August 1, 1972

Let $A$ be a Dedekind domain, $M = \Phi_{A}$, $C' \subset C \subset Q(M)$ full subcategories consisting of projective modules of ranks $< r$ and $\leq r$, respectively.

\[ j : C' \rightarrow C \]

the inclusion functor. Observe that $C'$ is acyclic in $C$, so in some sense, $C'$ is open in $C$.

Let $M \subset C$. Then

\[ (R^g \Lambda(M)) = H^g(j/M, \Lambda) \]

if we use contravariant functors. If $M \in C'$, the cat. $j/M$ has a final object, and the cohomology is trivial, so suppose $\text{rank}(M) = r$.

The category $j/M$ is equivalent to the ordered set of admissible layers in $M$ of rank $< r$, and which is the same for the $K$-module $M \otimes K$. We know that $j/M$ has the homotopy type of the suspension of the building of $M \otimes K$. Thus

\[ j/M \cong \text{bouquet of } S^{n-1} \]

and precisely

\[ H^g(j/M, \Lambda) = \begin{cases} 0 & g \neq n-1 \\ \text{Hom}(H_{n-1}(j/M), \Lambda) & g = n-1. \end{cases} \]

The homology group $H_{n-1}(j/M)$ is $H_{n-2}(\text{X}(M \otimes K))$.

More accurately: Let $\text{X}(M \otimes K)$ be the s.

complex
of chains $0 < V_0 < \ldots < V_p < \mathcal{M} \otimes K \ni \dim(V_p/V_0) < n$. Then

- $r = 1$, \( \mathcal{X}(\mathcal{M} \otimes K) = \emptyset \) two points
- $r = 2$, \( \mathcal{X}(\mathcal{M} \otimes K) = \sum P_i K \).

The point is that \( \tilde{H}_{r-1}(\mathcal{X}(V), \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module on which \( \text{Aut}(V) \) acts. It is called the Steinberg representation, and will denote it by \( \text{St}(V) \). It will perhaps be important to recall that if we fix a flag $0 < V_0 < \ldots < V_{r-1} < V$ in $V$ and let $B$ be the Borel subgroup associated to this flag, then as a $B$-module

\[
\text{St}(V) \cong \mathbb{Z}[B/T].
\]

Observe that \( M \mapsto \Lambda^r M \in \text{Pic}(A) \) determines a module of rank $r$ up to isomorphism. Let $M_\alpha$, $\alpha \in \text{Pic}(A)$ be representatives. Then we have an open-closed situation

\[
C_{r-1} \overset{\delta}{\longrightarrow} C_r \overset{\Lambda = \bigcup \alpha \text{Aut}(M_\alpha)}{\leftarrow} \bigcup \text{Aut}(M_\alpha)
\]

and we have seen that

\[
R^{n-1} j_*(\Lambda) = \Lambda_*(\alpha \mapsto (\text{St}(M_\alpha \otimes K), \Lambda))
\]

Thus we have a triangle

\[
\Lambda \longrightarrow Rj_*(\Lambda) \longrightarrow \bigoplus_{\alpha}(\text{Hom}(\text{St}(M_\alpha \otimes K), \Lambda))^{\otimes r+1}
\]
and a long exact sequence in cohomology
\[ \prod_{x} H^{q-2}(\text{Aut}_{x}(\mathcal{M}_{x} \otimes K)) \rightarrow H^{q}(\mathcal{C}_{n}, \Lambda) \rightarrow H^{q}(\mathcal{C}_{n-1}, \Lambda) \rightarrow \]

which homologically should amount to a long exact sequence
\[ \rightarrow H_{q}(\mathcal{C}_{n-1}, \mathbb{Z}) \rightarrow H_{q}(\mathcal{C}_{n}, \mathbb{Z}) \rightarrow \sum_{x \in \text{rel}(\mathcal{A})} H_{q-1}(\text{Aut}_{\mathcal{M}_{x}}, \text{St}(\mathcal{M}_{x} \otimes K)) \]

I can give a better derivation by using homology as follows.
\[ \mathcal{C}_{n-1} \xrightarrow{j} \mathcal{C}_{n} \xleftarrow{i} \bigcup_{x} \text{Aut}(\mathcal{M}_{x}) \]

If we use covariant functors
\[ L_{\mathcal{X}} j_{!}(\mathbb{Z})(N) = H_{\mathcal{X}}(j/N, \mathbb{Z}) = \mathbb{Z}[0] \]

if \( N \in \mathcal{C}_{n-1} \) and
\[ L_{\mathcal{X}} j_{!}(\mathbb{Z})(\mathcal{M}_{x}) = H_{\mathcal{X}}(j/M_{x}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 0, n-1 \\ \text{St}(\mathcal{M}_{x} \otimes K) & * = n-1 \end{cases} \]

hence we have a triangle
\[ \frac{1}{2} \tilde{\xi}(\alpha) \circ \text{St}(M_n \otimes K)^{[n-1]} \rightarrow \mathbb{L} \tilde{j}_!(\mathbb{Z}) \rightarrow \mathbb{Z} \]

which leads to the above exact sequence.
August 2, 1972

Review Borel-Serre, at least in the case of $\text{SL}_n$ over $\mathbb{Q}$. They complete

$$X = \text{SL}_n \mathbb{R}/\mathbb{O}_n$$

to a cornered manifold (non-compact) $\bar{X}$ by adjoining

$$\bar{\partial}X \sim \text{building belonging to the parabolics in } \text{SL}_n$$

Thus

$$\bar{\partial}X \sim \sqrt{\text{S}^{n-2}}$$

and

$$\hat{H}_{n-2}(\bar{\partial}X) = \text{St}(\mathbb{Q}^n)$$

Thus

$$\bar{\partial}(X, \bar{\partial}X) \sim \sqrt{\text{S}^{n-1}}$$

$$H_{n-1}(X, \bar{\partial}X) = \text{St}(\mathbb{Q}^n).$$

Now if $\Gamma \subset \text{SL}_n \mathbb{Z}$ is torsion-free of finite index, Borel-Serre prove $\Gamma$ acts freely on $\bar{X}$ and $\bar{X}/\Gamma$ is compact (topologically it is a manifold with $\partial$, from $C^\infty$-point of view it is a cornered manifold). One has spectral sequence for any $\Gamma$-module $M$

$$E^{2}_{pq} = H_p(\Gamma, H_q(X, \partial X; \tilde{M})) \Rightarrow H_{p+q}(X, \partial X; \tilde{M})$$

which degenerates. By Poincaré duality $+$ compactness of $\bar{X}$, we have

$$H_* (X, \partial X; \tilde{M}) \cong H_{d-*}(\bar{X}, \tilde{M}) = H_{d-*}(\Gamma, M)$$
and thus we get the Borel-Serre duality formula
\[ H_p(\Gamma, I \otimes M) = H^{(d-d_p)} P(\Gamma, M) \]
where
\[ d = \dim X \]
\[ l = \text{rank}_\mathbb{Q} G = n-1 \text{ for } St_n \]
\[ I = \text{Steinberg repn.} \]

But what is fascinating about the above is that we have a geometric interpretation of
\[ H_x(\Gamma, \text{St}(\mathbb{Q}^n)) \], namely
\[ H_x(\Gamma, \text{St}(\mathbb{Q}^n)) = H_{n-1+i} \left( \prod \overline{X}_\Gamma, \varpi \overline{x}_\Gamma; \mathbb{Z} \right) \]
This should be true for \( \Gamma = \text{GL}_n \mathbb{Z} \). But we also saw that
\[ H_{i,j}(C_n, C_{n-1}) = H_{i+j+1} \left( SL_n \mathbb{Z} , \text{St}(\mathbb{Q}^n) \right) \]
This suggests that there should be a close relation between
\[ (C_n, C_{n-1}) \text{ and } \left( \overline{X}_\Gamma, \varpi \overline{x}_\Gamma \right) \]
August 2, 1972

I want to achieve an understanding of homotopy inverse limits and especially how they relate to the descent problem in algebraic K-theory.

1. The Zariski descent problem in algebraic K-theory:

Let $X$ be a noetherian scheme. For each open set $U$ of $X$ we consider the abelian category $m(U) = \text{Modf}(\mathcal{O}_U)$.

These categories are the fibres of a fibred category $\mathcal{M}$ over the cat. Open$(X)$. $\mathcal{M}$ is a stack over the Zariski site of $X$.

Let

$$K_i(U) = K_i(m(U)) = \pi_{i+1} Q(m(U)).$$

These are presheaves on $X$; let $K_i$ be the associated sheaves. Then we should have

$$(K_i)_x = K_i(\text{Modf}(\mathcal{O}_X)).$$

One formulation of the Zariski descent problem is to construct a spectral sequence of the form

$$E_2^{pq} \Rightarrow H^p(X, K_q).$$

Slightly more general formulation: If $\mathcal{U} = \{U_i\}$ is a Zariski hypercovering of $X$, then there is a spectral sequence

$$E_2^{pq} = H^p(\nu \mapsto \text{H}^q K_{-q}(U_\nu)) \Rightarrow K_{-p-q}(X).$$
(2) \implies (1) by Verdier's theorem.

We can try to generalize (2) as follows. Suppose given a category \( I \) and a tensor \( P \) under \( I \) over \( X \). Thus for each object \( i \in I \), we have a sheaf \( P_i \) on \( X \), and for each map \( i' \to i \), a map \( P_i \to P_{i'} \).

We can try to generalize (2) as follows. Suppose \( I \) is a category and we have a contravariant functor \( i \mapsto U_i \) from \( I \) to sheaves on \( X \). Then we get a functor

\[
\begin{align*}
\text{Top}(X) & \xrightarrow{U_i} I^\vee \\
F & \mapsto F(U_i)
\end{align*}
\]

I want to assume for each \( x \), that the category of couples \((i, y) \in I \times U_i \_x\), is contractible; whence it should be the case that there is a spectral sequence

\[
E^{pq}_2 = H^p(I, i \mapsto H^q(U_i, F)) \Rightarrow H^{p+q}(X, F).
\]

for any abelian sheaf \( F \). Now (2) is a special case of a spectral sequence of the form

\[
E^{pq}_2 = H^p(I, i \mapsto K_{-q}(U_i)) \Rightarrow K_{-p-q}(X).
\]
Examples: of such functors $I^\circ \to \text{Top}(X)$.

a) Cribles. Take $I^\circ$ to be a covering crible $R$ with the evident functor $R \to \text{Open}(X)$. Given $x$, the category of couples $(U,y)$ is the directed set of open sets in $R$ containing $x$ (they exist since $R$ is covering). Thus the set of couples $(i,y)$ fibred over $I$ is filtering, hence contractible. In this case the spectral sequence in question is the Leray spectral sequence for the canonical morphism of toposi

$$\text{Top}(X) \to R^\wedge.$$ (I recall for any site, I can morph $R \to R^\wedge$ whose inverse image is "associated sheaf".)

b) Generalize a) by taking $R$ to be a presheaf whose associated sheaf is $e$, the final object of $\text{Top}(X)$. Again we have that the spectral sequence is the Leray spectral sequence for the morphism of toposi:

$$\text{Top}(X) \to R^\wedge.$$ Question: Can cohomology be computed using such presheaves $R$? Observe that the category of such presheaves (i.e. $\Rightarrow R \to e$ is covering) is cofiltering. Thus we can take the limit over $R$ in the spectral sequence.
\[ E_2^{p^+} = H^p(R, N^b(F)) \Rightarrow H^{p+q}(X, F) \]

and hopefully \( E_2^{p^+} \) will be zero in the limit, yielding an isomorphism

\[
\lim_{\to R} H^p(R, F) = H^p(X, F).
\]

c) Suppose \( X \) covered by two open sets \( U, V \) and consider the functor

\[
\begin{array}{ccc}
\text{I} & \rightarrow & U \cup V \\
\downarrow & & \downarrow \\
& & V
\end{array}
\]

Here \( I = \text{category of simplices in 1-simplex} \).

More generally we can consider \( n \) open sets.
August 4, 1972

descent problem

To understand flasque sheaves.

Let $X$ be a topological space, $F$ a sheaf (abelian) on $X$. One says $F$ is \textit{flasque} if whenever we have open sets $U \subset V$, then $F(V) \rightarrow F(U)$ is surjective. Let's make a list of possibilities:

1) $\forall U \subset V \quad U \cap V \rightarrow F(V) \rightarrow F(U)$

1') There is a covering $R$ of $X$ such that $U \cap V \subset R \rightarrow F(V) \rightarrow F(U)$.

2) \( H^1(X, F) = 0 \) for all locally closed $Z$ in $X$.

3) For any pointed sheaf of sets $S$, we have \( \textit{Ext}^+(\overline{Z} S, F) = 0 \) where \( \overline{Z} \) is the closure of $Z$.

3') For any abelian sheaf $G$ whose stalks are free over $\mathbb{Z}$ we have \( \textit{Ext}^+(G, F) = 0 \).

4) For any sheaf of sets $S$, we have...
\[ \text{Ext}^+ (ZS, F) = 0. \]

4') For any open set \( U \), \( H^+ (U, F) = 0 \).

Now we have the following relations:

1) \( \iff 1' \) See statement for local character of flasqueness.

2) If \( Z = U - V \), then

\[ F(U) \rightarrow F(V) \rightarrow H^4 (X, F) \rightarrow H^4 (U, F) \]

so \( 1) \Rightarrow 2) \) (using fact that flasque \( \Rightarrow H^4 (U, F) = 0 \) along \( U \)).

Clearly \( 3) \Rightarrow 2) \) since given \( V \subset U \), can take \( S \) to be a copy of \( U \) glued to \( X \) along \( V \), whence \( ZS = ZU + ZV \).

Conversely, given \( S \) and \( F \) flasque

\[ ZX \rightarrow ZS \rightarrow ZS \]

we have to prove \( H^+ (S, F) = 0 \) and that \( F(S) \rightarrow F(X) \).

The latter is clear as we have a map \( S \rightarrow X \), so we have to prove \( H^+ (S, F) = 0 \). But we can replace \( X \) by \( S \), so this results from the local character of flasqueness.

I know that \( 4') \neq 1) \). (Soft sheaves such as \( \mathbb{C} \) funs on a \( C^\infty \) manifold).
Proposition: An abelian sheaf \( F \) on a top space \( X \) is \textit{flasque} \( \iff \) for all open \( S \subseteq X \) we have \( H^+(S, f^*F) = 0 \).

Proof: \( (\Rightarrow) \) Let \( R \) be the crible over \( S \) such that an open \( U \subseteq S \) is in \( R \iff \) \( U \hookrightarrow X \) is an open immersion. Then for \( V \subseteq U \) in \( R \) we have 
\[
\Gamma(U, f^*F) = \Gamma(U, f_u^*F) = \Gamma(f_u(U), F)
\]
so by flasqueness of \( F \), \( \Gamma(U, f^*F) \to \Gamma(V, f^*F) \). So by the local character of flasqueness, \( f^*F \) is flasque on \( S \), whence
\[
H^+(S, f^*F) = 0
\]

\( (\Leftarrow) \). Given \( V \subseteq U \subseteq X \) and let \( S \) be the sheaf of sets
\[
S = U \sqcup V X
\]
Then we have a split exact sequence
\[
0 \to \mathbb{Z}_X \xrightarrow{\alpha} \mathbb{Z}_S \to \mathbb{Z}_U / \mathbb{Z}_V \to 0
\]
so
\[
0 = H^1(S, F) = \text{Ext}^1(\mathbb{Z}_S, F) = \text{Ext}^1(\mathbb{Z}_U / \mathbb{Z}_V, F).
\]
Since we have the long exact sequence
\[
H^0(U, F) \to H^0(V, F) \xrightarrow{\delta} \text{Ext}^1(\mathbb{Z}_U / \mathbb{Z}_V, F)
\]
it follows that \( F \) is flasque.
The preceding proposition works for a topos. Thus in SGA4, we have acyclic \(\Rightarrow\) flasque.

**Def.** A sheaf \(F\) (abelian) in a topos is flasque if

\[ H^+(S, F) = 0 \]

for all \(S\) in \(F\).

**Examples:**

1) \(G\)-sets. Then a \(G\)-module is flasque \(\Leftrightarrow\)

\[ H^+(H, M) = 0 \]

for all \(H < G\). Same as cohomological triviality.

2) I ordered set, \(E = \text{topos } I^\wedge\). \(F : I \to \mathbb{A}_{\text{ab}}\)

is flasque provided for every \(X \in I\) and crible \(U \subseteq I/\alpha\) we have

\[ F(\alpha) \to \lim_{\setminus U} F \]

(Here \(I\) is a topological space in which the open sets are crible, so what we are giving is the local criterion for flasqueness for all opens contained in an open of the form \(I/\alpha\)). Small example:

\[ \alpha \quad \beta \]

Then the condition amounts to

\[ F(\alpha) \to F(\alpha) \times F(\beta), \]

Another:

\[ \beta \quad \gamma \]

\[ U \quad V \]

then the condition is simply surjectivity:

\[ F(u) \to F(\alpha \cup U), \quad F(v) \to F(\beta \cup V). \]
3. Somthing similar for a category without loops?

Conjecture: Let $X$ be a top space and let $A$ be a stack over $X$. Assume that $V, V' \subseteq U$ the functor $A(U) \to A(V)$ is $h$-flat, meaning that $2$-base change $= h$-base change, i.e. any $2$-cartesian square

$$
\begin{array}{ccc}
C & \to & A(U) \\
\downarrow & & \downarrow \\
C' & \to & A(V)
\end{array}
$$

$(C \to C', A(U)_{equivalence})$ is $h$-cartesian. Then the conjecture asserts that the canonical functor

$$E(x, a) \Rightarrow hE(x, a)$$

is a sheaf.

Question: Given a top space $X$ and a functor from pairs $(U, V)$, $V \subseteq U$, of open sets, to long exact sequence

$$F^0(U, V) \to F^0(U) \to F^0(V) \Rightarrow F^0(V, U)$$

such that excision holds: $F^*(U, V) \Rightarrow F^*(U', V')$ if $U' \cap V' = U \cap V$. Does there then exist a spectral sequence

$$E^p_q = H^p(X, F^q) \Rightarrow F^{p+q}(X)?
It does not seem possible to produce the spectral sequence from such limited data even in the special case of a covering by 3 open sets.

\[ \bigcup U_0 \times \Delta^{(\dim \sigma)} \]

It seems one needs the skeleton of Segal's space which are not of the homotopy type of any open set.

---

Typical problem: Given A fibred over a group \( G \), when is \( \underline{\lim} A = \underline{\text{holim}} A \)? Want a kind of freeness condition which would hold in the Galois situation.
flashiness for simplicial sets:

Let $Y$ be a simplicial set, $F : (\Delta/Y)^\circ \to \mathcal{A}$ an abelian sheaf over $Y$, i.e. of the induced topos $\Delta^\circ/Y$. I claim that

$$F \text{ flasque } \iff \forall y \in \Delta/Y, \quad F(y) \to F(y).$$

Proof. Let $Z$ be a simplicial set over $Y$; I will show $H^1(Y, F) = 0$. Let $P$ be a torsor for $F$ over $Z$, and let $U$ be a simplicial subset over which $I$ sections of $P$, and such that $(U, s)$ is maximal. If $U < Z$, let $y$ be a simplex of $Z$ not in $U$ of least dimension. Then we have

$$\Delta(k)^\circ \to \Delta(k) \to \Delta(k)^\circ \to Z.$$

I claim that the arrow $\alpha$ is injective; this is a consequence of the Eilenberg-Zilber lemma which implies

$$\Delta(k)_m - \Delta(k)^\circ_m \to \mathbb{Z}_m \quad \forall m$$

as $y$ is a non-degenerate simplex of $Z$.

Now I section $s'$ of $F_y$; comparing restrictions to $\Delta(k)^\circ$ of $s$ and $s'$, we get an element $v \in F(y)$ which extends to $v' \in F(y)$ by hyp.
Thus modifying $s'$ via $Y'$, we get a section over the larger $U \Delta(k)^{A(k)} = U \circ \{y\} < Z$. Thus $U = Z$, by maximality, proving $H^1(Z, F) = 0$ for all $Z$.

Now given

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$$

with $F'$ satisfying the condition we have

$$0 \rightarrow F'(y) \rightarrow F(y) \rightarrow F''(y) \rightarrow 0$$

$$0 \rightarrow F'(y') \rightarrow F(y') \rightarrow F''(y') \rightarrow 0 \rightarrow H^1(y', F')$$

so $F$ satisfies the condition $\Leftrightarrow$ $F''$ does. From this it follows that for all $Z$, $H^1(Z, F) = 0$ and so $F$ is flasque.
August 10, 1972

Subdivision of a groupoid. If \( \mathcal{D} \) is a groupoid so is \( \text{Sd}\mathcal{D} \).

\[
\begin{align*}
\alpha &\quad \beta \\
\alpha^{-1} &\quad \beta^{-1}
\end{align*}
\]

\[\nu = \beta \alpha \nu\]
\[\Rightarrow \beta^{-1} \nu \alpha^{-1} = \nu\]

Thus the canonical functor

\[\text{Sd}\mathcal{D} \rightarrow \mathcal{D}\]

must be an equivalence of groupoids, the inverse functor being

\[
\begin{array}{ccc}
X & \rightarrow & X \\
\downarrow f & & \downarrow f \\
Y & \rightarrow & Y
\end{array}
\]

Consequence: I conjectured previously that for \( C \) finite

\[
[C, C'] = \lim_{m \to \infty} \pi_0 \text{Hom}(\text{Sd}^m C, C')
\]

The heuristic proof went as follows:

\[
\begin{align*}
\text{N Hom} (\text{Sd}^m C, C') &= \text{Hom} (\text{N(Sd}^m C), C') \\
&= \text{Hom} (\text{Sd}^m(NC), C') \\
&= \text{Hom} (NC, \text{Ex}^m(NC'))
\end{align*}
\]
Thus \( \lim_{m} \pi_0 \text{Hom} (\text{Sd}^m C, C') \Rightarrow \pi_0 \text{Hom} (\text{NE}, \text{Ex}^\infty (\text{NE}')) \)
\[= [C, C']. \]

(Recall \( N(\text{Sd} C) = \text{Sd}(\text{NE}) \) never established).

The last step seems to require \( \text{NE} \) to be finite, which signifies that \( \mathcal{A} \) is infinite and \( C \) has no loops.

The above shows that the conjecture is false unless \( \text{NE} \) is finite. In effect if \( G \) is a finite group then we have seen that \( \text{Sd} G \) and \( G \) are equivalent categories. But there are non-trivial \( x \in [G, K(\pi, n)] \), not coming from a functor, e.g. \( K(\pi, n) \) might be a simplicial complex.
August 8, 1972.

Consider the category of simplicial $G$-sets, where $G$ is a group. We try to make it into a model category by taking

- cofibrations = injective maps
- fibrations = fibrations of underlying s. sets
- weak equivalences = homotopy equivalences of underlying s. sets
- but not equivariant homotopy equivalence.

The corresponding fibrations will be called flexible maps to avoid confusion. Thus a map of simplicial $G$-sets $X \to Y$ is flexible provided it has the RLP

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

with respect to all injective $A \longrightarrow B$ which are homotopy equivalences.

Example: Let $Q \longrightarrow Q'$ be a map of s. $G$-sets whose underlying map of s. sets is a Kan fibration, and let $P$ be a free s. $G$-set. Claim

\[
\begin{array}{ccc}
\text{Ham}(P, Q) & \longrightarrow & \text{Ham}(P, Q') \\
\downarrow & & \downarrow \\
A \times P & \longrightarrow & Q \\
\downarrow & & \downarrow \\
B \times P & \longrightarrow & Q'.
\end{array}
\]

is a flexible map. In effect we have to check
but \( A \times P \to B \times P \) will be a \( G \)-cofibration in the strict sense (obtained by successive \( G \times \Delta(k) \to G \times \Delta(k) \) attaching), so the lifting has been proved in your earlier work. (Actually if \( P' \leq P \) is an injection of free \( s \cdot G \)-sets we have more generally that)

\[
\text{Hom}(P, Q) \to \text{Hom}(P, Q') \times \text{Hom}(P', Q)
\]

is a homotopy.

Consequences: suppose \( X \to Y \) is a homotopy.

1.) for any \( H \leq G \), \( X^H \to Y^H \) is a fibration.

This is the RLP lift

\[
G/H \times V(n,k) \to G/H \times \Delta(n).
\]

2.) When \( Y = pt \), then for any \( G \)-set \( S \), the map

\[
P_G \times S \to S
\]

is a homotopy. If \( S' \to S \) is a homotopy equivalence of \( G \)-sets, we can factor it

\[
S' \subset S' \times \Delta(1) \to S \to S
\]

where the second map is a strict \( G \)-equivariant homotopy equivalence. Thus since the first
Definition: Two maps of $G$-sets $X \rightarrow Y$ are strictly homotopic if they are in the same component of $\underline{\text{Hom}}(X, Y)$. (Analogous to equivariant homotopy of $G$-spaces.)

2) Suppose $A \rightarrow B$ is an injective heg.

Then so is

$$\left( A \times \Delta(n) \right) \cup \left( B \times \Delta(n) \right)^c \rightarrow B \times \Delta(n)$$

whence we can conclude for $X \rightarrow Y$ flaks that $\underline{\text{Hom}}(B, X) \rightarrow \underline{\text{Hom}}(A, X) \times \underline{\text{Hom}}(A, Y)$ is a heg + fibration. (Also when $A \rightarrow B$ is just injective it is a fibration.)

3) Suppose $S' \rightarrow S$ is a heg of $G$-sets + factor it

$$S' \subset \left( S' \times \Delta(n) \right) \cup S \rightarrow S.$$  

First map is an injective heg, second map a strict heg. Thus if $X \rightarrow pt$ is flakle

$$\underline{\text{Hom}}_G(S, X) \rightarrow \underline{\text{Hom}}_G(M_{pt}, X) \rightarrow \underline{\text{Hom}}_G(S', X)$$

In particular, applying this to the map $PG \times S \rightarrow S$, we have a heg

$$\underline{\text{Hom}}_G(S, X) \rightarrow \underline{\text{Hom}}_G(PG \times S, X)$$
for flask $X$. Special case $S = pt$

$$X^G \xrightarrow{\text{fixpts}} \text{Hom}_G(PG, X)$$

Example: Suppose we work with topological $G$-spaces. Then the condition that for all $G$-spaces $S$

$$\pi_0 \text{Hom}_G(S, X) \cong \pi_0 \text{Hom}_G(PG \times S, X)$$

is equivalent to

$$X \xrightarrow{\text{Hom}} (PG, X)$$

being a $G$-homotopy equivalence. If I remember correctly, Segal showed that a map of (reasonable) $G$-spaces $X \rightarrow Y$ is a $G$-homotopy equivalence iff for all $H \subset G$, $X^H \xrightarrow{\text{Hom}} Y^H$ is a homotopy equivalence. Maybe he assumed $G$ compact.

Conclude: If we work with reasonable $G$-spaces then flaskness for a $G$-space signifies that

$$X \xrightarrow{\text{Hom}} (PG, X)$$

is a $G$-homotopy equivalence, and this under suitable conditions (e.g. $G$ finite) means that for each $H \subset G$, $X^H = \text{homotopy H-fixed points of } X$.

More carefully: Suppose $X \rightarrow \text{Hom}(PG, X)$ is
a $G$-hep, and let $A \subseteq B$ be a injective hep of $G$-spaces. Then given

$$A \rightarrow X \rightarrow \text{Hom}(P_G, X)$$

and using the homotopy inverse, we know a $G$-homotopic map to $A \rightarrow X$ extends to $B$. Thus we have to establish the HEP

$$A \times I \cup B \times O \rightarrow X$$

but this is clear by standard arguments, i.e. $A \times I \cup B \times O$ has a nbd. which equivariantly deforms down, etc.

Characterization of flash hep's:

**Prop.** TFAE for a map of s. G-sets: $X \rightarrow Y$

(i) RLP wrt all injective $A \rightarrow B$.

(ii) RLP wrt maps

$$G/H \times \Delta(n) \subseteq G/H \times \Delta(n)$$

for all $n \geq 0$ and

$$\forall H \subseteq G, \quad X^H \rightarrow Y^H$$ is a filtration hep.
Proof: \( \text{Hom}_G(G/H \times Z, X) = \text{Hom}_Z(Z, X^H) \) for all s. set \( Z \), hence (ii) and (ii)' are equivalent. Clearly (i) \( \Rightarrow \) (ii) and to prove converse we use skeletal decomposition of \( A \rightarrow B \). The point is that if \( b \in B_k \) is a simplex of minimal dim. in \( B \) but not in \( A \), and \( H \) is the stabilizer of \( b \), then we have

\[
\begin{array}{ccc}
G/H \times \Delta(k) & \rightarrow & G/H \times \Delta(k) \\
\downarrow & & \downarrow \\
A & \rightarrow & A \cup b \subset B
\end{array}
\]

In effect, we must show-

\[
G/H \times \Delta(k)_m - G/H \times \Delta(k)_m \rightarrow B_m.
\]

i.e. that the elements \( g_i \cdot S \times b \), \( G = \coprod g_i \cdot H \) are distinct. The point is that \( S \times b \) clearly has stabilizer \( H \), and by E-Z lemma for \( G \subset B \) we know the orbits \( S \times G \cdot b \) are distinct. So clear.

Proposition: Suppose \( X \) is a simplicial \( G \)-set such that for all \( H \subset G \),

Proposition: \( X \) is a flat simplicial \( G \)-set iff (i) \( VH \subset G \), \( X^H \) is a Kan complex

(ii) \( X \rightarrow \text{Hom}(PG, X) \) is \( G \)-equivariant.
Proof: Necessity done already. Conversely, given an injective map \( A \rightarrow B \), we get from (i):

\[
\begin{align*}
A & \rightarrow X \\
\downarrow & \\
B & \rightarrow \text{Hom}(B, X)
\end{align*}
\]

and we have to establish an HEP:

\[
\begin{align*}
A & \rightarrow X^I \\
\downarrow & \\
B & \rightarrow X
\end{align*}
\]

But (i) \( \Rightarrow (X^I)^H \rightarrow X^H \) is a fibration \( \forall H \), so done by first proposition.

Special case: \( X = K(M, g) \) where \( M \) is a G-module. More generally, suppose \( X \) is a simplicial G-module. Then in testing

\[
\begin{align*}
A & \rightarrow X \\
\downarrow & \\
B
\end{align*}
\]

we can replace \( A \) by \( ZA \), and replace everything by chain complexes of G-modules

\[
\begin{align*}
C^N(A) & \rightarrow \text{Norm}(X) \\
\downarrow & \\
C^N(B)
\end{align*}
\]
By similar arguments to the above, flatness should amount to a $G$-sheaf

$$N \rightarrow \text{Hom}(P, M) \text{ truncated}$$

where $P \rightarrow \mathbb{Z}$ is a free $G$-resolution, here $N$ is a complex of $G$-modules, and the $\text{Hom}$ has to be truncated in degree 0:

$$\cdots \rightarrow \text{Hom}^{-1}(P, N) \rightarrow \mathbb{Z}^0 \text{Hom}(P, N) \rightarrow 0 \rightarrow 0 \cdots$$

so if $X = K(M, g)$, $N = \text{Ner}(X) = M[q]$, and we are considering the complex of cochains

$$0 \rightarrow M \rightarrow C^0(G, M) \rightarrow \cdots \rightarrow \mathbb{Z}^8(G, M) \rightarrow 0$$

conclude: if $M$ is a $G$-module, $K(G, g)$ is flat if the complex of

$$0 \rightarrow M \rightarrow C^0(G, M) \rightarrow \cdots \rightarrow \mathbb{Z}^8(G, M) \rightarrow 0$$

is homotopically trivial, that is, has a contracting homotopy.

$$\text{Map}(G, M) \rightarrow \text{Map}(G^2, M) \rightarrow \cdots \rightarrow \mathbb{Z}^8 \text{Map}(G, M) \rightarrow 0$$

$$I^0(G, M) \rightarrow I^1(G, M) \rightarrow \cdots \rightarrow \mathbb{Z}^8(G, M) \rightarrow 0$$

conclude: $K(M, g)$ is a flask as a $G$-set iff the complex

$0$
The sequence has an equivariant contracting homotopy. Observe that for \( g \geq 1 \) this implies \( M \) is a direct summand of \( I^0(G, M) = \text{Map}(G, M) \). I claim this means that the complex

\[
0 \to M \to I^0(G, M) \to I^1(G, M) \to \cdots \to Z^\infty(G, M) \to 0
\]

is contractible. Indeed to test contractibility of a complex all one has to show is that it remains exact for the functor \( \text{Hom}(J, \cdot) \) for all \( J \). This will be so for \( I^0(G, M) \), so if \( M \) is a direct summand of \( I^0(G, M) \), it's clear.

---

**Check:** Take \( PG = \text{New}_G \) of category \((G, G)\) with objects \( g \in G \) and a unique morphism \( g \to g' \). Then

\[
\text{Hom}(PG, K(M, 1)) = \text{Hom} (\text{New}(G, G), \text{New} M)
\]

\[
= \text{New} (\text{Hom}(G, G), M)
\]

Now any object of \( \text{Hom}(G, G), M \) consists of a function \( f : G^2 \to M \) such that

\[
f(g, g) = 0 \quad \forall g
\]

\[
f(g_1, g_2) + f(g_2, g_3) = f(g_1, g_3)
\]

i.e. \( f \in Z^1(G, M) \). And a natural transf. from \( f \to f' \) is a function \( h : G \to M \):

\[
f(g, g') + h(g') = f'(g, g') + h(g)
\]
$$\bullet \quad (f-f')(g,g') = h(g) - h(g').$$

Thus $\text{Hom}((G,G), M)$ is the category associated to the complex

$$I^0(G,M) \longrightarrow Z^1(G,M).$$

We have an evident map of the complex

$$M \longrightarrow 0$$

into this representing the functor

$$M \longrightarrow \text{Hom}((G,G), M).$$

To say this admits a retraction would signify that $M$ is a direct summand of $I^0(G,M)$.

Conclude: TFAE for a $G$-module:

(i) $K(M,1)$ is a flank simplicial $G$-set
(ii) $K(M,g)$ for all $g \in G$
(iii) $M$ is a direct summand of the (co)induced module $\text{Map}(G, M)$.

This is a very strong condition on $M$, and not the same as cohomological triviality, except if $G$ is finite.
Generalize the preceding to small categories.

Let $I$ be a small category; call objects of $\text{Hom}(I, \text{Sets})$ simply $I$-sets. We then can consider simplicial $I$-sets, which are the same thing as functors $i \mapsto X_i$ from $I$ to simplicial sets.

In my HA notes I made simplicial $I$-sets into a model category by calling the fibrations maps $X \rightarrow Y$ such that $X_i \rightarrow Y_i$ is a fibration for each $i$. Here I want to do something different.

Analogue of PG. $PI$ is the simplicial $I$-set with

$$(PI)_n = \text{Ar}_{n+1}I = \{ \leftrightarrow \}_{n+1 \text{ arrows}}$$

and evident left $I$ action. As a functor $I \rightarrow \Delta^+$ it sends

$$i \mapsto \text{New}(I/i).$$

Thus we see that

$$(\ast) \quad \frac{\text{Hom}(PI, X)}{I}$$

is what Kan-Drinfeld denote

$$\text{holim}(X).$$

Meaning of $\ast$.

$$\frac{\text{Hom}(PI, X)}{I} = \text{pr}_{2\times} \text{Hom}(\text{PI}, X)_{(I^e \times \Delta)^{\vee}}$$
Conjecture: A simplicial I-set is flasque provided the canonical map

$$X \rightarrow \text{Hom} (P_1, X)$$

is a I-hseq, and also $\text{Hom}_I (S, X)$ is a Kan complex for all I-sets $S$.

(unknown quantity: whether an injection $A \hookrightarrow B$ of simplicial I-sets can be broken down as in the prop on p. 5.)
August 9, 1972

Recall Postnikov systems: suppose \( X \rightarrow Y \) is a Kan fibration and define a tower

\[
X \rightarrow \cdots \rightarrow F^m X \rightarrow F^{m-1} X \rightarrow \cdots \rightarrow F^0 X \rightarrow Y
\]
as follows. \( F^m X \) is quotient of \( X \) by the equivalence relation: \( x, x' \in X \) are equivalent \( \iff \) they have same image in \( Y \) and the same \( m \)-skeleton. It is known that the above is a tower of fibrations. Moreover, the map \( F^m X \rightarrow F^{m-1} X \) may be factored canonically

\[
\begin{array}{ccc}
F^m X & \xrightarrow{\text{asph.}} & F^m X \\
\text{fib.} & & \text{fib.}
\end{array}
\]

by identifying simplices which are homotopic relative to \( X \) and \( (m-1) \)-skeletons. The minimal fibration has the fibre \( K(\pi_m, m) \), where \( \pi_m = \pi_m(\text{fibre of } X \rightarrow Y) \).

The preceding Postnikov factorizations are canonical and so can be applied as follows. Let \( X \) be a simplicial \( G \)-set which is a Kan complex. We have already seen that given a fibration of pointed simplicial \( G \)-sets

\[
F \rightarrow E \rightarrow B
\]
that we get a fibration of simplicial sets

\[
\underline{\text{Hom}}_G (P_G, F) \rightarrow \underline{\text{Hom}}_G (P_G, E) \rightarrow \underline{\text{Hom}}_G (P_G, B)
\]
More generally let $I$ be a small category and $X 	o Y$ a map of simplicial $I$-sets, such that $X_i 	o Y_i$ is a fibration for all $i$. Then we want to show the induced map

$$\text{Hom}_I(PI, X) \to \text{Hom}_I(PI, Y).$$

is a fibrations.

Recall $PI$ is the simplicial $I$-set, $n \mapsto A_{n+1}$; alternatively the functor $i \mapsto 	ext{Nerve}(I/i)$. We denote

$$\text{Hom}(PI, X)$$

the internal hom in $(I^\circ \times \Delta)^\wedge$. Then

$$\text{Hom}_I(PI, X) = \text{pr}_2^\ast \text{Hom}(PI, X).$$

From this formula it is clear that if $Y$ is a pointed $I$-set, whence a map $pt \to Y$, (pt= final ob. of $(I^\circ \times \Delta)^\wedge$) then

$$\text{Hom}_I(PI, pt \times_y X) \to \text{Hom}_I(PI, X)$$

is cartesian. (Better: $X \mapsto \text{Hom}_I(PI, X)$ commutes with lim's).

Let $A \to B$ be an injective map in $\Delta^\wedge$.

Then we have

$$\text{Hom}_0(B, \text{Hom}_I(PI, X)) = \text{Hom}_{(I^\circ \times \Delta)^\wedge}(\text{pr}_2^\ast B, \text{Hom}(PI, X))$$

= $\text{Hom}_{(I^\circ \times \Delta)^\wedge}(PI, \text{Hom}(\text{pr}_2^\ast B, X))$
\[ \text{Hom}(p_1^* B, X) = X^B \]

But since \( X \to Y \) is a fibration (object-wise)

\[
X^B \to X^A \times Y^A \quad Y^B
\]

is a fibration + heq (object-wise). Thus \( \star \) will be a fibration with fibre \( \text{Hom}_I(P^I, p^*_1 x YX) \) provided we show:

\[
\begin{array}{ccc}
\text{PI} & \to & Y \\
\downarrow & & \downarrow \\
X & \to & f
\end{array}
\]

whenever \( f \) is a fib + heq (object-wise).

The proof of this seems to require a skeletal induction.

**Example:** \( I = G \). Then \( PG = \text{Nerve}(G, G) \).

In order to construct the lifting we proceed by induction & constructing the lifting over \( PG(k) = \) the inverse image of the \( k \)-skeleton of \( BG = \text{Nerve} G \).

\[
\begin{array}{ccc}
\amalg G \times \Delta(k)^* & \to & \amalg G \times \Delta(k) \\
\uparrow & & \uparrow \\
PG(k-1) & \to & PG(k)
\end{array}
\]

Would work for any free simplicial \( G \)-set.

Concept of a free simplicial \( I \)-set \( P \). For each \( n \), \( P_n \) can be interpreted as an \( I \)-set, i.e. a covariant functor \( i \to P_n(i) \) from \( I \) to sets. We can speak of the degenerate \( I \)-subset \( P_n^\deg \subseteq P_n \)
and to say $P$ is free means that
\[ P_n = P_n^{\text{deg}} \rightarrow \text{representable functors} \]
so it's all pretty clear.

**Conclusion:** Suppose $X$ is a pointed simplicial $I$-set defined such that each $X_i$ is a Kan complex. Then the Postnikov tower
\[ X \rightarrow F^1X \rightarrow F^2X \rightarrow \cdots \rightarrow \text{pt} \]
of $X$ will give rise to a tower of fibrations
\[ \rightarrow \text{Ham}(PI, F^mX) \rightarrow \text{Ham}(PI, F^{m-1}X) \rightarrow \cdots \]
and hence to a spectral sequence (Kan-Bousfield style).

We can probably define a free simplicial $I$-set to be the total $s$-set of an $I$-tower
\[ \begin{array}{ccc}
\text{Obj}I & \xrightarrow{f} & P \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & \text{Ob}I \times P \\
\end{array} \]
such that each stalk $P_x$, $x \in B$ is representable.
Situation: The problem (where $I$ is a group $G$): given a category $A$ fibred over $G_{A}$, to find agreeable conditions which imply that there is a spectral sequence

$$E^{1}_{2} = H^{p}(G, \pi_{q} A) \Rightarrow \pi_{p-q} \left( \lim_{\rightarrow G} A \right)$$

(The spectral sequence lives in the range $0 \leq p \leq q$.)

Present program:

1. Replace $A$ by an equivalent pointed $G$-category, then by a pointed simplicial $G$-set, then by a pointed simplicial $G$-set $X$, satisfying the extension condition (forgetting the $G$-action).

2. Then we have the s.s.

$$\text{holim}_{G} X = \text{Hom}_{G}(PG, X)$$

and the spectral sequence of Bousfield-Kan

$$H^{p}(G, \pi_{q} A) \Rightarrow \pi_{p-q} \left( \text{holim}_{G} X \right).$$

Thus what we need now is something which will allow us to identify

$$\lim_{\rightarrow G} A \text{ and } \text{holim}_{G} X$$

up to homotopy. Now this seems hard, and
a better idea perhaps is to understand the spectral sequence and its construction with the hope that it could be done more directly working with $A$.

3. First approach was to form the Postnikov-tower of $X$ (defined because $X$ is a Kan complex)

$$
X \longrightarrow F_0^mX \longrightarrow F_1^{m-1}X \longrightarrow \cdots
$$

and then use the associated tower of fibrations

$$
\longrightarrow \text{Hom}_q(PG,F^mX) \longrightarrow \text{Hom}_q(PG,F^{m-1}X) \longrightarrow \cdots
$$

to construct the spectral sequence.

This approach has the defect of requiring one to work with $X$. It might be better to understand:


Review B-K theory:

Examples. 1. If $P, X$ two simplicial sets, then

$$
pq \longmapsto \text{Hom}(P_p,X_q) = Y_{pq}
$$

is a cosimplicial $A$-set. The total $s$-set is

$$
\text{Tot}(Y) = \text{Hom}_\Delta(\Delta, Y) = \text{pr}_2* \text{Hom}(\Delta, Y)
$$

where $\Delta : p \longmapsto \Delta(p)$. (Here $\Delta$ is an efficient version of $PD$.)
Thus \[
\text{Ham}(\Delta(n), \text{Tot}(X)) = \text{Ham}((\Delta \times \Delta) \wedge (p_2^* \Delta(n), \text{Ham}(\Delta, Y)) \wedge \text{Ham}(\Delta \times p_2^* \Delta(n), Y))
\]

An \(n\)-simplex of \(\text{Tot}(p \mapsto \text{Ham}(\Delta, P_p, X))\) is a compatible family of maps
\[
\Delta(p) \times \Delta(n) \rightarrow \text{Ham}(P_p, X)
\]

i.e. a map
\[
P \times \Delta(n) \rightarrow X.
\]

Thus we have
\[
\text{Tot}(p \mapsto \text{Ham}(P_p, X)) = \text{Ham}((\Delta \wedge P_p, X)).
\]

2. Let \(P, X\) be simplicial \(G\)-sets, and calculate
\[
\text{Tot}(p \mapsto \text{Ham}_G(P_p, X))
\]
An \(n\)-simplex is a compatible family of \(G\)-maps
\[
P \times \Delta(p) \times \Delta(n) \rightarrow X
\]

so we get
\[
\text{Tot}(p \mapsto \text{Ham}_G(P_p, X)) = \text{Ham}_G(P, X).
\]
Skeleta: Denote by $P^{[s]}_s$ the $s$-skeleton of the simplicial set $P$. B-K define

$$\text{Tot}_s(Y) = \frac{\text{Hom}(\Delta^{[s]}, Y)}{\hat{A}}$$

$$= \text{Ker} \left\{ \prod_p \text{Hom}(\Delta(p)^{[s]}, Y_p) \right\}$$

Thus an $n$-simplex of $\text{Tot}_s(p \mapsto \text{Hom}(P_p, X))$ is a compatible family

$$\Delta(p)^{[s]} \times \Delta(n) \to \text{Hom}(P_p, X)$$

i.e.

$$P_p \times \Delta(p)^{[s]} \times \Delta(n) \to X$$

i.e. a map

$$P^{[s]} \times \Delta(n) \to X.$$

Thus

$$\text{Tot}_s(p \mapsto \text{Hom}(P_p, X)) = \text{Hom}(P^{[s]}, X).$$

Similarly we have for a simplicial $G$-sets

$$\text{Tot}_s^G(p \mapsto \text{Hom}_G(P_p, X)) = \text{Hom}_G(P^{[s]}, X)$$

where here $P^{[s]}_s$ is the inverse image of the $s$-skeleton of the orbits $P/G$. 
Conclusion: Since problems with extension condition can perhaps be circumvented by suitable subdivision, it appears that the essential problem of whether

\[ \lim_{G} A \quad \text{and} \quad \lim_{G} 'A' \]

coincide cannot be treated by Brouwer-Kan methods.

It seems that the correct yoga is this: On the category of \( G \)-spaces (polyhedra?) we have defined an analogue of KR-theory. We must then prove that

\[ KR(X) \rightarrow KR(PG \times X). \]

It seems that this requires something like periodicity.

You should determine why this works in the Zariski case.
August 12, 1972

The barycentric subdivision of a simplicial set

start with

\[ \text{Sd } \Delta(n) = \text{the nerve of the category of simplices} \]

\[ = \text{the s. set belonging to the barycentric subdivision of } \Delta(n). \]

Thus a simplex of \( \text{Sd } \Delta(n) \) is a chain

\[ \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k \]

of non-empty subsets of \( \{0, \ldots, n\} \).

Note that any map \( \Delta(m) \rightarrow \Delta(n) \) induces a map \( \{0, \ldots, m\} \rightarrow \{0, \ldots, n\} \) and hence a map

\[ \text{Sd } \Delta(m) \rightarrow \text{Sd } \Delta(n), \]

whence we have a functor

\[ \begin{array}{ccc}
\Delta & \rightarrow & \text{Sd} \\
\downarrow & & \downarrow \\
\Delta^\wedge & \rightarrow & \Delta^\wedge
\end{array} \]

Claim \( \text{Sd} \) extends to \( \text{Sd}^\wedge \) in a unique way compatible with arbitrary \( \lim \)'s.

Proof: Define \( \text{Ex} : \Delta^\wedge \rightarrow \Delta^\wedge \) by the formula
\[ \text{(*)} \quad \text{Hom}(\Delta(n), \text{Ex}(X)) = \text{Hom}(\text{Sd } \Delta(n), X). \]

Then define \text{Sd} as we must the formula

\[ \text{Sd } Y = \lim_{\Delta(n) \to Y} \text{Sd } \Delta(n) \quad \text{(Kan extension)} \]

the ind. limit being taken over the cat, \(\Delta / Y\).

Passing to the limit in (*), we get

\[ \text{Hom}(Y, \text{Ex } X) = \text{Hom}(\text{Sd } Y, X) \]

which proves (**), defines a functor \(\text{Sd}\) with a right adjoint, hence compatible with arbitrary \(\lim\)'s.

Now, let us use the skeletal decomposition of \(X\)

\[ \begin{array}{ccc}
\coprod_{k \in \text{sk}_{k-1} X} X_k & \xrightarrow{\text{coeq}} & \coprod_{k \in \text{sk}_k X} X_k \\
\downarrow \text{coeq} & & \downarrow \text{coeq} \\
\coprod_{k \in \text{sk}_k X} X_k & \xrightarrow{\text{coeq}} & \coprod_{k \in \text{sk}_k X} X_k \\
\end{array} \]

\[ X = \bigcup_{k} \text{sk}_k X. \]

Then we get cocartesian squares

\[ \begin{array}{ccc}
\coprod_{k} \text{Sd } \Delta(k) & \xrightarrow{\text{coeq}} & \coprod_{k} \text{Sd } \Delta(k) \\
\downarrow & & \downarrow \\
\text{Sd } (\text{sk}_{k-1} X) & \xrightarrow{\text{coeq}} & \text{Sd } (\text{sk}_k X) \\
\end{array} \]
Let $z$ be a non-degenerate simplex of $sd(X)$. Then there is a least $k$ such that $z \in sd(\Delta(k))$. There is then a unique non-deg. $k$-simplex $x: \Delta(k) \to X$ such that $z$ is in the image of $sd \Delta(k) \to sd X$, and not in the image of the composite map

$$sd \Delta(k) \subset sd \Delta(k) \xrightarrow{sd(x)} X$$

Better, there is a unique non-deg. simplex $x: \Delta(k) \to X$ such that $z$ is in the image of the map

$$sd \Delta(k) \xrightarrow{\text{Sk} \Delta(k)} \xrightarrow{sd(x)} X$$

the injectivity and uniqueness of $x$ being evident from (+). thus we see that $z$ may be identified with a chain of faces

$$x_0 \xleftarrow{<} x_1 \xleftarrow{<} \cdots \xleftarrow{<} x_8 = x$$

(< means proper face).

Conclude: If $X$ has the property that any face of a non-degenerate simplex is non-degenerate, then $sd X$ is the nerve of the category of non-degenerate simplices of $X$ (full subcat of $\Delta/X$ consisting of non-deg. simplices).

In general, there is a map

$$\text{Nerve (non-deg. of } X) \xrightarrow{\cong} sd X$$

which is injective, but not necessarily onto, e.g.
if \( X = \Delta(2) / \Delta(2) \), then there are two only non-degenerate simplices forming a category:

\[
\begin{array}{ccc}
\circ & \xrightarrow{d_0} & \circ \\
\circ & \xrightarrow{d_1} & \circ \\
\circ & \xrightarrow{d_2} & \circ \\
\end{array}
\]

with different homotopy type.

Idea: To modify \( X \) in a fashion analogous to replacing a category by a category without loops.

Given a simplicial set \( X \), let \( X' \) be the simplicial subset of \( X \times \Delta(\infty) \) consisting of pairs \((x, y)\) of the same dimension such that: If \( y = y' \circ y \) with \( y \) a surjective map, then \( x = \eta^* x' \) for some \( x' \). In other words, the non-degenerate simplices of \( X' \) are the pairs \((x, y)\) with \( y \) non-degenerate.

I want to prove that \( X' \to X \) is a deg. It will suffice to show for each non-degenerate simplex \( z : \Delta(n) \to X \), that the fibre \( \tilde{X} \) is contractible.

\( \tilde{X}_z \) consists of pairs \((y, y)\) such that \( (\phi^* z, y) \in \tilde{X} \). Suppose \( y = y' \circ y' \) with \( y \) surj. and \( y' \) non-deg. Then we have \( y^* z = \eta^* x' \) for some \( x' \). Write \( y = \varepsilon' y' \), whence

\[
\eta^* \varepsilon' z = \eta^* z
\]

\[
\eta^* \eta^* w
\]

Can't conclude \( \phi = \eta y \) because the face of \( z \) might be degenerate.
August 13, 1972

Why any p-set is replaceable by a simplicial complex.

Conjecture (perhaps proved by Whitehead): The geometric realization of a p-set is triangulable in the following way: The image of $\Delta^{n} \times \Delta^{n}$ is homeomorphic with the realization of the second barycentric subdivision modulo the

Suppose given a non-degenerate simplex $\Delta^{n} \times \Delta^{n}$. Then the claim is that if we identify vertices of the 2nd barycentric subdivision of $\Delta^{n}$ according to $\partial R$, we have a simplicial complex.

Ex. 1.

Ex. 2.
The point of the construction: Ordered simplicial complexes (compatible linear orderings on simplices) form a full subcategory of $\Delta^\wedge$ closed under products, subobjects, and containing $\Delta(k)$. What one should show is that if $R$ is an equivalence relation on $K$, then in the category the quotient of the 2nd barycentric subdivision of $K$ by $R$ exists.

Remains to understand the reduced subdivision functor $\text{Sdr}$ on simplicial sets, the one related to subdivision of a category. Any non-deg. simplex of $\text{Sdr}(X)$ would consist of a non-deg. simplex $\tau \in X_k$ + a simplex in $\text{Sdr}(\Delta(k)^n) \to \text{Sdr}(\Delta(k)^n)$ which would be a chain of intervals

$$\tau_0 < \tau_1 < \ldots < \tau_n \in \mathbb{Z} [k]$$

with $\tau_n = [0, k]$. Thus any non-deg. simplex of $\text{Sdr}(X)$ is uniquely representable as a chain

$$\tau_0 \to \tau_1 \to \ldots \to \tau_n$$

where $\tau_n$ is non-deg. in $\Delta X$, and where each arrow is an interval face. Which is proper.

Observe it will not generally be the case that the face of a non-degenerate simplex is non-degenerate. However if $X = \text{Nerve}(C)$, then any faces of a non-degenerate simplex is non-degenerate.
August 13, 1972: Proof that \( \text{Sd} \text{(Nerv)} = \text{Nerv} \text{(Sd)} \).

Reduced subdivision:

\[
\text{Sd} \Delta(n) = \text{Sd} \text{ Nerv}([n]) = \text{Nerv} \text{ (Sd)[n]}]
\]

Thus a non-deg. simplex in \( \text{Sd} \Delta(n) \) is a chain of intervals

\[ \tau_0 < \ldots < \tau_q \]

of the ordered set \([n]\). Now the observation is that

\[ \Delta(n) \rightarrow \text{Sd} \Delta(n) \]

is a functor

\[ \Delta \rightarrow \Delta^\wedge \]

hence extends to a pair of adjoint functors

\[ \Delta^\wedge \rightleftarrows \Delta \]

given by formulas:

\[ \text{Hom} (\Delta(n), \text{Exr} X) = \text{Hom} \text{ (Sd} \Delta(n), X) \]

\[ \text{Sd} \text{x} (X) = \lim_{\Delta(n) \rightarrow X} \text{Sd} \Delta(n). \]

Since \( \text{Sd} \) commutes with \( \lim \)'s it will be compatible with skeletal decomposition, so we have a cocartesian square.
\[ \text{II} \, \text{Schr} \, \Delta(n)^* \quad \longrightarrow \quad \text{II} \, \text{Schr} \, \Delta(n) \]

\[ \downarrow \quad \downarrow \]

\[ \text{Schr} \, (sk_{n-1}X) \quad \longrightarrow \quad \text{Schr} \, (sk_nX) \, . \]

\[ \text{II} \, \text{taken over} \quad X_n^\text{un}. \]

First have to check \( \text{Schr} \, \Delta(n)^* \subseteq \text{Schr} \, \Delta(n) \).

Start with cartesian square

\[ \begin{array}{ccc}
\text{II} & \text{II} & \text{II} \\
\downarrow & \downarrow & \downarrow \\
\text{Schr} \, \Delta(n-1) \times_{\Delta(n)} \text{Schr} \, \Delta(n-1) & \longrightarrow & \text{Schr} \, \Delta(n-1) \\
\downarrow & \downarrow & \\
\partial \Delta(n-1) & \longrightarrow & \Delta(n) \\
\end{array} \]

Suppose can prove

(?) \( \text{Schr} \, (\partial_i \Delta(n-1) \times_{\Delta(n)} \partial_j \Delta(n-1)) = \text{Schr} \, (\partial_i \Delta(n-1)) \times \text{Schr} \, (\partial_j \Delta(n)) \).

Abbreviate the square to

\[ \begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y \\
\end{array} \]

so that (?) implies

\[ \begin{array}{ccc}
\text{Schr}(X \times_Y X) & \longrightarrow & \text{Schr}(X) \\
\downarrow & & \downarrow \\
\text{Schr}(X) & \longrightarrow & \text{Schr}(Y) \\
\end{array} \]

is cartesian. Therefore follows that \( \text{if } F \, \text{is finite} \, \text{then} \, F \times_F X \, \text{is finite} \).
If \( Z = \text{Im} (X \to Y) \), then we have

\[
X \times_Y X \xrightarrow{\sim} X \to Z
\]

and so since \( Sdr \) is right exact

\[
Sdr(X \times_Y X) \to Sdr(X) \to Sdr(2)
\]

Thus we conclude that

\[
Sdr \, Z = \text{Im} (Sdr \, X \to Sdr \, Y)
\]
i.e.

\[
Sdr \, \Delta(n) \to Sd \, \Delta(n) \quad \text{as desired.}
\]

So consider (?). OKAY for \( i = j \) because

\[
Sdr \, \Delta(n-1) \to Sd \, \Delta(n). \quad \text{The left side is } Sdr \, \Delta(n-2).
\]

Suppose we have a chain of integers

\[
\sigma_0 \preceq \sigma_1 \preceq \cdots \preceq \sigma_d
\]

A simplex on the right may be identified with a chain of 1-simplices (possibly degenerate)

\[
\sigma_0 \preceq \sigma_1 \preceq \cdots \preceq \sigma_d
\]
in \( \Delta(n) \), i.e. arrows in \([n]\), such that each \( \sigma_i \)
also a 1-simplex in \( \partial_i \Delta(n-1) \) and \( \partial_j \Delta(n-1) \), i.e. neither
source or target of \( \sigma_i \) equals \( i \) or \( j \). Thus \( \sigma_i \)
is an arrow in the full subcategory

\[
\{ 0, 1, \ldots, i, \ldots, n \} \subset [n], \quad \text{so it's all clean}
\]
Conclude that $\text{Sdr}$ preserves injections.

Returning to skeletal decomposition:

$$\text{Sdr}(\Delta(n))^* \subset \text{Sdr}(\Delta(n))$$

$$\text{Sdr}(\text{sk}_n X) \subset \text{Sdr}(\text{sk}_n X)$$

Let $z$ be a simplex of $\text{Sdr}(X)$. Let $n$ be least such that $z \in \text{Sdr}(\text{sk}_n X)$. Since the above diagram shows

$$\text{Sdr}(\text{sk}_n X) - \text{Sdr}(\text{sk}_{n-1} X) \equiv \text{Sdr}(\Delta(n) - \text{Sdr}(\Delta(n))^*)$$

we see there is a unique non-deg. $n$-simplex $x$ and a unique simplex $y$ of $\text{Sdr}(\Delta(n) - \text{Sdr}(\Delta(n))^*)$ such that

$$\text{Sdr}(\Delta(n) - \text{Sdr}(\Delta(n))^*) \subset \text{Sdr}(\text{sk}_n X)$$

$$y \xrightarrow{x} z$$

If $x$ is a chain

$$x_0 \subset x_1 \subset \cdots \subset x_n$$

of 1-simplices in $\Delta(n)$, the fact that it is not in $\text{Sdr}(\Delta(n))^*$ means that every $i$, $0 \leq i \leq n$ is either source or target of some $x_i$. In particular we must have $x_{\delta} = (0 \leq \delta \leq n)$.

Thus we arrive at a canonical form for any
Observe that the map in question is in effect a diagram of the form

\[
\begin{array}{ccc}
\text{Sh}(\text{New}(C)) & \xrightarrow{\Delta \text{New}} & \text{New}(\text{Sh}(C)) \\
\end{array}
\]

hence a map

\[
\text{Sh}(\text{New}(C)) = \text{New}(\text{Sh}(C))
\]

For each \( [n] \in C \), we have a map

\[
\text{Sh}(\text{New}(C)) = \text{New}(\text{Sh}(C))
\]

so we have a pullback of \( \Delta \text{New} \) and a pullback of \( \Delta \). Thus, we have

\[
\Delta \text{New} \xrightarrow{\text{pullback}} \text{New}(\text{Sh}(C))
\]

which may be identified with a pair \((x,y)\) in \( \text{Sh}(\text{New}(C)) \).
are the identity maps. Now this comes from a functor \([2g+1] \to C\).

We know prove injectivity. Given a simplex \((x, \mathcal{F})\) in \(\text{Sch}(\text{NewC})\), we shall identify its image in \(\text{New}(\text{SchC})\) and show the image determines \(x\) and \(\mathcal{F}\). So let \(x\) be the diagram

\[
\begin{array}{ccccccc}
X_0 & \to & X_1 & \to & \cdots & \to & X_n \\
\downarrow & & \downarrow & & \ddots & & \downarrow \\
\end{array}
\]

in \(C\) where no map is an identity. Suppose

\[
\mathcal{F}: \quad \sigma_0 \subset \cdots \subset \sigma_\delta.
\]

\[
\sigma_i = (\lambda_i \leq \mu_i)
\]

\[
\lambda_0 \leq \cdots \leq \lambda_\delta \leq \mu_\delta \leq \cdots \leq \mu_0
\]

By the condition that \(\mathcal{F} \in \text{Sch}(\Delta(n))\), we know that the sequence \(\lambda_0, \ldots, \mu_0\) exhausts \(0, 1, \ldots, n\). The image of \((x, \mathcal{F})\) is then the diagram

\[
\begin{array}{ccccccc}
X_{\lambda_0} & \to & X_{\lambda_1} & \to & \cdots & \to & X_{\lambda_\delta} \\
\downarrow & & \downarrow & & \ddots & & \downarrow \\
X_{\mu_0} & \to & X_{\mu_1} & \to & \cdots & \to & X_{\mu_\delta} \\
\end{array}
\]

But this diagram determines \((x, \mathcal{F})\); it is the unique non-degenerate simplex associated to

\[
X_{\lambda_0} \to \cdots \to X_{\mu_0}
\]

And it determines the sequence \(\lambda_0, \ldots, \mu_0\), hence it determines \(x\).
Conclude:

$$\text{Sdr}\ (\text{New}\cdot C) \sim \text{New}\cdot (\text{Sdr} C)$$

Application:

$$\text{Sdr}\ (\Delta(p) \times \Delta(q)) = \text{Sdr}\ \text{New}(\Delta(p) \times \Delta(q))$$

$$= \text{New}((\text{Sdr}\ \Delta(p) \times \text{Sdr}\ \Delta(q))$$

$$= \text{Sdr}\ \Delta(p) \times \text{Sdr}\ \Delta(q)$$

Conclude, taking limit over $\Delta(p) \in \Delta/X$, $\Delta(q) \in \Delta(Y)$,

$$\text{Sdr}\ (X \times Y) = \text{Sdr}(X) \times \text{Sdr}(Y)$$

More generally

$$\text{Sdr}(X \times \text{New}\cdot C) = \text{Sdr}(X) \times \text{Sdr}(Y)$$

Observe however that $\text{Sdr}$ cannot commute with fibred products, since then the adjoint functors

$$\Delta^\wedge \leftrightarrow \text{Sdr} \quad \text{Ex}$$

would constitute a morphism of topoi, which would mean that

$$\Delta^m \dashrightarrow (\text{Sdr} \Delta^m)_n$$

would be pro-representables.
Summary. We originally wanted the formula
\[ \text{Sd} = \text{Ner} - \text{Sd} \text{C} \]
or better a good theory of Sd, in order to do the $\text{Ex}^\infty$ theory nicely on
the category level. In particular
\[
[\text{C}, \text{C}'] = \lim_{\Delta} \text{Hom}(\text{Sd} \text{N} \text{C}, \text{Ex}^\infty \text{N} \text{C})
\]
\[
= \lim_{\Delta} \text{Hom}(\text{Sd} \text{m} \text{N} \text{C}, \text{N} \text{C})
\]
\[
= \lim_{\Delta} \text{Hom}_\text{cat}(\text{Sd} \text{m} \text{C}, \text{C}')
\]
provided $\text{N} \text{C}$ is finite. The point which might be
useful later is that certain constructions
turn out nicely. Example: I conjecture that when $\text{C} \to \text{C}$ is
\[
\text{Ex}^\infty(\text{N} \text{C}') \to \text{Ex}^\infty(\text{N} \text{C})
\]
should be a Kan fibration, not just a q-fib.

It remains to establish that $\text{Ex}^\infty(\text{N} \text{C})$
is a Kan complex, among other things. This requires
explicitly retracting $\Delta(n)$ to $\text{V}(n, j, k)$ after subdividing.

Idea: Use instead $\Delta(n) \times 0 \cup \Delta(n) \times 1 \subset \Delta(n+1)$
$\Delta(n) \times 1 \cup \Delta(n) \times \Delta(1)$

This suffices (see Gabriel-Zisman).
Mumford conjecture  

Let $k = \overline{F}_p$ and let $V$ be a representation of $B = \text{Borel subgroup of } \text{GL}_n \text{ over } k$. I want to compute  

$$H^*(B(k), V) = \lim_{k_d < k} H^*(B(k_d), V)$$  

where $k_d$ denotes the subfield of $k$ with $p^d$ elements. First of all  

$$H^*(B(k_d), V) \rightarrow H^*(U(k_d), V)$$  

as $T(k_d)$ is prime to $p$. Secondly, by Borels fixed point theorem $\exists$ a flag in $V$ stable under $B$  

$$V = V_0 \supset V_1 \supset \ldots \supset V_N = 0$$  

hence a spectral sequence  

$$E_1^{pq} = H^{p+q}(B(k_d), V_p/V_{p+1}) \Rightarrow H^{p+q}(U(k_d), V).$$  

This gives an estimate  

$$P.S. \{ H^*(U(k_d), V) \} \ll P.S. \{ H^*(U(k_d)) \} \cdot P.S. \{ V \}$$  

where the Poincare series is defined to be  

$$\sum t^n [H^n(U(k_d), V)] \in \Gamma(T(k_d))^\times.$$
$R(T(k_d)) = \text{the character ring of } T(k_d).$

To simplify suppose $p=2$ and $n=2$. Then

$$\text{H}^* (U(k_d)) = S \left[ \bigoplus_{a=0}^{d-1} L^a \right]$$

where $L$ is the one-diml repn. of $k_d^*$ on $k$. So

$$\text{P.S.} \left\{ \text{H}^* (U(k_d)) \right\} = \prod_{a=0}^{d-1} \frac{1}{(1 - t L^a)}$$

Note: $\text{P.S.} \left\{ V \right\} = \left( \bigoplus_{n=0}^{\infty} \left( \bigwedge \alpha \cdot \bigwedge \alpha \right) \right)$

where the $\alpha$ are integers $\geq 0$.

Now $\text{P.S.} \left\{ V \right\}$ is a fixed sum of characters $L_x$ of $T(k_d)$. If $H^1$ contains an invariant, then we have that

$$L_x = L^a$$

for some $x$ and $a$. 
August 26, 1972

\[ C \text{ cat, } \text{Sub} \ C = \text{cat with objects } (X \to Y) \in C \text{ and maps } (X' \to Y') \to (X \to Y) \text{ being degreens.} \]

\[
\begin{array}{c}
X' \leftarrow X \\
\downarrow \\
V' \rightarrow Y
\end{array}
\]

What is \( \text{Sub(Sub} \ C) \)? Objects are squares.

\[
\begin{array}{c}
U \leftarrow X \\
\downarrow \\
V \rightarrow Y
\end{array}
\]

Thus objects are

\[
\begin{array}{c}
X \rightarrow U \\
\downarrow \\
X' \rightarrow U'
\end{array}
\]

\[
\begin{array}{c}
\uparrow \\
V \rightarrow Y
\end{array}
\]

\[
\begin{array}{c}
\downarrow \\
V' \rightarrow Y'
\end{array}
\]

\[\Rightarrow \text{ we get a chain of length 8.}\]
August 27, 1972

**C** proj n.s. curve over $k = \Gamma(C, \mathcal{O}_C)$ finite, $F =$ fin. field

$\infty$ point of $C$, $\mathcal{O} = \mathcal{O}_\infty$

$A = \Gamma(C-\infty, \mathcal{O}_C)$

$M$ f.g. proj. $A$-module, $\Gamma = \text{Aut}(M)$

$X = \bigoplus (M \otimes F)$ building consisting of $\mathcal{O}$-lattices in the $F$-vector space $M \otimes F$. A vertex $L$ of $X$ represents an extension of $M$ to a vector bundle over $C$.

The problem is to compactify $X/\Gamma$. What Borel-Serre do in the arithmetic case is to define

$$X(V) = \bigsqcup_{0 \neq W \subset W'} X(W'/W)$$

with a suitable topology. Perhaps I can do the same thing here.

There are perhaps problems associated with $GL_2$. For example, Nagao's theorem

$$GL_2(k[X]) = GL_2(k) \times B_2(k[X])$$

shows that $H_1(GL_2, k[X]) = k^* \oplus k[X]$ which is not finitely generated.
What I want to do is understand whether these compactifications might be relevant to K-theory.

Review: Recall that we have this filtration

\[ \cdots \subset F_{n-1}Q(p_A) \subset F_nQ(p_A) \subset \cdots \]

and that it leads to a long exact sequence

\[ \rightarrow H_qF_{n-1} \rightarrow H_qF_n \rightarrow \bigoplus_{\alpha} H_{q-r}(\text{Aut}(M_\alpha), \text{St}(M_\alpha \otimes F)) \rightarrow \]

where \( M_\alpha \) runs over isomorphism classes of brfg. \( A \)-modules. (Thus \( \alpha \in \text{Pic } A \)).

I refresh my memory: One has

\[ F_{n-1} \xrightarrow{i} F_n \xleftarrow{i^\ast} \text{Aut}(M) \]

and for any \( M \) of rank \( r \) we know that \( j/M \) is equivalent to the suspension of the building \( X(M \otimes F) \).
\( \text{GL}_n \quad \phi_{12} \)

\( \text{GL}_n \supset \text{GL}_{n-1} \quad H_x(\Gamma, \text{St}(V)) \)

\( \text{GL}_n \quad \text{equivariant cohomology with compact supports} \)

\( \text{GL}_n \leftarrow \text{GL}_{n-1} \leftarrow \text{GL}_{n-2} \leftarrow \)

I am interested in \( H_x(\Gamma, \text{St}(V)) \). Homology vanishes

\( \text{Hom}_G(\Gamma, \text{St}(V)) \)

\( G \), group
consider the category of \( G \)-sets
and the crible form of those which whose stabilizers are the unipotent groups. Note a subobject of the final object. Can we make sense of relative colb.

cat of \( G/U \), \( U \) unipotent
cat of \( G/E \). Result is

\[
H^*_x(G) \to H^*_x(G/U)
\]

\[
\text{Hom}_G(G/P, G/P) = \{ gP \mid gP^p = gP \} = \text{Norm}(P)/P.
\]

\[
\text{Hom}_G(G/P, G/P)
\]
Take cochains \( \tilde{c} \) in \( C^*(G, \Lambda) \) which vanish in all \( p \)-subgroups. Then one wants the relative cohomology

\[
\bigcup_{P \subset G} BP \subset BG
\]

But if Sylow \( p \)-subgroup \( P \) is normal in \( G \) get

\[
H^*(BG, BP)
\]

this is the reduced cohomology of \( G \) with \( Z' \) coeffs.

But for \( Z_p \) it is wrong, since we should expect \( 0 \)

\[
H^*(BG, \mathbb{F}_p) = H^*(BP, \mathbb{F}_p)^{G/P}
\]

In the case of \( GL_2(\mathbb{F}_p) \) one has a \( P \) for each line, so that

\[
\bigcup_{P} BP = \bigvee_{P} BP
\]

and the homology is gigantic

\[
H_\ast(\Gamma, St)
\]

part of homology \( \Gamma \) which is primitive.

for \( \pi_1 \), if we restrict the Steinberg to a parabolic \( P \), then \( St \) is free over the unipotent part so it collapses the homology.

\[
H^*(P, St^\wedge) = H^*(P/U, St(P/U)) \cong \text{tensor product}
\]
\[ H^\ast (GL_2, St) \]

\[ B(B) \rightarrow B(G) \rightarrow H^\ast (GL_2, St(k^2)) \]

\[ H^\ast (GL_2, St(k^2)) \rightarrow H^\ast (G_1 \times G_1) \rightarrow \ldots \]

Goal was fairly simple. To modify \( GL_2 \mathbb{A} \)
so that its homology

\[ GL_2(\mathbb{K}) \]

act on parahoric building of \( \mathbb{K}^2 \).

Simple orbits

The category of \( p \)-subgroups of \( G \)

Consider the cat of irreducible \( G \)-sets whose stabilizers

are \( p \)-groups. One object for each conjugacy
class of \( p \)-groups, and funny maps.

\[ G/H \]
The parahoric building $X(V)$ has three kinds of simplices:

- $0$-simplex $L$
- $1$-simplex $L < L'$, $\dim L/L' = 1$
- $2$-simplex $L_0 < L_1 < L_2$

There are maps:

$$P \xrightarrow{\text{in}} \text{mod } P \xrightarrow{\text{twisted}} \text{G}$$

What about $H_\ast(P) \rightarrow H_\ast(G) \rightarrow \ast$?

$\mod P$ cohomology

$H^\ast(G) \hookrightarrow H^\ast(P)$

$H^\ast(GL_2, H^\ast(C)) \hookrightarrow H^\ast(P, H^\ast(C))$

$P \times P^{-1} = T \times C$

$$P \times P^{-1} \xrightarrow{\text{in}} P \xrightarrow{\alpha \iota \alpha}$$

$x \in G$. But we can modify $x$ outside by an element of $P$, but one can only change $\alpha x$ by an element of the cent $(P \times P^{-1})$. 
The idea, it would appear, is to systematically introduce the "Steinberg homology" into the calculations.

To prove Moore's theorem:

$$
\oplus_{m} K_{2i-1}(\mathbb{Q}/m) \to K_{2i-1} \to K_{2i-1} F \to 0
$$

want this to be zero.

Thus A Dedekind, we want

$$
K_{2i-1}(\mathbb{A}/\mathfrak{m}) \to K_{2i-1} \mathbb{A}
$$
to be zero. Representation

$$
\oplus_{m} R_{A/m}(G) \to R_{A}(G) \to R_{F}(G) \to 0
$$

$$
R_{A/m}(G) \to R_{A}(G) \to R_{F}(G) \to 0
$$

for $G$ prime to $p$.

Steinberg homology for $GL_2$.

$H^*_x(GL_2, St(k^n))$
Now we let $\text{GL}_n K$ acts on $X(K^n)$
and an orbit will be $L_0 < \ldots < L_q$ $q < n$
and there are two kinds:
\[ \dim(L_q/L_0) < n \]
\[ \dim(L_q/L_0) = n. \]

we have integers
\[ \pi L_0 \subset L_0 < L_1 < L_q \]
\[ n_0, n_1, \ldots, n_q \]
where $n_i \geq 0$ $i > 0$

and stabilizer is essentially
\[ G_{n_0} \times G_{n_1} \times \cdots G_{n_q} \]
so our $E_1$ term is as follows: Take
\[ R \stackrel{\otimes \tilde{R} \otimes \cdots \otimes \tilde{R}}{\longrightarrow} R \]

But the boundary is more complicated: In addition to deleting the $L_i$ we must produce
don deletes $L_0$, so it adds $n_0 + n_1$
but $d_q$ deletes $L_q$, so it adds $n_q$ to $n_0$
i.e. this means we have
\[ (R \otimes \tilde{R} \otimes \cdots \otimes \tilde{R} \otimes R) \otimes R \]

and so we get
\[ \text{Tor}^R_{n}(R, R) = R \otimes \text{Tor}^R_{n}(N, N) \]
for the algebra. So in degree $n$ we should have
\[ \bigoplus \text{H}_x(G_{\cdot i}) \otimes \text{Tor}^R_{n}(N, N)_{n-i} \]
which is most messy.
Review the building

$GL(V)$ acts on $X(V)$, $[V : k] = n$, $k$ local field
with valuation ring $A$, residue field $k_0$.

Want then to understand the orbit spectral sequence
for the mod $l$ cohomology, $l \neq \text{char } k$.

$$\bigoplus_{n \geq 0} H^*_x(GL_n(k)) = R$$

It will be very important to understand the Steinberg
homology

$$H^*_x(GL_n(k), St(k^n))$$

$$\mathcal{B}_G \xrightarrow{\text{II}} \bigoplus_{i,j \geq 0} \mathcal{B}G_{ij} \xrightarrow{\text{II}} \mathcal{B}G_n \xrightarrow{\text{I}} \ast$$

$$\widetilde{R} = \bigoplus_{n > 0} H^*_x G_n$$

$$(\widetilde{R} \otimes \widetilde{R})_n \xrightarrow{\sim} \widetilde{R}_n$$

The point is that

$$H^*_x(F_n, F_{n-1}) = Tor^*_x((\Lambda, \Lambda)_n = H^*_{x-n}(G_n, St(k^n))$$
\[ n = 2, \quad h = n_0 + n_1 + \ldots + n_6 \quad \text{with} \quad g \leq 2 \]

- \( g = 0 \)
- \( g = 1 \)
- \( g = 2 \)

---

\[ h = 3, \quad g = 0 \]

- \( g = 0 \)
- \( g = 1 \)
- \( g = 2 \)
- \( g = 3 \)

---

\[ n = 4 \]

- \( g = 0 \)
- \( g = 1 \)
- \( g = 2 \)
- \( g = 3 \)
- \( g = 4 \)
Rank 2 bundles
compute $H_x(\text{Aut } E, St(E \otimes K))$

- $E$ indecomposable — no good
- $E$ decomposable — should be possible

assume $E = L \oplus L'$ with $\deg L \gg \deg L'$

then $\text{Aut}(E) = k^* \text{Han}(L', L)$

$B = \begin{pmatrix} k^* & k^* \\ k^* & k^* \end{pmatrix}$

subgroup of the Steinberg group

Recall that Steinberg restricted to the Borel subgroup is $Z[B/B_T]$

If $k = \mathbb{F}_2$, for example, $k^* = 1$, so $Z[B/B_T] = Z[u]$ is the induced representation, direct sum of infinitely many copies of the regular rep. of $\text{Aut}(E)$. \[ H_x(\text{Aut } E, St(E \otimes K)) \] is infinite.

The Vinodin game

Valodin

$\begin{array}{c}
\text{Valodin} \\
\text{fibre} \\
\text{Serre's} \\
\text{idea}
\end{array} 
\begin{array}{c}
BU + BU' \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
\text{fibre} \\
\text{Serre's} \\
\text{idea}
\end{array} 
\begin{array}{c}
B(\text{Borel}) \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
BGL_2, \quad \text{cohomology} \\
BGL_2 \\
BGL_2 \times BGL_2 \\
\text{background}
\end{array}$
\[
0 \rightarrow L \rightarrow E \rightarrow L' \rightarrow 0
\]

\[
h^0(E) - h^1(E) = \deg E + 2(1-g)
\]

\[
h^0(E(n)) \geq \deg E + 2n + 2(1-g) > 0
\]

\[
\Rightarrow \deg_L L(n) > 0 \quad \text{as } L \text{ is maximal.}
\]

\[
\deg L + n > 0
\]

\[
2n > \underbrace{2(g-1)}_{F = F_0} - \deg E
\]

\[
n > g - \frac{1}{2} \deg E
\]

\[
\deg L + g - \frac{1}{2} \deg E \geq 0
\]

\[
\deg L + g - \frac{1}{2} (\deg L + \deg L') \geq 0
\]

\[
\frac{1}{2} (\deg L - \deg L') + g \geq 0
\]

**i.e.**

\[
\deg L' - \deg L \leq 2g
\]

If indecomposable, then

\[
H^q(\Hom(L', L)) \geq 0
\]

\[
H^0(\omega \otimes L \otimes L')
\]

\[
2g - 2 - \deg L + \deg L' > 0
\]

\[
\deg L' - \deg L \geq -(2g-2)
\]
The problem. To find for each \( n \) a map

\[ BGL_n \rightarrow X_n \]

such that

i) exact sequences add

\[ BGL_{p}\lambda \rightarrow BGL_{p+\lambda} \rightarrow X_{p+\lambda} \]

\[ \downarrow \quad \downarrow \quad \quad \quad \quad \downarrow \]

\[ BGL_p \times BGL_q \rightarrow X_p \times X_q \]

\[ \rightarrow X_{p+q} \]

ii) For \( \ell \) prime to the characteristic, the map

\[ BGL_n \rightarrow X_n \]

should induce iso's on mod \( \ell \) homology

iii) \( X_n \) should be nice with respect to stability

iv) \( X_n \) should have no mod \( p \) cohomology for a

finite field of characteristic \( p \).
The stable splitting theorem has to be understood at the same time as stability theorem.

Operations on extensions:

\[ 0 \to V_0 \to V \to 1 \to 0 \]

Sum, product, symm \( S^n V \).

The point perhaps is that \( G \) is a perfect group and \( BG^+ \) has no p-torsion. The problem is that \( G \) acts on \( V_0 \) so the cohomology is non-trivial, but

\[
\begin{align*}
0 & \to V_0 \to V \to 1 \\
& \to S^n V \to 1 \to 0 \\
0 & \to I \to SV \to k[T] \to 0 \\
I/I^2 & \cong V_0 \\
I/I^2 & \to SV/I^2 \to k \to 0
\end{align*}
\]

Conjecture:

\[ 0 \to S^p V_0 \to S^p V \to 1 \to 0 \]

Okay for \( p = 2 \):

\[ S^2 V_0 < \frac{V_0 \cdot V}{S^2 V} \to S^2 V \to 1 \]
Conjecture is clear

\[
V_0 \xrightarrow{sp^{-1}} V \xrightarrow{} S^p V \xrightarrow{} S^p 1 \xrightarrow{} 0
\]

\[
o \rightarrow V_0^{(p)} \xrightarrow{} V^{(p)} \xrightarrow{} 1^{(p)} \xrightarrow{} 0.
\]

This shows that the operation comes from a map

\[
V_0^{(p)} \xrightarrow{} V_0 sp^{-1} V.
\]

which dies in

\[
p=2
\]

\[
\]

\[
H^1(S_2 V_0) \leftrightarrow H^1(V_0^{(2)})
\]

\[
\]

\[
H^1(V_0 V) \xrightarrow{} \nabla
\]

\[
H^1(V_0) \leftarrow \xrightarrow{\circ} H^1(V_0)
\]

No the thing to prove is that the class if it dies in \(H^1(V_0 V)\) then it dies in \(H^1(S_2 V_0)\).

(Oakay if \(V_0\) has no invariants)

\[
S_2 V_0 \rightarrow \begin{array}{c}
V_0 V
\end{array} \rightarrow V_0
\]

\[
V_0 \subset V \rightarrow \frac{1}{2}
\]

\[
x_0 \neq 0
\]

Wrong because

\[
\begin{array}{c}
s_2 \quad x=x+y
\end{array}
\]

\[
s_2 \quad y=y
\]

\[
S_2 \quad \text{dents split here.}
\]

\[
S_2 \quad \text{doesnt split here.}
\]
\[
\begin{align*}
\mathcal{F} \Rightarrow \mathcal{F}_{\mathcal{G}}
\end{align*}
\]
August 31, 1972

depose $C$ is a proj. n.s. curve over $k$ finite, $k = H^0(C, O_C)$.

Let $\infty$ be a point of $C$ and

$$\Lambda = H^0(C-\infty, O_C)$$

the coordinate ring of the affine curve $C-\infty$. Suppose

$k$ is of char. $p$.

Let $M$ be a proj. $\Lambda$-module of rank $r$. I wish to prove that

$$H_i(\text{Aut } M, \mathbb{Z}[\frac{1}{p}])$$

is a finitely generated abelian group for each $i$.

Suppose I know this is true for all proj. $\Lambda$-modules of rank $< r$.

Let us be given a simple

$$0 < (M_0 < \cdots < M_\gamma) < M$$

of the building of $M$ (i.e. of $F \otimes M$, $F$ = fin. field). Let

$\Gamma_0$ be the stabilizer of $\sigma$.

Then if we choose complements $M_j$ in $M_{j+1}$, we have an exact sequence

$$1 \rightarrow \Gamma_0' \rightarrow \Gamma_0 \rightarrow \prod_{j=0}^{\gamma+1} \text{Aut}(M_j/M_{j-1}) \rightarrow 1.$$ 

It is clear that $H_0(\Gamma_0', \mathbb{Z}[\frac{1}{p}]) = \mathbb{Z}[\frac{1}{p}].$ By induction, this reduces to the fact that the additive group $\text{Hom} (M, N)$ will have trivial homology because it is a $k$-module.

Now by our induction hypothesis, the big product on the right has fin. type homology over $\mathbb{Z}[\frac{1}{p}]$, hence so-
does $\Gamma$ for any simplex $\sigma$ of $X(M)$.

Now there are only finitely many orbits of Aut(M) on $X(M)$. In effect once we give two filtrations

$$0 < M_1 < \ldots < M_n = M$$

$$0 < M_1' < \ldots < M_n' = M$$

with $M_i/M_{i-1} \cong M_i'/M_{i-1}'$ for all $i$, then these simplices are conjugate under $\Gamma = \text{Aut}(M)$. Once the jump ranks are fixed, the iso. classes are in 1-1 corres. with elts. of $\text{Pic} \Lambda$ which is finite.

We conclude therefore by the induction hypothesis that

$$H_i^\Gamma(X(M), \mathbb{Z}[\frac{1}{p}])$$

is finitely generated for each $i$. So now the problem is to show that the relative group

$$H_i^\Gamma(\text{pt}_\Gamma, X(M), \mathbb{Z}[\frac{1}{p}])$$

is finitely generated. Then we have that

$$H_i^\Gamma(\mathbb{Z}[\frac{1}{p}])$$

is f.g. and we can continue the induction.

Notice:

$$0 \rightarrow \overline{C}(X(M)) \rightarrow C(X(M)) \rightarrow \mathbb{Z} \rightarrow 0$$

and

$$H_i^\Gamma(\overline{C}(X(M), \mathbb{Z}) = \begin{cases} 
0 & i \neq r-2 \\
\text{St}(\text{F}\otimes M) & i = r-2 
\end{cases}$$
Therefore the relative group is:

\[ H_i \left( \text{st}_\Gamma, X(m)_\Gamma; \mathbb{Z}\left[ \frac{1}{\rho} \right] \right) = H_{i-1} \left( \Gamma, \overline{C(X(m))} \otimes \mathbb{Z}\left[ \frac{1}{\rho} \right] \right) \]

\[ = H_{i-\lambda+1} \left( \Gamma, \text{St}(M \otimes F) \otimes \mathbb{Z}\left[ \frac{1}{\rho} \right] \right) \]

*Conjecture* (possibly proved by Serre)

\[ H_i \left( \text{Aut}(M), \text{St}(M \otimes F) \right) \text{ is f.g. } \forall i \]