$. Make $G\mathcal{H}$ into an ordered set using inclusion. ~~Following Wagoner~~ ~~Basic Wagoner~~. Basic object in Wagoner-Volodin theory is:

$$B(G\mathcal{H}) = \text{classifying space of the ordered set } G\mathcal{H} \text{ of left cosets of groups in } \mathcal{H}.$$

Example: If $\mathcal{H}$ consists of a single group H, then $B\mathcal{H} \sim G/H$.

Next we want to give an alternative description.

First note that if $g_0H_0 \subset \cdots \subset g_pH_p$ is a p-simplex in $\mathrm{Nerv}(G\mathcal{H})$, then $g_iH_i = g_0H_i$. Hence

$$\mathrm{Nerv}_p(G\mathcal{H}) = \coprod_{H_0 \subset \cdots \subset H_p \in \mathrm{Nerv}_p(\mathcal{H})} G/H_0$$

and so we see that we can also describe $B\mathcal{H}$ as the classifying space of the cofibred category over $\mathcal{H}$ associated to the functor $H \longmapsto G/H$ ~~~~. If you want, $G\mathcal{H}$ is this cofibred category.

Next Wagoner considers the bisimplicial set ($N\mathcal{H}$ in his notation)

$$p,q \longmapsto \coprod_{H_0 \subset \cdots \subset H_p} \ \coprod_{gH_0 \in G/H_0} (gH_0)^{q+1}$$

With respect to g this is contractible; ~~the~~ also

$$\coprod_{gH_0 \in G/H_0} (gH_0)^{g+1} = G \times_{H_0} H_0^{g+1}$$

So we get a heq

$$B(G\mathcal{H}) \longleftarrow \left| \coprod_{H_0 \subset \cdots \subset H_p} G \times_{H_0} H_0^{g+1} \right|$$

However we have a cartesian square

$$\begin{tikzcd} \coprod_{H_0 \subset \cdots \subset H_p} G \times_{H_0} H_0^{g+1} \arrow[r] \arrow[d] & G^{g+1} \arrow[d] \\ \coprod_{H_0 \subset \cdots \subset H_p} H_0^{g} \arrow[r] & G^{g} \end{tikzcd}$$

where the vertical arrows are quotient maps for the G action. Thus we get a cart. square

$$\begin{tikzcd} \left| \coprod_{H_0 \subset \cdots \subset H_p} G \times_{H_0} H_0^{g+1} \right| \arrow[r] \arrow[d] & EG \arrow[d] \\ \left| \coprod_{H_0 \subset \cdots \subset H_p} BH_0 \right| \arrow[r] & BG \end{tikzcd}$$

of G-torsors.

Thus we arrive at the following:

Prop: The ordered set $G\mathcal{H}$ of cosets of the family $\mathcal{H}$ is the fibre of the map from the telescope of the functor $H \mapsto BH$ to BG.

October 9, 1973:

Summary:

$$
\begin{tikzcd}
\left| p \mapsto \coprod\limits_{H_0 \subset \cdots \subset H_p} G/H_0 \right| \arrow[d, phantom, "\|"] & \left| p,q \mapsto \coprod\limits_{H_0 \subset \cdots \subset H_p} G \times_{H_0} H_0^{q+1} \right| \arrow[l, "\text{heq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart. square of } G \text{ torsors}"] & \left| q \mapsto G^{q+1} \right| \arrow[d] \\
B(G\mathcal{H}) & \left| p,q \mapsto \coprod\limits_{H_0 \subset \cdots \subset H_p} H_0^{q} \right| \arrow[r] \arrow[d, phantom, "\|"] & \left| q \mapsto G^{q} \right| \arrow[d, phantom, "\|"] \\
& \left| p \mapsto \coprod\limits_{H_0 \subset \cdots \subset H_p} BH_0 \right| & BG
\end{tikzcd}
$$

So that $B(G\mathcal{H})$ is the fibre of $\left| p \mapsto \coprod\limits_{H_0 \subset \cdots \subset H_p} BH_0 \right| \longrightarrow BG$.

Additional point: Assume $\mathcal{H}$ closed under finite intersections. Then we have augmentation

$$\coprod_{H_0 \subset \cdots \subset H_p} BH_0 \longrightarrow \bigcup_{H \in \mathcal{H}} BH$$

and for each $x \in \bigcup BH$ the cat of $H \ni x \in BH$ is contractible. Thus in this case

$$\left| p \mapsto \coprod_{H_0 \subset \cdots \subset H_p} BH_0 \right| \xrightarrow{\text{heq.}} \bigcup_{H \in \mathcal{H}} BH$$

<u>Category interpretation</u>: One can form the cofibred category $(G\mathcal{H})_G$ over G with fibre $G\mathcal{H}$. Then ~~[illegible]~~

$$B(G\mathcal{H})_G \sim \left| p \mapsto \coprod_{H_0 \subset \cdots \subset H_p} BH_0 \right|$$

K-theory: In A^n one has the standard basis $e_1, \ldots, e_n$. It is probably better to ~~[illegible]~~ start with $A[X]$ X a set. Then what are is interested in are the following subgroups. Choose a filtration

$$P: \quad \phi \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

and let U_P be the subgroup ~~[illegible]~~ of $\mathrm{Aut}(A[X])$ "centralizing" the flag

$$(*) \qquad 0 \subset A[X_1] \subset \cdots \subset A[X_k] = A[X]$$

where centralizing means that $\theta \in U_P$ normalizes the flag (preserves it) and acts trivially on the quotients.

Wagener calls such a flag $(*)$ <u>semi-standard</u> and he defines

$$K_q^{BN}(A, n) = \pi_{q-1}\left(\widehat{GL(A,n)}\right) \qquad q \geq 1$$

where $\widehat{GL(A,n)}$ = ordered set of cosets of the subgroups U_P where U_P not trivial (P ~~[illegible]~~ is <u>proper</u> in his terminology.)

Vilodin uses same construction, but his family of subgroups are all finite intersections of the family U_P where P is a full semi-stand. flag. Let U_n ~~[illegible]~~ be the standard strictly upper triang. unip. group (in $GL_n(A)$), then $U_P = \pi U_n \pi^{-1}$ $\pi \in \Sigma_n$.

WRONG

Assertion: The family of finite intersections of $\{\pi U \pi^{-1}, \pi \in \Sigma_n\}$ is the family of U_P where P runs over all semi-standard flags.

Proof. A permutation $\pi \in \Sigma_n$ can be interpreted as a new linear ordering on $X = \{1, \ldots, n\}$. The subgroup U_n is $U_n = I + N_n$ where N_n is the set of matrices ~~[illegible]~~ $\ni$

$$\alpha \in N_n \iff (i \leq j \Rightarrow \alpha_{ij} = 0)$$

Thus

$$\bigcap_a \pi_a U_n \pi_a^{-1} = I + \bigcap_a \pi_a N_n \pi_a^{-1}$$

where

$$\alpha \in \bigcap_a \pi_a N_n \pi_a^{-1} \iff (\, i \leq j \text{ wrt some } \pi_a \Rightarrow \alpha_{ij} = 0).$$

Thus $\alpha \in$ ———— $\iff (\alpha_{ij} \neq 0 \Rightarrow i > j$ for all $\pi_\alpha)$.

and it's clear we get a filt.

$$\phi \subset X_1 \subset X_2 \subset \cdots$$

such that $x \in X_{n+1} - X_n \Rightarrow$ $x >$ all elements of X_n for all the π_α.

CLEAR.

NO: π_1 $1 < 2 < 3$ then $1 < 2$ for both orderings, but

π_2 $1 < 2,\ 3 < 1$ $2 \sim 3$ and $1 \not< 3$

Therefore the only different between the Wagoner and Vilodin theory for Gl_n seems to be ~~the~~ that the trivial group is allowed in Vilodin's.

In stabilizing $Gl_n \subset Gl_{n+1}$ Wagoner sends ~~[illegible]~~ semi-standard flag

$$\phi \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

into

$$\phi \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset X_k \cup \{n+1\} = X \cup \{n+1\}$$

of the sort that in the limit ~~the~~ as $n \mapsto \infty$ there is no difference.

Wagoner's map from $\widehat{Gl_n(A)}$ to the building. The building recall is the ordered set of ~~flags~~ "proper" subbundles of A^n, and so if we subdivide we get the ordered set of flags

$$0 < E_1 < \cdots < E_k < A^n$$

with $k \geq 1$. But one can send a coset gU_P into gP so one gets a map.

(bottom page 2 before

But to make this more clear, one should introduce the ~~cat~~ cofibred cat C over $\mathcal{H}$ assoc. to the functor $H \mapsto H$ as group. Thus C has objects $H \in \mathcal{H}$ and a map $H \longrightarrow H'$ is an element of H' if $H \subset H'$, otherwise there is no map. Denote by

$$\cancel{(H\subset H', h')}\ (h', H\subset H') : H \longrightarrow H'$$

this arrow. Then

$$(h'', H'\subset H'')(h', H\subset H') = (h''h', H\subset H'')$$

We have a functor $f: C \longrightarrow G$ sending H to the unique object $\cdot$ and $(h', H\subset H')$ to $h' \in G$. As usual replace C by the equivalent cofibred cat over G whose objects are pairs (H,g) $g \in G$ in which a map

$$(H,g) \longrightarrow (H',g') \qquad f(H) \xrightarrow{g} \cdot \xleftarrow{g'} f(H')$$

is a map $(h', H\subset H')$ such that $g'h' = g$. ~~The fibre over $\cdot$~~

$$\begin{tikzcd}
(H,g) \arrow[r, "h'"'] \arrow[rr, bend left, "h''h'"] & (H',g') \arrow[r, "h''"'] & (H'',g'')
\end{tikzcd}$$

$$g = g'h' \qquad g' = g''h'' \qquad g = g''h''h'$$

~~(What is the fibre over $\cdot$? It has objects (H,g) but only maps $(H,g) \longrightarrow (H',g)$)~~

Actually the thing we have just described is the fibre over $\cdot$. Observe that

$$\begin{array}{ccc}
(H,g) & \longmapsto & gH = g'h'H \\
\downarrow h' & & \\
(H',g') & \longmapsto & g'H' \ ?
\end{array}$$

and conversely if $gH \subset g'H'$ then $g = g'h'$.

Conclusion: Given a family of subgroups $\mathcal{H}$ in G, Wagoner + Vilodin consider the ~~[illegible]~~ ordered set $G\mathcal{H}$ of ~~[illegible]~~ cosets of the subgroups $\mathcal{H}$. One has then a fibration

$$G\mathcal{H} \longrightarrow (G\mathcal{H})_G \longrightarrow G$$

But what is interesting is the fact that by taking $\mathcal{H}$ to be certain unipotent subgroups of $G = GL(A)$, the category $(G\mathcal{H})_G$ is acyclic.

Thus Vilodin takes

$$\mathcal{H} = \{\pi U \pi^{-1}\}$$

where π runs over all permutation matrices (finite?) and U is the group of upper triangular with 1 on the diagonal.

X set with a ~~partial ordering~~ transitive reflexive relation R.

Let $x, y \in X$ be such that $(y,x) \notin R$.

Put

$$R' = R \cup \{(u,v) \in X^2 \mid uRx,\ yRv\}.$$

Claim R' transitive (will write aRb as $a \le b$).

Let $(a,b) \in R'$, $(b,c) \in R'$.

Case 1: $a \le b$, $b \le c$. Then $a \le c \Rightarrow (a,c) \in R'$

Case 2: $a \le b$, $b \le x, y \le c$. Then $a \le x, y \le c \Rightarrow (a,c) \in R'$

Case 3: $a \le x, y \le b$, $b \le c$. Then $a \le x, y \le c \Rightarrow (a,c) \in R'$

Case 4: $a \le x, y \le b$, $b \le x, y \le c$. Then $y \le x$ which is contrary to assumption.

Claim R' anti-reflexive if R is:

Let $(a,b) \in R'$, $(b,a) \in R'$

Case 1: $a \le b$, $b \le a \Longrightarrow a = b$.

Case 2: $a \le b$, $b \le x, y \le a \Rightarrow y \le x$ imp.

Case 3: $a \le x, y \le b$, $b \le a \Rightarrow y \le x$ imp.

Case 4: $a \le x, y \le b$, $b \le x, y \le a \Rightarrow y \le x$ imp.

Thus if R is a maximal partial ordering of X, then $(y,x) \notin R \Rightarrow (x,y) \in R$ and so R is a linear ordering. By Zorn every partial ordering can be refined to a linear ordering. Moreover every partial ordering R is the intersection of those linear orderings refining it, since given $(y,x) \notin R$ we can enlarge R so that $x < y$ in R'. $\Rightarrow$ $(y,x) \notin$ any linear ordering refining R'.

<u>Application:</u> Let $X = \{1, \dots, n\}$ and consider the ~~[illegible]~~ $(n \times n)$-matrices over A. Given a partial ordering R on X one gets a subring

$$M_R = \{ (a_{ij}) \mid a_{ij} \neq 0 \Rightarrow (i,j) \in R \}.$$

Since $i<j$, $j \leq k \Rightarrow i<k$ etc. one sees that ~~the~~

$$\begin{aligned} M_R &\longrightarrow A^n \\ (a_{ij}) &\longmapsto a_{ii} \end{aligned}$$

is a surjective homomorphism with kernel which is nilpotent. Precisely, if the ordering R is enlarged to a total ordering R', then clearly after a permutation $R' =$ standard ordering and M_R is a subring of upper triangular matrices.

If $R_1 \subset R_2$

$$M_{R_1} \subset M_{R_2} \qquad \text{since} \left\{ \begin{array}{l} a \in M_{R_1} \ \& \\ a_{ij} \neq 0 \Rightarrow (i,j) \in R_1 \subset R_2 \end{array} \right\}$$

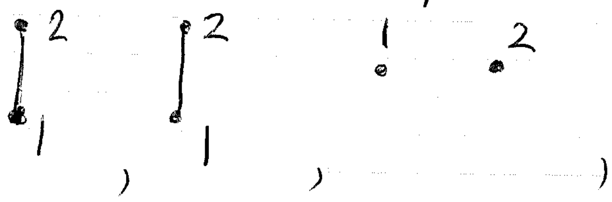
and

$$M_{\bigcap R_\nu} = \bigcap M_{R_\nu}$$

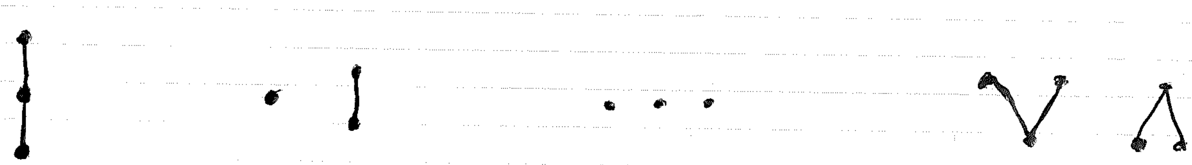
since if $a \in \bigcap M_{R_\nu}$ & $a_{ij} \neq 0 \Rightarrow (i,j) \in R_\nu$ all ν.

Therefore it ~~[illegible]~~ is clear that every ring M_R is obtained by taking a finite intersection of $\pi T \pi^{-1}$ $T =$ triang., $\pi \in \Sigma_n$.

<u>Example</u>: $n=2$ Possible partial orderings:



<u>$n=3$</u>.



6 of these   6 of these   1 of these   3   3

It seems that the graphs arising from semi-standard flags have the property that incomparability is an equivalence relation

October 15, 1973. Grassmannian - Shubert geometry.

Suppose to fix the ideas that E is ~~a~~ vector bundle over a manifold X and $\operatorname{rank}(E) = r$. Let $V \subset \Gamma(X, E)$ be a space of sections spanning E, say $\dim(V) = N$. Then we get a map

$$f : X \longrightarrow Gr^{r}(V) = \{A^{N-r} \subset V^N\}$$

induced E from the quotient bundle.

Let $W^m \subset V^N$ and denote by Z_W the cycle in $Gr^r(V)$

$$Z_W = \{A \mid W \cap A \neq 0\}.$$

(This is a simple kind of Shubert cycle). Set

$$\tilde{Z}_W = \{(A, \ell) \mid \ell \text{ is a line in } A \cap W\}$$

This is a manifold because it fibres

$$\tilde{Z}_W \longrightarrow \mathbb{P}(W)$$

with fibre $Gr^r(V/\ell)$ over ℓ. In particular

$$\dim \tilde{Z}_W = m - 1 + r(N - r - 1)$$

so $$\operatorname{codim}(Z_W \text{ in } Gr^r(V)) \text{ is } r - m + 1.$$

This fits morally because ~~~~ I remember that for $m=1$ this cycle represents $c_r(E)$, and for $m = r$ that it represents $c_1(E)$.

Now what I would like to know now is whether we can always choose W so that the maps

$$\begin{array}{ccc} & & \tilde{Z}_W \\ & & \downarrow \\ X & \xrightarrow{f} & Gr^r(V) \end{array}$$

are transversal. The way to do this is to

$$\begin{tikzcd}
 & F \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & X \arrow[d, "f"] \\
\tilde{Z}_W \arrow[r] \arrow[d] & \bigcup_W \tilde{Z}_W \arrow[r] \arrow[d] & Gr^r(V) \\
pt \arrow[r, "W"] & Gr_m(V) &
\end{tikzcd}$$

show that $\bigcup_W \tilde{Z}_W$ is smooth over $Gr^r(V)$ whence (1) is trans.-cart and we can form F. Then all we have to do is choose W so that it is a regular value for the map from F to $Gr_m(V)$.

But

$$\bigcup_W \tilde{Z}_W = \left\{\left(\ell \substack{\subset A^{N-r} \subset \\ \subset W^m \subset} V^N\right)\right\}$$

is a Grassmannian bundle

$$\left\{\left(\ell \substack{\subset A^{N-r} \subset \\ \subset W^m \subset} V^N\right)\right\} \longrightarrow \left\{\ell \subset A^{N-r} \subset V^N\right\}$$

over the projective bundle

$$\left\{\ell \subset A^{N-r} \subset V^N\right\} \longrightarrow \left\{A^{N-r} \subset V^N\right\}$$

of the subbundle over $Gr^r(V)$. So we win.

Now the next question is as follows.

Suppose now we wish to find two ~~pa~~ subspaces

$$W_1^{m_1} \, , \, W_2^{m_2} \quad \text{in } V$$

~~[illegible]~~ which are good in the preceding sense and also such that $W_1^{m_1} \subset W_2^{m_2}$. Moreover I would like to know that if $W_1^{m_1}$ is already chosen to be good, then I can find a good $W_2^{m_2}$ containing it.

By the transversality thm. arg. enough to show

$$\bigcup_{W_2 \supseteq W_1} \tilde{Z}_{W_2} \longrightarrow Gr^r(V)$$

is transversal to $f, X \longrightarrow Gr^r(V)$, provided $\tilde{Z}_{W_1}$ is trans. to f. First note that

$$\bigcup_{W_2 \supseteq W_1} \tilde{Z}_{W_2} = \left\{ \begin{matrix} l \subset A \\ l \subset W_2 \end{matrix} \;\middle|\; W_2 \supseteq \underset{\text{fixed}}{\underline{W_1}} \right\}$$

(where $\dim(A) = N-r$, $\dim l = 1$, $\dim(W_2) = m_2$); this is a manifold. To show transversal to f I break this up into two strata

$$\text{Open strata} = \left\{ \begin{matrix} l \subset A \\ l \subset W_2 \end{matrix} \;\middle|\; \begin{matrix} W_2 \supset W_1 \\ l \cap W_1 = 0 \end{matrix} \right\}$$

This part is smooth over $Gr^r(V)$ since if we fix A we can fibre over $\{ l \subset A \mid l \cap W_1 = 0 \}$ which is open in $P_1 A$, with fibre $\{ W_2 \supset l \oplus W_1 \}$; so it is a Grassmannian bundle. On the other hand I have the

$$\text{Closed strata} = \left\{ \begin{matrix} l \subset A \\ W_2 \end{matrix} \;\middle|\; W_2 \supset \underset{\text{fixed}}{W_1} \supset l \right\}$$

This fibres over $\{ l \subset A \mid l \subset W_1 \} = \tilde{Z}_{W_1}$ with fibre $\{ W_2 \mid W_2 \supset W_1 \}$

and by assumption $\tilde{Z}_{W_1}$ is transversal to X.

What exactly does it mean for a subspace W to be good? We must compute the tangent space to $\tilde{Z}_W$ at a point $\ell \subset A$.

First consider

$$\bigcup_W \tilde{Z}_W = \{\ell \subset A\}$$

Then what is the tangent space at $(\ell \subset A)$. First one recalls that the tangent space to $Gr^r(V)$ at A is $\mathrm{Hom}(A, V/A)$. Thus the tangent space we are after, note it $T_{\ell\subset A}$ has to fit in an exact sequence

$$\begin{tikzcd}
 & & \mathrm{Hom}(\ell, A/\ell) \arrow[d] \arrow[rr, bend left] & & \\
0 \arrow[r] & \mathrm{Hom}(A/\ell, V/A) \arrow[r] \arrow[dr, bend right] & T_{\ell\subset A} \arrow[rr] \arrow[d] & & \mathrm{Hom}(\ell, V/\ell) \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{Hom}(A, V/A) \arrow[rr] & & \mathrm{Hom}(\ell, V/A) &
\end{tikzcd}$$

In addition one has

So it is more or less clear that

$$T_{\ell\subset A} = \mathrm{Hom}(A, V/A) \times_{\mathrm{Hom}(\ell, V/A)} \mathrm{Hom}(\ell, V/\ell)$$

So we have

$$T(\tilde{Z}_W)_{(\ell\subset A)} = \mathrm{Hom}(A, V/A) \times_{\mathrm{Hom}(\ell, V/A)} \mathrm{Hom}(\ell, W/\ell)$$

and as $\mathrm{Hom}(A, V/A) \twoheadrightarrow \mathrm{Hom}(\ell, V/A)$, one has

$$\mathrm{Cokernel}\left\{T(\tilde{Z}_W)_{(\ell\subset A)} \to T(Gr^r V)\right\} = \mathrm{Coker}\{H(\ell, W/\ell) \to \mathrm{Hom}(\ell, V/A)\}$$
$$= \mathrm{Hom}(\ell, V/A+W).$$

So we get

Prop. $\tilde{Z}_W$ is transversal to X at the point x if the canonical map from $T(X)_x$ to

$$\mathrm{Hom}(\ell, E(x)/\mathrm{Im}\{W \xrightarrow{ev_x} E(x)\})$$

is surjective for each $\ell \subset \mathrm{Ker}\{W \to E(x)\}$.

To be clearer recall that one has can. map

$$T(X)_x \otimes \underset{\substack{\| \\ W \cap A(x)}}{\mathrm{Ker}\{W \to E(x)\}} \longrightarrow \underset{\substack{\| \\ V/A(x)+W}}{\mathrm{Coker}\{W \to E(x)\}}$$

Now suppose $\dim(W) = 1$. Then we only worry about $x \ni$ ~~...~~ if $W = \mathbb{C}s$, then $s(x)=0$. In this case $ds : T(X)_x \twoheadrightarrow E(x)$ must be onto and so s must be transversal to zero.

Now suppose that $W_1 = \mathbb{C}s_1$ where s_1 is trans. to the zero section, and $W_2 = \mathbb{C}s_1 + \mathbb{C}s_2$ is good.

No problem if s_1, s_2 independent at x.

If $s_1(x) \neq 0$ yet s_1 and s_2 are dependent, ~~say up to translation that $s_2(x) = 0$~~ then I have to know that

$$T(X)_x \otimes \underset{\substack{\text{assume spanned} \\ \text{by } s_2 - \lambda s_1}}{\mathrm{Ker}\{W \to E(x)\}} \longrightarrow \underset{\substack{\| \\ E(x)/\mathbb{C}s_1(x)}}{\mathrm{Coker}\{W \to E(x)\}}$$

So this condition clearly amounts to having s_2 mod s_1 a good section $E/\mathbb{C}s_1$

But assume now $s_1(x) = 0$ and $s_2(x) \neq 0$. Then no problem because $ds_1(x)$ maps $T(X)_x$ onto $E(x)$ already.

Last suppose $s_1(x) = s_2(x) = 0$. Then we have $T(X)_x \otimes W \longrightarrow E(x)$ derivative and the condition is that each $w^{\neq 0}$ in W is transversal to zero at x.

So we can reformulate the preceding as follows

~~Prop.~~ ~~$\tilde{Z}_W$ is transversal to $X \xrightarrow{f} Gr^r(V)$ at x provided if we choose $W' \subset W$ complem. to the evaluation $W \to E(x)$ and replace W by W/W' and E by E/W' in a nbd. of x, then each non-zero element of~~

~~Prop.~~ ~~$\tilde{Z}_W$ is transversal to $X \xrightarrow{f} Gr^r(V)$ at x if each non-zero w in $\mathrm{Ker}\{W \xrightarrow{ev_x} E(x)\}$ is transversal to zero w~~

<u>Prop</u> $\tilde{Z}_W$ is trans. to $X \xrightarrow{f} Gr^r(V)$ at x if on choosing W' comp to $\mathrm{Ker}\{W \xrightarrow{ev_x} E(x)\}$, then each non-zero element w of this kernel gives rise to a section of E/W' transversal to zero at x.

October 16, 1973

Serre's theorem on stability:

Let E be a vector bundle over an affine variety X, to fix the ideas, I will assume $X = \mathrm{Spec}(A)$, $A = \Gamma(X, \mathcal{O}_X)$.

We consider the problem of constructing a non-vanishing section. Suppose $\mathrm{rang}(E) \geq n$, and construct a sequence $s_1, \dots, s_n$ of sections of E as follows.

Choose s_1 to be non-zero at the generic points of X. Let Z be the closed set where s_1 vanishes, and $U = X - Z$, let $P_1, \dots, P_a$ be the generic points of U, and $Q_1, \dots, Q_b$ the generic points of Z. If $n \geq 2$ then we can find s_2 which is independent of s_1 at $P_1, \dots, P_a$ and non-zero at $Q_1, \dots, Q_b$.

Now let D_1 be the closed set where s_1, s_2 are dependent and let $D_0 \subset D_1$ be the set where $s_1 = s_2 = 0$. Let $\{P_i\}$ be the set of generic points of $X - D_1$, $D_1 - D_0$, D_0. If now $n \geq 3$, then we can choose s_3 so as to be independent of s_1, s_2 at the points $\{P_i\}$. It is clear how to continue this process.

Next compute codimensions. Since s_1 is non-zero at the generic points of X, we know that $\mathrm{codim}(Z, X) \geq 1$.

$$\mathrm{cod}\,(D_0(s_1), X) \geq 1. \qquad D_0(s_1) = \{x,\ \mathrm{rank}\, s_1(x) \leq 0\}$$

Now what about $D_1(s_1, s_2) =$ the set D_1 above. Then because s_2 is independent of s_1 at the gen. pts of X we have

$$\mathrm{cod}\,(D_1(s_1, s_2), X) \geq 1$$

$D_0(s_1, s_2) \subset D_0(s_1)$ but it misses the generic points, hence

$$\operatorname{cod}\left(D_0(s_1, s_2), D_0(s_1)\right) \geq 1$$

$$\Longrightarrow \operatorname{cod}\left(D_0(s_1, s_2), X\right) \geq 2.$$

Next $D_2(s_1, s_2, s_3)$ has codim ≥ 1 since $D_2(s_1, s_2, s_3) \subset D_2(s_1, s_2) = X$ and by construction it misses the generic points.

$D_1(s_1, s_2, s_3) \subset D_1(s_1, s_2)$. If P is a generic point of $D_1(s_1, s_2)$, either $P \in D_0(s_1, s_2)$ so $\operatorname{cod}(P) \geq 2$, or else $P \notin D_0(s_1, s_2)$ and so $P \notin D_1(s_1, s_2, s_3)$. Thus

$$\operatorname{cod}\left(D_1(s_1, s_2, s_3), X\right) \geq 2$$

Similarly if Q is a generic point of $D_0(s_1, s_2, s_3)$, then either $Q \in D_{-1}(s_1, s_2, s_3) = \phi$, or $Q < P$ gen. pt of $D_0(s_1, s_2)$ hence

$$\operatorname{cod}\left(D_0(s_1, s_2, s_3), X\right) \geq 3.$$

By induction we have

$$\operatorname{cod}\left(D_q(s_1, \cdots, s_{n-1}), X\right) \geq n-1-q.$$

where $D_q(s_1, \cdots, s_{n-1}) = \{x \mid \operatorname{rank}(s_1(x), \cdots, s_{n-1}(x)) \leq q\}$. Now we choose s_n so that it is independent of $s_1, \cdots, s_{n-1}$ at the generic points of the sets

$$D_q(s_1, \cdots, s_{n-1}) - D_{q-1}(s_1, \cdots, s_{n-1}) \qquad 0 \leq q$$

So now let Q be a point of $D_q(s_1, \cdots, s_n) \subset D_q(s_1, \cdots, s_{n-1})$, and P a gen. pt. of $D_q(s_1, \cdots, s_{n-1}) \ni Q \leq P$
Either P is contained in $D_{q-1}(s_1, \cdots, s_{n-1})$ and so by induction

$$\operatorname{cod}(Q, X) \geq \operatorname{cod}(P, X) \geq n-q$$

or else P is not in $D_{q-1}(s_1, \cdots, s_{n-1})$, so s_n

is ind of $s_1, \dots, s_{n-1}$ at P and $s_1, \dots, s_{n-1}$ have rank q at P, $\Rightarrow$ $P \notin D_q(s_1, \dots, s_n) \Rightarrow Q < P$ so

$$\operatorname{cod}(Q, X) \geq 1 + \operatorname{cod}(P, X) \geq 1 + n - 1 - q = n - q.$$

Thus the induction marches.

Review: Suppose $s_1, \dots, s_{n-1}$ constructed so that

$$\operatorname{cod} D_q(s_1, \dots, s_{n-1}) \geq n-1-q$$

one then arranged s_n to be independent at the gen. points of $D_q(s_1, \dots, s_{n-1}) - D_{q-1}(s_1, \dots, s_{n-1})$ for all q. We can do this by Chinese remainder theorem and the fact that $n \leq \operatorname{rank}(E)$.

Second step of the theorem is to improve the beginning of $s_1, \dots, s_n$ by elementary transformations.

$\underline{n=2}$. Assume $\operatorname{cod} D_1(s_1, s_2) \geq m$, $\operatorname{cod} D_0(s_1, s_2) \geq m+1$. Then consider where a section of the form $s_1 + f s_2$, $f \in \Gamma(X, \mathcal{O}_X)$ vanishes. If $(s_1 + f s_2)(P) = 0$, then $P \in D_1(s_1, s_2)$ so codim $P \geq m$. Let $\{P_i\}$ be the generic points of $D_1(s_1, s_2) - D_0(s_1, s_2)$. Then I can choose f so that $s_1 + f s_2$ doesn't vanish at the $\{P_i\}$. Thus if P is a point where $s_1 + f s_2$ vanishes, and if Q is a gen. point of $D_1(s_1, s_2)$ containing P, either $Q \in D_0(s_1, s_2)$ whence $\operatorname{cod} P \geq m+1$, or else $Q \notin D_0(s_1, s_2)$ hence Q is one of the P_i, ~~which is impossible~~ $P < Q$ $\Rightarrow \operatorname{codim}(P) \geq 1 + \operatorname{cod} D_1(s_1, s_2) \geq 1 + m$.

It would be better to take s_1, s_2 gen. ind. yet $D_0(s_1, s_2)$ of codim ≥ 2. Then look at the gen points P_i of the set where s_1, s_2 are dependent, but not both zero, and arrange

$s_1 + f s_2$ not to vanish at the P_i. Then it is clear that the vanishing set has codim ≥ 2.

$\underline{n=3}$. Assume cod $D_2(s_1, s_2, s_3) \geq 1$

$D_1(\quad) \geq 2$

$D_0(\quad) \geq 3$

and I am going to consider $D_1(s_1 + f_1 s_3, s_2 + f_2 s_3) \subset D_2(s_1, s_2, s_3)$
Let $\{P_i\}$ be the generic points of $D_2(s_1, s_2, s_3) - D_1(s_1, s_2, s_3)$, and arrange that $s_1 + f_1 s_3$, $s_2 + f_2 s_3$ be independent at these points. Let $\{P_i'\}$ be the generic points of $D_1(s_1, s_2, s_3) - D_0(\ )$, and arrange that $s_1 + f_1 s_1$, $s_2 + f_2 s_3$ have rank 1 at these points. Let $s_1' = s_1 + f_1 s_3$, $s_2' = s_2 + f_2 s_3$ and consider

$$P \in D_1(s_1', s_2') \subset D_2(s_1, s_2, s_3)$$

and let Q be a gen. pt. cont. P of $D_2(s_1, s_2, s_3)$. Thus if $Q \in D_1(s_1, s_2, s_3) \Rightarrow \operatorname{cod}(P) \geq 2$. And if $Q \notin P_1(s_1, s_2, s_3)$, then $Q = P_i$ and so $P < P_i \Rightarrow \operatorname{cod}(P) \geq 2$. $\therefore$

$$\therefore \quad \operatorname{cod} D_1(s_1', s_2') \geq 2.$$

And

$$P \in D_0(s_1', s_2') \subset D_1(s_1, s_2, s_3)$$

and Q be a gen. pt. of $D_1(s_1, s_2, s_3)$ cont. P.
If $P \in D_0(s_1, s_2, s_3)$, then cod $P \geq 3 \Rightarrow$ cod $Q \geq 3$. Otherwise $Q \in \{P_i'\}$ and s_1' s_2' not both zero at $P \Rightarrow P < Q \Rightarrow \operatorname{cod} P \geq 1 + \underset{=3}{2}$.

$$\therefore \quad \operatorname{cod} D_0(s_1', s_2') \geq 3$$

Rest is similar. One can now do the $n=2$ case to get a $s_1' + g s_2'$ which vanishes in cod ≥ 3.

October 22, 1973 λ-operations.

The problem to treat now is to construct products and λ-operations on K-theory for a scheme X.

How to handle products. We first have to give a spectrum $B_1, B_2, \ldots$ such that

(i) it is an Ω-spectrum : $B_n = \Omega B_{n+1}$

(ii) $B_1 \sim Q(P(X))$.

(iii) it is a ring spectrum, so that there are maps

$$B_p \wedge B_q \longrightarrow B_{p+q}$$

satisfying the conditions of associativity, unity.

In effect, from (i),(ii) we have

$$K_q(X) = \pi_{q+1}(Q(P(X)))$$
$$= \pi_{q+1}(B_1) = \pi_{q+k}(B_k) \qquad \text{for any } k > 0.$$

whereas from (iii) we get pairings

$$\pi_i(B_p) \otimes \pi_j(B_q) \longrightarrow \pi_{i+j}(B_p \wedge B_q) \longrightarrow \pi_{i+j}(B_{p+q}).$$

so we get: $i = a+p$, $j = b+q$ maps

$$K_a(X) \otimes K_b(X) \longrightarrow K_{a+b}(X).$$

Construction of $B_1, B_2, \ldots$. This makes sense for a general exact category $\mathcal{M}$; in fact (i) and (ii) do.

B_1 will be a simplicial groupoid: $p \mapsto B_{1,p}$.

An object of $B_{1,p}$ will be roughly an object of $\mathcal{M}$ equipped with an admissible filtration of length p:

$$0 \subset M_1 \subset \cdots \subset M_p = M.$$

Precisely, for each $0 \le i \le j \le p$ we get an obj. M_{ij} of $\mathcal{M}$.

and if $0 \le i \le j \le k \le p$, we get an exact sequence

$$0 \to M_{ij} \to M_{ik} \to M_{jk} \to 0.$$

such that the following hold:

α) (identity condition)

$$M_{ii} \simeq 0$$

$$0 \to M_{ii} \to M_{ij} \to M_{ij} \to 0$$

these arrows are the identity

$$0 \to M_{ij} \to M_{ij} \to M_{jj} \to 0$$

β) (assoc. condition). If $i \le j \le k \le l$, then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_{ij} \arrow[r] \arrow[d] & M_{ik} \arrow[r] \arrow[d] & M_{jk} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{ij} \arrow[r] \arrow[d] & M_{il} \arrow[r] \arrow[d] & M_{jl} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & M_{kl} \arrow[r] \arrow[d] & M_{kl} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

commutes.

~~This is the same as a functor from the subcategory of $[p] = \{0,1,\dots,p\}$ to $Q(m)$.~~

$$\cancel{\mathrm{Sub}[p] \to Q(m)}$$

$$\cancel{i \le j \mapsto M_{ij}}$$

This is the same as a functor
$$\mathrm{Sub}[p] \longrightarrow Q(\mathcal{M})$$
which carries an arrow, $(i \le j) \to (i \le k)$ into an injection $M_{ij} \rightarrowtail M_{ik}$, and an arrow $(j \le k) \longrightarrow (i \le k)$ into a surjective $M_{jk} \twoheadleftarrow M_{ik}$, and an object $(i \le i)$ into a zero object. (Psychology: One starts with $[p] \longrightarrow$ admissible injections $M_0 \subset \dots \subset M_p$ in $\mathcal{M}$, and then extends it to $\mathrm{Sub}[p]$ by sending $i \le j$ to M_j/M_i.

Quite generally a good thing to consider would be a functor from the arrow category $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$ to $\mathcal{M}$ such that for every 2-simplex
$$i \xrightarrow{u} j \xrightarrow{v} k$$
we get an exact sequence
$$0 \longrightarrow M_u \longrightarrow M_{vu} \longrightarrow M_v \longrightarrow 0.$$
What I therefore seem to think is basic is a functor f from $\mathcal{C}$ to Admis monos. in $\mathcal{M}$ extended to a functor from $\mathrm{Ar}(\mathcal{C})$ to $\mathcal{M}$ by putting $M_u = \mathrm{Coker}\, f(u)$.

So for any small category $\mathcal{C}$ I want to introduce a groupoid $\Theta(\mathcal{C};\mathcal{M})$ which consists of such functors $u \longmapsto M_u$ from $\mathrm{Ar}(\mathcal{C})$ to $\mathcal{M}$, and in which the morphisms are isos. of such functors.

Clearly contravariant in $\mathcal{C}$ so therefore we get a simplicial groupoid
$$p \longmapsto \mathrm{Iso}\,\Theta([p],\mathcal{M})$$

Put $\mathcal{M}^{(p)} = \Theta([p], \mathcal{M})$, and notice that $\Theta(\mathcal{C}, \mathcal{M})$ is an exact category also hence we can form

$$\mathcal{M}^{(p,q)} = \Theta([p], \Theta([q], \mathcal{M}))$$

which is a bisimplicial groupoid. Now what is an object of $\mathcal{M}^{(p,q)}$. We can think of it as a p-filtered object in $\mathcal{M}^{(q)}$, hence it is a (p,q)-bifiltered object

$$\begin{array}{ccccccccc}
0 & \subset & F_{p,1}M & \subset \cdots \subset & F_{p,q-1}M & \subset & F_{pq}M & = & M \\
 & & \cup & & \cup & & \cup & & \\
 & & & & & & F_{p-1,q}M & & \\
 & & & & & & \cup & & \\
 & & & & & & \vdots & & \\
 & & \cup & & & & \cup & & \\
0 & \subset & F_{11}M & \subset \cdots \subset & F_{1,q-1}M & \subset & F_{1,q}M & & \\
 & & \cup & & & & \cup & & \\
 & & 0 & & & & 0 & &
\end{array}$$

such that $F_{ij}M = 0$ if i or $j = 0$, such that

$$F_{p,j}M \cap F_{i,q}M = F_{ij}M$$

and such that $F_{ij}M \longrightarrow F_{i'j'}M$ is an admissible injection for each $(i,j) \leq (i',j')$

Thus we can identify up to equivalence $\mathcal{M}^{(p,q)}$ with the groupoid consisting of objects equipped with two cleanly intersecting filtrations, one of length p and the other of length q.

Now what has to be done is to identify the realization of the simplicial groupoid $p \mapsto \mathrm{Iso}\,\mathcal{M}^{(p)}$ with the category $Q(\mathcal{M})$.

October 24, 1973

On buildings — after a conversation with Serre.

The ordering on the Weyl group. Let Σ_n be the symmetric group of degree n. Thus it is the group of autos of $\{1, 2, \ldots, n\}$. Think of $1, 2, \ldots, n$ as n-cards, and a permutation $\sigma 1, \sigma 2, \ldots, \sigma n$ as a shuffling of these cards. Notation is confusing. It would be better to think of σ as an operation which one performs on the deck of n cards.

The simple operations

$s_1 \quad s_2 \quad s_{n-1}$

where s_i interchanges the i-th and $(i+1)$-th card in the deck.

Given a permutation w, put

$$\ell(w) = \text{the number of pairs of cards which are in the wrong order.}$$

$$= \text{number of } (i,j) \quad 1 \le i < j \le n \text{ such that } wi > wj$$

Now notice that if we apply an operation s_i to w,

$$\ell(s_i w) = \begin{cases} \ell(w) + 1 & \text{if the } i \text{ and } i+1 \text{ cards are in order} \\ \ell(w) - 1 & \text{if the } i \text{ and } i+1 \text{ cards are not in order} \end{cases}$$

since $\ell(w) = 0 \iff w = id$, it should now be clear that $\ell(w)$ is the minimal number of factors in a expression of w as a product of the s_i. Thus if

$$w = s_{i_1} \cdots s_{i_m}$$

then

$$l(s_{i_1} \cdots s_{i_m}) \leq 1 + l(s_{i_2} \cdots s_{i_m})$$
$$\geq m + l(1) = m.$$

and on the other hand if we choose $s_{i_1}, \dots, s_{i_m}$ so that w, $s_{i_1}w$, $s_{i_2}s_{i_1}w$, ... have decreasing length, then $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ with $m = l(w)$.

The Coxeter complex: It is the simplicial complex one obtains by intersecting the Weyl chambre decomposition of the real space spanned by the roots with the unit sphere. Thus it has one $(l-1)$-simplex for each element of W.

In the above example one takes the real vector space $\{t \in \mathbb{R}^n \mid \sum t_i = 0\}$ with $W = \Sigma_n$ permuting the coordinates. ~~Better to a permutation w one can~~ Better: Consider the boundary of the simplex with vertices $\{1, \dots, n\}$ with natural W action. Then simplex in its barycentric subdivision is a sequence $\phi \neq \alpha_0 < \alpha_1 < \cdots < \alpha_q < \{1, \dots, n\}$, and in particular, if $q = n-2$, then $\mathrm{card}(\alpha_j) = j-1$ and so this $(n-2)$-simplex may be identified with linear ordering of $\{1, \dots, n\}$. Thus in this example the Coxeter complex C is the simplicial complex of chains $\alpha_0 < \cdots < \alpha_q$ of proper subsets of $\{1, 2, \dots, n\}$.

Now Serre's proof of the connectivity amounts to the following. One filters C by subcomplexes C_m = union of those $(n-2)$-simplices indexed by w such that $l(w) \leq m$. Then $C_0 = \Delta(n-2)$ is contractible, and

$$C_m = C_{m-1} \cup \{\sigma_w \mid w \text{ of length } m\}.$$

Suppose then that $l(w) = m$. Then consider each face of σ. Observe that an $(n-3)$-simplex $\phi < \alpha_0 < \cdots < \alpha_{n-3} <$ has one jump card $\alpha_i - \alpha_{i-1} = 2$, hence it belongs to exactly 2 $(n-2)$-simplices. ~~It is clear that a (n-2) face of σ_w is in C_{m-1}~~

What one has to see is that σ_w intersects C_{m-1} along of unions of codim 1 faces corresp to the $i \ni l(s_i w) < l(w)$, and that no two of the σ_w of length m intersect except on C_{m-1}.

Now take the building $T(V)$ of a vector space V over a field. Thus it consists of chains $0 < W_0 < \cdots < W_g < V$ of proper subspaces. Fix a standard flag

$$0 < E_1 < \cdots < E_n = V \qquad E_i = \overline{\{e_1, \ldots, e_i\}}$$

and then one gets a map from $T(V)$ to the Coxeter complex: ~~Namely given W one considers~~ Given a full flag $0 < W_1 < W_2 < \cdots < W_n = V$, this filtration of V induces a filtration of $gr^E(V)$. Thus one has a bifiltration.

$$\begin{array}{ccccc} E_{n-2} & < & E_{n-1} & < & V \\ & & & & \vee \\ & & & & V_{n-1} \\ & & & & \vee \\ & & & & V_{n-2} \end{array}$$

Recall that if we have $\{F_p'V\}, \{F_q''V\}$ then

$$F_p'(F_qV) = F_p'V \cap F_qV$$

$$F_p'(gr_qV) = F_p'V \cap F_qV / F_p'V \cap F_{q-1}'V$$

$$gr_p'(gr_qV) = F_p'V \cap F_qV / F_p'V \cap F_{q-1}V + F_{p-1}'V \cap F_qV$$

Therefore a subspace W^d of V gives rise to a subset of $1, \dots, n$ of card d namely

$$\{q \mid F_qV \cap W > F_{q-1}V \cap W\}.$$

And if $W < W'$ then because

$$\begin{array}{ccc} F_qV \cap W & \subset & F_qV \cap W' \\ \cup & & \cup \\ F_{q-1}V \cap W & \subset & F_{q-1}V \cap W' \end{array}$$

is cartesian $F_{q-1}V \cap W < F_qV \cap W \Rightarrow$ same for W'.

Therefore I am now certain that I have a map from the building of $T(V)$ to the Coxeter complex. Namely I take a proper subspace W of V and send it to the subset of $\{1, \dots, n\}$ consist of $q \ni F_qV \cap W > F_{q-1}V \cap W$.

Since one has this map the basic combinatorics of the contraction remain the same. One observes again that if one puts $T(V)_k$ = subcomplex which is the union of the $(n-2)$-simplices of length $\leq k$, then attaching an $(n-2)$ simplex of length $k+1$ goes the same way.

October 25, 1973

Some lemmas about classifying spaces.

I) New proof of Thm. B.

Thm. B: $f: \mathcal{C} \to \mathcal{C}'$ $\ni$ $\forall Y' \to Y$ in $\mathcal{C}'$ $f/Y' \to f/Y$ heq
$\Rightarrow$ f/Y h-theoretic fibre of f over Y.

Proof: Consider the bisimplicial set

$$J(f): (p,q) \longmapsto \{X_0 \to \cdots \to X_p, fX_p \to Y_0 \to \cdots \to Y_q\}$$

which has augmentations to $N\mathcal{C}$ and $N\mathcal{C}'$.
Realizing first wrt q get

$$|J(f)| = \Big| p \longmapsto \coprod_{X_0 \to \cdots \to X_p} B(fX_p \backslash \mathcal{C}') \Big|;$$

as $fX_p \backslash \mathcal{C}'$ contractible, one concludes that the augmentation of $J(f)$ to $N\mathcal{C}$ induces a heq

$$|J(f)| \longrightarrow B\mathcal{C}$$

which we denote a^h. Realizing wrt p get

$$|J(f)| = \Big| q \longmapsto \coprod_{Y_0 \to \cdots \to Y_q} B(f/Y_0) \Big|$$

By hyp. of the thm. + Segal's lemma one ~~...~~ knows the cart. square

$$\begin{tikzcd} B(f/Y) \arrow[r] \arrow[d] & {|J(f)|} \arrow[d, "a^v"] \\ \{Y\} \arrow[r] & B\mathcal{C}' \end{tikzcd}$$

(horizontal maps are fibres over vertex Y) is h-cartesian.
~~...~~ Observe that if $f = id_{\mathcal{C}'}$ then a^v is a heq. Thus we get the diagram

$$(2)\qquad \begin{tikzcd} B(f/Y) \arrow[r] \arrow[d] & {|J(f)|} \arrow[r, "heq"] \arrow[d] & B\mathcal{C} \arrow[d, "Bf"] \\ B(\mathcal{C}'/Y) \arrow[r] \arrow[d, "heq"] & {|J(id_{\mathcal{C}'})|} \arrow[r, "heq"] \arrow[d, "heq"] & B\mathcal{C}' \\ \{Y\} \arrow[r] & B\mathcal{C}' & \end{tikzcd}$$

from which we conclude that

$$(1)\qquad \begin{tikzcd} f/Y \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{C}'/Y \arrow[r] & \mathcal{C}' \end{tikzcd}$$

is homotopy-cartesian.

To write this up it would perhaps be good to first state the theorem as asserting (1) is homotopy cartesian. Then give (2).

<u>Remark</u>: This proof doesn't differ from your earlier one because in fact the two bisimplicial sets
$$p,q \mapsto \{X_0 \to \cdots \to X_p,\ fX_p \to Y_0 \to \cdots \to Y_q\}$$
$$p,q \mapsto \{X_0 \to \cdots \to X_p,\ fX_p \to Y_q \to \cdots \to Y_0\}$$
have the same geometric realization.

II) Coscinded categories.

Let $f : \mathcal{C} \longrightarrow \mathcal{C}'$ be coscinded ~~[illegible]~~, i.e. $Y \longmapsto f^{-1}(Y)$ is a functor from $\mathcal{C}'$ to cat and $\mathcal{C}$ is the total category. Then we want to compare the telescope

$$\Big| \, p \longmapsto \coprod_{Y_0 \to \cdots \to Y_p} B f^{-1}(Y_0) \Big|$$

with $B\mathcal{C}$. The method is to observe that above we established a heq

$$\Big| \, p \longmapsto \coprod_{Y_0 \to \cdots \to Y_p} B(f/Y_0) \Big| \xrightarrow{\;`a^{h}\;} B\mathcal{C}$$

On the other hand we have heqs

$$f/Y \longrightarrow f^{-1}(Y)$$
$$(X, fX \xrightarrow{u} Y) \longmapsto u_* X$$

which are functorial in Y since $v_*(u_* X) = (vu)_* X$.
Thus we get a heq

$$\begin{array}{c} \Big| \, p \longmapsto \coprod_{Y_0 \to \cdots \to Y_p} B(f/Y_0) \Big| \\ \big\downarrow \\ \Big| \, p \longmapsto \coprod_{Y_0 \to \cdots \to Y_p} B f^{-1}(Y_0) \Big| \end{array}$$

III. Simplicial category. Let $p \mapsto \mathcal{C}_p$ be a simp. category. We want to compare $|p \mapsto B\mathcal{C}_p|$ with $B\mathcal{C}$, where $\mathcal{C}$ is the scinded category over Δ assoc. to $p \mapsto \mathcal{C}_p$. Now ~~if~~ $f : \mathcal{C} \longrightarrow \Delta$ is the projection we have

$$\left| (p,q) \longmapsto \{p \to fX_0,\ X_0 \to \cdots \to X_q \text{ in } \mathcal{C}\} \right| = \left| q \mapsto \coprod_{X_0 \to \cdots \to X_q} \Delta_{fX_0} \right| \xrightarrow{\text{heq}} B\mathcal{C}$$

$$\| $$

$$\left| p \longmapsto B(p \backslash f) \right|$$

$$\Big\downarrow \text{heq}$$

$$\left| p \longmapsto B\mathcal{C}_p \right|$$

so done.