fourth part: comparison theorem

Def: of $\overline{Q}(\mathcal{M}) \xrightarrow{q} Q(\mathcal{M})$

L1 [redacted] q fibred with fibre $\mathcal{E}_M$ over M, description of base change

L2 [redacted] $\overline{Q}(\mathcal{M})$ is contractible.
(because $\overline{Q}(\mathcal{M})$ is equivalent to the subdivision of the cat of admiss. monos. in $\mathcal{M}$, which has initial obj 0)

Suppose exact sequences split in $\mathcal{M}$, $\mathcal{S} = \mathcal{E}_0 = \mathrm{Iso}(\mathcal{M})$.

Def. $\mathcal{S}$ action on [redacted] $\overline{Q}(\mathcal{M})$.

L3 [redacted] $\mathcal{S}$-action on $\overline{Q}(\mathcal{M})$ is cartesian over $Q(\mathcal{M})$.

L4. $\forall u: M \to M'$ in $Q(\mathcal{M})$, u^*: [redacted] $\mathcal{E}_M \to$ [redacted] $\mathcal{E}_{M'}$ induces a heq $\mathcal{S}^{-1}\mathcal{E}_{M'} \to \mathcal{S}^{-1}\mathcal{E}_M$.

[redacted] (Proof. [redacted] Reduce to case of $i_{M!}: 0 \rightarrowtail M$, $j_M^!: 0 \twoheadleftarrow M$, whence we have the functors [redacted]

$$\mathcal{S}^{-1}\mathcal{E}_0 \longrightarrow \mathcal{S}^{-1}\mathcal{E}_M \longrightarrow \mathcal{S}^{-1}\mathcal{E}_0$$

which in preceding section have shown to be homology isos. But we know these spaces are H-spaces $\ni$ π_0 is a group.)

Proof of comparison thm: By L3 + earlier stuff $\mathcal{S}^{-1}\overline{Q}(\mathcal{M})$ fibres over [redacted] $Q(\mathcal{M})$ with fibres $\mathcal{S}^{-1}\mathcal{E}_M$. L4 $\xRightarrow{\text{ThB}}$ $\mathcal{S}^{-1}\mathcal{E}_M =$ h-fibre of $\mathcal{S}^{-1}\overline{Q}(\mathcal{M})$ [redacted] $\to Q(\mathcal{M})$ over M. But $\mathcal{S}^{-1}\overline{Q}(\mathcal{M})$ cofibres over $\mathcal{S}^{-1}(pt)$ with $\overline{Q}(\mathcal{M}) \sim$ ~~pt.~~ for fibres, so $\mathcal{S}^{-1}\overline{Q}(\mathcal{M})$ is contractible. $\therefore$ $\mathcal{S}^{-1}\mathcal{S} \sim \Omega Q(\mathcal{M})$ basepoint at 0.

Comparison thm (outline)

Recall what is an exact cat. $\mathcal{M}$

The cat $Q(\mathcal{M})$ $\quad i_!, j^!$ recall

Definition of $\overline{Q}(\mathcal{M})$ and $q: \overline{Q}(\mathcal{M}) \to Q(\mathcal{M})$.

Define $\mathcal{E}_M$ to be the groupoid whose objects are the admiss epis $(u: E \twoheadrightarrow M)$ in $\mathcal{M}$ with target M, and in which a map from $(u: E \twoheadrightarrow M)$ to $(u': E' \to M)$ is an isomorphism $E \xrightarrow{\sim} E'$ compatible with u and u'.

Given a map $\phi: M' \to M$ in $Q(\mathcal{M})$ we propose to define a functor

$$\phi^*: \mathcal{E}_M \longrightarrow \mathcal{E}_{M'}.$$

If ϕ is injective, $\phi = i_!$ where $i: M' \rightarrowtail M$, then define using pull-back:

$$(i_!)^*(E \to M) = (M' \times_M E \xrightarrow{pr_1} M)$$

and if ϕ is surjective, $\phi = j^!$ where $j: M \twoheadrightarrow M'$, then we put

$$(j^!)^*(E \xrightarrow{u} M) = (ju: E \longrightarrow M').$$

In general if $\phi = j^! i_!$:

$$M' \overset{j}{\twoheadleftarrow} M_0 \overset{i}{\rightarrowtail} M$$

we put $\phi^* = (j^!)^*(i^!)^*$ so

$$\phi^*(E \longrightarrow M) = (M_0 \times_M E \xrightarrow{pr_1} M_0 \xrightarrow{j} M').$$

$\overline{Q}(\mathcal{M})$ will be a fibred cat over $Q(\mathcal{M})$ with fibre $\mathcal{E}_M$ over M and base change functors ϕ^*. Thus an object of $\overline{Q}(\mathcal{M})$ is an admissible epi $(u: E \twoheadrightarrow M)$ in $\mathcal{M}$. A morphism from $(u':E'\to M')$ to $(u:E\to M)$ will consist of a morphism ϕ from M' to M in $Q(\mathcal{M})$ and an isom. of u' with $\phi^*(u)$, in other words a comm. diagram

$$(*)\qquad \begin{tikzcd} E' \arrow[r, "\simeq"] \arrow[d, "u'"'] & M_0\times_M E \arrow[r] \arrow[d] & E \arrow[d, "u"] \\ M' & M_0 \arrow[l, two heads] \arrow[r, tail] & M \end{tikzcd}$$

where the square at the right is cartesian. To define composition, we first Notice that if $i:E'\to E$ is the map given by the top row of the above diagram, then ~~i is an admissible monom. such that a) $i(\mathrm{Ker}\,u')\supset \mathrm{Ker}(u)$, b) the inclusion $\mathrm{Ker}(u)\to i(\mathrm{Ker}\,u')$ is an admissible monomorphism. Let us call [illegible] an admissible monomorphism~~

a) i is an admiss mono.

b) $i(\mathrm{Ker}\,u')\supset \mathrm{Ker}\,u$

c) the inclusion $\mathrm{Ker}\,u\to i(\mathrm{Ker}\,u')$ is an admiss mono.

~~Let us say that a monomorphism $i:E'\to E$ induces a map $M'\to M$ in $Q(\mathcal{M})$ if a), b), c) hold~~

Furthermore the map $\phi: M'\to M$ given by the bottom row is isomorphic to the one represented by:

$$(**)\qquad M'=E'/\mathrm{Ker}\,u' \cong iE'/i(\mathrm{Ker}\,u') \twoheadleftarrow iE'/\mathrm{Ker}\,u \rightarrowtail E/\mathrm{Ker}\,u = M.$$

~~In general we~~ We will say that $i:E'\to E$ induces a map from M' to M in $Q(\mathcal{M})$ if conditions a), b), c) hold; and we will call $(**)$ the map ~~[illegible]~~ induced by i. Thus it is clear that a map from $(u':E'\to M')$ to $(u^{\#}:E^{\#}\to M^{\#})$ in $Q(\mathcal{M})$ may be identified with a map $i:E'\to E$ which induces a map from M' to M in $Q(\mathcal{M})$.

In terms of this description of maps in $\overline{Q}(\mathcal{M})$, ~~it is~~ ~~to~~ we can define composition of the maps "i". It is clear then that $\overline{Q}(\mathcal{M})$ is a well-defined category.

<u>Lemma:</u> Let $\mathcal{I}$ be the category having the same objects as $\mathcal{M}$ ~~[illegible]~~ in which a morphism is an admissible mono. $M' \hookrightarrow M$ in $\mathcal{M}$. Then ~~the~~ ~~[illegible]~~ we have an equivalence of categories

$$\overline{Q}(\mathcal{M}) \longrightarrow \mathrm{Sub}(\mathcal{I})$$

$$(u: E \to M) \longmapsto (\mathrm{Ker}(u) \rightarrowtail E).$$

Consequently the category $\overline{Q}(\mathcal{M})$ is contractible.

<u>Proof:</u> The first statement is ~~[illegible]~~ immediate. Since $\mathcal{I}$ has an initial obj 0, it is contractible. Hence so is $\mathrm{Sub}(\mathcal{I})$ (ref.), whence ~~[illegible]~~ also $\overline{Q}(\mathcal{M})$.

Definition of $\bar{Q}(\mathcal{M})$ and $q: \bar{Q}(\mathcal{M}) \to Q(\mathcal{M})$.

$\mathcal{M}$ exact category.

Recall $Q(\mathcal{M})$ is the cat. with same objects as $\mathcal{M}$ ~~by~~ in which a morphism from M' to M is defined to be ~~an~~ isom of M' with an admissible subquotient of M. Equivalently a morphism from M' to M is an equivalence class of diagrams

$$M' \twoheadleftarrow M_0 \rightarrowtail M$$

where ~~the equivalence relation is~~ two such diag are considered equivalent if they are isomorphic, the isomorphism ~~on~~ M' and M being the identity.

Given M in $\mathcal{M}$ let $\mathcal{E}_M$ denote the groupoid whose objects are the admissible epis $u: E \to M$ in $\mathcal{M}$ with target M, and in which a map from $u: E \to M$ to $u': E' \to M$ is an isom $w: E \xrightarrow{\sim} E'$ compatible with u and u'. Given a map in $Q(\mathcal{M})$

$$\phi: M' \overset{j}{\twoheadleftarrow} M_0 \overset{i}{\rightarrowtail} M$$

and $(u: E \to M)$ in $\mathcal{E}_M$ put

$$\phi^*(u: E \to M) = \left(M_0 \times_M E \xrightarrow{j\,pr_1} M'\right).$$

In this way we obtain a functor

$$\phi^*: \mathcal{E}_M \longrightarrow \mathcal{E}_{M'}$$

assoc to any map in $Q(\mathcal{M})$.

The category $\bar{Q}(\mathcal{M})$ will be a fibred cat over $Q(\mathcal{M})$ having $\mathcal{E}_M$ as fibre over M and base change functors as above. Specifically, an obj of $\bar{Q}(\mathcal{M})$ is an admiss epi $u: E \to M$. Given two objects $(u': E' \to M')$ and $(u: E \to M)$, an a map $i: E' \xrightarrow{\text{in } \mathcal{M}} E$ let us say that i induces a map $M' \to M$ in $Q(\mathcal{M})$ if 1) i is an admissible mono., 2) $i(\mathrm{Ker}\, u') \supset \mathrm{Ker}\, u$, 3) the inclusion $\mathrm{Ker}\, u \subset i(\mathrm{Ker}\, u')$ is an admissible monom.

In this case we obtain from i a map in $Q(\mathcal{M})$

$$M' \xleftarrow{\sim} E'/\mathrm{Ker}\, u' \simeq i(E')/i\,\mathrm{Ker}(u') \twoheadleftarrow E/i\,\mathrm{Ker}(u') \rightarrowtail E/\mathrm{Ker}\, u = M.$$

Define a map in $\bar{Q}(\mathcal{M})$ from $(u': E' \to M)$ to $(u: E \to M)$ to be a map $i: M \to M'$ in $\mathcal{M}$ which induces a map $M' \to M$ in $Q(\mathcal{M})$. It is clear that we then obtain a well-defined category $\bar{Q}(\mathcal{M})$ and a functor

$$q: \bar{Q}(\mathcal{M}) \longrightarrow Q(\mathcal{M}), \quad q(u: E \to M) = M.$$

<u>Lemma 1</u>: q is fibred, $q^{-1}(M) = \mathcal{E}_M$, $\phi^* = \phi^*$ as above.

<u>Proof</u>: Given $(u': E' \to M') \xrightarrow{i} (u: E \to M)$ a map in $\mathcal{E}_M$ mapped by q into

$$\phi: M' \twoheadleftarrow M_0 \rightarrowtail M$$

clearly we have

Possibility:

$$\begin{tikzcd}
 & M_0 \times_M E \arrow[r] \arrow[d] \arrow[l, "\sim"'] & E \arrow[d, "u"] \\
E' \arrow[ddr] & M_0 \arrow[r] \arrow[d] & M \\
 & M' &
\end{tikzcd}$$

$$\begin{tikzcd}
E' \arrow[d] \arrow[r, phantom, "\simeq"] & M_0 \times_M E \arrow[r] \arrow[d] & E \arrow[d] \\
M' & M_0 \arrow[l] \arrow[r, tail] & M
\end{tikzcd}$$

Localization

A, S m.s. contained in center A, $S^{-1}A$ = the loc. of A w.r.t. S, $\gamma : A \to S^{-1}A$ = the can. hom., $\gamma^* : K_q A \to K_q(S^{-1}A)$ the hom. induced by γ.

$\mathcal{H}_S(A)$ = full subcat of $P_\infty(A)$ cons. of $M \ni S^{-1}M = 0$.

Define

$$c : K_q(\mathcal{H}_S(A)) \longrightarrow K_q A$$

to be the homo induced by the inc. $\mathcal{H}_S(A) \subset P_\infty(A)$ followed by the isomorphism $K_q A \simeq K_q(P_\infty(A))$ obtained from the resolution thm.

<u>Theorem</u>: Assume the elements of S are non-zero divisors in A. Then there is a long exact sequence

$$\longrightarrow K_{q+1}(S^{-1}A) \xrightarrow{d} K_q(\mathcal{H}_S(A)) \xrightarrow{c} K_q A \xrightarrow{\gamma^*} K_q(S^{-1}A) \longrightarrow$$

where d is a canonical homomorphism, functorial in the pair (A,S), to be defined below.

The proof will occupy the rest of section.

$\mathcal{H}^n_S(A)$ = full subcat of $P_n(A)$ cons. of $M \ni S^{-1}M = 0$.

<u>Lemma 1</u>: $K_q \mathcal{H}^1_S(A) \xrightarrow{\sim} K_q \mathcal{H}^2_S(A) \xrightarrow{\sim} \cdots \xrightarrow{\sim} K_q \mathcal{H}_S(A)$.

Pf: By res. thm. the point being that any M in $\mathcal{H}^n_S(A)$ is a quotient of $(A/sA)^n$ for some $s \in S$, and because s is a non-zero-divisor $A/sA \in \mathcal{H}^1_S(A)$.

In following we put $\mathcal{H} = \mathcal{H}^1_S(A)$ and identify $K_*(\mathcal{H})$ with $K_*(\mathcal{H}_S(A))$, and c with the homo.

$$K_q(\mathcal{H}) \xrightarrow{c} K_q(A)$$

induced by inclusion of $\mathcal{H}$ in $P_1(A)$.

$\mathcal{V}$ = full subcat of $\mathcal{P}(S^{-1}A)$ cons. of $V \simeq S^{-1}P$, $P \in \mathcal{P}(A)$

$\mathcal{P} = \mathcal{P}(A)$.

Lemma 2: Inclusion $\mathcal{V} \subset \mathcal{P}(S^{-1}A)$ induces isos

$$K_q\mathcal{V} \xrightarrow{\sim} \begin{cases} \operatorname{Im}\{K_0A \to K_0(S^{-1}A)\} & q = 0. \\ K_q(S^{-1}A) & q > 0. \end{cases}$$

~~Proof: Let $\mathcal{S} = \mathcal{I}so(\mathcal{P})$ act on $\mathcal{I}so(\mathcal{V})$ and $\mathcal{I}so(\mathcal{P}(S^{-1}A))$ in obvious way. The action is cofinal, so from comparison thm. get~~

~~$\mathcal{S}^{-1}\mathcal{I}so(\mathcal{V}) \to \Omega Q(\mathcal{V})$~~

~~and similarly for $\mathcal{P}(S^{-1}A)$.~~

~~$K_0\mathcal{V} = (\pi_0\mathcal{S})^{-1}\pi_0\mathcal{I}so(\mathcal{V})$~~

Proof: To prove the assertion for $q=0$, suffices to show $K_0\mathcal{V}$ injects into $K_0(S^{-1}A)$, since it obviously has the same image as K_0A. Let $\mathcal{S} = \mathcal{I}so(\mathcal{P})$ act on $\mathcal{I}so(\mathcal{V})$ and $\mathcal{I}so(\mathcal{P}^{-1}A)$ in the obvious way: $P \# V = S^{-1}P \oplus V$. The action is cofinal, so

$$K_0\mathcal{V} = (\pi_0\mathcal{S})^{-1}\cdot\pi_0\mathcal{I}so(\mathcal{V})$$

and similarly for $\mathcal{P}(S^{-1}A)$. Since $\pi_0\mathcal{I}so(\mathcal{V})$ injects into $\pi_0\mathcal{I}so(\mathcal{P}(S^{-1}A))$, it follows from exactness of localization that $K_0\mathcal{V}$ injects into $K_0(S^{-1}A)$.

From the comparison thm. we have a heq

$$\mathcal{S}^{-1}\mathcal{I}so(\mathcal{V}) \sim \Omega Q(\mathcal{V})$$

and similarly for $\mathcal{P}(S^{-1}A)$. Thus to prove the assertion of the lemma for $q>0$, we have to show that the induced map

(*) $$\mathcal{S}^{-1}\mathcal{I}so(\mathcal{V})_0 \longrightarrow \mathcal{S}^{-1}\mathcal{I}so(\mathcal{P}(S^{-1}A))_0$$

is a homotopy equivalence, where the subscript zero denotes the connected component of the basept. Since these spaces are comm. H-spaces, hence simple, it suffices by Whitehead to show this map is a homol. isomorp. But

$$H_*(S^{-1}\mathcal{I}so(\mathcal{P})_0) = \varinjlim H_*(\mathrm{Aut}(S^{-1}P))$$

where ~~[illegible]~~ the limit is taken over $\mathrm{Trans}(\pi_0 S)$. By cofinality, this is

$$\varinjlim_n H_*(\mathrm{Aut}(S^{-1}A^n)) = H_*(GL(S^{-1}A))$$

which is also the homology of the right side of $(*)$, so done.

Denote by

$$\lambda : Q(\mathcal{P}) \longrightarrow Q(\mathcal{V})$$

the functor induced by the loc. functor $P \longmapsto S^{-1}P$ from $\mathcal{P}$ to $\mathcal{V}$. We are going to show that the homotopy-theoretic fibre of λ may be identified with $Q(\mathcal{H})$. By the above lemmas, the homotopy exact sequence of λ will then furnish the exact sequence of the Theorem.

To effect the required identification we will construct a diagram of categories as follows:

$$\begin{tikzcd}
Q(\mathcal{H}) & \mathcal{D} \arrow[l, "f"'] \arrow[r, "h"] \arrow[dr, "g"'] & \mathcal{C} \arrow[r, "\cancel{\lambda'}\mu"] \arrow[d, "g'"] & \bar{Q}(\mathcal{V}) \arrow[d, "\varepsilon"] \\
& & Q(\mathcal{P}) \arrow[r, "\cancel{\lambda}\mu"'] & Q(\mathcal{V})
\end{tikzcd} \qquad (\quad)$$

We begin with the construction of $\mathcal{D}$.

(To be rewritten: I want to state that we will show f, h are heqs, ~~and the map gf^{-1} is negative of c.~~ And that the square is h-cartesian, whence $\mathcal{C}$ is the h-theoretic fibre of λ. Will get then from the homotopy sequence of λ ~~long~~ exact sequence

$$\longrightarrow K_*\mathcal{H} \xrightarrow{g_*f_*^{-1}} K_*\mathcal{P} \xrightarrow{\lambda_*} K_*\mathcal{V} .$$

Then will identify $g_*f_*^{-1}$ with negative of c, proving thm.)

Given $M \in Q(\mathcal{H})$, $P \in Q(\mathcal{P})$, let $\mathcal{D}_{M,P}$ be the groupoid consisting of A-module surjections $L \to M \times P$ with $L \in \mathcal{P}$ in which the morphisms are isomorphisms over $M \times P$. Given morphisms

$$\phi : M' \twoheadleftarrow M_0 \rightarrowtail M \quad , \psi : P' \twoheadleftarrow P_0 \rightarrowtail P$$

in $Q(\mathcal{H})$ and $Q(\mathcal{P})$ resp., and an object $(L \to M \times P)$ in $\mathcal{D}_{M,P}$, then

$$(\phi,\psi)^*(L \to M \times P) = ((M_0 \times P_0) \times_{(M \times P)} L \longrightarrow M_0 \times P_0 \longrightarrow M' \times P')$$

is an object of $\mathcal{D}_{M',P'}$. In this way we obtain a functor

$$(\phi,\psi)^* : \mathcal{D}_{M,P} \longrightarrow \mathcal{D}_{M',P'}$$

associated to the each morphism in $Q(\mathcal{H}) \times Q(\mathcal{M})$. We let $\mathcal{D}$ be the fibred category over $Q(\mathcal{H}) \times Q(\mathcal{M})$ with fibre $\mathcal{D}_{M,P}$ over (M,P) and with these base change functors. Specifically, an object of $\mathcal{D}$ is a surjection $L \to M \times P$ with $L, P \in \mathcal{P}$, $M \in \mathcal{H}$, and a morp. from $(L' \to M' \times P')$ to $(L'' \to M'' \times P'')$ is an injection $L' \to L''$ which induces a morphism $M' \times P' \to M'' \times P''$ in $Q(\mathcal{P}_1)$ which is the product of morphisms $M' \to M''$ in $Q(\mathcal{H})$ and $P' \to P$ in $Q(\mathcal{P})$. (Compare proof of res. thm.)

Denote by $f : \mathcal{D} \to Q(\mathcal{H})$ and $g : \mathcal{D} \to Q(\mathcal{P})$ the functors sending $(L \to M \times P)$ to M and P resp.

<u>Lemma 3</u>: f is a heq.

<u>Proof</u>: f is fibred since it is the composition $pr_1 (f,g)$ of two fibred functors, hence it suffices to show that $f^{-1}(M)$ is contractible for each M in $Q(\mathcal{H})$. Let R_M be ~~following~~ cat. whose objects are epis. $L \twoheadrightarrow M$ with $L \in P$, and in which a map $(L' \twoheadrightarrow M) \to (L \twoheadrightarrow M)$ is an adm. inj. $L' \rightarrowtail L$ in P compat. with maps to M. Have equivalence

$$f^{-1}(M) \longrightarrow Sub(R_M)$$

$$(L \twoheadrightarrow M \times P) \longmapsto \left(\begin{array}{ccc} Ker(L \twoheadrightarrow P) & \longrightarrow & L \\ & \searrow \quad \swarrow & \\ & M & \end{array} \right)$$

so enough to show R_M is contractible. This you do by cone const. using fact $\exists\, (L_0 \twoheadrightarrow M) \in R_M$.

$$(L \twoheadrightarrow M) \longrightarrow (L \oplus L_0 \twoheadrightarrow M) \longleftarrow (L_0 \twoheadrightarrow M).$$

(This will appear in proof of res. thm.)

~~Lemma: The map $Q(\mathcal{H}) \xrightarrow{f^{-1}} Q(P)$ is the category in the negative of~~

<u>Lemma 4</u>: The homom.

$$K_g \mathcal{H} = \pi_{g+1} Q(\mathcal{H}) \underset{\sim}{\xleftarrow{(f_*)^{-1}}} \pi_{g+1} \mathcal{D} \xrightarrow{g_*} \pi_{g+1} Q(P) = K_g P$$

is the negative of the homom. c defined above.

<u>Proof</u>: (This step will really appear in the proof of the resolution theorems). We have a comm. diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[rrr, "(L \to M \times P) \mapsto L"] \arrow[d, "{(f,g)}"'] & & & \bar{Q}(\mathcal{P}_1) \arrow[d] \\
Q(\mathcal{N}) \times Q(\mathcal{P}) \arrow[r, "i \times j"] & Q(\mathcal{P}_1) \times Q(\mathcal{P}_1) \arrow[rr, "\oplus"] & & Q(\mathcal{P}_1)
\end{tikzcd}$$

where i and j are the inclusions. Since $\bar{Q}(\mathcal{P}_1)$ is contractible, we have

$$0 = i_* f_* + j_* g_* \; : \; \pi_{g+1}\mathcal{D} \longrightarrow \pi_{g+1} Q(\mathcal{P}_1)$$

and since $j_* c = i_*$ by definition of c

$$\begin{tikzcd}
Q(\mathcal{P}) \arrow[rr, "j_*", "\sim"'] & & Q(\mathcal{P}_1) \\
& Q(\mathcal{N}) \arrow[ul, "c"] \arrow[ur, "i_*"'] &
\end{tikzcd}$$

$$\therefore \quad 0 = (j_*)^{-1} i_* + g_* (f_*)^{-1}$$ as was to be shown.

~~Given an object $X = (u,v) : L \to M \times P$ of $\mathcal{D}$, let $h(X)$ denote the objects of $\mathcal{C}$ given by~~

Next we define the category $\mathcal{C}$ to be the pull-back of $\bar{Q}(\mathcal{V})$ by the functor $\lambda : Q(\mathcal{P}) \to Q(\mathcal{V})$, ~~Recall that $\bar{Q}(\mathcal{V})$ is a fibred categ. over $Q(\mathcal{V})$ such th~~ so that the square in the diag ( ) is cartesian. Recall that $\bar{Q}(\mathcal{V})$ is a fibred cat over $Q(\mathcal{V})$ having as fibre over W the groupoid denoted $\mathcal{E}_W$, which consists of all admiss. epis $V \twoheadrightarrow W$ in $\mathcal{V}$ and W-isomorphisms between them. Thus $\mathcal{C}$ is a fibred cat over $Q(\mathcal{V})$ having as fibre over P the groupoid $\mathcal{E}_{S^{-1}P}$.

If X is an object of $\mathcal{D}$ given by a surjection $L \to M \times P$ with components (u,v), let $h(X)$ be the object of $\mathcal{C}$ given by $S^{-1}v : S^{-1}L \to S^{-1}P$. It is easily seen that we obtain in this way a functor

$$h : \mathcal{D} \longrightarrow \mathcal{C}$$

over $Q(\mathcal{P})$, i.e. such that $g'h = g$ (notation as in (*)).

Lemma 5: h is a heq.

Pf: Both $\mathcal{D}$ and $\mathcal{C}$ being fibred over $Q(\mathcal{P})$, it is enough to show the induced map $h_P : g^{-1}(P) \to \mathcal{E}_{S^{-1}P}$ on the fibres over a given object P of $Q(\mathcal{P})$ is a heq. Let $\mathcal{L}_P$ denote the cat whose objects are surjections $L \to P$ in $\mathcal{P}$ in which morphisms are comm. triangles

$$\begin{tikzcd} L' \arrow[r] \arrow[dr] & L \arrow[d] \\ & P \end{tikzcd}$$

such that $L' \to L$ is injective with cokernel in $\mathcal{H}$. Then have equivalence

$$g^{-1}(\ _P) \longrightarrow \mathrm{Sub}(\mathcal{L}_P) \quad , \quad (L \to M \times P) \longmapsto \begin{tikzcd} \mathrm{Ker}(L\to M) \arrow[r] \arrow[dr] & L \arrow[d] \\ & P \end{tikzcd} .$$

The functor h_P is the composition of this equivalence with the functors

$$\mathrm{Sub}(\mathcal{L}_P) \xrightarrow{\text{target}} \mathcal{L}_P \xrightarrow{\ w\ } \mathcal{E}_{S^{-1}P}$$

where $w(L \to P) = (S^{-1}L \to S^{-1}P)$. Since target is a heq, we have only to show w is a heq.

By Th. A, enough to show w/X is a contractible cat for any object X of $\mathcal{E}_{S^{-1}P}$. Let X be given by the admiss. epim $v : V \to S^{-1}P$. An object of w/X consists of an epi. $u : L \to P$ in $\mathcal{P}$ together with an isom. $\theta : S^{-1}L \xrightarrow{\sim} V$

$$\begin{tikzcd} S^{-1}L \arrow[r, "\sim"] \arrow[dr, "S^{-1}(u)"'] & V \arrow[d, "v"] \\ & S^{-1}P \end{tikzcd}$$

s.t. $v\theta = S^{-1}(u)$; and a morp. from $(u : L \to P, \theta)$ to $(u' : L' \to P, \theta')$ is an injection $i : L \to L'$ such that $u'i = u$, $\theta' S^{-1}(i) = \theta$.

$$\begin{tikzcd} L \arrow[r] \arrow[dr] & L' \arrow[d] \\ & P \end{tikzcd} \qquad \begin{tikzcd} S^{-1}L \arrow[r] \arrow[rr, bend left, "\sim"] \arrow[d] & S^{-1}L' \arrow[r, "\sim"] \arrow[dl] & V \arrow[dll, "v"] \\ P & & \end{tikzcd}$$

Let us mean by a *lattice* in V an A-submodule L such that $L \in \mathcal{P}$ and $S^{-1}L = V$. Denote by $\mathrm{Lat}(X)$ the set of those lattices L in V such that $v(L) = P$, where P is identified with a lattice in $S^{-1}P$, and order $\mathrm{Lat}(X)$ by inclusion. Then it is easily seen that by associating to an object $(u: L \to P,\ \theta: S^{-1}L \xrightarrow{\sim} V)$ of w/X the lattice $\theta(L)$ in V, we obtain an equivalence of cats

$$w/X \longrightarrow \mathrm{Lat}(X).$$

We are therefore reduced to proving:

Lemma 6: For any $X = (v: V \twoheadrightarrow S^{-1}P)$ in $\mathcal{E}_{S^{-1}P}$, $\mathrm{Lat}(X)$ is a non-empty directed set, and hence it is a contractible category.

Pf: Because v is an admissible epi in $\mathcal{V}$, we have $V \simeq \mathrm{Ker}(v) \oplus S^{-1}P$, where $\mathrm{Ker}(v) \in \mathcal{V}$, and hence $\mathrm{Ker}(v) \simeq S^{-1}L_0$ with $L_0 \in \mathcal{P}$. Thus we can assume $V = S^{-1}(L_0 \oplus P)$ with $v = S^{-1}(pr_2)$. Then $L_0 \oplus P \in \mathrm{Lat}(X)$ showing it is non-empty. Further, $v^{-1}(P)$ is the union of the lattices $s^{-1}L_0 \oplus P$ as s runs over S. Hence if L, L' are two elements of $\mathrm{Lat}(X)$, then because they are fin. gen, they are contained in $s^{-1}L_0 \oplus P$ for some s, showing $\mathrm{Lat}(X)$ is directed. This proves Lemmas 6 and 5.

<u>Lemma 7</u>: The square
$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "g'"] & \bar{Q}(\mathcal{V}) \arrow[d, "g"] \\ Q(\mathcal{P}) \arrow[r, "\mu"'] & Q(\mathcal{V}) \end{tikzcd}$$
in ( ) is homotopy-cartesian.

<u>Pf</u>: Let $\mathcal{S} = \mathrm{Iso}(\mathcal{P})$ act on $\bar{Q}(\mathcal{V})$ by $T\#(W \overset{u}{\twoheadrightarrow} V) = (S^{-1}T \oplus W \xrightarrow{u\,pr_2} V)$; ~~and give $\mathcal{C}$ the induced action. The action is cartesian~~ we know from the comp. thm. that $\mathcal{S}^{-1}\bar{Q}(\mathcal{V})$ is fibred over $Q(\mathcal{V})$ having fibre $\mathcal{S}^{-1}\mathcal{E}_V$ over V and that the base change functors are heq's so that $\mathcal{S}^{-1}\mathcal{E}_V$ may be identified with the h-fibre of $\mathcal{S}^{-1}\bar{Q}(\mathcal{V}) \longrightarrow Q(\mathcal{V})$. The $\mathcal{S}$-action on $\bar{Q}(\mathcal{V})$ induces one on $\mathcal{C}$
$$T\#(W \twoheadrightarrow S^{-1}P) = (S^{-1}T \oplus W \longrightarrow S^{-1}P)$$
which is cart over $Q(\mathcal{P})$, so $\mathcal{S}^{-1}\bar{Q}(\mathcal{V})$ is fibred over $Q(\mathcal{P})$ with fibre $\mathcal{S}^{-1}\mathcal{E}_{S^{-1}P}$ over P. Thus
$$\begin{tikzcd} \mathcal{S}^{-1}\mathcal{C} \arrow[r] \arrow[d] & \mathcal{S}^{-1}\bar{Q}(\mathcal{V}) \arrow[d] \\ Q(\mathcal{P}) \arrow[r, "\mu"] & Q(\mathcal{V}) \end{tikzcd}$$
is cartesian, as well as h-cartesian since the fibres coincide with h-fibres. Since $\bar{Q}(\mathcal{V})$ is contractible, so is $\mathcal{S}^{-1}\bar{Q}(\mathcal{V})$. Remains to show $\mathcal{C} \to \mathcal{S}^{-1}\mathcal{C}$ is a heq, and for this it suffices to show $T\#? : \mathcal{C} \to \mathcal{C}$ is a heq for any $T \in \mathcal{S}$. (reference)

But we can make $\mathcal{S}$ act on $\mathcal{D}$ by
$$T\#(L \overset{u}{\twoheadrightarrow} M \times P) = (T \oplus L \xrightarrow{u\,pr_2} M \times P)$$
and this action is compatible with the functor h and also f, where $\mathcal{S}$ acts trivially on $Q(\mathcal{H})$. Thus

f heq $\Rightarrow$ $T\#? : \mathcal{D} \to \mathcal{D}$ homotopic to identity

h heq $\Rightarrow$ $T\#? : \mathcal{C} \to \mathcal{C}$ ______.

so ~~we~~ have finished the proof of thm.

<u>Description of $d: K_q(S^{-1}A) \to K_{q-1}(\mathcal{H})$</u>, $q > 0$.

$$\begin{tikzcd}
 & S^{-1}\mathcal{D}_0 \arrow[r, "h_0"] \arrow[dl, "f_0"'] \arrow[d, "\wr"] & S^{-1}\mathcal{E}_0 \arrow[d, "\wr"] \\
Q(\mathcal{H}) & S^{-1}\mathcal{D} \arrow[l, "f"'] \arrow[r] & S^{-1}\mathcal{C}
\end{tikzcd}$$

~~Thm~~ $S^{-1}\mathcal{E}_0 \sim \Omega Q(\mathcal{V})$. Above diagram shows that the boundary map d is the composition $(f_0)_*(h_0)_*^{-1} : \pi_q(S^{-1}\mathcal{E}_0)$ to $\pi_q Q(\mathcal{H})$.

Thus

$\mathcal{E}_0 = \mathrm{Iso}(\mathcal{V})$

~~Do cat consisting of~~

replace $\mathcal{D}_0$ by Subdivision of the cat $\mathcal{L}_0$ consisting of $L \in P$ with maps $L \hookrightarrow L' \ni S^{-1}L \xrightarrow{\sim} S^{-1}L'$. Then we have a heq

$$\mathrm{Sub}(\mathcal{L}_0) \xrightarrow{\text{targ.}} \mathcal{L}_0 \xrightarrow{S^{-1}} \mathcal{E}_0$$

and a functor $\mathrm{Sub}(\mathcal{L}_0) \longrightarrow Q(\mathcal{H})$, $(L \hookrightarrow L') \mapsto L'/L$.

Other possible descriptions: Preceding proof shows $d: K_q(S^{-1}A) \longrightarrow K_{q-1}(\mathcal{H})$ for $q>0$ can be ident. ~~with~~ ~~map on homotopy induced by~~ as follows

First $K_q(S^{-1}A) = \pi_{q+1} Q(\mathcal{V}) = \pi_q(\mathcal{S}^{-1}\mathcal{E}_0)$ and we have a comm. diag.

$$\begin{tikzcd}
 & \pi_q(\mathcal{S}^{-1}g^{-1}(0)) \arrow[r, "(\mathcal{S}^{-1}h_0)_*"] \arrow[dl, "(\mathcal{S}^{-1}f_0)_*"'] \arrow[dd] & \pi_q(\mathcal{S}^{-1}\mathcal{E}_0) \arrow[dd] \\
\pi_q Q(\mathcal{H}) & & \\
 & \pi_q(\mathcal{S}^{-1}\mathcal{D}) \arrow[ul, "\sim"] \arrow[r, "(\mathcal{S}^{-1}h)", "\sim"'] & \pi_q(\mathcal{S}^{-1}\mathcal{E})
\end{tikzcd}$$

~~with~~ the vertical isos. being induced by inclusion of fibre over $0 \in Q(\mathcal{P})$. Thus one sees that d is the map of hom. groups induced by

$$Q(\mathcal{H}) \longleftarrow \mathcal{S}^{-1}g^{-1}(0) \xrightarrow[\text{heq}]{} \mathcal{S}^{-1}\mathcal{E}_0 .$$

Concretely this means that

?

Variants:

Relation with the localization thm. for abelian cats.

Here I would like to justify the description of the boundary homo. given in (I).

~~$A \to SA$~~

A noeth. domain, fraction field F

$\mathcal{P}$ = torsion-free f.t. A-modules

$\mathcal{H}$ = torsion A-modules.

$\mathcal{V} = \mathcal{P}(F)$.

Then by resolution thm. $Q(\mathcal{P}) \longrightarrow Q(\mathrm{Modf}(A))$ heq. and by same proof I should be able to identify $Q(\mathcal{H})$ with fibre of $Q(\mathcal{P}) \longrightarrow Q(\mathcal{V})$.

Boundary map describable in terms of lattices.

$\boxed{\text{Generalization to schemes:}}$

Theorem: X quasi-compact scheme, Y closed subscheme ~~which is a Cartier divisor~~, $U = X - Y$. Let $\mathcal{H}_Y(X)$ denote the full subcat of quasi-coherent sheaves on X which have support contained in Y and which admit a finite resolution by vector bundles on X. Suppose Y is a Cartier divisor in X (i.e. ~~locally~~ the ideal defining Y is invertible ~~generated by a non-zero divisor~~), and that U is affine. Then there is a long exact sequence

$$\to K_{q+1}U \longrightarrow K_q(\mathcal{H}_Y(X)) \longrightarrow K_q X \longrightarrow K_q U \to \cdots$$

Check that same proof works:

$\mathcal{H} = \mathcal{H}_Y^1(X)$. $\boxed{K_* \mathcal{H} \xrightarrow{\sim} K_* \mathcal{H}_Y(X)}$?

$F \in \mathcal{H}_Y(X) \Rightarrow \exists \quad E \twoheadrightarrow F \qquad E \in \mathcal{P}(X)$

Because F finite type with support in Y we know locally $I^n F = 0$, I = ideal defining Y. Since X quasi-compact $\Rightarrow$ $I^n F = 0$ some n. Then have

$$0 \to F' \to E/I^n E \to F \to 0$$

and since $I^n E = I^{\otimes n} \otimes E$ is a vector bundle $\Rightarrow$ $E/I^n E \in \mathcal{P}_1(X)$, hence $E/I^n E \in \mathcal{H}$. Then $F' \in \mathcal{H}_Y^{n-1}(X)$ and so resolution works.

Lat(X) $\quad X = (V : V \twoheadrightarrow j^* P) \qquad P$ v.b. on X.

ord. set of $L \subset j_* V \ni$ $\quad V = j^* L_0 \oplus j^* P \qquad L_0$ v.b. on X.

But have $\quad L_0 \subset I^{-1} L_0 \subset I^{-2} L_0 \subset \ldots\ldots$

$$\bigcup_n I^{-n} L_0 = j_* V$$

Now if $L \subset j_* V$

What is Lat(X). a lattice in V consists of a sub O_X-module $L \subset j_* V \ni$ i) $L \in \mathcal{P}(X)$
ii) $j^* L = j^* j_* V = V$.

L is in Lat(X) means that the ~~comp.~~ map (1) is onto.

$$\begin{tikzcd} L \arrow[r, phantom, "\subset"] \arrow[drr, bend right, "(1)"'] & j_* V \arrow[r, "j_*(v)"] & j_* j^* P \\ & & P \arrow[u] \end{tikzcd}$$

or better that

$$j_*(v)(L) = \mathrm{Im}(P \subset j_* j^* P).$$

Since $j_*(v)^{-1}(P) = \bigcup_n I^{-n} L_0 \oplus P$ rest is clear!!!!

obsolete

Localization thm. outline.

$A, S, S^{-1}A$, $\mathcal{H} \subset \mathcal{P}_1(A)$. Recall $\mathcal{P}_{\infty}(A) \subset \mathcal{P}_1(A)$ induces isos on K-groups, so ~~we~~ we obtain a homomorphism

Somewhere should relate $\mathcal{H}$ to $\mathcal{H}_{S,\infty}(A) = M \in \mathcal{P}_\infty(A) \ni S^{-1}M = 0$.

(1) $$t : K_*(\mathcal{H}) \longrightarrow K_*(A)$$

which we might call the transfer.

Theorem: If S consists of non-zero divisors, $\exists$ a long exact sequence

$$\cdots\cdots \xrightarrow{\partial} K_1\mathcal{H} \xrightarrow{t} K_1(A) \longrightarrow K_1(S^{-1}A) \xrightarrow{\partial} K_0\mathcal{H} \xrightarrow{t} K_0(A) \longrightarrow K_0(S^{-1}A)$$

∂ suitably defined.

Introduce notation: $\mathcal{P} = \mathcal{P}(A)$, $\mathcal{V}$ = full subcat of $\mathcal{P}(S^{-1}A)$ consisting of V isom to $S^{-1}P$ with $P \in \mathcal{P}$. $\mathcal{V}$ is an exact cat $\ni$ all exact sequences split, so I know $\Omega Q(\mathcal{V})$, and so I can see that

$$Q(\mathcal{V}) \longrightarrow Q(\mathcal{P}(S^{-1}A))$$

so $K_i(\mathcal{V}) \xrightarrow{\sim} K_i(S^{-1}A)$ for $i>0$ and $\hookrightarrow$ $i=0$.

is up to heq the covering space ~~of~~ of $Q(\mathcal{P}(S^{-1}A))$ belonging to the subgroup $\operatorname{Im}\{K_0A \to K_0(S^{-1}A)\}$. So the ~~proof~~ proof consists in identifying the homotopy fibre of the functor

$$\lambda : Q(\mathcal{P}) \longrightarrow Q(\mathcal{V})$$

induced by $P \longmapsto S^{-1}P$ with $Q(\mathcal{H})$. Once this is done the desired exact sequence is the homotopy sequence of λ.

~~Method~~ We will ~~be to~~ construct a diagram of categories

$$\begin{tikzcd}
Q(\mathcal{H}) & \mathcal{D} \arrow[l, "f"'] \arrow[r, "g"] & \mathcal{C} \arrow[r] \arrow[d, "g'"'] & \bar{Q}(\mathcal{V}) \arrow[d, "g"] \\
 & & Q(\mathcal{P}) \arrow[r, "\lambda"'] & Q(\mathcal{V})
\end{tikzcd}$$

~~where the functors f, g are heq's, and where the map $g'gf^{-1}$ in the hom-cat induces the negative of the transfer map in K_0~~

such that f, g are heq's and such that the map $g'gf^{-1}$ in the hom-cats induces the negative of the transfer. We will show that the square at the left is h-cartesian using the proof of the comp. thm. Since $\bar{Q}(\mathcal{V})$ is contractible, this identifies $\mathcal{C}$ with the h-fibre of λ, and so we win.

~~Def~~ Construction of $\mathcal{D}$:

Let $M \in \mathcal{H}$, $P \in \mathcal{P}$ and denote by $\mathcal{D}(M,P)$ the following groupoid. An object is an epim. $L \twoheadrightarrow M \times P$ of A-modules where $L \in \mathcal{P}$. A map $(L \twoheadrightarrow M\times P) \to (L' \twoheadrightarrow M \times P)$ is an isom $L \simeq L'$ over $M \times P$.

Given $\phi : M' \twoheadleftarrow M_0 \rightarrowtail M$ and $\psi : P' \twoheadleftarrow P_0 \twoheadrightarrow P$ define

$$(\phi, \psi)^* : \mathcal{D}(M,P) \longrightarrow \mathcal{D}(M', P')$$

by sending $L \twoheadrightarrow M \times P$ into the composite epimorphism

$$(M_0 \times P_0) \times_{(M\times P)} L \longrightarrow M_0 \times P_0 \twoheadrightarrow M' \times P'$$

$\mathcal{D}$ will be the fibred category over $Q(\mathcal{M}) \times Q(\mathcal{P})$ with fibres $\mathcal{D}(M,P)$ and base change functors $(\phi,\psi)^*$. A moments reflection shows that $\mathcal{D}$ admits the following explicit description: An object of $\mathcal{D}$ is an epi $(L \twoheadrightarrow M \times P)$ as above. A map from $(L \to M\times P)$ to $(L' \to M' \times P')$ is an injection $i : L \longrightarrow L'$ $\boxed{\text{which induces maps}}$ $M \longrightarrow M'$ and $P \longrightarrow P'$ in $Q(\mathcal{H})$ and $Q(\mathcal{P})$ (I mean precisely that $i\, \mathrm{Ker}(L \to M) \supset \mathrm{Ker}(L' \twoheadrightarrow M')$, and that this inclusion is admissible (no condition here), & one wants $\mathrm{Ker}(L' \to P') \subset i\, \mathrm{Ker}(L \to P)$ to be $\mathcal{P}$-admissible. $\boxed{}$ to be defined in advance.)

Define $f : \mathcal{D} \longrightarrow Q(\mathcal{H})$, $(L \twoheadrightarrow M \times N) \mapsto M$

<u>Lemma 1:</u> f is a heq.

Proof. f is fibred (compos. of two fibred functors $\mathcal{D} \to Q(\mathcal{H}) \times Q(\mathcal{P}) \longrightarrow Q(\mathcal{H})$), so enough to show $f^{-1}(M)$ cont. $\forall M \in \mathcal{H}$. $f^{-1}(M)$ is category with objects $L \twoheadrightarrow M \times P$ in which a map $(L \twoheadrightarrow M \times P) \longrightarrow (L' \to M \times P')$ is an injection $L' \to L'$ which induces $P \to P'$. ~~Have equiv.~~

$\cancel{f^{-1}(M) \simeq \mathrm{Sub}(R_M)}$

Let R_M be cat whose objects are epis $L \twoheadrightarrow M$, $L \in \mathcal{P}$, and in which a map $(L \twoheadrightarrow M) \to (L' \twoheadrightarrow M)$ is an admiss.

mono. $L \rightarrowtail L'$ in $\mathcal{P}$, which is over M. Then have equivalence of cats.

$$f^{-1}(M) \longrightarrow \mathrm{Sub}(\mathcal{R}_M)$$

$$(L \twoheadrightarrow M \times P) \longmapsto (\mathrm{Ker}(L \to P) \rightarrowtail L)$$

Since $\mathrm{Sub}(\mathcal{C})$ & $\mathcal{C}$ are h.eq. reduce to

<u>Lemma 2</u>: $\mathcal{R}_M$ is contractible.

Fix $(L_0 \overset{u_0}{\twoheadrightarrow} M)$ in $\mathcal{R}_M$. Then have natural transf. of functors

$$(L \overset{u}{\twoheadrightarrow} M) \longrightarrow (L \oplus L_0 \overset{u+u_0}{\twoheadrightarrow} M) \longleftarrow (L_0 \overset{u_0}{\twoheadrightarrow} M)$$

(<u>cone construction</u>).

Define $\mathcal{C}$ so the square

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "q'"'] & \bar{Q}(\mathcal{V}) \arrow[d, "q"] \\
Q(\mathcal{P}) \arrow[r, "\lambda"'] & Q(\mathcal{V})
\end{tikzcd}$$

is cartesian. Thus $\mathcal{C}$ is fibred over $Q(\mathcal{P})$, the fibre $\mathcal{C}_P$ being groupoid of admss. epis $V \twoheadrightarrow S^{-1}P$, and the base change functor wrt $\phi: P' \twoheadleftarrow P_0 \rightarrowtail P$ being

$$(V \twoheadrightarrow S^{-1}P) \longmapsto (S^{-1}P_0 \times_{S^{-1}P} V \twoheadrightarrow S^{-1}P_0 \twoheadrightarrow S^{-1}P')$$

Define

$$g : \mathcal{D} \longrightarrow \mathcal{C} \quad , \quad (L \twoheadrightarrow M \times P) \longmapsto (S^{-1}L \longrightarrow S^{-1}P)$$

<u>Lemma 3</u>: g is a heq.

<u>Proof</u>: $\mathcal{D}$, $\mathcal{C}$ are fibred over $Q(\mathcal{P})$ and g is a functor over $Q(\mathcal{P})$, so (new version of Th. A) it is enough to show g is a heq on the ~~each~~ fibres over a given $P \in Q(\mathcal{P})$. Let $\mathcal{D}_P$, $\mathcal{C}_P$ be the resp. fibres.

$\mathcal{C}_P$ = groupoid of admiss. epis $V \twoheadrightarrow S^{-1}P$.

$\mathcal{D}_P$: objects are $L \twoheadrightarrow M \times P$; a map $(L \twoheadrightarrow M \times P) \to (L' \twoheadrightarrow M' \times P)$ is an inj $L \to L'$ over P which induces a map $M \to M'$.

The functor $g_P : \mathcal{D}_P \to \mathcal{C}_P$ sends $(L \twoheadrightarrow M \times P)$ to $(S^{-1}L \to S^{-1}P)$

Let $\mathcal{I}_P$ be the cat consisting of ~~in which a map $L \to L'$ is an injection $\ni L'/L \in$~~ epis $L \twoheadrightarrow P$ with $L \in \mathcal{P}$ in which a map $(L \twoheadrightarrow P) \longrightarrow (L' \twoheadrightarrow P)$ is a injection $L \hookrightarrow L'$ over P $\ni$ $L'/L \in \mathcal{H}$. Then can factor g_P:

$$\mathcal{D}_P \longrightarrow \mathrm{Sub}(\mathcal{I}_P) \xrightarrow{\ t\ } \mathcal{I}_P \xrightarrow{\ l\ } \mathcal{C}_P$$

$$(L \twoheadrightarrow M \times P) \longmapsto \left(\begin{array}{c} \mathrm{Ker}(L \to M) \hookrightarrow L \\ \searrow \quad \swarrow \\ P \end{array} \right) \qquad\qquad (L \twoheadrightarrow P) \longmapsto (S^{-1}L \twoheadrightarrow S^{-1}P)$$

$$\left(\begin{array}{c} L' \hookrightarrow L \\ \searrow \quad \swarrow \\ P \end{array} \right) \longmapsto (L \twoheadrightarrow P)$$

First functor is an equivalence of categories.

Will show contractibility of $\mathcal{G}_P/v$. So let $v : V \twoheadrightarrow S^{-1}P$ be a given object of $\mathcal{C}_P$. An object of $\mathcal{G}_P/v$ is a pair consisting of an object $L \to M \times P$ of $\mathcal{D}_P$ and an isom. $\theta : S^{-1}L \xrightarrow{\sim} V$ over $S^{-1}P$; morphisms are such that it is a fibred category over $\mathcal{D}_P$. ~~Up to~~ ~~Up to isomorphism we can iden~~ ~~Define Lat~~ Need some preliminaries:

Define a lattice L in $V \in \mathcal{V}$ to be a A-submod $\ni S^{-1}L = V$, and $\ni L \in \mathcal{P}$. Let $\mathrm{Lat}(v : V \twoheadrightarrow S^{-1}P)$ be the set of lattices in $V \ni vL = P$, ordered by inclusion.

So now given an object $(L \twoheadrightarrow M \times P,\ \theta : S^{-1}L \xrightarrow{\sim} V)$ of $\mathcal{G}_P/v$ we can associate to it the image θL which is an element of $\mathrm{Lat}(v)$. Also $\theta(\mathrm{Ker}\, L \to M) \in \mathrm{Lat}(v)$, and so we get a functor

$$\mathcal{G}_P/v \longrightarrow \mathrm{Sub}(\mathrm{Lat}(v))$$

$$\begin{array}{c}(L \to M \times P,\ S^{-1}L \xrightarrow[\sim]{\theta} V) \\ \searrow\ \swarrow \\ S^{-1}P\end{array} \longmapsto (\theta(\mathrm{Ker}\, L \to M),\ \theta L)$$

which one sees easily is an <u>equivalence</u> of categories. Since $\mathrm{Sub}(\mathrm{Lat}(v))$ is heq $\mathrm{Lat}(v)$ we are reduced to

<u>Lemma 4</u>: $\mathrm{Lat}(v)$ is a non-empty directed set (hence a contractible category).

Proof. Since $v : V \to S^{-1}P$ admiss epi, $V \simeq \mathrm{Ker}(v) \oplus S^{-1}P$ where $\mathrm{Ker}(v) \in \mathcal{V}$. $\Rightarrow \exists\ L_0$ lattice in $\mathrm{Ker}(v)$, so $L_0 \oplus P \subset \mathrm{Ker}(v) \oplus S^{-1}(P) \simeq V$ is an elt of $\mathrm{Lat}(v)$, showing it is $\neq \emptyset$. Since $v^{-1}(P) = \bigcup_{s \in S} s^{-1}L_0 \oplus P$, and any lattice is fin. gen, $\Rightarrow$ any $L_1, L_2 \in \mathrm{Lat}(v)$ are contained in $s^{-1}L_0 \oplus P$ for some s, showing $\mathrm{Lat}(v)$ is directed!

Now I have to identify the map in the h-cat

$$Q(\mathcal{H}) \xrightarrow{f^{-1}} \mathcal{D} \xrightarrow{g} \mathcal{C} \xrightarrow{g'} Q(\mathcal{P})$$

with negative of transfer. Set $\mathcal{P}_1 = \mathcal{P}_1(A)$ so that transfer is

$$Q(\mathcal{H}) \hookrightarrow Q(\mathcal{P}_1) \xleftarrow[\text{heq}]{} Q(\mathcal{P})$$

So it is enough to note we have

$$\begin{tikzcd}
\mathcal{D} \arrow[rrr, "\text{commutativity}"] \arrow[d] & & & \overline{Q}(\mathcal{P}_1) \arrow[d] \\
Q(\mathcal{H}) \times Q(\mathcal{P}) \arrow[r, phantom, "\subset"] & Q(\mathcal{P}_1) \times Q(\mathcal{P}_1) \arrow[rr, "\oplus"] & & Q(\mathcal{P}_1)
\end{tikzcd}$$

$$(L \rightarrowtail M\times P) \longmapsto (L \rightarrowtail M\times P) \longmapsto M\times P$$

$$(L \rightarrowtail M\times P) \longmapsto (M,P) \longmapsto (M,P) \longmapsto M\times P$$

which implies that the sum of the homos

$$\begin{tikzcd}
& K_i(\mathcal{H}) \arrow[dr] \arrow[d, "t"] & \\
K_i(\mathcal{D}) \arrow[ur, "f_*"] \arrow[rr] \arrow[dr, "g'_* g_*"'] & {} & K_i(\mathcal{P}_1) \\
& K_i(\mathcal{P}) \arrow[ur, "\sim"] &
\end{tikzcd}$$

is zero. $\therefore$

$$t f_* = - g'_* g_*$$

$$-t = (g' g f^{-1})_* : K_i\mathcal{H} \to K_i\mathcal{P}$$

as claimed.

Last lemma: The square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \bar{Q}(\mathcal{V}) \\ \downarrow & & \downarrow \\ Q(\mathcal{P}) & \xrightarrow{\lambda} & Q(\mathcal{V}) \end{array}$$

is h-cartesian, or in other ~~words~~ words (as $\bar{Q}(\mathcal{V})$ is contractible) $\mathcal{C}$ is the h-fibre of λ.

Proof. $S = \mathrm{Iso}(\mathcal{V})$. Make S acts on $\bar{Q}(\mathcal{V})$ as in proof of comparison thm.

$$I \# (V' \overset{v}{\twoheadrightarrow} V) = (I \oplus V' \xrightarrow{v\,pr_2} V)$$

I know from the proof of that theorem that $S^{-1}\bar{Q}(\mathcal{V}) \to Q(\mathcal{V})$ is fibred ~~and base changes are hegs~~ with fibre $S^{-1}\mathcal{E}_V$ over V, and that base changes are hegs. Consider induced S-action on $\mathcal{C}$. Again can conclude $S^{-1}\mathcal{C} \to Q(\mathcal{P})$ fibred with fibres $S^{-1}\mathcal{E}_{S^{-1}P}$ over P and all basechanges are hegs. So then have

$$\begin{array}{ccc} S^{-1}\mathcal{C} & \longrightarrow & S^{-1}\bar{Q}(\mathcal{V}) \\ \downarrow & & \downarrow \\ Q(\mathcal{P}) & \longrightarrow & Q(\mathcal{V}) \end{array}$$

is h-cartesian. Enough to show then that in

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \bar{Q}(\mathcal{V}) \\ \downarrow & & \downarrow \\ S^{-1}\mathcal{C} & \longrightarrow & S^{-1}\bar{Q}(\mathcal{V}) \end{array}$$

vertical maps are hegs. But this recall will follow (

(because $S^{-1}X$ fibres over $S^{-1}(pt)$ which is contractible) if we can show that $\forall T \in S$ $S\#$? is a heq on $\mathcal{C}$ (resp. $\overline{Q}(V)$ - this case is trivial). Thus reduced to

<u>Lemma</u>: $\forall T \in S$, $S\# : \mathcal{C} \longrightarrow \mathcal{C}$ is a heq.

<u>Proof</u>: Let $T = S^{-1}P_0$, $P_0 \in \mathcal{P}$ and make $\mathcal{P}$ act on $\mathcal{D}$ by

$$P\#(L \twoheadrightarrow M \times P) = (P \oplus L \longrightarrow M \times P).$$

Then one verifies that

$$\begin{tikzcd} & \mathcal{D} \arrow[r] \arrow[d, "P\#"] \arrow[dl] & \mathcal{C} \arrow[d, "T\#"] \\ Q(\mathcal{P}) & \mathcal{D} \arrow[r] \arrow[l] & \mathcal{C} \end{tikzcd}$$

commutes. Since all horizontal arrows are heqs, it follows $T\#$ is a heq (in fact $\sim$ identity). Done

Fundamental thm. (Outline):

A ring, t indeterminate

$$\begin{tikzcd} A \arrow[r, "(\ )"] \arrow[d, "(\ )"'] & A[t^{-1}] \arrow[d, "(\ )"] \\ A[t] \arrow[r, "(\ )"'] & A[t,t^{-1}] \end{tikzcd}$$

Composition: $A \longrightarrow A[t] \xrightarrow{t \mapsto 1} A$ is id, so

$$K_q A[t] \simeq K_q A \oplus NK_q(A) \quad , \quad NK_q(A) = K_q A[t]/K_q A.$$

Nil(A) = full subcat of $A[t]$-modules M such that $M \in \mathcal{P}(A)$ and such that $t^n = 0$ on M some n.

Have functors

$$\begin{array}{ll} \mathcal{P}(A) & P \\ \downarrow & \downarrow \\ \mathrm{Nil}(A) & P \text{ with } t \text{ acting as } 0 \\ \downarrow \text{ forget } t\text{-action.} & \\ \mathcal{P}(A) & \end{array}$$

so get direct sum decomp

$$K_q(\mathrm{Nil}(A)) = K_q A \oplus \mathrm{Nil}_q(A).$$

~~where $\mathrm{Nil}_q(A)$ = ...~~

~~Fund. Thm: ① $\exists$ can. isom $NK_q(A) \simeq \mathrm{Nil}_{q-1}(A)$ $q \geq 1$.~~

~~i.e. $K_q A[t] \cong K_q A \oplus \mathrm{Nil}_{q-1}(A)$.~~

~~② Sequence~~

~~$0 \longrightarrow K_q A \longrightarrow$~~

Fundamental thm:

Fundamental theorem

(1) For $q \geq 1$ there is a canonical isomorphism $NK_q(A) = Nil_{q-1}(A)$, hence $K_q(A[t]) = K_q A \oplus Nil_{q-1}(A)$.

(2) The sequence

$$0 \longrightarrow K_q A \longrightarrow K_q A[t] \oplus K_{q-1} A[t^{-1}] \longrightarrow K_q A[t,t^{-1}]$$

is exact for $q \geq 0$. For $q \geq 1$ there is a canonical map $\partial : K_q A[t,t^{-1}] \longrightarrow K_{q-1} A$ such that

$$K_q A[t] \oplus K_q A[t^{-1}] \longrightarrow K_q A[t,t^{-1}] \xrightarrow{\partial} K_{q-1} A \longrightarrow 0$$

is exact. Furthermore ∂ has a canonical section, so there are canonical isos

$$K_q A[t,t^{-1}] = K_q A \oplus Nil_{q-1}(A) \oplus Nil_{q-1}(A) \oplus K_{q-1} A$$

Remarks: "Canonical" implies functorial wrt A.

(2) says K_q is a contracted functor with $LK_q = K_{q-1}$ in Bass's terminology. He proves this for $q=1$, but the result is new already for K_2.

(2) For $q \geq 1$ there is a canonical epi $\partial : K_q A[t,t^{-1}] \to K_{q-1} A$ such that the sequence

$$0 \to K_q A \to K_q A[t] \oplus K_q A[t^{-1}] \to K_q A[t,t^{-1}] \xrightarrow{\partial} K_{q-1} A \to 0$$

is exact. Furthermore ∂ has a canonical section, so there are canon. isom

$$K_q(A[t,t^{-1}]) = K_q A \oplus$$

also $\Rightarrow$ $0 \to K_0 A \to K_0 A[t] \oplus K_0 A[t^{-1}] \xrightarrow{\text{exact}} K_0(A[t,t^{-1}])$

Remarks: Canon $\Rightarrow$ natural. Proved by Bass for $q=1$. Says K_q is a contracted functor with $LK_q = K_{q-1}$. new for K_2.

Outline of fundamental thm:

recall K_* of $\mathbb{P}^1_A$

Put $X = \mathbb{P}^1_A$. Define $\mathrm{Mod}(X)$, $\mathcal{P}(X)$

$$\begin{tikzcd} \mathrm{Mod}(X) \arrow[r] \arrow[d] & \mathrm{Mod}\, A[t^{-1}] \arrow[d] \\ \mathrm{Mod}(A[t]) \arrow[r] & \mathrm{Mod}\, A[t,t^{-1}] \end{tikzcd}$$

Define $\mathcal{O}_X(n)$ $\qquad T_0, T_1$

$$h_n : \mathcal{P}(A) \longrightarrow \mathcal{P}(X), \qquad h_n(M) = \mathcal{O}_X(n) \otimes_A M$$

Recall:

Theorem: $(K_*A)^2 \xrightarrow{\sim} K_*X$, $\quad (a,b) \longmapsto h_{0*}(a) + h_{-1*}(b)$.

$$(h_n)_* - 2(h_{n-1})_* + (h_{n-2})_* = 0.$$

~~Define~~

Define $\mathcal{H}'$ = full subcat of $\mathcal{P}_1(X)$ consisting of $M = (M^+, M^-, \theta)$ s.t. $M^- = 0$ and M^+ killed by t^n.

Define $\mathcal{D}'$ = fibred cat over $Q(\mathcal{H}') \times Q(\mathcal{P}(X))$ with fibre $\mathcal{D}'_{(M,P)}$ = groupoid of epis $L \to M \times P$ in $\mathrm{Mod}(X)$ where $L \in \mathcal{P}(X)$. $\mathcal{V}'$ = full subcat of $\mathcal{P}(A[t^{-1}])$ cons. of V which extend to an object of $\mathcal{P}(X)$. Claim then we have a fibration

$$\mathcal{D}' \longrightarrow Q(\mathcal{P}(X)) \longrightarrow Q(\mathcal{V}')$$

and a heq $\mathcal{D}' \longrightarrow Q(\mathcal{H}')$ and that resulting h-map $Q(\mathcal{H}') \longrightarrow Q(\mathcal{P}(X))$ is negative of Cartan map.

Define $S = \{t^n\}$ in $A[t]$

$\mathcal{H} = \mathcal{H}_{S,1}(A[t])$ modules in $P_1(A[t])$ killed by same t^n

$\mathcal{H}'$ = modules M in $P_1(X)$ $\ni$ $M^- = 0$, M^+ killed by same t^n.

<u>Lemma 1</u>: $\mathcal{H}' \longrightarrow \mathcal{H}$, $M \longmapsto M^+$ equivalence.

<u>Pf</u>: Have to check that if $N \in \mathcal{H}$, then $\tilde{N} = (N, 0, ..)$ is in $P_1(X)$. But $N = P/P'$ where P extends to a v.b. $\tilde{P}$ on X (can take $P = A[t]^k$); ~~[illegible]~~ if $\tilde{P}'$ = submod of $\tilde{P}$ with $\tilde{P}'^+ = P'$, $\tilde{P}'^- = \tilde{P}^-$, then $\tilde{P}' \in P_1(X)$, and $N = \tilde{P}/\tilde{P}' \in P_1(X)$.

Define $c' : K_*(\mathcal{H}') \longrightarrow K_*(X)$.

<u>Lemma 2</u>: We have a map of long exact sequences

$$(*)\quad \begin{tikzcd} \arrow[r] & K_q(\mathcal{H}') \arrow[r, "c'"] \arrow[d, "\wr"] & K_q(X) \arrow[r, "(\ )"] \arrow[d] & K_q(A[t^{-1}]) \arrow[r, "\partial'"] \arrow[d] & K_{q-1}(\mathcal{H}') \arrow[r] \arrow[d, "\wr"] & {} \\ \arrow[r] & K_q(\mathcal{H}) \arrow[r, "c"] & K_q(A[t]) \arrow[r, "(\ )"] & K_q(A[t, t^{-1}]) \arrow[r, "\partial"] & K_{q-1}(\mathcal{H}) \arrow[r] & {} \end{tikzcd}$$

where ~~[illegible]~~ ∂ is the differential of the exact sequence of the local. thm. applied to $A[t]$ and $S = \{t^n\}$, and where ∂' is defined so that the right square commutes.

<u>Pf</u>. ~~[illegible]~~ Define $\mathcal{D}, \mathcal{P}, \mathcal{V}, \mathcal{D}', \mathcal{P}', \mathcal{V}'$, so that we have diagram

$$\begin{tikzcd} Q(\mathcal{H}') \arrow[d] & \mathcal{D}' \arrow[l, "a'"'] \arrow[r, "b'"] \arrow[d] & Q(\mathcal{P}') \arrow[r, "\mu'"] \arrow[d] & Q(\mathcal{V}') \arrow[d] \\ Q(\mathcal{H}) & \mathcal{D} \arrow[l, "a"'] \arrow[r, "b"] & Q(\mathcal{P}) \arrow[r, "\mu"] & Q(\mathcal{V}) \end{tikzcd}$$

In the proof of the loc. thm. showed $\begin{matrix} a \text{ heq} \\ b \text{ h-fibre of } \mu . \end{matrix}$

We propose now to construct a map of long exact sequences

$$\begin{tikzcd}
\to K_q(\mathcal{H}') \arrow[r, "c'"] \arrow[d, "\delta"] & K_q(X) \arrow[r, "(j^-)^*"] \arrow[d, "(j^+)^*"] & K_q(A[t^{-1}]) \arrow[r, "\partial'"] \arrow[d, "(i^+)^*"] & K_{q-1}\mathcal{H}' \to \arrow[d, "\delta"] \\
\to K_q(\mathcal{H}) \arrow[r, "c"] & K_q(A[t]) \arrow[r, "(i^-)^*"] & K_q(A[t,t^{-1}]) \arrow[r, "\partial"] & K_{q-1}\mathcal{H} \to
\end{tikzcd}$$

where the the bottom row results from loc. thm. applied to ring $A[t]$ and m.s. $S = \{t^n\}$. I recall that this exact sequence results as follows: $\mathcal{V}$ = full subcat of $P(A[t,t^{-1}])$ cons. of P isom to $S^{-1}L$, $L \in P(A[t])$; $\mathcal{D}$ = fibred cat over $Q(\mathcal{H}) \times Q(P(A[t]))$ with $\mathcal{D}_{(M,P)}$ = groupoid

---

straight-forward modification shows analogue holds for top row, ~~so~~ done.

Define $\mathrm{Nil}(A) \longrightarrow \mathcal{H}'$ using exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \otimes_A N \xrightarrow{T_1 \otimes 1 - T_0 \otimes t} \mathcal{O}_X \otimes_A N \longrightarrow \tilde{N} \longrightarrow 0$$

$t =$

and use this to prove:

<u>Lemma 3</u>: $K_q(\mathrm{Nil}(A)) \xrightarrow{\sim} K_q(\mathcal{H}') \xrightarrow{\sim} K_q(\mathcal{H})$

<u>Lemma 4</u>:

$$\begin{tikzcd}
K_q(\mathrm{Nil}(A)) \arrow[r] \arrow[d] & K_q(\mathcal{H}') \arrow[d, "c'"] \\
K_q(A) \arrow[r, "h_{0*} - h_{-1,*}"] & K_q(X)
\end{tikzcd}$$

Also $c: K_q\mathcal{H} \to K_q A[t]$ is zero

Now apply Thm computing $K_*(X)$ and one sees that from $(*)$ in Lemma 2 we obtain a map of exact sequences $\boxed{\text{FOR } q \geq 1}$

$$(**)\qquad \begin{tikzcd} 0 \arrow[r] & K_q A \arrow[r, "()"] \arrow[d, "()"] & K_q A[t^{-1}] \arrow[r, "d'"] \arrow[d, "()"] & Nil_{q-1}(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_q A[t] \arrow[r, "()"] & K_q A[t,t^{-1}] \arrow[r, "d"] & K_{q-1}(Nil(A)) \arrow[r] & 0 \end{tikzcd}$$

Lemma 5: The two rows in $(**)$ have canonical splittings, functorial in A:

~~Proof: top row split by homo~~

~~$K_q A[t^{-1}] \longrightarrow K_q A \qquad t \longmapsto 1.$~~

~~Let s' be the corresponding section of d'.~~

~~Computato~~

$$\cancel{\begin{array}{c} K_q A \oplus Nil_{q-1}(A) \\ \Big\downarrow \scriptstyle ((\,)(\,),\ (\,)s') \\ K_q A[t,t^{-1}] \\ \Big\downarrow \scriptstyle ((t\mapsto 1)_*,\ pr\cdot d) \\ K_q A \oplus Nil_{q-1}(A) \end{array}}$$

Proof: Top row split by
$(t \mapsto 1)_* : K_q(A[t^{-1}]) \longrightarrow K_q(A)$.

Let s' be corresp. section of d', and $\pi : K_{q-1} Nil(A) \to Nil_{q-1} A$ the canonical projection. Computation shows that the composition

$$\begin{array}{c}
K_q A \oplus Nil_{q-1}(A) \\
\Big\downarrow {\scriptstyle (\ ),\ \delta'} \\
K_q A[t^{-1}] \\
\Big\downarrow {\scriptstyle (\ )} \\
K_q A[t,t^{-1}] \\
\Big\downarrow {\scriptstyle ((t\mapsto 1)_*,\ \pi d)} \\
K_q A \oplus Nil_{q-1} A
\end{array}$$

is the identity which shows $(\ ): K_q A[t^{-1}] \longrightarrow K_q A[t,t^{-1}]$ has a left inverse. Interchanging t, t^{-1}, the same holds for $(\ )$, showing <u>bottom</u> sequence of $(**)$ splits.

Fund. thm. follows:

$$K_q A[t] \cong K_q A \oplus Nil_{q-1} A$$

$$\begin{aligned} K_q A[t,t^{-1}] &\cong K_q A[t] \oplus K_{q-1} Nil(A) \\ &\cong K_q A \oplus Nil_{q-1} A \oplus K_{q-1} A \oplus Nil_{q-1} A. \end{aligned}$$

Applications of the fundamental thm.

Regular coherent rings

Recall that a ring A is called coherent if every fin. gen. ideal is f.p or equivalently if the category of f.p. Amodules, to be denoted $Mod_{fp}(A)$, is an abelian category. A is called reg. coherent if in addition every f.p. modules has finite proj. dimension, which implies $\mathcal{P}_\infty(A) = Mod_{fp}(A)$.

The following result was proved ~~[illegible]~~ for A regular noeth. in [ ], and ~~[illegible]~~ by Gersten [ ] ~~[illegible]~~ assuming $A[t]$ is regular coherent.

__Theorem:__ A regular coherent $\Rightarrow$

$$K_q A = K_q A[t]$$

$$K_q A[t,t^{-1}] = \begin{cases} K_0 A & q=0 \\ K_q A \oplus K_{q-1} A & q>0 \end{cases}$$

__Proof:__ ~~A regular coherent~~ means $Modfp(A)$ ~~is an abelian~~ category and that $P_\infty(A) = Modfp(A)$.

Set $Nil(M)$ = exact cat of objects of M with a nilpotent endo, so that $Nil(A) = Nil(P(A))$. Then have square

$$\begin{tikzcd} P(A) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & Nil(P(A)) \arrow[d, phantom, "\cap"] \\ P_\infty(A) \arrow[r, phantom, "\subset"] & Nil(P_\infty(A)) \end{tikzcd}$$

where horizontal map associates ~~to~~ M the object M with the zero endomorphism. (Better have canonical maps $M \subset Nil(M) \longrightarrow M$). ~~In the above square~~ ~~the vertical~~ Also should note that $Nil(P_\infty(A)) = \mathcal{H}_S(A[t])$.

In the above square the vertical arrows induce isos on K-groups by the resolution theorem.

But now if A is regular coherent we know $P_\infty(A) = Modfp(A)$ is an abelian category, as well as $Nil(P_\infty(A))$. Thus the devissage theorem implies

$$K_* P_\infty(A) = K_* Nil(P_\infty(A))$$

so we conclude $K_* A = K_* Nil(A)$, i.e. $Nil_*(A) = 0$.

This yields the formulas above for $q > 0$, using the fundamental theorem.

To prove $K_0A = K_0A[t] = K_0A[t,t^{-1}]$, it will suffice in view of the exact sequence

$$0 \longrightarrow K_0 A \longrightarrow K_0(A[t]) \oplus K_0(A[t^{-1}]) \longrightarrow K_0(A[t,t^{-1}])$$

to show that $K_0(A[t^{-1}]) \longrightarrow K_0(A[t,t^{-1}])$ is surjective, or equivalently that $K_0(A[t]) \longrightarrow K_0(A[t,t^{-1}])$ is surj. To do this we will prove that given $P' \in \mathcal{P}(A[t,t^{-1}])$ then $P' = S^{-1}L'$, where $L' \in \mathcal{P}_\infty(A[t])$ and $S = \{t^n\}$. This will suffice because L' has a finite resolution by modules $P_i \in \mathcal{P}(A[t])$, so

$$cl(P') = \sum (-1)^i \, cl(S^{-1}P_i) \in K_0(A[t,t^{-1}])$$

showing $cl(P')$ comes from $K_0(A[t])$.

Choose $P'' \in \mathcal{P}(A[t,t^{-1}])$ so that $P' \oplus P''$ is free over $A[t,t^{-1}]$, hence of the form $S^{-1}L$ where L is free over $A[t]$. Let π', π'' be the resp. projections of $P' \oplus P''$ onto P', P'' and put $L' = \pi'(P')$, $L'' = \pi'(P'')$. Then $S^{-1}L' = P'$, and L', L'' are fin. gen. over $A[t]$, hence $\exists\, n \ni$

$$L \subset L' \oplus L'' \subset t^{-n}L$$

Then $J = t^{-n}L/(L' \oplus L'')$ is finitely presented over $A[t]$, hence f.p over A, so it is an object of $Nil(\mathcal{P}_\infty(A))$ by the reg. coherence of A. But $Nil(\mathcal{P}_\infty(A)) = \mathcal{H}_S(A[t]) \subset \mathcal{P}_\infty(A[t])$. Thus J is in $\mathcal{P}_\infty(A[t])$, so $L' \oplus L'' \in \mathcal{P}_\infty(A[t])$, and hence $L' \in \mathcal{P}_\infty(A[t])$. (More generally if $L' \oplus L'' \in \mathcal{P}_n(B)$, then

$L', L'' \in \mathcal{P}_n(B)$ as one sees easily using induction on n).

(Begin proof with general facts

1) $\mathcal{P}_n(A)$ Karoubian

2) $\mathcal{M} \longrightarrow \mathcal{N}il(\mathcal{M}) \longrightarrow \mathcal{M}$

3)
$$\begin{array}{ccccc} \mathcal{P}(A) & \subset & \mathcal{N}il(\mathcal{P}(A)) & & \\ \cap & & \cap & & \\ \mathcal{P}_\infty(A) & \subset & \mathcal{N}il(\mathcal{P}_\infty(A)) & = & \mathcal{H}_S(A[t]) \end{array}$$
vertical maps induce isos in K_q $\quad S = \{t^n\}$.)

<u>Remarks</u> 1) Preceding established for reg. noeth rings in [Quillen], then extended to rings A such that $A[t]$ is regular coherent in [Gersten].

2) <u>Conjecture</u>: $K_q A = K_q(A[t_1, \ldots, t_n])$ for A regular coherent, and all the $K_n(A) = 0$ for $n < 0$.

~~Corollary: (Gersten). A regular noetherian & comm and if $A[X]$ is the ring of non-comm.~~

<u>Corollary</u>: If $A = A_0 \oplus A_1 \oplus \cdots$ graded ring such that A is regular coherent, then $K_* A_0 = K_* A$.

<u>Proof</u>: Clearly $K_* A_0$ is a direct summand of $K_* A$, so it is enough to show that the homomorphism $\rho: A \longrightarrow A_0 \longrightarrow A$ induces the identity on $K_* A$. ~~But ρ and id are homotopic by the~~ Since

A is regular coh, we have the homotopy property $K_*A = K_*(A[t])$, hence it suffices to note that the element $h : A \longrightarrow A[t]$, given by $h(a) = at^n$ if $a \in A_n$, constitutes a homotopy joining ρ to id_A.

Rings of characteristic p

Theorem: Let p be a prime number such that $pA = 0$. Then

$$K_i A \otimes \mathbb{Z}[p^{-1}] \xrightarrow{\sim} K_i(A[t]) \otimes \mathbb{Z}[p^{-1}]$$

Proof: We first show that it suffices to establish the above isomorphism for $i > 0$, or equivalently by FT to show $\mathrm{Nil}_q(A)$ is a p-torsion group for all $i \geq 0$. In effect, we know (FT) that $K_1 A$ is naturally a direct summand of $K_1(A[x,x^{-1}])$. Hence if we establish that the inclusion

$$A[x,x^{-1}] \longrightarrow A[x,x^{-1}][t] = A[t][x,x^{-1}]$$

induces isos. on $K_1 \otimes \mathbb{Z}[p^{-1}]$, it follows that $A \to A[t]$ induces isos. on $K_0 \otimes \mathbb{Z}[p^{-1}]$.

If n is an integer ≥ 1, let $\mathrm{Nil}^n(A)$ be the full subcategory of $\mathrm{Nil}(A)$ consisting of modules killed by t^n. Then we have the functors

$$\mathcal{P}(A) \xrightarrow{\text{put } t=0} \mathrm{Nil}^n(A) \xrightarrow{\text{forget } t} \mathcal{P}(A)$$

and hence a splitting

$$K_i(\mathrm{Nil}^n(A)) = K_i A \oplus \mathrm{Nil}_i^n(A)$$

where this defines $\mathrm{Nil}_i^n(A)$. Since $\mathrm{Nil}(A)$ is the union of the $\mathrm{Nil}^n(A)$, we have

$$\mathrm{Nil}_*(A) = \varinjlim_n \mathrm{Nil}^n_*(A).$$

Hence to prove $\mathrm{Nil}_*(A)$ is p-torsion, it suffices to prove the following.

<u>Lemma:</u> If $q = p^n$ $\mathrm{Nil}^q_*(A)$ is killed by q.

<u>Proof:</u> Let F be the functor from $\mathrm{Mod}(A[t])$ to itself given by restricting scalars to $A[t^q]$ and then extending to $A[t]$:

$$F(M) = A[t] \otimes_{A[t^q]} M = \left(A[t] \otimes_{A[t^q]} A[t]\right) \otimes_{A[t]} M$$

Since

$$A[t] \otimes_{A[t^q]} A[t] = A[t', t''] \big/ A[t', t''](t' - t'')^q$$

with $t' = t \otimes 1$, $t'' = 1 \otimes t$, it is clear that on filtering this ring by powers of the ideal $I = A[t', t''](t' - t'')$, we obtain a filtration $\{I^j F\}$ of the functor F such that $\mathrm{gr}(F)$ is the q-fold direct sum of the identity.

~~Clearly the~~

~~Now consider F restricted to $\mathrm{Nil}^q(A)$. Clearly it is an exact functor from $\mathrm{Nil}^q(A)$ to itself, and~~

Now the restrictions of F, $I^{\delta}F$, $gr(F)$ to $Nil^{\varepsilon}(A)$ are exact functors from $Nil^{\varepsilon}(A)$ to itself. Applying the basic additivity result [~~used above in proof of fun~~ ref), one knows F and $gr(F)$ induce the same map on K-groups, hence $F_* =$ mult by q on $K_*(Nil^{\varepsilon}(A))$. On the other hand, it is clear that F factors thru the "forget-t" functor ( ), hence any ~~[illegible]~~ element of $K_*(Nil^{\varepsilon}(A))$ killed by the "forget-t" functor is also killed by q, which proves the lemma and the theorem.

Preceding admits the following generalization

Theorem ': ~~Let~~ If n ~~be~~ is an integer such that $nA = 0$, ~~then~~

$$K_i A \otimes \mathbb{Z}[n^{-1}] \xrightarrow{\sim} K_i(A[t]) \otimes \mathbb{Z}[n^{-1}].$$

Proof: ~~Breaking~~ Writing A as a product of its different p-primary components, and using the fact that K-groups commute with products one reduces to the case where $n = p^r$, ~~p~~ p prime. Then pA is a nilpotent ~~two sided~~ ideal in A, so it suffices to prove the following

Lemma. Let I be a nilpotent (2-sided) ideal in A such that $p^r I = 0$ for some r. Then

$$K_*(A) \otimes \mathbb{Z}[p^{-1}] \xrightarrow{\sim} K_*(A/I) \otimes \mathbb{Z}[p^{-1}].$$

For dim. zero, one has $K_0 A = K_0(A/I)$ (see Bass ____)

Proof. Immediate reduction to the case $I^2 = 0$, in which case we have an exact sequence of groups

$$0 \longrightarrow N \longrightarrow GL_m(A) \longrightarrow GL_m(A/I) \longrightarrow 0$$

where N is isomorphic to the group of matrices with coefficients in I under addition. Thus $p^r N = 0$, so the homology of N tensored with $\mathbb{Z}[p^{-1}]$ is trivial. It follows that $GL_m(A) \longrightarrow GL_m(A/I)$ induced isos. on homology tensored with $\mathbb{Z}[p^{-1}]$ for all m, hence the same is true for the map of H-spaces $BGL(A)^+ \longrightarrow BGL(A/I)^+$.

Rings of characteristic p

Theorem: Let n be an integer and A a ring such that $nA = 0$. Then

~~[illegible]~~

$$K_q A \otimes \mathbb{Z}[n^{-1}] \xrightarrow{\sim} K_q(A[t]) \otimes \mathbb{Z}[n^{-1}].$$

~~Assertion trivial if $n=0$, and If $n \neq 0$, then writing A as the product of its p-primary components and using the fact that K_q commutes with products, one sees it suffices to assume that $n = p^d$~~

Using the fact that K_q commutes with products one reduces to the case where $n = p^d$, p a prime.

From homotopy theory one ~~knows that the [illegible] of homotopy groups tensored [illegible]~~ then has

$$\pi_q(BGL(A)^+) \otimes \mathbb{Z}[p^{-1}] \xrightarrow{\sim} \pi_q(BGL(A/I)^+) \otimes \mathbb{Z}[p^{-1}]$$

proving the lemma.

Remarks: Preceding implies that for a ring killed by n the Karoubi-Villamayor K-groups ~~[illegible]~~ and the K-groups differ only by n-torsion.

Question: Is ~~[illegible]~~ $K_{-1}(A)$, p-torsion for a ring killed by p?