April 5, 1973

To understand Serre's theorem:

E vector bundle over \( X \) (affine, say). Want to construct a section \( s \) of \( E \) transversal to the zero section. Transversal means \( \forall x \) either \( s(x) \neq 0 \) or \( s(x) = 0 \) and the image of \( s \) in \[
m_x E / m_x^2 E = (m_x / m_x^2) \otimes E(x)
\]
gives rise to a surjective map
\[
(m_x / m_x^2)^* \twoheadrightarrow E(x).
\]

If this is the case then for each \( x \) where \( s(x) = 0 \), we have \( \dim (m_x / m_x^2) \geq \text{rank } E \), so \( \text{codim } (x) \geq \text{rank } E \).

Start by constructing sections \( s_1, s_2, \ldots, s_n \), inductively so that for all \( j \):
\[
\text{codim } \{ x \mid \text{rank } (s_1(x), \ldots, s_n(x)) \leq n-j \} \geq j
\]

\( n=1 \): Choose \( s_1 \) to be \( \neq 0 \) at generic points.

\( n=2 \): Having chosen \( s_1 \) we choose \( s_2 \) so that
\[
x \in X_0 \Rightarrow \text{rank } (s_1(x), s_2(x)) \geq 2
\]
\[
x \in X_1 \Rightarrow \text{rank } (s_1(x), s_2(x)) \geq 1.
\]

\( n=3 \): \[
x \in X_0 \Rightarrow \text{rank } (s_1(x), s_2(x), s_3(x)) \geq 3
\]
\[
x \in X_1 \Rightarrow \quad \geq 2
\]
\[
x \in X_2 \Rightarrow \quad \geq 1
\]
Inductive step: Assume \( a_1, \ldots, a_{n-1} \) chosen so that

\[ \forall p \geq 0 \quad x \in X_p \implies \text{rank } (a_1(x), \ldots, a_{n-1}(x)) \geq n-1-p. \]

Consider closed set \( Z_p \) where \( \text{rank } (a_1(x), \ldots, a_{n-1}(x)) \leq n-1-p \).

It doesn't contain any \( x \in X_j, j < p \) so \( Z_p \cap X_p \) is finite and if \( A_n \) is chosen independent of \( a_1, \ldots, a_{n-1} \) at these points, then

\[ x \in X_p \implies \text{rank } (a_1(x), \ldots, a_{n-1}(x)) \geq n-p \]

This gives you a finite set of open conditions at a finite set \( \bigcup Z_p \cap X_p \), so \( A_n \) can be chosen as desired.

But actually if we are working with a fixed space of sections over an alg. closed field, then one should think of the preceding as a 'flag', because replacing \( a_i \) by any combination \( a_i + \sum_{j<i} \lambda_j a_j \) doesn't not affect the conclusion.
The next step is to take a suitable linear combination.

\[ \text{Rank } (E) \geq 2. \text{ We have chosen } a_1, a_2 \text{ so that} \]
\[ \text{rank } (a_1, a_2) \geq 2 \text{ on } X_0 \]
\[ \geq 1 \text{ on } X_1. \]

Then \( a_1 + a_2 \) is non-zero on \( X_0 \) and finitely many points of \( X_1 \). At the bad points we can choose \( a \) so that it doesn’t vanish, because \( a_1, a_2 \) do not simultaneously vanish on \( X_1 \).

\[ \text{Rank } (E) \geq 3. \text{ We have chosen } a_1, a_2, a_3 \text{ so that} \]
\[ \text{rank } (a_1, a_2, a_3) \geq 3 \text{ on } X_0 \]
\[ \geq 2 \text{ on } X_1 \]
\[ \geq 1 \text{ on } X_2. \]

By preceding step, we can assume \( s_1 \neq 0 \) on \( X_1 \). Then \( (s_1, s_2 + a s_3) \) are dependent contains a finite subset of \( X_1 \) and because \( s_2 \) and \( s_3 \) are not simultaneously \( \text{ind.} \) on \( s_1 \), we can choose \( a \) so that \( (s_1, s_2 + a s_3) \) are \( \text{ind.} \) on \( X_1 \). Thus we can find \( s_1, s_2 \) \( \text{ind.} \) on \( X_1 \) such that \( s_1 \neq 0 \) on \( X_1 \). Now then \( s_1 + bs_2 \) vanishes on finitely many points of \( X_2 \) but it might vanish identically.

So go back to the choice of \( a \). We have \( s_1 \neq 0 \) on \( X_1 \) and so can arrange \( a \) so that \( s_2 + a s_3 \) is \( \text{ind.} \) of \( s_1 \) on \( X_1 \). But \( s_1 \neq 0 \) on \( X_2 \), we know \( s_2, s_3 \) are not both zero, so we can arrange \( a \) so that \( s_2 + a s_3 \neq 0 \) on those points of \( X_2 \) where \( s_1 = 0 \). So can assume that \( s_1, s_2 \) are \( \text{ind.} \) on \( X_1 \) and that they have rank \( \geq 1 \) on \( X_2 \). Then we consider \( s_1 + bs_2 \), which vanishes only
at finitely many points of $X_2$.

So the result is to replace $(a_1, a_2, a_3)$ by

$$(a'_1, a'_2, a'_3) = (a_1 + a_{12} a_2 + a_{13} a_3, a_2 + a_{23} a_3, a_3)$$

so that $a'_1$ non-zero on $X_{\leq 2}$,

$a'_1, a'_2$ independent on $X_{\leq 1}$

$a'_1, a'_2, a'_3$ ind. on $X_{\leq 0}$. 
Given a vector bundle of rank $n$ over a variety $X$ over an alg. closed field $k$, suppose $E$ generated by a space $V \to \Gamma(E)$. $X(E)$ the building (= simplicial complex associated to the ordered set of proper subbundles of $E$ - assume $X$ irreducible). Suppose $K$ is a finite subcomplex of $X(E)$.

Better fix an integer $d$ and consider $X_d(E)$ the subcomplex of $X(E)$ consisting of chains $0 < F_0 < \cdots < F_r \hookrightarrow E$ where $\text{rank}(F_r) \leq d$. Let $K$ be a finite subcomplex of $X(E)$. Then for each vertex $F$ of $K$, the vector bundle $E/F$ is gen. by $V$, and so for a Zariski dense subset of $v \in V$, $v$ spans a trivial subbundle of $E$ transversal to $F$. (Forgot to say that we are assuming $d < \text{rank}(E) - \text{dim}(X)$ so that $\text{rank}(E/F) \geq \text{rank}(E) - d > \text{dim}(X)$). So it is clear that we can find $v_1, v_2, \ldots, v_r$ spanning a trivial subbundle of rank $r$ of $E$ transversal to all $F$ in $K$ provided $d + r \leq \text{rank}(E) - \text{dim}(X)$.

Now having chosen $v_1, v_2, \ldots, v_r$ let me consider the subcomplex of $X_d(E)$ consisting of those $F$ which are well placed with respect to $v_i$ in some sense.

Let $L \subset E$ be a line bundle. Call $F$ a subbundle $F$ well placed with respect to $L$ if either $L \subset F$ or if $L \to E/F$ is a subbundle. Thus want

$$0 < L \cap F \subset L + F \subset E$$
Let $H$ be a $k$-subbundle of rank $d$ with respect to $Y$. Try to prove that $F$ is an $X$-equivalent of rank $d+1$. If it is possible to use induction on $d$, then...

A local ring $O_C$ is well-placed with respect to the flag. Let $V$ be a fiber of $F$. We are concerned with the case $n=2$. Now let $Y \subset X(v)$. Denote the subbundle of $V$ by $V$. To show that $F$ is well-placed, we need to verify that $F$ is well-placed with respect to $V$. Use induction on $d$. To show that $F$ is well-placed with respect to $V$, conclude that...

To be a diagram of...
Let $E$ be a vector bundle of rank $n$ over $X$ connected and let $0 \subset E_1 \subset \cdots \subset E_n = E$
be a full flag for $E$. Call a subbundle $F$ of $E$
adapted to the flag if for each $i$, $E_i \cap F$ and $E_i + F$
are subbundles of $E$, equivalently $E/E_i + F$ is loc.
free (since one has exact sequences

$E_i \cap F \to E_i \to E_i + F \to E/E_i + F \to 0$)

Let $X(E)$ be the simplicial complex whose simplices
are chains of subbundles $F_0 < \cdots < F_n$ with $0 \neq F_0$,
$F_n \neq E$ and $X(E)'$ the full subcomplex whose vertices are
those $F$ adapted to the flag. I want to show
that $X(E)'$ is a bouquet of $(n-2)$-spheres.

First point: suppose $F$ is adapted to $\{E_i\}$ and
$F'$ is a subbundle of $F$. Claim $F'$ adapted to $\{E_i\}$ iff
$F'$ adapted to $\{F \cap E_i\}$. Proof.

$$F' + (F \cap E_1) = F' + (F_n E_i)$$

$\text{F} \cap \text{F} \cap (F' + E_i) = F' + (F \cap E_i)$

shows that $F' + (F \cap E_i)$ is a subbundle of $F$ iff $F' + E_i$
is a subbundle of $E$. 
Now let $\mathcal{H}$ be the set of $F$ in $\mathcal{X}(E)'$ such that $F+E_i$ is not in $\mathcal{X}(E)'$. Note that if $F$ is adapted to $\{E_i\}$ so is $F+E_a$ because $F+E_a+E_i = F+E_i$ for $i \geq a$. Thus if $F+E_1$ is not in $\mathcal{X}(E)'$ it must be that $F+E_1 = E$, and since $E_1$ is a line bundle and $F < E$, we must have $F \oplus E_1 = E$. Thus $\mathcal{H}$ is the set of subbundles of $E$ of rank $n-1$ which are adapted to $\{E_i\}$ and such that $F \oplus E_1 = E$.

Let $Y \subset \mathcal{X}(V)'$ be the full subcomplex having the vertices not in $\mathcal{H}$. Then for $F \in Y$ we have the retraction

$$F \leq F+E_1 \geq E_1$$

so $Y$ is contractible.

Given $H \in \mathcal{H}$, what is its link? The ordered set of $0 < F < H$ which are adapted to $\{E_i\}$, or equivalently (by the preceding point) which are adapted to $\{E_i \cap H\}$. Now $O = E_1 \cap H \subset E_2 \cap H \subset \ldots \subset E_n \cap H = H$
is a full flag since $H$ has rank $n-1$. Thus the link of $H$ is the complex of proper subbundles in $H$ adapted to the flag $\{E_{i+1} \cap H\}_{i=1}^{n-1}$. By induction the links will be a bouquet of $(n-3)$-spheres, so I can conclude as before that $\mathcal{X}(E)'$ is a bouquet of $(n-2)$-spheres.
Application: Let \( A \) be a local ring with an infinite residue field \( k \), and suppose \( A \) is a \( k \)-alg. \( E = A \otimes_k V \). To prove \( X(E) \) is a bouquet of \((n-2)\)-spheres \( n = \dim_k(V) \). Suffices to show any finite subset \( S \) of \( X(E) \) is adapted to some flag in \( E \), for then have
\[
S \subset X(E) \subset X(E)
\]
showing that \( X(E) \) has no non-trivial homotopy groups in degrees \( < n-2 \).

So let \( F \subset A \otimes_k V \) be a subbundle with quotient \( Q \).
Let \( 0 \subset V_1 \subset \cdots \subset V_n = V \) be a full flag in \( V \). The generic situation is where the composite
\[
V_i \subset V \longrightarrow k \otimes_k Q
\]
is an isomorphism, \( q = \text{rank}(Q) \). If this is the case, then I claim \( F \) is adapted to the flag
\[
\{ A \otimes_k V_i \}.
\]
Recall that if elements \( z_1, z_2 \in Q \) are such that their images in \( k \otimes_k Q \) are independent, then \( A \longrightarrow Q \) is a subbundle (because can extend to a map \( A^g \longrightarrow Q \) which is an isom after \( k \otimes_k Q \), hence before).

Thus it follows that for \( i \leq q \)
\[
A \otimes_k V_i \longrightarrow A \otimes_k V \longrightarrow Q = A \otimes_k V / F
\]
is a subbundle injection, so \( (A \otimes_k V_i + F) \) is a subbundle of \( A \otimes_k V \). For \( i \geq q \) it is pure, so \( A \otimes_k V_i + F = A \otimes_k V \).

Thus we can consider for each \( F \in S \) the subset of flags in \( V \) such that \( V_i \oplus k \otimes_k F = V \), \( q = n - \text{rank}(F) \), and these form a Zariski dense subset of all flags. Since
k is infinite, there exists a flag such that each $F_i$ is adapted with respect to it.

But suppose now that $A$ is local with residue field $k$ infinite, and let $E$ be a vector bundle of rank $n$. Given a finite set of subbundles $F$ of $E$, we can since $k$ is infinite, find a full flag $\{V_i\}$ in $k \otimes E$, such that

$$V_{\delta(r)} \oplus k \otimes F = k \otimes E$$

$$\delta(F) = n - \text{rank}(F).$$

for each $F$ in $S$. Now lift the flag $\{V_i\}$ to a flag $\{E_i\}$ in $E$. Again it follows that

$$E_i + F$$

is a subbundle of $E$ for each $i$, so $F$ is adapted to $\{E_i\}$. Thus I seem to have proved

**Proposition.** If $A$ is a local ring with infinite residue field (not nec. noeth. or commutative), then $X(A^n)$ has homotopy type of a bouquet of $(n-2)$-spheres.

As before this gives a stability result for the $A$-category.
April 9, 1973. (more stability)

A local ring residue field \( k \), \( E \) a free \( A \)-module of rank \( n \), \( X(E) \) the ordered set of proper subbundles \( F \) of \( E \). To show \( X(E) \sim V S^{n-2} \).

\( n=2 \), \( X(E) \) discrete.
\( n=3 \), have to check \( X(E) \) connected. But given two lines \( L_1, L_2 \) we know that either \( L_1 \) and \( L_2 \) are independent or that we can find \( L_3 \) independent of \( L_1, L_2 \) separately. Precisely, look at the lines \( L_1 \otimes k, L_2 \otimes k \) in \( E \otimes k \), choose an independent line and lift it to \( L_3 \).

\( n \geq 4 \). Fix a line \( L \) and let \( \mathcal{H}_L \) be the set of complementary "hyperplanes," and let \( Y \) be the full subcategory of \( X(E) \) consisting of \( F \) not in \( \mathcal{H}_L \). Then drawing a picture we have the picture

\[ \text{Link}(H) \]

where \( \text{Link}(H) = X(H) \). By induction I know that \( \text{Link}(H) \sim V S^{n-3} \) whence from

\[ \begin{align*}
\bigwedge H_{n-2} \big( \text{Link}(H) \big) & \rightarrow H_{n-2} (Y) \\
& \rightarrow H_{n-2} (X) \xrightarrow{\delta} H \big( \bar{H}_{n-3} \big( \text{Link}(H) \big) \big) \\
& \rightarrow 0
\end{align*} \]

This \( X \sim V S^{n-2} \) (at least ignoring \( \pi_1 \)).

Now let \( Y' \subset Y \) consist of \( F \) such that \( F \) is not independent of \( L \), i.e. \( \not\exists k \otimes \mathcal{H}_L \).
(Observe that $H_1$ depends only on $k$. Thus $H \otimes k$ is complementary to $k \otimes k \otimes k$.) The same is true for $\lambda$. Now we can retract $Y$ to $Y'$ by sending $F$

$F \leq F' \quad \text{if} \quad F \in Y'$

$F \leq F + L \quad \text{if} \quad F \in Y' \setminus Y$

This is well-defined because if $F_1 \leq F_2$?

Doesn't work, because we can have $F_2$ dependent on $L$, $F_1$ independent, and $F_1 + L \not\leq F_2$.

Wait: $\text{Link}(H) = X(H)$ contracts within $Y$ because $F \leq F + L$ ($F \subset H \Rightarrow E/F + L \cong H/F$ so $F + L$ is a sub-bundle). Thus we know that $\text{Link}(H) \to Y$ is null-homotopic, and so

$X \cong V^S S^{n-2} \iff Y \cong V S^{n-2}$
April 10, 1973  (more stability)

Example: Let $S$ be a set and consider the simplicial complex $K(S, n)$ whose simplices are

$$(s_0, i_0), \ldots, (s_q, i_q)$$

with $0 \leq i_0 < \ldots < i_q \leq n$. Claim $K(S, n)$ is $(n-1)$-connected (begins in dim. $n$).

Use induction on $n$. For $n = 0$, it is clear. For $n = 1$ we have a connected graph so it is also clear.

Fix $s_0 \in S$. The link of $(s_0, 0)$ is clearly $K(S, n-1)$. The result of removing all the vertices $(s_0, 0)$ for $s \neq s_0$ is a cone with vertex $(s_0, 0)$. Thus

$$K(S, n) = \bigvee_{s \neq s_0} K(S, n-1)$$

so the induction works.

$$\text{rank } \tilde{H}_n(K(S, n)) = (m-1) \cdot \text{rank } \tilde{H}_n(K(S, n-1))$$

$$\implies \text{rank } \tilde{H}_n(K(S, n)) = (m-1)^{n+1}$$

Check Euler char:

no. of $0 < \ldots < i_q$\hspace{1cm} \frac{(n+1) \ldots (q)}{(q+1)!} = \binom{n+1}{q+1}

no. of $q$-simplices is\hspace{1cm} \binom{n+1}{q+1}(m-1)^{q+1}

So

$$\chi = -\sum_{q=0}^{\infty} (-1)^q \frac{(n+1)^q (m-1)^{q+1}}{(q+1)!} = +1 - \frac{(1-m)^{n+1}}{1+1} = 1 + (-1)^m \binom{m-1}{m+1}$$

Let $M = \prod_{n \geq 0} BG_n$ be a top monoid associated to the family $GL_n$ or $\Sigma^n$ in $\text{Grp}$, and $e$ the base point of $BG_1$. Multiplying by $e$ on the left or right defines an embedding $BG_{n-1} \to BG_n$ unique up to homotopy (more or less) and so we can speak of the cofibre $BG_p/BG_{p-1}$.

Have the spectral sequence

$$E_{pq} = H_{p+q}(BG_p/BG_{p-1}) \Rightarrow H_n(BG_{\infty})$$

which results from filtering $BG_{\infty}$ via $BG_p$. This spectral sequence has products because the $H$-space structure on $BG_{\infty}$ induces maps

$$(BG_p/BG_{p-1}) \wedge (BG_q/BG_{q-1}) \to BG_{p+q}/BG_{p+q-1}$$

Example: $BG_p = BU_p$. Then

$$BG_p/BU_{p-1} = MU_p$$

since $BU_{p-1}$ is the canonical sphere bundle over $BU_p$.

In general it does not seem to be the case that $BG_p/BG_{p-1}$ forms a spectrum in a natural way. However it does once one fixes a map

$$S^i \to BG_e/BG_{e-1}$$

for some $e$, $i$. The question becomes whether one gets
any interesting cohomology theories in this way.

Question: From the calculations for a finite field, one leads to conjecture that the fibres

\[ \mathbb{B}G_{n-1}^+ \to \mathbb{B}G_n^+ \]

is a Moore space of type \( \mathbb{Z}/(q^{n-1})\mathbb{Z}, 2n-1 \)?

Recall that the \( \mathbb{Q} \) category is an \( H \)-space with multiplication given by direct sum. Clearly we get

\[ \mathbb{Q}_p \times \mathbb{Q}_q \to \mathbb{Q}_{p+q} \]

for each \( p, q \) hence we get maps

\[ (\mathbb{Q}_p/\mathbb{Q}_{p-1}) \wedge (\mathbb{Q}_q/\mathbb{Q}_{q-1}) \to (\mathbb{Q}_{p+q}/\mathbb{Q}_{p+q-1}) \].

Now recall that \( \mathbb{Q}_1/\mathbb{Q}_0 = \bigoplus p \mathbb{Q}/\mathbb{Q}_p \), hence there is a canonical map

\[ S^1 \to \mathbb{Q}_1/\mathbb{Q}_0 \]

so that \( \{\mathbb{Q}_p/\mathbb{Q}_{p-1}\} \) is a spectrum in a canonical way, in fact a ring spectrum.

Now recall that we have a co-cart. square

\[
\begin{array}{ccc}
\Sigma X_n G_n & \to & G_n \\
\downarrow & & \downarrow \\
Q_{n-1} & \to & Q_n
\end{array}
\]
so that $Q_n/Q_{n-1}$ is the Thom space of the bundle over $BG_n$ with fibre the suspension of $X_n$, which is a wedge of $(n-1)$-spheres.

Now it should be possible to exhibit a $G_p \times G_q$ equivalent map

$$\Sigma X_\ell \vee \Sigma X_\delta \longrightarrow \Sigma X_{\ell+\delta}$$

in fact given vector spaces $V$ let $J'(V)$ be the ordered set of proper layers in $V$, and $J'(V)$ the ordered set of all layers. Then

$$J'(V) \times J'(W) \longrightarrow J'(V \oplus W)$$

$$(V_0, V_i), (W_0, W_i) \longrightarrow (V_0 \oplus W_0, V_i \oplus W_i)$$

carries $J(V) \times J(W) \cup J'(V) \times J(W)$ into $J(V \oplus W)$, so it induces a map

$$\frac{J'(V)}{J(V)} \vee \frac{J'(W)}{J(W)} \longrightarrow \frac{J'(V \oplus W)}{J(V \oplus W)}.$$

Since $J'(V)$ is contractible, this is a map

$$\Sigma J(V) \vee \Sigma J(W) \longrightarrow \Sigma J'(V \oplus W).$$

Better one has only to note that

$$J(V) \times J'(W) \cup J'(V) \times J(W) = J(V) \ast J(W)$$

$$\frac{J(V) \times J(W)}{J(V) \times J(W)}.$$
up to homotopy.

In particular we have

\[ J(V) \times J'(k) \cup J'(V) \times J(k) \]

\[ J(V) \times J(k) \]

\[ \overset{\cong}{\longrightarrow} \]

\[ J(V) \times I \cup CJ(V) \times I \]

\[ J(V) \times I \]

is hom to \( \Sigma J(V) \):

\[ \begin{array}{c}
\text{I} \\
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\text{J(V)} \\
\end{array} \]
The $\mathbb{Q}$-category in classical $K$-theory: I recall that it is a simplicial groupoid:

$$M \times M \rightleftarrows M \rightleftarrows \text{pt}$$

where

$$M = \coprod_{n \geq 0} BG_n$$

so it is

$$\coprod_{a, b \geq 0} BG_{a,b} \rightleftarrows \coprod_{a} BG_a \rightleftarrows \text{pt}$$

Filtering by the total degree as before we see that $Q_p/Q_{p-1}$ is the simplicial space

$$\begin{array}{ccc}
\text{pt} & \downarrow & \text{pt} \\
\coprod_{a+b=p} BG_{a,b} & \rightleftarrows & BG_p \\
\end{array}$$

whose homology we can compute dimensionwise as before. Thus

$$\bigoplus_p H_*(Q_p/Q_{p-1}) \leftarrow \text{Tor}_{Z}^R(Z, Z)$$

where

$$R = \bigoplus_{n \geq 0} H_*(B G_n).$$

Now take $G_n = BU_n$, and recall

$$\bigoplus H_*(BU_n) = Z[b_0, b_1, \ldots]$$
where $b_i \in H_{2i}(BU_1)$ is the dual basis to $c_i^*$. Thus

$$\text{Tor}^R_q(\mathbb{Z}, \mathbb{Z}) = \Lambda [\tilde{b}_0, \tilde{b}_1, \ldots]$$

where $\tilde{b}_i \in \text{Tor}^R_1(\mathbb{Z}, \mathbb{Z})_{2i}$ is the image of $b_i$ in the indecomposable space of $R$. So this means that

$$Q_1/Q_0 = \Sigma^1 BU_1$$

has the generators. Precisely we can say that

$$\bigoplus_{p>0} H_x(Q_p/Q_{p-1})$$

is an exterior algebra with generators $\tilde{b}_0, \tilde{b}_1, \ldots$ where $\tilde{b}_i \in H_{2i+1}(Q_1/Q_0)$.

Note that the least degree element of $H_x(Q_p/Q_{p-1})$ is $\tilde{b}_0 \ldots \tilde{b}_{p-1}$ which has degree $\sum_{i=0}^{p-1} (2i+1) = 2 \frac{p(p-1)}{2} + p = p^2$.

The spectrum $\{Q_p/Q_{p-1}\}$ has homology

$$\lim_{\overset{\to}{p}} H_x(Q_p/Q_{p-1}) = 0$$

and so it represents the trivial genus homology theory.
April 14, 1973

Let k be a field $G = PGL_2(k) = \text{group of } \text{automorphism of } P^1_k$. I want to compute the low dimensional homology $C_2$ of $G$.

Let $G$ act on $P^1_k$ and consider the complex of chains on $P^1_k$ considered as a simplicial complex in which every finite non-empty subset is a simplex. We get an exact sequence of $G$-modules.

$\longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$

\[ C_0 = \mathbb{Z}[G/B] \]  $B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ fixes $z = \infty$.

$C_1 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[N]} \mathbb{Z}^{\text{sign}}$ modulo scalar.

$N = \text{normalizer of } \text{torus } T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ modulo scalar.

$C_2 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[\Sigma_3]} \mathbb{Z}^{\text{sign}}$

$G$ acts transitively on triples of points in $P^1_k$.

The stabilizer of $0,1,\infty$ is $\Sigma_3$ generated by the transpositions $z \mapsto 1-z$, $z \mapsto \frac{1}{z}$.
\[ C_3 = \prod_{\{z_0, \ldots, z_3\}} \mathbb{Z} \] 

\( G \) doesn't act transitively on 3-simplices. Let \( \tilde{C}_3 \) be the group of linear combinations of ordered 3-simplices. Thus
\[ \tilde{C}_3 \cong \mathbb{Z} \left[ \mathbb{Z}^4 \right] \] 

\[ \tilde{C}_3 = \bigoplus_{i \neq 0,1} \mathbb{Z}[G] \]

because any \((z_0, z_1, z_2, z_3)\) is uniquely \( G \)-conjugate to one of the form \((0, 1, \infty, z)\).

Now I will work with coefficients such that 2, 3 are invertible. Look at coefficients such that \( B, T \) have same homology. Then
\[ H_*(G, C_0) = H_*(T) \]
\[ H_*(G, C_1) = H_*(\mathbb{Z} \otimes \mathbb{Z} \otimes N, \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}) = \left( H_*(T) \otimes \mathbb{Z} \otimes \mathbb{Z} \right)^{\mathbb{Z}_4_2} = \{ x \in H_*(T) \mid \omega x = -x \} \] 

and so the map
\[ H_*(G, C_1) \to H_*(G, C_0) \]
is the inclusion of the anti-invariant elements of \( H_*(T) \).
and so the cokernel must be the coinvariants.

\[ H^*_G(G, C_2) = H^*_G(\Sigma_3, \mathbb{Z}^{\otimes n}) = 0. \]

What are the orbits of \( G \) on 3-simplices \( \{z_1, z_2, z_3, z_4\} \)? On ordered sets get \( (0, 1, \infty, z) \) with \( z \neq 0, 1, \infty \). Have then an action of \( \Sigma_4 \)

\[
z 1 \rightarrow \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}
\]

so then \( z_4 \rightarrow \frac{z_4 - z_1}{z_4 - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_4} = \gamma \)

Now permute \( z_1, z_2, z_3, z_4 \) and see how \( \gamma \) changes

\[
\begin{align*}
(0, -1, \infty, z) & \rightarrow z \\
(0, 1, z, \infty) & \rightarrow 1 - z \\
(0, \infty, 1, z) & \rightarrow \frac{z}{z - 1} \\
(0, z, 1, \infty) & \rightarrow \frac{z - 1}{z} \\
(0, \infty, z, 1) & \rightarrow \frac{1}{1 - z} \\
(0, z, \infty, 1) & \rightarrow \frac{1}{z} \\
\end{align*}
\]
Thus the Klein group acts trivially. (This well-known, I think. Thus Attijah told me one gets a surjection \( \Sigma_4 \rightarrow \Sigma_3 \) with kernel the Klein group by letting \( \Sigma_4 \) act on line pairs

\[
\begin{align*}
(1 & 0 & \infty & z) \rightarrow 1 - z \\
(1 & 0 & z & \infty) \rightarrow z \\
(0 & 1 & z & \infty) \rightarrow \\
(z & 0 & 1 & \infty) \rightarrow \\
(\infty & 0 & 1 & z) \rightarrow \\
(\infty & z & 0 & 1) \rightarrow z \\
(0 & \infty & 1 & z) \rightarrow z \\
(z & \infty & 1 & 0) \rightarrow z \\
(z & 0 & \infty & 1) \rightarrow z
\end{align*}
\]

Thus the cross-ratio changes into

\[
\frac{x-1}{1-z} = \frac{x-1}{x} \cdot \frac{x}{z-1}
\]
and there should be an invariant function of degree \(G\) in \(z\): (at least one by Luroth, and all others are related by \(G\))

\[
(z - \lambda)(\frac{1}{z} - \lambda)(1-z - \lambda)(1-\frac{1}{z} - \lambda)(\frac{1}{1-z} - \lambda)(\frac{2}{z-1} - \lambda)
\]

I don't know if there is a particularly simple one.

In any case let's go back to \(H_k(\mathbb{Z}, C_3)\). We have that this is a direct sum \(\mathbb{Z}_3\) over \(k = \{0, 1, 2\}\) modulo \(\Sigma_3\) of the cohomology of the stabilizer of \((0, 1, \infty, z)\) which is at least the Klein group. The bad points for the \(\Sigma_3\) action are \(z\)

\[
z = \frac{1}{2}, \quad z = \pm 1, \quad z = \frac{1}{3}, \quad z = \frac{1}{2}, \quad z = \frac{3}{2}, \quad z = \frac{1 + \sqrt{3}}{2}, \quad z = \frac{3}{2}
\]

One orbit is \([-1, \frac{1}{2}, 2]\) and stabilizer \(\mathbb{Z}/2\). Other is \([\frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}]\) and stabilizer is \(\mathbb{Z}/3\).
Also \([0,1,\infty]\) stabilizer \(\mathbb{Z}/2\).

The simplest perhaps is
\[
\omega = \frac{(z^2 - z + 1)^3}{z^2(z - 1)^2}
\]

has triple zeroes at \(\frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}\) and double poles at \(0, 1\).

Clearly \(\omega\) is unchanged under \(z \rightarrow 1 - z, z \rightarrow \frac{1}{z}\) hence under \(\Sigma_3\) as these transpositions generate.

By sending \(z \rightarrow \omega\) I get a bijection of the \(\Sigma_3\) orbits on \(k - \{0,1\}\) with \(k - \{0,1\}\).

Now when \(z = 2\), \(\omega = \frac{(4 - 2 + 1)^3}{4} = \frac{27}{4}\).

The stabilizer is \(\Sigma_2\) which acts non-trivially on the sign representation. So this doesn't contribute to the cohomology. When \(z = \frac{1 + \sqrt{3}}{2}\), \(\omega = 0\) and the stabilizer is \(\mathbb{Z}/3\) which acts trivially on the sign rep. Thus \(\omega = 0\) contributes along with all \(\omega \neq \frac{27}{4}\). So this suggests we change \(\omega\) to
\[
\overline{\omega} = \frac{(z + 1)^2 (z - \frac{1}{2})^2 (z - \frac{\sqrt{3}}{2})^2}{z^2(z - 1)^2}
\]

Thus it seems that
\[
H_*(G, C_3) = \frac{1}{\omega \neq 0} A [0,1]
\]

where \(A\) is the coefficient group (assuming 2,3 invertible).
Tuples $(z_1, z_2, z_3, z_4, z_5)$ and try to understand relations between cross-ratios.

$(0, 1, \infty, a, b)$

$(1, \infty, a, b) \quad \mapsto \quad \frac{b-1}{b-a}$

$(0, \infty, a, b) \quad \mapsto \quad \frac{b}{b-a}$

$(0, 1, a, b) \quad \mapsto \quad \frac{b}{b-a} \cdot \frac{1-a}{1}$

$(0, 1, \infty, b) \quad \mapsto \quad b$

$(0, 1, \infty, a) \quad \mapsto \quad a$

$(a, 0, 1, \infty, b)$

$(0, 1, \infty, b) \quad \mapsto \quad b$

$(a, 1, \infty, b) \quad \mapsto \quad \frac{b-a}{1-a}$

$(a, 0, \infty, b) \quad \mapsto \quad \frac{b-a}{-a}$

$(a, 0, 1, b) \quad \mapsto \quad \frac{b-a}{b-1} \cdot \frac{1}{a}$

$(a, 0, 1, \infty) \quad \mapsto \quad \frac{\infty-a}{\infty-1} \cdot \frac{1}{a} = \frac{1}{a}$

TOO COMPLICATED.
Explore more abstractly:

The point perhaps to keep in mind is that what I am trying to do is to understand the cohomology of $GL_2(k)$ via that of the algebraic group $GL_2$ which is known. Thus suppose $k = \mathbb{C}$ and we have mod $\ell$ coefficients. Then the subgroups $B, T$ have the same cohomology as the corresponding algebraic groups. This takes care of the dimensions 0, 1, 2, but once one hits dim. 3 there appears to be a problem.

$GL_n(k[t])$ of vector bundles on curves

Let $k$ be a field, $\Gamma = GL_n(k[t]) = \text{Aut}(M)$, $M=k[t]^n$ and let $X$ be the building at $\infty$ of $k(t) \otimes_{k[t]} M = V$. Thus $X$ is the simplicial complex whose $g$-simplices are chains of lattices

$$L_0 \prec \cdots \prec L_g$$

in $V$ for the d.v.r. $\mathcal{O}_x = k[t]_{\infty}^x$, $m_{L_\infty} = \frac{1}{t}$, such that $m_{L_\infty}|_{L_0} = 0$ (equivalently $L_\infty \subset L_0$). Can also think of such a lattice as an extension of $M$ to a vector bundle $E$ of rank $n$ on $P^1$.

We know $X$ is contractible, hence the chains on $X$ form a $\Gamma$-resolution of $\mathbb{Z}$, and we obtain a spectral sequence relating the homology of $\Gamma$ with the homology of $X/\Gamma$ with coefficients in the local system of isotropy homology. We now have to compute the $\Gamma$-orbits on the $g$-simplices and the stabilizers.

Can think of $\mathcal{L}$ as a pair $(E, M \to j^*E)$.

The $(E_0^x)$ $(E_\infty^x)$ are $\Gamma$-conjugate means that

We think of $L \subset V$ as $E_\infty \subset V$ with $M = \Gamma(M^1, E) \subset E_{\infty} = V$. For $L$, $L'$ to be $\Gamma$-conjugate means the vector bundles $E, E'$ are isom.
and the stabilizer of \( L \) is simply the group of automorphisms of the bundle \( E \).

We know every vector bundle \( E \) on \( \mathbb{P}^n \) is isomorphic to
\[
\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)
\]
with \( k_1 \leq \ldots \leq k_n \). Let
\[
\lambda_i(E) = k_{i+1} - k_i
\]
for \( i = 1, \ldots, n-1 \); these are roots in some sense.

I want now to compute the group of automorphs of this bundle. Now I know that if I write
\[
E = \bigoplus_{k} \mathcal{O}(k)^{n_k} \quad \sum n_k = n
\]
then the subbundle \( \bigoplus_{k \neq p} \mathcal{O}(k)^{n_k} = F_p E \) is intrinsic.

Better,
\[
\text{End}(\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)) = \text{ring of matrices } \{ \text{Hom}(\mathcal{O}(k_i), \mathcal{O}(k_j)) \}
\]
\[
= \left\{ \begin{array}{c} \Gamma(\mathcal{O}(k_i - k_j)) \end{array} \right\}
\]
\[
= \begin{pmatrix} M_{n_p}(k) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}
\]
I need some notation. Thus given $k_1 < \cdots < k_n$, I need to know the size of the blocks, and the jumps. Thus I want to know the blocks and the degrees. So suppose we put

$$d_1 = \cdots = d_1 < d_2 = \cdots = d_2 < \cdots = d_n = k_1 < \cdots < k_n$$

So that

$$\DeclareMathOperator{End}{End} \End \left( O(k_1) \oplus \cdots \oplus O(k_n) \right) = \begin{pmatrix} 1 \end{pmatrix}$$

where $(i,j)$-th block consists of polynomials of degree $d_j - d_i$ and of size $s_i \times s_j$

and the auto group is the set of matrices such that the diagonal entries are invertible.

Observe that

$$\text{Aut}(E) \longrightarrow \text{Aut}(E \otimes K(\infty)) = GL_n(K)$$

Next we want to compute the homo-

$$\text{Aut}(E) \longrightarrow \text{Aut}(E \otimes K(\infty))$$

so we first have to understand
\[ \Gamma(\mathcal{O}(k)) \rightarrow \Gamma(\mathcal{O}(k) \otimes k(\infty)) \]

The former is homog. polys of degree \( k \) in \( t_0, t \), where \( t = t_1/t_0 \). At \( z = \infty \), \( t_0 \) is zero and we set \( t_k \) as the base of \( \Gamma(\mathcal{O}(k) \otimes k(\infty)) \). Thus, if we think of \( \Gamma(\mathcal{O}(k)) \) as polys in \( z = t^{-1} \) of degree \( \leq k \), the above map takes the constant term. So it is now clear that

\[
\text{Aut}(E) \rightarrow \text{Aut}(E \otimes k(\infty)) \cong GL_n(k)
\]

simply evaluate the polynomial matrices at \( z = 0 \). It is therefore clear that

\[
\text{Im} \left( \text{Aut}(E) \rightarrow \text{Aut}(E \otimes k(\infty)) \right) = \text{the parabolic subgroup fixing the canonical filtration of } E \otimes k(\infty)
\]

i.e. matrices

\[
\begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}
\]

At this point we understand the \( \Gamma \)-classes of vertices and their stabilizers. Now I recall that a \( g \)-simplex \( L_0 < \cdots < L_g \) is simply a vertex \( L \) together with the flag

\[ 0 = L_0 < L_1 < \cdots < L_g = L/t^{-1} L \]

Thus a \( g \)-simplex is simply a vector bundle \( E \) with
a flag \( F: 0 \leq E_0 < \cdots < E_8 = E \otimes k(\infty) \).

We are therefore interested in determining the classes of such flags under \( \text{Aut}(E) \). This leads to

**Problem:** Given an \( n \)-dimensional vector space \( V \) over \( k \) with a filtration
\[
V > W_1 > W_2 > \cdots > W_k = 0
\]

having jumps \( s_i = \dim (W_i / W_{i-1}) \), \( i = 1, \ldots, k \)

let \( P \) be the corresponding parabolic subgroups of \( \text{Aut}(V) \). Classify the classes of flags
\[
0 \leq V_0 < V_1 < \cdots < V_8 = V
\]

under the action of \( P \).

Change notation. Start with

\[ (\star) \quad 0 < V_1 < \cdots < V = V \quad \text{dim} \ V = n \]
given and fixed, and \( P = \text{Aut}(0 < V_1 < \cdots < V_{k-1} < V) \). Now suppose given a subspace \( W \) of dimension \( p \). To determine its \( P \)-class one has only to give the dimensions of the filtration

\[
0 \leq V_1 \cap W \leq \cdots \leq V_k \cap W = W
\]

Thus if the jumps in (\star) are \( s_1, s_2, \ldots, s_k \)
then the P-class of a subspace is a sequence of jumps $t_1, \ldots, t_k$ with $0 \leq t_i \leq 2^i$.

To simplify things take the case where all $\beta_i = 1$. Then a subspace is determined by a sequence $t_i = 0$ or 1, and a flag

$0 \leq W_0 < W_1 < \cdots < W_8 = V$

is determined by an increasing family

$t(0) \leq t(W_0) < t(W_1) < \cdots < t(W_8) = t(V)$

where

$t(W) = \text{the sequence } (\dim(W \cap V_1), \dim(\frac{W \cap V_2}{W \cap V_1}), \ldots)$

Thus it seems that what we are getting is that any P class of simplices may be identified with a simplex in the following simplicial complex: It has for vertices sequences $\vec{r}_0, \ldots, \vec{r}_n$ and a simplex is an increasing sequence $\vec{r}_0 < \cdots < \vec{r}_n$ for the product ordering such that each component of $\vec{r}_n - \vec{r}_0$ is either 0 or 1.

Thus what this seems to be is the product of the ordered simplicial complex

\[ \cdots \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \]

divided out by the action of $\Sigma_n$. 
So the next point is to understand the stabilizers. Thus given a $q$-simplex I want to understand its stabilizer.

Again consider the case where all $x_i = 1$. The stabilizer of $\text{Aut}(E)$ maps onto the Borel subgroup of $\text{Aut}(E \otimes k(x))$. Now one knows that the mod $l$ cohomology of $\text{Aut}(E)$ is the same as that of the torus. Thus it seems that all of the simplices with top vertex $E$ have same mod $l$ stabilizer homology. But now when we come to higher $x_i$ the situation is even messier.
Proposition: Let \( C \) be a complete n.s. curve of genus \( g \) over \( k \) alg. closed, and let \( E \) be a vector bundle of rank \( n \) over \( C \). Assume \( E \) has a flag
\[
(*) \quad 0 < E_1 < \cdots < E_n = E
\]
with quotients \( L_i = E_i / E_{i-1} \) satisfying
\[
\deg(L_{i-1}) - \deg(L_i) > 0
\]
Then (*) is the unique maximal flag in \( E \).

Proof. Recall that a maximal flag is one such that \( E_1 \) is a line bundle of maximum possible degree in \( E \), \( E_2 / E_1 \) is of max. deg. in \( E / E_1 \), etc.
Suffices to show that if \( L \) is a sub-line bundle of \( E \) of maximum degree, then \( L = E_1 \). (Induction on \( n \)).
But then \( \deg(L) \geq \deg(E_1) > \deg(L_i) \) for all \( i \geq 2 \).
so one sees that the map
\[
L \rightarrow E_n \rightarrow E_n / E_{n-1} \quad \text{is zero}
\]
so \( L \leq E_{n-1} \), etc. until finally that \( L \leq E_1 \), whence \( L = E_1 \) as the degrees are equal. DONE
Given a vector bundle $E$ over $C$, we consider
\[ \mu_1(E) = \sup \left\{ \frac{\deg(F)}{\text{rank}(F)} \mid 0 < F \leq E \right\} \]
where $F$ runs over subbundles of $E$. Actually, we shall maybe eventually want to consider the polygon obtained by plotting the points $(\deg(F), \text{rank}(F))$ in the plane and taking the shaded area.

So we get a sequence of slopes.
\[ \mu_1 > \mu_2 > \ldots \]

Suppose that we consider two subbundles (non-zero) $F_1, F_2$ with slope $= \mu_1(E)$. Then have an exact sequence of vector bundles:
\[ 0 \rightarrow F_1 \cap F_2 \rightarrow F_1 \oplus F_2 \rightarrow F_1 + F_2 \rightarrow 0 \]

hence
\[ \text{deg}(F_1 \cap F_2) + \text{deg}(F_1 + F_2) = \text{deg}(F_1) + \text{deg}(F_2) \]
\[ \text{rank}(F_1 \cap F_2) + \text{rank}(F_1 + F_2) = \text{rank}(F_1) + \text{rank}(F_2) \]
Let \( F_1 + F_2 \) be the smallest subbundle of \( E \) containing \( F_1 + F_2 \), whence
\[
d(F_1 + F_2) \leq d(F_1 + F_2)
\]
with equality iff the two are equal, and
\[
r(F_1 + F_2) \leq r(F_1 + F_2).
\]

Then by assumption,
\[
d(F_1 \cap F_2) \leq \mu_1(E) r(F_1 \cap F_2)
\]
\[
d(F_1 + F_2) \leq d(F_1 + F_2) \leq \mu_1(E) r(F_1 + F_2)
\]
so adding get
\[
d(F_1) + d(F_2) \leq \mu_1(E) (r(F_1) + r(F_2)) = d(F_1) + d(F_2)
\]
since \( F_1 \cap F_2 \) have slope \( \mu_1(E) \). Thus all the preceding inequalities must be equalities and so we see that
\[
d(F_1 \cap F_2) = \mu_1(E) r(F_1 \cap F_2)
\]
\[
F_1 + F_2 = \overline{F_1 + F_2} \text{ is a subbundle of } E
\]
\[
F_1 + F_2 \text{ has slope } \mu_1(E).
\]

So if \( F_1 \) is a \( \mu_1 \)-subbundle with slope \( \mu_1(E) \) having the maximum rank, then \( F_2 \subset F_1 \). Thus get

**Proposition:** There is a unique \( \mu_1 \)-subbundle of \( E \) maximal of slope \( \mu_1(E) \), and it is semi-stable of that slope.
Proposition: \( \text{If } E \text{ is semi-stable and } \deg(E) < 0, \) then \( H^0(E) = 0. \)

Proof: If \( H^0(E) = 0 \), then \( E \) has a sub-line bundle of degree \( \geq 0 \), contradicting \( \frac{\deg(L)}{1} \leq \frac{\deg(E)}{\text{rg}(E)} < 0. \)

Cor: \( E \) semi-stable and \( \deg(E) > \text{rg}(E) \cdot (2g-2) \) \( \Rightarrow H^1(E) = 0. \)

Proof: \( \mathcal{O} \otimes E^v \) is also semi-stable. (Check: Any subbundle of \( E^v \) is of the form \( F^\perp \) for some subbundle \( F \) of \( E \), and \( F^\perp = (E/F)^v \).

\[
\begin{align*}
\deg(F^\perp) &= \deg(F) - \deg(E) \\
\text{rg}(F^\perp) &= \text{rg}(E) - \text{rg}(F)
\end{align*}
\]

\[
\mu(E) = \frac{\deg(E)}{\text{rg}(E)} \quad \mu(E^v) = \frac{-\deg(E)}{\text{rg}(E)} = -\mu(E).
\]

\[
\mu(F^\perp) = \frac{\deg(F^\perp)}{\text{rg}(F^\perp)} = \frac{\deg(F) - \deg(E)}{\text{rg}(F^\perp)}
\]

\[
\leq \frac{\mu(E) \text{rg}(F) - \mu(E) \text{rg}(E)}{\text{rg}(E) - \text{rg}(F)} = -\mu(E) = \mu(E^v)
\]

so OK. Actually the way to see this is to note that \( E \) semi-stable is equivalent to \( \mu(E/F) \geq \mu(E) \) for any proper quotient bundle.
\(\mu(E)\)
\[\mu(E/F) \leq -\mu(E) = \mu(E)\]
and hence
\[\mu(E^\vee) = -\mu(E/F) \leq -\mu(E) = \mu(E)\]
as desired. So \(\Omega \otimes E^\vee\) is semi-stable with
\[\deg(\Omega \otimes E^\vee) = rg(E)(2g-2) - \deg(E) < 0\]
so
\[H^1(E)\text{ dual to } H^0(\Omega \otimes E^\vee) = 0.\]

**Proposition:** If \(\mu_1(E) < 0\), then \(H^0(E) = 0\).
and if \(\mu_{\min}(E) > 2g-2\), then \(H^1(E) = 0\).

Recall that \(\mu_1(E) \geq \mu_2(E) \geq \ldots \geq \mu_{\min}(E)\).
The same proof works as before. Thus, by definition
for any line bundle \(L \subseteq E\), we have
\[\deg(L) \leq \mu_1(E)\]
so \(\deg(L) < 0\) if \(\mu_1(E) < 0\). Similarly

\[\mu_\min\]

for any \(F\) we have
\[\frac{\deg(E/F)}{rg(E/F)} \geq \mu_{\min}(E)\]
so if this is > \(\deg(2) = 2g - 2\), then can't have \(H^0(\Omega \otimes 2) \neq 0\).
For a consistent notation put
\[ \mu_{\text{max}}(E) = \mu_1(E) \]
so that
\[ \mu_{\text{max}}(E) \geq \mu(E) \geq \mu_{\text{min}}(E) \]
with equalities for semi-stable bundles.

Recall that \( H^1(E) = 0 \) and \( E \) is gen. by \( H^0(E) \)
if \( H^1(E(-P)) = 0 \) for all points \( P \). Thus we see
that if
\[ \mu_{\text{min}}(E(-P)) = \mu_{\text{min}}(E) - 1 > 2g - 2 \]
then \( H^1(E) = 0 \) and \( E \) is gen. by \( H^0(E) \). Thus we get

**Proposition:** If \( \mu_{\text{min}}(E) \geq 2g - 2 \), then
\( H^1(E) = 0 \) and \( E \) is generated by \( H^0(E) \).

**Corollary:** If the ground field is finite, then there are only finitely many isomorphism classes of vector bundles with given rank, \( \mu_1 \) and \( \mu_{\text{min}} \).

**Proof.** Tensoring with a line bundle we can suppose \( \mu_{\text{min}} \) large enough so the preceding proposition applies. Thus we know \( H^0(E) \) by Riemann-Roch, and so it suffices to show that \( \mu_{\text{grass}} \) over \( C \) have finitely many rational points of given degrees. This must be usual Hilbert scheme nonsense. 

Can do directly as follows. Choose a very ample line bundle \( L \) e.g. \( O(2g + 1) \).
Then if $H^1(E \otimes L^{-1}) = 0$, we have $\alpha$ by general regularity considerations and exact sequence

$$L^{-1} \otimes k_0 T_1(E) \to O \otimes k_0 T_0(E) \to E \to 0$$

with

$T_0(E) = H^0(E)$

$T_1(E) = H^0(\text{Hom}(L^{-1} \otimes k_0 T_0(E), E))$

and if $E$ is sufficiently positive

so suppose $H^1(E \otimes L^{-1}) = 0$. Then $E$ is regular with respect to the embedding defined by $L$, so we have

$$0 \to Z \to O \otimes H^0(E) \to E \to 0$$

and

$$O \otimes H^0(Z \otimes L) \to Z \otimes L \to 0$$

exact. Moreover we know the dimensions of $H^0(E)$ and $H^0(Z \otimes L)$ from the dimensions of $H^0(E \otimes L)$ which can be determined by R-R. Thus we have a presentation:

$$L^{-1} \otimes k_0 T_1(E) \to O \otimes k_0 H^0(E) \to E \to 0$$

where the dimensions of $T_1(E)$ and $H^0(E)$ are known, hence there are only finitely many possibilities for $\alpha$, the result is now clear.
Corollary: There are only finitely many stable vector bundles of given rank and degree, when the ground field is finite.

Proposition: Let $E_v$ be a sequence of vector bundles of the same rank and degree. TFAE
a) $\mu_{\text{max}}(E_v) \to \infty$
b) $\mu_{\text{min}}(E_v) \to -\infty$
c) Let $\delta(E_v)$ be the maximal degree of a sub-line bundle of $E_v$. Then $\delta(E_v) \to +\infty$.

Proof: From the picture

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (3,0) node[midway,above] {$\mu_{\text{max}}$};
\draw[->] (0,0) -- (1,2) node[pos=0.5,above] {$(1,\mu_{\text{max}})$};
\draw[->] (0,0) -- (2,3) node[pos=0.5,above] {$(\text{rg } E, \text{deg } E)$};
\end{tikzpicture}
\end{center}

one sees that the polygon contains the point $(1, \mu_{\text{max}})$, hence

$$\mu_{\text{min}} \leq \frac{\text{deg } E - \mu_{\text{max}}}{\text{rg } E - 1}$$

whence a) $\Rightarrow$ b). Converse similar.

c) $\Rightarrow$ a) because $\delta(E) \leq \mu_{\text{max}}(E)$.
a) $\Rightarrow$ c). Recall from Serre’s course the
Lemma: If \( L_1, \ldots, L_n \) are the quotients of a maximal flag in \( E \), then
\[
\deg(L_i) - \deg(L_i) \leq 2g.
\]

Proof: Enough to consider the case \( n = 2, i = 1 \), since tensoring \( E \) with a line bundle doesn't change \( \deg(L_2) - \deg(L_1) \), so can suppose
\[
\deg(E) = 2g - 1 + \varepsilon \quad \varepsilon = 0 \text{ or } 1
\]
Then \( \Omega \Rightarrow \)
\[
h^0(E) \geq \deg(E) + 2(1-g) = 1 + \varepsilon \geq 1
\]
so \( E \) has a sub-line-bundle of degree \( > 0 \), so
\[
\deg(L_1) \geq 0
\]
Thus
\[
\deg(L_2) - \deg(L_1) = \deg E - 2 \deg(L_1)
\]
\[
\leq \deg(E) = 2g - 1 + \varepsilon \leq 2g
\]
as claimed.

Thus once we give \( \deg(L_i) \) in a maximal flag, we get \( \delta(E) \). Then for any subbundle \( F \), \( \delta(F) \leq \delta(E) \), and so if \( F \) has a maximal flag with quotients \( L_1, \ldots, L_n \), the best the degree of \( F \) can be is...
\[ \deg(L_i) = \left[ \deg(L_i) - \deg(L_{i-1}) \right] + \cdots + \left[ \deg(L_2) - \deg(L_1) \right] + \delta(F), \]
\[ \leq (i-1)2g + \delta(F). \]
\[ \deg(F) = \sum_{i=1}^{r} \deg(L_i) \leq 2g \sum_{i=1}^{r} (i-1) + r \delta(F) \]
\[ \deg(F) \leq 2g \frac{r(r-1)}{2} + r \delta(F) \]
\[ \frac{\deg(F)}{r} \leq g(r-1) + \delta(F) \]

which shows that if \( \delta(E_v) \) remains bounded so does \( m_{\max}(E_v) \), whence (a) \( \Rightarrow \) (c).

Suppose \( C \) is of genus 1. Then

Now I want to understand the limiting behavior of a sequence \( E_v \) of the same rank and degree which go to infinity.
Proposition: Given an exact sequence
\[ 0 \to E' \to E \to E'' \to 0. \]

if \[ \mu_{\min}(E') \geq \mu_{\max}(E'') + 4g \]
then the sequence splits.

Proof: Recall \[ \mu_{\min}(E' \otimes L) = \mu_{\min}(E') + \deg L, \]
and whether the sequence splits or not is unchanged by tensoring with a line bundle. Since
\[ \mu_{\min}(E') - 2g \geq \mu_{\max}(E'') + 2g \]
we can by tensoring with a suitable line bundle assume that
\[ \mu_{\min}(E') \geq 2g \]
\[ \mu_{\max}(E'') + 2g - 2 < 0 \]

The first inequality implies that \( E' \) is generated by \( H^0(E') \), hence \( \mathcal{O}^n \to E'' \) so
\[ \text{Ext}^1(E'', \mathcal{O}^n) \to \text{Ext}^1(E'', E'). \]

But \( \text{Ext}^1(E'', \mathcal{O}) = H^1(E'', \mathcal{O}) \) dual to \( H^0(E'' \otimes \mathcal{O}) \)
and
\[ \mu_{\max}(E'' \otimes \mathcal{O}) = \mu_{\max}(E'') + 2g - 2 < 0 \]
so \( H^0(E'' \otimes \mathcal{O}) = 0 \) and the sequence splits as claimed.
Suppose now that $E$ is indecomposable of rank $n$ with slopes $\mu_1 > \mu_2 > \ldots > \mu_n$. Then

$$\mu_1 - \mu_2 \leq 4g - 1$$

\[ \mu_{r-1} - \mu_r \leq 4g - 1 \]

$$\mu_1 - \mu_{n-1} \leq n(4g-1) \leq n(4g-1)$$

$$\mu_1 \leq n(4g-1) + \mu_n \leq n(4g-1) + \frac{\deg E}{n}$$

Thus concludes

**Proposition:** The set of vector bundles of a given rank and degree form a limited family. So for a finite no. when the ground field is finite.

---

The $4g - 1$ can be improved to $2g - 2$. The point:

$$\text{Ext}^1(E', E') \neq 0 \iff \text{Hom}(E', E' \otimes \Omega) \neq 0 \Rightarrow \exists f : E' \to E' \otimes \Omega \neq 0 \Rightarrow$$

$$\mu_{\min}(E') \leq \mu(Coinf) \leq \mu(I\text{nf}) \leq \mu(E' \otimes \Omega) = \mu_{\max}(E') + 2g - 2.$$
April 18, 1973. Localization

Let \( A \) be a Dedekind domain with quotient field \( F \), let \( \mathfrak{m} \) be a maximal ideal, and let \( B \) be the Dedekind ring obtained by removing \( \mathfrak{m} \) from \( A \). Then \( B = \bigcap A_{\mathfrak{m}'} \subseteq F \).

Let \( M \) be a vector bundle over \( B \), let \( X \) be the building consisting of \( A_{\mathfrak{m}} \)-lattices \( L \) in \( F \otimes_B M \). Equivalently, \( X \) is the building of extensions of \( M \) to a vector bundle \( E \) over \( A \). Formulas:

\[
E = M \otimes A \subseteq F \otimes_B M
\]
\[
L = E_{\mathfrak{m}} = A_{\mathfrak{m}} \otimes A E
\]
\[
M = B \otimes A E
\]

Then \( \Gamma = \text{Aut}(M) \) acts on \( X \) which is contractible.

The \( \Gamma \)-classes of vertices of \( X \) are the same as iso. classes of \( E \) extending \( M \). One knows that a vector bundle over a Dedekind domain is determined by its rank and first Chern class. Thus

iso. classes of \( E \) = \{ \alpha \in \text{Pic}(A) \mid \alpha \rightarrow c(M) \in \text{Pic}(B) \}

In virtue of the exact sequence

\[
0 \rightarrow A^* \rightarrow B^* \otimes A \rightarrow \mathbb{Z} \rightarrow \text{Pic}(A) \rightarrow \text{Pic}(B) \rightarrow 0
\]

the iso classes are the cosets in \( \text{Pic}(A) \) for the cyclic group \( [m] \). This cyclic group is finite iff \( \exists f \in m \cap B^* \), which
is the case in the standard arithmetic examples.

A \( q \)-simplex in \( X \) is the same as a lattice \( L_\mathfrak{g} \) together with a filtration

\[
0 \leq \bar{L}_0 < \bar{L}_1 < \cdots < \bar{L}_\mathfrak{g}
\]

where \( \bar{L}_\mathfrak{g} = L_\mathfrak{g} \otimes \mathbb{A}^{\mathfrak{m}} \), \( k = A/m \). Thus a \( \Gamma \)-class of lattices in the \( q \)-simplexes is the same as an isom. class of pairs \( (E,F) \) where \( E \) is a bundle over \( A \) extending \( M \), and where \( F \) is a filtration

\[
0 \leq \bar{E}_1 < \cdots < \bar{E}_i < \bar{E} = A/m \otimes E
\]

Thus to determine the \( \Gamma \)-classes we have to determine the action of \( \text{Aut}(E) \) upon the set of such flags.

Now we know that

\[
\text{Aut}(E) \cong GL_n(k)
\]

and that \( SL_n(k) \) acts transitively on flags with same dimension, so what we want is

**Lemma:** \( \text{Aut}(E) \to \text{Aut}(\bar{E}) \) is onto the elementary subgroup.

Assuming this it follows that as \( \Gamma \)-class of \( q \)-simplexes is described by an \( x \in \text{Pic}(A) \) over \( \mathbb{C} \), together with a sequence of positive integers

\[
\begin{align*}
&n_1, \ldots, n_{\mathfrak{g}-1} \\
&\mathfrak{E}_1 < \cdots < \mathfrak{E}_{\mathfrak{g}-1} < \mathfrak{E}
\end{align*}
\]

such that \( \sum n_i \leq n = \text{rank}(M) \).
To prove the lemma, suppose given an exact sequence
\[ 0 \to k \to E \to W \to 0 \]
Now, modifying slightly the same theorem, one can find a section \( s \) of \( E \) such that \( s(k) = k \) and such that \( s \) is unimodular whence we get an exact sequence
\[ 0 \to A \to E \to E' \to 0 \]
which reduces to (1) modulo \( m \). Splitting (2) we have

\[ \text{Act}(2) \xrightarrow{\text{red. mod } m} \text{Act}(1) \]

\[
\begin{pmatrix}
A^* \\ O
\end{pmatrix}
\begin{pmatrix}
E' \\ \text{Aut}(E')
\end{pmatrix}
\to
\begin{pmatrix}
k^* \\ O
\end{pmatrix}
\begin{pmatrix}
W \\ \text{Aut}(W)
\end{pmatrix}
\]

and since \( E' \to W \), it is clear that every elementary automorphism in \( \text{Aut}(E) \) lifts to one in \( \text{Aut}(E) \). **DONE**.

So now have an explicit description of the \( \Gamma \)-orbits on the simplices of \( X \). It is clear that the stabilizer of
\[ E_0 < \ldots < E_g \]
corresponding

is simply the parahorique subgroup of $\text{Act}(E)$.

Now this situation to which we arrive I considered before when trying to compute the mod $p$ cohomology of $\text{GL}_n$ of a local field of res. char.$p$. Instead of getting a “clear” relation between the cohomology of $\text{GL}_n(F)$, $\text{GL}_n(k)$, $\text{GL}_n(A)$ as we do in the localization theorem, we get this confused picture with the parahorique groups.
Splitting then.

A ring, \( W \) fixed f.t. \( A \)-module, \( E_w \) the groupoid of epis \( P \to W, \ Q \in \text{Epi}(A) \), with iso over \( W \). Then have a forgetful functor

\[
k : E_w \to E_0 \quad (P \to W) \mapsto P.
\]

Also have an action of \( E_0 \) on \( E_w \) compatible with \( k \)

\[
\text{Q} \# (P \xrightarrow{u} W) = (Q \oplus P \xrightarrow{0+u} W).
\]

Let \( \mathcal{S} \) be the monoid \( \pi_0 E_0 \); it acts on \( H_*(E_w) \) and \( H_*(E_0) \). To prove:

Thus:

\[
\tilde{k} : \mathbb{I}^{-1} H_*(E_w) \to \mathbb{I}^{-1} H_*(E_0).
\]

Put \( \pi_0 E_w = \mathbb{S} J \), so

\[
H_*(E_w) = \frac{H}{j \in J} H_*(\text{Aut}(E_j))
\]

where \( E_j \) is a rep for \( j \in J \). The action of \( \mathbb{S} \) on \( H_*(E_w) \) is as follows. If \( \lambda \) is rep by \( \mathbb{S} Q_i \), then multip. by \( i \) \( \lambda \) (denoted \( \lambda_i \)) is

\[
H_*(\text{Aut}(E_j)) \to H_*(\text{Aut}(Q_i \# E_j))
\]

\[
\lambda_i \to H_*(\text{Aut}(E_{i \# j}))
\]

was last map induced by any isom of \( Q_i \# E_j \) with \( E_{i \# j} \).

Thus to invert \( \mathbb{I} \), what we are doing is this:

\[
\mathbb{I}^{-1} H_*(E_w) = \lim_{\text{trans}(I)} \{ i \mapsto H_*(E_w), (i \mapsto i') \mapsto \lambda_{i'} \}
\]
But we can form over $\text{Trunc}(\mathbf{I}) = \langle \mathbf{I}, \mathbf{I} \rangle$ the cofibred cat $\langle \mathbf{I}, \mathbf{I} \times \mathbf{J} \rangle$, and we have

$$I^{-1}H^*_x(E_W) = \lim \{(i,j) \mapsto H^*_\mathbf{I}(\text{Aut}(E_i))\}_{(i,j) \in \langle \mathbf{I}, \mathbf{I} \times \mathbf{J} \rangle}$$

Let $I^{-1}J = \prod_0 \langle \mathbf{I}, \mathbf{I} \times \mathbf{J} \rangle = \text{set of couples}$

Then clearly

$$I^{-1}H^*_x(E_W) = \prod_{x \in I^{-1}J} \lim \{(i,j) \mapsto H^*_\text{Aut}(E_i)\}_{(i,j) \in \langle \mathbf{I}, \mathbf{I} \times \mathbf{J} \rangle}$$

Recall $I$ operates in $E_W$.

**Lemma:** 

$s_1 + s_2 = k s_1 \# s_2$

**Proof:** $s_i$ rep by $u_i : P_i \to W$. Then $s_1 + s_2$ rep by $P_1 \oplus P_2 \xrightarrow{u_1 + u_2} W$

$k s_1 + s_2$ --- $P_1 \oplus P_2 \xrightarrow{0 + u_2} W$

so want $P_1 \oplus P_2$

\[
\begin{array}{ccc}
\text{(id + 0, id + id)} & \downarrow \\
P_1 \oplus P_2 & \xrightarrow{u_1 + u_2} & W \\
\end{array}
\]

\[
\begin{array}{ccc}
P_1 \oplus P_2 & \xrightarrow{0 + u_2} & W \\
\end{array}
\]

$u_2 \varphi = u_1$

now $u_1 \varphi$ exists as $P_1$ is proj and $u_2$ onto.
The problem now is this: Given $f_0 + f = f'$ there are two maps

$$H_\ast(\text{Aut } E_j) \rightarrow H_\ast(\text{Aut}(E_j'))$$

The former is induced by $\# k f_0$, the latter with $- f_0$.

And although I see that the objects $(Q \oplus P \xrightarrow{0+u\chi} E)$ and $(Q \oplus P \xrightarrow{u+\chi} W)$ are isom., I don't see that these representations are conjugate, no matter how big $Q$ is. In fact they aren't, since the former has no invariant mapping into the latter.

Therefore your generalization doesn't work.

Take a vector bundle $M$ over $\mathbb{Z}$ of rank $n$ and form the formal series

$$\sum_{L \in M} \frac{1}{\text{card}(M/L)^s}$$

where $L$ runs over all lattices contained in $M$.

Problem: Compute this series.

Now the first thing to notice is that $M/L$ can be split into its primary components, hence we get an Euler product:

$$\sum_{L \in M} \frac{1}{\text{card}(M/L)^s} = \prod_p \sum_{M/L} \frac{1}{\text{card}(M/L)^s}$$

and that lattices such that $M/L$ is a $p$-group may be identified with lattices in $M \otimes \mathbb{Z}_p$ contained within $M \otimes \mathbb{Z}_p$. Thus we are down to a local problem.

Local problem: Let $A$ be a discrete valuation ring with quotient field $F$ and residue field $k$. Assume $k$ has $q$ elements. Calculate the sum:

$$\sum_{L \subset A} \frac{1}{\text{card}(A/L)^s}$$
To avoid biasing things, fix a lattice $M$ in $F^n$. Want to compute the lattices $L \subset M$ with given $\text{card}(M/L)$. Let $\pi$ generate the maximal ideal of $A$. Any lattice $L \subset M$ determines $\pi$. 

Try $n = 2$. Then let $p$ be greatest such that $L \subset \pi^p M$ and $q$ least such that $\pi^{p+q} M \subset L$.

The integers $p, q$ being given, one sees that $L$ is completely determined by giving a line in $M/\pi^p M$, that is, the line $\pi^{-p} L/\pi^p M \subset M/\pi^p M$. $\sim (A/\pi^p A)^2$.

How many such lines? No of unimodular vectors is $(q^2 - 1)(q^4)^{n-1}$.

Number of units is $(q^2 - 1) q^{n-1}$ if $n \geq 1$.

Thus get

$$\frac{q^2 - 1}{q - 1} q^{n-1} = q^2 + q^{n-1}$$ if $n \geq 1$.

and it seems I want

$$\sum_{p \geq 0, n \geq 1} \frac{q^2 - 1}{q - 1} q^{n-1} \cdot (q^{2p} \cdot q^r)^{-1} + \sum_{p \geq 0} (q^{2p})^{-1} = \left[(1 - q^{-1})(1 - q^{1-r})\right]^{-1}$$.
Better method: Let $M$ have the basis $e_1, e_2$ and consider the trace of the filtration

$$0 < Ae_1 < A^2$$

on $L$: $0 < L e_1 < L$. Then we get a basis for $L$ of the form

$$\prod^1 e_1, \quad \alpha_1 e_1 + \prod^1 e_2$$

where the class of $\alpha$ in $A/\prod^1 A$ is unique. Thus we are interested in the sum

$$\sum_{j, k \geq 0} q^j (q^{j+k}-1) = \left[(1-q^{-1})(1-q^{-2})^{-1}\right]^{-1}$$

as before.

In general, if $M = A e_1 + \ldots + A e_n$ are filters:

$$0 < A e_1 < A e_1 + A e_2 < \ldots < M$$

$$0 < L_1 < L_2 < \ldots < L$$

and find a basis

$$\prod^k e_1$$

$$\prod^k e_2 + \alpha_{21} e_1$$

$$\prod^k e_n + \alpha_{n-1} e_{n-1} + \ldots + \alpha_{n1} e_1$$

where $\alpha_{ij}$ is determined in $A/\prod^k A$, and one can compute that the sum is

$$\frac{n!}{\prod_{i=0}^{n-1} (1-q^{-i-1})^{-1}} \quad (Weyl's \ book, \ p. \ 194)$$

Cohomology computations

$k$ finite field, $q = \text{card } k$, $l$ prime $l \neq q$.

I want to compute

$$H_*(\text{GL}_n(k), st(k^n))$$

where $st(k^n)$ is the Steinberg module mod $l$.

If $X$ is the building of $k^n$ we have an exact sequence

$$0 \rightarrow st(k^n) \rightarrow C_{n-2}(X) \rightarrow \cdots \rightarrow C_0(X) \rightarrow \mathbb{F}_l \rightarrow 0$$

with $C_p(X) \equiv \text{mod } p$ defines on $X$. This exact sequence holds for $n = 1$ if we define $st(k^1) = \mathbb{F}_l$. Since

$$C_{p-1}(X) = \prod_{0 < w_1 < \cdots < w_p < V} \mathbb{F}_l$$

we have

$$H_*(\text{GL}_n, C_{p-1}(X)) = \prod_{\sum d_i = n, \quad d_i > 0} H_*(\text{GL}_{d_1, \ldots, d_{p+1}})$$

where $\text{GL}_{d_1, \ldots, d_{p+1}}$:

```
 \begin{array}{cccc}
   x & x & x & x \\
   x & x & x & x \\
   x & x & x & x \\
\end{array}
```

Since $l \neq q$ we have

$$H_*(\text{GL}_{d_1, \ldots, d_{p+1}}) = H_*(\text{GL}_{d_1}) \otimes \cdots \otimes H_*(\text{GL}_{d_{p+1}}).$$
This on applying $H_\ast(GL_n)$ to the complex

$$K_n: e \rightarrow C_{n-1}(X) \rightarrow \ldots \rightarrow C_0(X) \rightarrow F_e \rightarrow 0$$

degrees: $n$ 1 0

we get

$$\ldots \rightarrow \bigoplus_{a+b=n} H_\ast(GL_a) \oplus H_\ast(GL_b) \rightarrow H_\varepsilon(GL_n) \rightarrow 0$$

which is the degree $n$ part of the bar construction

$$\ldots \rightarrow \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0$$

where

$$R = \bigoplus_{n>0} H_\varepsilon(GL_n), \quad \overline{R} = \bigoplus_{n>0} H_\varepsilon(GL_n)$$

Now we have seen

$$R = P[\varepsilon, \xi_0, \xi_1, \ldots] \otimes \wedge [\eta_1, \ldots]$$

where $\varepsilon$ base for $H_\varepsilon(GL_1)$

$\xi_j$ base for $H_{2j-1}(GL_n)$ $j \geq 1$

$\eta_j$ base for $H_{2j}(GL_n)$ $j \geq 1$

The homology of the above bar construction is

$$\text{Tor}_R^R(k,k) = \wedge [\varepsilon, \xi_1, \ldots] \otimes \mathbb{R}[\overline{\eta}_1, \ldots]$$
where $\overline{e}, \overline{e}_1, \ldots, \overline{e}_i, \ldots$ is the obvious base for

$$\text{Tor}^R(k, k) = \frac{R}{R^2}$$

\[\text{If } r = 1, \text{ any monomial } \overline{e}^\alpha \overline{e}_1^\beta \]

is of degree $(|\alpha| + |\beta|) n$. Thus $\text{Tor}_1$ occurs only in degree 1. The point is that $R$ is a graded algebra and its generators are homogeneous of degree 1. Thus $\text{Tor}_1$ is homogeneous of degree $\beta$, so we find that given $n$, the complex

\[0 \rightarrow H_\ast(Gl_n, C_{n-2}) \rightarrow \ldots \rightarrow H_\ast(Gl_n) \rightarrow 0\]

has exactly one homology group which is in degree $n$. But we have

$$K_n \cong \text{st}(k^\ast)[a]$$

so there is a spectral sequence.

$$E^1_{pq} = H_q(Gl_n, (K_n)_p) \Rightarrow H_{p+q}(Gl_n, \text{st}(k^\ast))$$

0 for $p \neq n$

so the spectral sequence degenerates, yielding an
isomorphism \( \Phi \)

\[ H_x(\text{GL}_n, \text{st}(k^n)) = T\Omega^R_n(k, k) \]

monomials of degree \( n \) into \( \frac{\bar{x}}{\bar{y}} \).

Example: \( n = 1 \), where

\[ H_x(\text{GL}_1, \text{st}(k^1)) = H_x(\text{GL}_1) \]

has base \( \frac{\bar{x}}{\bar{y}}, \frac{2i}{\bar{y}i}, \frac{2i-1}{\bar{y}i} \).

Then one has a product

\[ H_x(\text{GL}_1, \text{st}(k^1)) \otimes H_x(\text{GL}_1, \text{st}(k^1)) \rightarrow H_x(\text{GL}_2, \text{st}(k^2)) \]

and maybe divided powers whereby one generated the latter.
April 24, 1973

Localization via Serre's methods

Let \( A \) be a complete discrete valuation ring with residue field \( k \) and quotient field \( F \). Let \( l \) be a prime no. \( \neq \text{char}(k) \). I want to understand the continuous homology of \( \text{Gl}_n(F) \) mod \( l \).

If \( V \) is a vector space over \( F \), let \( \mathcal{J}(V) \) be the ordered set of layers \( (L_0, L_1) \) in the ordered set of lattices in \( V \) such that \( mL_1 \subset L_0 \). Let \( \text{Aut}(V) \) act on \( \mathcal{J}(V) \) and form the associated cofibered category over \( \text{Aut}(V) \).

We may identify this with the category of pairs \( (L_0, L_1) \) of lattices free \( A \)-modules such that \( L_1/L_0 \) is a \( k \)-mod with maps \( (L_0, L_1) \xrightarrow{\phi} (L_0', L_1') \)
defined to be an embedding \( L_1 \leftarrow \phi \leftarrow L_1' \)

\[ L_0' \subset \phi L_0 \subset \phi L_1 \subset L_1' \]

Category to be denoted \( (\mathcal{J}(V), \text{Aut}(V)) \). Now there is an evident functor

\[ (\mathcal{J}(V), \text{Aut}(V)) \twoheadrightarrow \mathbb{Q} \subset \mathbb{Q}(k\text{-mod}) \]

\[ (L_0, L_1) \twoheadrightarrow L_1/L_0 \]

where \( n = \text{rank}(V) \). This functor is fibred, the fibre over a \( k \)-module \( W \) being the groupoid of surjections \( L \twoheadrightarrow W \).
where $L$ is a free $A$-module of rank $n$, and their isomorphisms. Thus we get a spectral sequence

$$E^2_{pq} = H_p(Q_n, W) \longrightarrow H_q(GL_n(F))$$

where \( \begin{pmatrix} \equiv 1 & \equiv 0 \\ * & * \end{pmatrix} \) \subset GL_n(A) is the subgroup of matrices of $A$ which induce the identity on $k^n/k^{n-d}$. Actually it should be the group

\[
\begin{pmatrix}
* & \equiv 0 \\
\equiv 1 & *
\end{pmatrix}
\]

So now the question to ask is whether it might be the case that as \( n \to \infty \) the group

\[
\begin{pmatrix}
\equiv 1 & * \\
\equiv 0 & *
\end{pmatrix}
\]

has the same homology as $GL_n(A)$. 
Fix a \( k \)-module \( W \) and let \( L_W \) denote the groupoid consisting of surjections

\[
E \xrightarrow{p} W
\]

where \( E \) is a free \( A \)-module (f.g.). Let \( L \) be the groupoid of free \( A \)-modules. \( L_W \) has operation

\[
(E \xrightarrow{w} W) \downarrow (E' \xrightarrow{w'} W) = (E \times E' \xrightarrow{w \times w'} W)
\]

(Also one has \( E \oplus E' \xrightarrow{w} W \)). The kernel functor

\[
k : L_W \longrightarrow L
\]

\[E \times W \longrightarrow \ker(p)\]

is compatible with the operations. In addition \( L \) acts on \( L_W \) by

\[
L \times (E \xrightarrow{p} W) = (L \oplus E \xrightarrow{p \oplus 0} W)
\]

and

\[k(L \times E) = L \oplus kE.\]

We have the basic identity

\[
(E) \downarrow (E) = (E \times E \xrightarrow{w} W)
\]

\[= kE \times E\]

Hence if we fix \( E_0 \), we have
\[ E \perp E \perp E_0 \cong (kE \times E) \perp E_0 \]
\[ \cong kE \times (E \perp E_0) \]
\[ \cong kE \times (E_0 \perp E) \]
\[ \cong (kE \times E_0) \perp E \]

so if \( \theta \) is an exponential char. class for representations over \( LW \) we have

\[ \theta(E) \theta(E) \theta(E_0) = \theta(kE \times E_0) \theta(E) \]

and so if \( \theta(E) \) is invertible, then

\[ \theta(E) = \theta(kE \times E_0) \theta(E_0)^{-1} \]

so it is now clear that \( k \) induces a map

\[ H_\ast(LW) \longrightarrow H_\ast(L) \]

which becomes an isomorphism after localization.

But the connected components are easily seen to be

\[ \begin{array}{ccc}
1 & \longrightarrow & GL_\infty(A) \\
\cong 0 & \longrightarrow & GL_\infty(A) \\
L \longrightarrow & L \times E_0 \\
H_\ast(L)[\pi_0 L]^{-1} \longrightarrow & H_\ast(LW)[\pi_0 L]^{-1} \longrightarrow & H_\ast(L)[\pi_0 L]^{-1} \\
L \times E_0 \longrightarrow & L \oplus kE_0
\end{array} \]
Thus we can conclude that the inclusion

$$GL_n(A) \xrightarrow{\subset} \left\{ \begin{array}{c}
\equiv 1 \\
\equiv 0
\end{array} \right\}$$

induces an isomorphism in the limit as $n \to \infty$.

---

**Alternative proof.** Define operation on $L_W$

$$(E_{P_1} \Rightarrow W) \oplus (E_{P_2} \Rightarrow W) = E \oplus E \xrightarrow{P+P'} W$$

and the functor

$$t: L_W \longrightarrow L$$

$$(E_{P_1} \Rightarrow W) \longmapsto E$$

compatible with operation and with the action

$$L \times (E_{P_1} \Rightarrow W) = L \otimes E \xrightarrow{P+P'} W.$$ 

Have

$$0 \rightarrow E \xrightarrow{(id,-id)} E \oplus E \xrightarrow{\oplus} E \xrightarrow{\sim} 0$$

$$\xrightarrow{\circ} \downarrow_{P+P'} \xrightarrow{P} W$$

giving a canonical isomorphism in $L_W$

$$E \oplus E \cong \pm E \times E$$

It follows that we have for any invertible
exponential char. class $\Theta$ that

$$E \oplus E \oplus E_0 \cong (tE \times E) \oplus E_0$$

$$\cong E \oplus (tE \times E_0)$$

so

$$\Theta(E) = \Theta(tE \times E_0) \Theta(E_0)^{-1}$$

To finish we notes that

$$\Theta \longmapsto (L \longmapsto \Theta(L \times E_0) \Theta(E_0)^{-1})$$

makes inv. exp. classes $\Theta$ for $L_w$ to those for $L$ and that

$$\Phi \longmapsto (E \longmapsto \Phi(tE))$$

goes the other way, and clearly these are inverses of each other. (Check:

$$\Theta(L_1 \times E_0) \Theta(E_0)^{-1} \cdot \Theta(L_2 \times E_0)^{-1}$$

$$= \Theta((L_1 \times E_0) \oplus (L_2 \times E_0)) \Theta(E_0)^{-1}$$

$$= \Theta((L_1 \oplus L_2) \times E_0 \oplus E_0) \Theta(E_0)^{-2} = \Theta((L_1 \oplus L_2) \times E_0) \Theta(E_0)^{-1}$$

OKAY.)

It would seem that we also have a new proof of the splitting theorem for exact sequences.
Now given a $k$-module $W$ we consider the groupoid of surjections

$$E \twoheadrightarrow W$$

and all automorphisms including auto. of $W$. We can operate:

$$L \times (E \twoheadrightarrow W) = (L \oplus E \xrightarrow{ppr} W)$$

as before, and hence stabilize, getting the group

$$\Gamma_{d,\infty} = \left( \begin{array}{c|c} \ast & \ast \\ \hline \equiv 0 & \ast \\ \end{array} \right)$$

Now I have a homomorphism over $GL_d(k)$

$$\Gamma_{d,\infty} \longrightarrow GL_{d+\infty}(A) \times GL_d(k)$$

$$\downarrow$$

$$GL_d(k)$$

hence to show the horizontal arrow induces a bijection it suffices to show the inclusion

$$\left( \begin{array}{c|c} \equiv 1 & \ast \\ \hline \equiv 0 & \ast \\ \end{array} \right) \subset GL_{d+\infty}(A)$$

is a homology isomorphism, which I have proved above.
Thus it is clear that we can define the transfer in this situation, namely we lift the representation in $\text{GL}_d(k)$ to $\Gamma_d^n$, so as to be trivial in $\text{GL}_{d+\infty}(A)$, and then look at the map to the kernel.

So in the limit we get a spectral sequence

$$E^2_{pq} = H_p(Q(k), H_q(\text{GL}(A))) \Longrightarrow H_{p+q}(\text{GL}(F))$$

which is what one expects. Can you make a proof out of this construction for the localization theorem?
April 28, 1973

Problem: Given a commutative ring \( A \), I know how to decompose \( K_i: A \otimes \mathbb{Q} \), \( i \geq 0 \) into eigenspaces for the Adams operations. The problem is to explicitly construct a space representing the K-theory of \( A \) of a given weight.

Let \( k = \overline{F} \) and let \( l \) be a prime number \( \neq p \). Then \( \overline{F}^p \) is base extension by Frobenius in characteristic \( p \). We can consider the effect of Frobenius on the different spaces we have been led to consider in the K-theory of \( k \). In this situation we have that \( B(k)^p \) is the \( \mathbb{Q}/\mathbb{Z} \)-version of \( BU_{1/p}^l \):

\[
B(k) \longrightarrow BU_{1/p}^l \longrightarrow BU_{\mathbb{Q}}.
\]

So we expect that \( \text{gr}_i B(k)^p \) should be an Eilenberg–Maclane space of type \((\mathbb{Q}/\mathbb{Z}, 2i-1)\). Thus its mod \( l \) homology should be fairly complicated, and not so easy to recognize.

Classical approach: form connected K-theory with periodicity operator \( \beta \), then take the relative term of multiplying by \( \beta \).

Suppose \( f: E \to B \) is a proper submersion of smooth manifolds. The key point is to define a transfer map

\[
\eta^0(E) \to \eta^0(B)
\]

for any GCT \( \eta \). In the case where \( f \) is orientable for \( \eta \), this transfer coincides with the map

\[
\eta \mapsto f_* \left( \eta(\tau_f) \cdot \eta \right) \quad \forall \eta \in \eta^0(E).
\]

Definition of (i). Choose an embedding

\[
\begin{array}{ccc}
E & \xymatrix{ \ar[r]^-i & } & B \times \mathbb{R}^N \\
& \downarrow f & \\
& B & \ar[ll]^-{p = p^1}
\end{array}
\]

and choose tubular nbds \( N \) for \( j, i \) so that we get a diagram

\[
\begin{array}{ccc}
E & \xymatrix{ \ar[r]^-\iota & } & B \times \mathbb{R}^N \\
& \downarrow f & \\
& B & \ar[ll]^-{p = p^1}
\end{array}
\]

\[
\begin{array}{ccc}
E & \xymatrix{ \ar[r]^-{j \times \text{id}} & } & B \times \mathbb{R}^N \\
& \downarrow f & \\
& B & \ar[ll]^-{p = p^1}
\end{array}
\]
Choose a splitting of the vector bundle surjection where we have

$$ E \xrightarrow{\text{0-section}} E \times \mathbb{R}^N \leftarrow \text{subbundle surjection} \quad \nu \xrightarrow{\text{framed}} B $$

and thus

$$ h^0(E) \xrightarrow{\text{susp} \nu} h^N_{p/B}(E \times \mathbb{R}^N) \xrightarrow{\text{res}.} h^N_{p/B}(\nu) \rightarrow h^N_{p/B}(B \times \mathbb{R}^N) \cong h^0(B) $$

When the map $f$ is orientable for $h^0$, then we have

$$ h^0(E) \xrightarrow{\sim} h^d_{p/B}(\mathcal{E}) \xrightarrow{\sim} h^N_{p/B}(\nu) \xrightarrow{\sim} h^0(B) $$

with $\mathcal{E} = E \times \mathbb{R}^N$.
Another version: Suppose to simplify that \( f: E \to B \) is a differentiable fibre bundle with compact fibres. Let \( S \) be a generic section of the tangent bundle along the fibres and \( Z \) its zero submanifold. Then have

\[
\begin{array}{c}
Y \xrightarrow{i} E \\
\downarrow g \downarrow \downarrow f \\
\downarrow B
\end{array}
\]

\[ v_i = \tau_f \]

and so \( g \) is canonically framed. Hence we get

\[ \text{tr: } h^0(E) \xrightarrow{i^*} h^0(Y) \xrightarrow{g_*} h^0(B) \]

and we have the formula

\[ g_* i^* (f^* b) = g_* 1 \cdot b \quad \forall b \in h^0(B) \]

where \( g_* 1 \in h^0(B) \) is a class which augments to \( X(F) \), \( F \) the fibre of \( f \).

Now for the Adams conjecture one considers a contractible principal \( G \)-bundle (\( G \) compact Lie group) \( P \to B \), and forms the associated bundle

\[ P/N \to B \]

where \( N \) is the normalizer of a maximal torus \( T \) in \( G \). One knows (classically) \( X(G/T) = \text{order of } W \).
hence \( X(G/N) = 1 \), and so applying the preceding transfer theory we find that

\[
h^0(B) \rightarrow h^0(P/N)
\]

image is a direct summand for any GCT. Now since spherical fibrations lead to a GCT by Boardman-Vogt, this reduces the Adams conjecture to the case of a bundle with axes, where it can be done by Adams' methods.

---

**Strong splitting principle:** Given a vector bundle \( E \) over \( X \), there exists a space \( f: Y \rightarrow X \) such that in the \( S \)-category \( X \) is a direct factor of \( Y \), and such that \( f^*(E) \) has axes.

---

General case: Suppose we have \( f: E \rightarrow B \) proper and we choose an embedding

\[
E \hookrightarrow B \times \mathbb{R}^N
\]

Then we have defined \( f_1: h^0(E) \rightarrow h^0(B) \) which is \( h^0(B) \)-linear, hence

\[
f_1 f^*(b) = f_1 1 \cdot b \quad f_1 1 \in h^0(B).
\]
and it would seem from the definition that \( f^* \) would be compatible with transversal basechange, which implies that \( f^* \) augments to \( X(\mathbb{G}_m) \) for every regular point \( b \in B \). This implies that the Euler classes of the different fibres of \( f \) are the same, which one knows isn't the case.

Conclude this construction makes sense only for fibre bundles and not for a proper map between manifolds. What is missing is that we need to take a generic section along the fibres of the tangent bundle along the fibres. For a general map this bundle is only a virtual bundle, so it doesn't have an Euler class (except mod 2).
April 26, 1973

K-theory for $\mathbb{Z} \times \mathbb{R}$

Recall that we decided long ago while looking at the $\Gamma$ function that a vector bundle $E$ over $\mathbb{Z} = \mathbb{Z} \times \mathbb{R}$ should be a vector bundle $M$ over $\mathbb{Z}$ together with a positive definite quadratic form $q$ on $M$. One sets

$$\Theta_E = \sum_{x \in M} e^{-\beta \pi \theta(x)}$$

to measure the "number" of sections of $E$.

Poisson summation formula:

$$\sum_{m \in M} f(x+m) = \sum_{\lambda \in \Lambda'} a_{\lambda} e^{2\pi i \langle x, \lambda \rangle}$$

where

$$a_{\lambda} = \frac{1}{\text{vol}(V/M)} \int_{M} \sum_{m \in M} f(x+m) e^{-2\pi i \langle x, \lambda \rangle} \, dx$$

$$= \frac{1}{\text{vol}(V/M)} \int_{V} f(x) e^{-2\pi i \langle x, \lambda \rangle} \, dx$$

$$f(\lambda)$$

so

$$\sum_{m \in M} f(x+m) = \frac{1}{\text{vol}(V/M)} \sum_{\lambda \in \Lambda'} f(\lambda) e^{2\pi i \langle x, \lambda \rangle}$$

Now, taking $f(x) = e^{-\pi \theta(x)}$ and $dx = dx_1 \cdots dx_n$, where

$$q(x) = \sum x_i^2$$
we know that \( f(\lambda) = e^{-\pi g(\lambda^*)} \)

where

if \( g(x) = b(x, x) \) then \( b(x, \lambda^*) = \langle x, \lambda \rangle \)

do we get

\[
\sum_{m \in M} e^{-\pi g(m)} = \frac{1}{\text{vol}_q(V/M)} \sum_{\lambda \in M'} e^{-\pi g(\lambda)}
\]

which is the analogue of the Riemann–Roch formula:

\[
\frac{\theta_E^{\lambda}}{\theta_{E^*}^{\lambda}} = d_E
\]

\[E = (M, g) \quad E^* = (M', g^*)\]

\[d_E = \frac{1}{\text{vol}(V/M)} \left( \uparrow \rightarrow \infty \quad \text{as} \quad g \downarrow 0 \right).\]

------

Exact sequence of vector bundles over \( \mathbb{Z} \):

An exact sequence of vector bundles over \( \mathbb{Z} \)

(1) \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \)

is by definition an exact sequence

\( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \)

of vector bundles over \( \mathbb{Z} \) together with an exact sequence of quadratic spaces.
\[(\ast) \quad 0 \rightarrow M'_R \rightarrow M''_R \rightarrow M^*_R \rightarrow 0\]

which means that \( \varrho \) on \( M_R \) induces \( \varrho' \) and \( \varrho'' \) in the evident way. Motivation for the definition is as follows. We know that

\[\text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R})/O_n\]

is the set of isomorphism classes of rank \( n \) vector bundles. Thus

\[\text{GL}_{a+b}(\mathbb{Z}) \backslash \text{GL}_{a+b}(\mathbb{R})/O_a \times O_b\]

should be the set of isomorphism classes of exact sequences with ranks \( a, b \).

Notice that

\[\text{GL}_{a+b}(\mathbb{R})/O_a \times O_b \cong \text{GL}_{a+b}(\mathbb{R})/O_{a+b}\]

(think of triangular matrices with positive diagonal entries) hence an exact sequence \((\ast)\) really amounts to giving \( M'_R \) as the orthogonal direct sum of \( M^*_R \) and \( M''_R \).

Now suppose we are given an exact sequence \((\ast)\) of \( \mathbb{Z} \)-bundles of rank \( n \); and choose an isomorphism

\[M_R = \mathbb{R}^n \quad \varrho = \sum_{i=1}^{n} x_i^2\]

\[M'_R = \mathbb{R}^a \quad \varrho' = \sum_{i=1}^{a} x_i^2\]

Then I would like to compare \( \Theta_E \) with \( \Theta_E', \Theta_E'' \) and hopefully prove

\[\Theta_E \leq \Theta_E', \Theta_E''\].
Any \( m \in \mathcal{M} \) determines \( p m \in M'' \). \( p : \mathcal{M} \to M'' \).

\[
\Theta_E = \sum_{m \in \mathcal{M}} e^{-\pi g(m)} = \sum_{m'' \in M''} \sum_{m \in \mathcal{M}} e^{-\pi g(m)}
\]

Given \( m'' \) fix \( s(m'') \in \mathcal{M} \Rightarrow p s(m'') = m'' \). Then

\[
\Theta_E = \sum_{m'' \in M''} \sum_{m' \in \mathcal{M}} e^{-\pi g(s(m'') + m')}
\]

\[
= \sum_{m'' \in M''} e^{-\pi g(m'')} \sum_{m \in \mathcal{M}} e^{-\pi g(s(m'')) - m'' + m'}
\]

Since \( s(m'') - m'' \in \mathcal{M}'_R \) what you want to know therefore is that \( \forall z \in \mathcal{M}'_R \)

\[
\sum_{m' \in \mathcal{M}'} e^{-\pi g(z + m')} \leq \sum_{m' \in \mathcal{M}'} e^{-\pi g(m')}
\]

(with equality iff \( z \in \mathcal{M}' \) maybe)
What happens on the line: \( M = \mathbb{Z} \).

\[
f(x) = \sum_{m \in \mathbb{Z}} e^{-\frac{\pi}{\alpha} (x+m)^2}
= \sum_{n} a_n e^{2\pi i \langle x, n \rangle}
\]

\[
a_n = \int_{0}^{1} \sum_{m} e^{-\frac{\pi}{\alpha} (x+m)^2} e^{-2\pi i n x} \, dx
= \int_{-\infty}^{\infty} e^{-\frac{\pi}{\alpha} x^2} e^{-2\pi i n x} \, dx
= \int_{-\infty}^{\infty} e^{-\pi \left( \frac{1}{\sqrt{\alpha}} x + \frac{i n}{\sqrt{\alpha}} \right)^2} \, dx
= e^{-\frac{\pi n^2}{\alpha}} \int_{-\infty}^{\infty} e^{-\pi \left( \frac{x}{\sqrt{\alpha}} \right)^2} \, dx
= \frac{e^{-\frac{\pi n^2}{\alpha}}}{\sqrt{\alpha}}
\]

\[
\sum_{m \in \mathbb{Z}} e^{-\frac{\pi}{\alpha} (x+m)^2} = \sum_{n \in \mathbb{Z}} \frac{e^{-\frac{\pi n^2}{\alpha}}}{\sqrt{\alpha}} \cos(2\pi n x)
\]

From this we see that

\[
f''(0) = -\sum_{n} \frac{e^{-\frac{\pi n^2}{\alpha}}}{\sqrt{\alpha}} (2\pi n)^2 < 0
\]

and so \( f \) has a local maximum at \( x = 0 \), which lends support to our contention.
Curiosity: Differentiate the Fourier expansion
\[ \sum_{m \in \mathbb{Z}} e^{-\pi (x+m)^2} = \sum_{\lambda \in \mathbb{Z}} e^{-\pi \lambda^2} e^{2\pi i \lambda x} \]

\[ \sum_{m \in \mathbb{Z}} e^{-\pi (x+m)^2} (-2\pi i)(x+m) = \sum_{\lambda \in \mathbb{Z}} e^{-\pi \lambda^2} 2\pi i \lambda e^{2\pi i \lambda x} \]

\[ \sum_{m \in \mathbb{Z}} e^{-\pi (x+m)^2} [4\pi^2 (x+m)^2 - 2\pi] = \sum_{\lambda \in \mathbb{Z}} e^{-\pi \lambda^2} (2\pi i \lambda)^2 e^{2\pi i \lambda x} \]

Let \( x = 0 \)

\[ \sum_{m \in \mathbb{Z}} e^{-\pi m^2} (4\pi^2 m^2 - 2\pi) = \sum_{\lambda \in \mathbb{Z}} e^{-\pi \lambda^2} (-4\pi^2 \lambda^2) \]

\[ \sum_{m \in \mathbb{Z}} 4\pi^2 m^2 e^{-\pi m^2} = \pi \sum_{m \in \mathbb{Z}} e^{-\pi m^2} \]

\[ \sum_{m \in \mathbb{Z}} e^{-\pi \delta(x+m)} \leq \sum_{m \in \mathbb{M}} e^{-\pi \delta(m)} \quad \forall x \in \mathbb{M} \]

with equality iff \( x \in \mathbb{M} \).

Proof: We have the Fourier expansion
\[ \sum_{m \in \mathbb{M}} e^{-\pi \delta(x+m)} = \frac{1}{\text{vol}(\mathbb{M}/\mathbb{R})} \sum_{\lambda \in \mathbb{M}} e^{-\pi \delta(\lambda)} e^{2\pi i \langle x, \lambda \rangle} \]
Now take real parts
\[ \sum_{m \in M} e^{-\pi \phi(x+m)} = \frac{1}{2\pi \phi(M R/M)} \sum_{\lambda \in M'} e^{-\pi \phi(\lambda)} \cos(2\pi \langle x, \lambda \rangle) \]

Let now use the fact that \( \cos(2\pi \langle x, \lambda \rangle) \leq 1 \) with equality for all \( \lambda \leftrightarrow \langle x, \lambda \rangle \in \mathbb{Z} \) all \( \lambda \leftrightarrow x \in M' = M \).

So returning to page 4 we find that
\[ \theta_E = \sum_{m'' \in M''} e^{-\pi \phi(m'')} \sum_{m' \in M'} e^{-\pi \phi'(m') - m'' + m'} \]
\[ \leq \sum_{m'' \in M''} e^{-\pi \phi''(m'')} \sum_{m' \in M'} e^{-\pi \phi'(m')} = \theta_{E''} \theta_E', \]
with equality iff \( \phi'(m') \in M' \), i.e. we could take \( m'' = m' \) which means that we can find for each \( m'' \in M'' \) a rep. \( s(m'') \in M \) with \( \phi(s(m'')) = \phi''(m'') \). Thus the sequence actually splits as an orthogonal direct sum.

Thus we have proved

\[ \text{Prop. For any exact sequence (1) of } \mathbb{Z} \text{-bundles we have } \theta_E \leq \theta_{E''} \theta_E', \]
with equality iff the sequence splits, i.e. \( M_n(M_R) \xrightarrow{\phi} M'' \).
Remark: The above proposition is somewhat surprising from the finite field viewpoint, where
\[ \Theta_E = 0 \]
and it is quite easy to have \( \Theta_E = \Theta_E', \Theta_E'' \) without the sequence splitting.

The preceding proposition ought to be true for a number fields.

Consider now what happens when we remove a prime \( p \) from \( \mathbb{Z} \). Bundles over \( \mathbb{Z} - \{p\} \) should be pairs consisting of a \( \mathbb{Z}[p] \)-module \( M \) (free fin.) and a \( g \) on \( M_R \). The notion of exact sequence should be the same as before.

Questions: To what extent do exact sequences of bundles over \( \mathbb{Z} - \{p\} \) split, and to what extent is a vector bundle determined by its rank and first Chern class?

Given
\[
0 \to M' \to M \to M'' \to 0
\]
vector bundles over \( \mathbb{Z}[p] \) and \( g \) in \( M \), we can choose a splitting \( s: M'' \to M_R \), \( ps = id \). Then we have the picture:
and $(M')^*$ can be interpreted as the graph of a map from $M'' \to M'_R$. Because we are over $\mathbb{Z}^{[\frac{1}{p}]}$, this map can be approximated by a map $M'' \to M'$ as close as one wants. Thus we can approximate the given exact sequence by split exact sequences, but not every sequence splits.

Similarly to any vector bundle $E$ we can associate

$$\Lambda^n E = (\Lambda^* M, \Lambda^n q)$$

so that getting a line bundle whose isomorphism class is an element of

$$\text{Pic} = \mathbb{R}^+ / \{p^n | n \in \mathbb{Z}\}$$

Now choosing in $M$ a vector of length close to 1, (i.e., a line which is close to being a trivial line bundle), then continuing the process to get a flag, we see that $E$ is approximately an orthogonal direct sum of trivial line bundles $\oplus \Lambda^n E$. But another way
$G_n(\mathbb{Z}\left[\frac{1}{p}\right])$ acts densely on the fibres of the map

$$GL_n(R)/O(R) \xrightarrow{\text{disc}} R^+ \xrightarrow{} R^+/\{p^n\},$$

which is as close as we can get to having that a vector bundle up to isomorphism is determined by its rank and first Chern class.

In some sense, then we get a family of "virtual" subgroups of $G_n(\mathbb{Z}\left[\frac{1}{p}\right])$ in Mackey’s sense, since it is probably true that $G_n(\mathbb{Z}\left[\frac{1}{p}\right])$ acts ergodically on the fibres. The meaning of all this, especially the relation with $L_2(G\Gamma)$ deserves elaboration.

---

Real problem: If $M = \mathbb{Z}\left[\frac{1}{p}\right]^n$, I know that $\Gamma = \text{Aut}(M)$ acts "pseudo-transitively" on the set of possible extensions of $M$ to a vector bundle on $\mathbb{Z}\left[\frac{1}{p}\right]$ with prescribed first Chern class. Can you find what might be thought of as the cohomology of the stabilizers of this "transitive" action.
Problems: If I believe that the correct gadget is a $\mathbb{Z}$-bundle $M$ with pos. def. form $g$, then I want a localization situation.


What I lack at the moment is a way of going from a $\mathbb{Z}$-bundle $M$ to $Q(pos. def. real$ quad. forms)$.

Thus we can consider the symmetric space $X$ of all forms $g$ on $M_\mathbb{R}$. The problem is to modify this so as to get a map to $Q(\infty)$.

Actually it may be unreasonable to expect there to be a $Q(\infty)$. Thus we have a cartesian situation

\[
\begin{array}{ccc}
\tilde{\mathbb{Z}} & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{O}_{\text{disc}} & \rightarrow & \mathbb{R}_{\text{disc}}
\end{array}
\]

and there is no obvious reason why the horizontal arrow is a localization, and hence has an identifiable relative term.

Question: Given $g_1$ and $g_2$ on a real vector space one can simultaneously diagonal them. Is the simplicial complex $\mathcal{G}$ whose simplices are
chains \( q_0 \leq \cdots \leq q_n \) of simultaneously diagonalizable forms a contractible complex?