Continuation of the descent problem for a Galois extension $F \subset E$ with Galois group $\pi$.

The naive hope is for a spectral sequence of the form

$$E_2^{pq} = H^p(\pi, K(E)) \Rightarrow K_{p-q}(F).$$

The motivation: Let $X \to Y$ be a Galois covering with group $\pi$ and let $h^\ast$ be a generalized cohomology theory. Then the canonical map

$$\varphi : P_\pi \times \pi X \to Y$$

is a sheaf (fibre bundle with fibre $P_\pi$), hence

$$h^\ast(Y) = h^\ast(P_\pi \times \pi X).$$

We can consider $Y$ as being fibred (up to homotopy) over $B\pi$ with fibre $X$. Thus from skeletal decomposition of $B\pi$, we get a spectral sequence

$$E_2^{pq} = H^p(\pi, h^q(X)) \Rightarrow h^{p+q}(Y).$$

There might be convergence difficulties, but not if $B\pi$ is a finite dim. CW complex.

However: Let us consider cases which are known. Thus take $F = \varphi^\ast$, $E = F$. Then

$$\pi = \hat{\mathbb{Z}}.$$
and the $E_2$-term appears:

\[
\begin{array}{cccc}
\delta = 0 & Z & 0 & \mathbb{Q}/\mathbb{Z} \\
\delta = -1 & (K_1E)^T & 0 & 0 \\
\delta = -2 & 0 & 0 & 0
\end{array}
\]

\[H^1(\mathbb{Z}, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \]

\[H^2(\mathbb{Z}, \mathbb{Z}) = H^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \]

In other words, the $\mathbb{Q}/\mathbb{Z}$-term destroys the effect. Next suppose $F = \mathbb{F}_\delta$, $E = \mathbb{F}_\delta$ so that $\Pi$ is cyclic of order $d$. Then we have

\[
\begin{array}{cccc}
\mathbb{Z} & 0 & \mathbb{Z}/d\mathbb{Z} & 0 \\
(K_1E)^T & (K_2E)^T & (K_3E)^T & (K_4E)^T \\
0 & 0 & 0 & 0
\end{array}
\]

by the periodicity of the cohomology of the cyclic group $\mathbb{Q}/\mathbb{Z}$.

\[H^1(\Pi, A) = \frac{\text{Ker} \{ N : A \to A \} }{\text{Im} \{ \sigma - 1 \} } \]

\[H^2(\Pi, A) = \frac{\text{Ker} \{ \sigma - 1 \} }{\text{Im} \{ N \} } \]

Now, for the $K_i$ of finite fields we know that $N$ is surjective onto the invariants, whence
$\sigma - 1$ must map onto the kernel of $\Delta$. Thus everything is OK except for the $E_2^{2,0} = H^2(\mathbb{Z}/dz, \mathbb{Z}) = \mathbb{Z}/dz$ terms.
July 9, 1972.

Let \( k \) be an algebraically closed field of characteristic \( p \) and \( k_0 = \{ x \in k \mid x^p = x \} \) the finite subfield with \( q = p^d \) element. I wish to understand the non-commutative ring

\[
R = k[F]
\]

where \( F \) is an indeterminate \( \Rightarrow Fx = x^8F \) for all \( x \in k \).

Elements of \( R \) are uniquely expressible as polynomials

\[
a_0 + a_1 F + \ldots + a_n F^n \quad a_i \in k
\]

Thus \( R \) is a graded ring without zero divisors (consider the highest degree terms).

**Ideal structure:** If \( L \) is a non-zero left ideal in \( R \), let \( f \) be a monic polynomial of least degree contained in \( L \). Then \( L = Rf \) by division algorithm, so every left ideal is principal.

Conclude

1) \( R \) left month (every left ideal \( f,g \))
2) \( R \) left regular (every monogenic \( R \)-module of form \( R/Rf \), so either free, or of projective dim 1:

\[
0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/Rf \longrightarrow 0
\]

as \( R \) has no zero divisors.

Thus \( R \) being a graded left regular ring \( \Rightarrow K(k) \xrightarrow{\phi} K(R) \).
(Remark: The preceding holds for any endomorphism \( \sigma \) of \( R \) instead of \( x \cdot x^1 \) and hence only used \( R \) field.

The preceding holds for right modules for an auto \( \sigma \). Otherwise, it is not possible to find a monic \( f \)

\[ a_n F^n = F^n a_n \]

to get a monic \( F \).)

Suppose \( I \) is a 2-sided ideal. Then if \( f \) is a monic poly of minimal degree \( \text{in} \; I \), we have

\[ I = Rf = fR \]

Let

\[ f = F^n + a_{n-1} F^{n-1} + \cdots + a_0 \]

Then

\[ x \cdot F^n = F^n + x \cdot a_{n-1} F^{n-1} + \cdots + x \cdot a_0 \]

so by uniqueness of \( f \), can conclude \( a_i = 0 \).

2-sided ideals are:

\[ R \cdot F^n \quad n \geq 0 \]

Module structure: Let \( M \) be a finitely generated \( R \)-module and choose a presentation for \( M \)

\[ R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0 \]

\[ \left( \begin{array}{c} a_{11} \\ a_{12} \\ \vdots \\ a_{nj} \end{array} \right) \]
with $g$ minimal. Assuming $M$ is not free, so that $a_{ij} \neq 0$, we can choose the presentation such that $a_{ij} \neq 0$ and such that the degree of $a_{ij}$ is minimal. Can suppose $a_{ij}$ is monic. Then necessarily by division algorithm

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} \\
  \vdots \\
\end{pmatrix}
$$

we must have $a_{ij} \in R_{a_{ij}}$, $a_{ij} \in a_{ij} R$, so performing these obvious row- and column operations, we can replace the matrix by

$$
\begin{pmatrix}
  a_{11} & 0 & 0 \\
  0 & \cdots & \cdots \\
  0 & \cdots & \cdots \\
\end{pmatrix}
$$

Whence $M = R/R_{a_{11}} \oplus M'$. Conclude

1. Every f.g. $M$ sum of monogenic modules.
2. Every torsion-free f.g. $M$ is free.

Torsion (means $\forall m, a_m$ annihilates nonzero) are the same as $R$-modules which are f.g. over $k$. 

f.g. $R$-module
This is because for \( f \) monic of degree \( n \), \( R/\mathfrak{m}f \) is free of rank \( n! \) with basis \( 1, \ldots, f^{n-1} \).

**Structure of torsion module.** A torsion \( R \)-module is simply a \( k \)-vector space \( V \) of finite dimension endowed with an operator \( F : V \to V \) satisfying

\[
F(xv) = x\delta Fv, \quad x \in k, v \in V.
\]

We have a decreasing sequence of \( R \)-submodules (recall \( RF \) is a 2-sided ideal)

\[
V > FV > F^2V > \ldots \ldots
\]

hence by Fitting's lemma, there is a unique splitting

\[
V = V' \oplus V''
\]

such that \( F \) is nilpotent (resp. bijective) on \( V' \) (resp. \( V'' \)).

**Basic lemma:** If \( V \) is a f.d. \( k \)-v.s. \( / k \) with an \( F \) which is an auto, then

\[
k \otimes_k V^0 \sim V
\]

where \( V^0 = \{v^0 \mid Fv = 0\} \).

**Proof.** Bijectivity: Let \( e_i, i \in I \) be a basis for \( V^0 \) and let

\[
\sum x_i e_i = 0
\]

be a primordial relation (set of \( i \neq \sum x_i e_i = 0 \) is minimal + one \( x_i = 1 \)). Comparing this relation with its translate under \( F \), one
sees $\chi_i^0 = \chi_i$, contradicting independence of the $\chi_i$.

Surjectivity: First we show $V \neq 0 \Rightarrow V^0 \neq 0$.

Can suppose $V$ simple $R$-module, hence $V \simeq R/Rf$, where

$\mathbf{f} = \mathbf{f}^m + \cdots$ is a monic polynomial of degree $n$, say.

Claim $n=1,$ will show $\mathbf{f} = g(F-\lambda)$ for a suitable $\lambda$.

Have identity

$$F^m = (F^{m-1} + \lambda^i \delta^{m-1} F^{m-2} + \cdots + \lambda^i \delta^{m-1} \cdots \delta^0 F - \lambda) + \lambda^i \delta^{m-1} \cdots \delta^0$$

Hence if

$$\mathbf{f} = \sum_{m=0}^{n} a_m F^m$$

Then

$$\mathbf{f} = g(F-\lambda) + \left\{ \frac{\lambda^i \delta^{n-1} \cdots \delta^1}{a_{m-1} \delta^{m-2} \cdots \delta^1} + a_0 \right\}$$

Better we have that the remainder is

$$\pi(\lambda) = \lambda^i \delta^{n-1} \cdots \delta^1 + a_{n-1} \delta^{n-2} \cdots \delta^1 + \cdots + a_0$$

and since $k$ is algebraically closed, there exists a root of this polynomial.

Thus must have $\mathbf{f} = F - \lambda$, so $V$ is 1-dimensional.

Thus, must have $\mathbf{f} = \mathbf{F} - \lambda$, so $V$ is 1-dimensional.

For some $i \neq 0$, $Fv_i = \lambda v_i$. Now changing $v$ to $x v_i$ and arranging $x$ so that $F(x v_i) = \lambda (x v_i)$, i.e. $x \delta^1 = \lambda,$ we see $\mathbf{V}^0 \neq 0.$
Suppose then that \( W = k \otimes_k V^0 < V \). As \((V/W) \neq 0\) we have a \( v \in V, v \notin W\), such that \( Fv - v = \omega_\psi = \sum \gamma_i e_i \) where \( e_i \) is a basis of \( W^0 = V^0 \). To find 

\[
F(v - \sum x_i e_i) = v - \sum x_i e_i
\]

i.e. \( Fv - v = \sum (x_i - x_\psi) e_i \).

\[Fv = \sum \gamma_i e_i,\]

Can be done since \( x_i - x_\psi = y_i \) has roots. Done with basic lemmas.

**Remark**: Above holds for \( k \) separably closed and probably for any strictly local ring in char. p.

(Yes, see Oct. 18, 1971 report attached below.)

**Cor.** Category of f.g. (resp. arbitrary) \( R = k[F] \)-modules on which \( F \) acts invertibly is equivalent to the category of f.g. (resp. arb.) \( k \)-modules.

**Cor.** Any torsion \( R \)-module on which \( F \) acts invertibly is an injective \( R \)-module.

**Proof.** Have to show

\[
\text{Hom}_R(R, V) \rightarrow \text{Hom}_R(L, V)
\]

for any left ideal \( L \) in \( R \). Can suppose \( V \) f.g. and \( L \neq 0 \), whence \( L = RF \). Can suppose \( V = k \) with \( Fx = x^2 \). Then have
But the bottom sequence splits \( \cong \) by the equivalence with \( k_0 \)-modules, so \( \varphi \) extends.
July 14, 1972

Homotopy of sets, again

Let $C$ be a small category such that (*) the only endos. and isos. in $C$ are the identity maps. (Equivalently: for any diagram

$$
\begin{array}{ccc}
  X & \xrightarrow{f} & Y \\
  \downarrow{g} & & \downarrow{g}
\end{array}
$$

in $C$ we have $Y = X$ and $f = g = \text{id}_X$.)

Let us consider a simplex in $\text{New}(C)$

$$
X_0 \longrightarrow \cdots \longrightarrow X_p
$$

For this to be non-degenerate means none of the arrows are the identity. If two of the vertices coincide, say $X_j = X_k$ with $j < k$, then (*) can't hold. In effect, if $k = j + 1$ then the arrow $X_j \to X_k$ would be an endo-, hence the identity; and if $k \geq j + 2$ we have maps

$$
X_j \longrightarrow X_{j+1} \longrightarrow X_k
$$

so $X_j \to X_{j+1}$ would be the identity. Thus

(*) $\Rightarrow$ all vertices of a non-degenerate simplex are distinct.

The converse is also true since

$$
X \xrightarrow{f} X \text{ would be non-degenerate if } f \neq \text{id}_X
$$

$$
X \xrightarrow{f} Y \xrightarrow{g} X \text{ if } f \neq \text{id}_X \neq g.
$$
Conclude: Suppose \( C \) satisfies (*) Then.

The category \((\Delta/\text{New\,}C)^{nd}\) of \(\Delta/\text{New\,}C\) consisting of non-degenerate simplices is an ordered set, and it is fibred over \(\Delta^+\) (= subset of injective maps in \(\Delta\)).

Observe that the last vertex map

\[
(\Delta/\text{New\,}C)^{nd} \longrightarrow C
\]

is cofibred, and the fibre has an initial element.

Deligne's construction: Given \( C \) satisfying (*), Deligne considers finite subcategories \( F \) of \( C \) having final objects. These form an ordered set under inclusion and there is a functor

\[
I \longrightarrow C
\]

sending \( F \) to its final object. The functor is pre-cofibred, the fibres being ordered sets with initial element. Note that non-degenerate simplices are special cases of such functors \( F \), i.e.,

\[
(\Delta/\text{New\,}C)^{nd} \subset I
\]

Advantage of Deligne's construction: The ordered set \( I \) is directed when \( C \) is filtering.
The way to replace a category $C'$ by a $C$ satisfying (*) is to let $C'$ be the subcategory of $C \times \mathbb{N}$ with same objects where

$$
\text{Hom}_{C'}((X',m'),(X,m)) = \begin{cases} 
\emptyset & m' > m \\
\{ \text{id}_X \} & m' = m, X' \neq X \\
\text{Hom}(X',X) & m' = m, X' = X \\
 & m' < m
\end{cases}
$$

Then $C' \to C$ is pre-cofibred. Given $(X,m)$

$$
X \xrightarrow{f} Y
$$

then $f^*_X(X,m) = \begin{cases} 
(X,m) & \text{if } f = \text{id}_X \\
(Y,m+1) & \text{if } f \neq \text{id}_X
\end{cases}
$

The fibre over $X$ is $\mathbb{N}$ which has an initial object. Thus $C^\mathbb{N} \to C'$ is a bg.

Now let $C^\mathbb{N}$ be an arbitrary small category and let $I$ be the set of diagrams in $C \times \mathbb{N}$ of the form

$$
(X_0,n_0) \to (X_1,n_1) \to \ldots \to (X_p,n_p)
$$

with $n_0 < n_1 < \ldots < n_p$. Then $I$ is an ordered
set and we have a functor

\[ I \rightarrow C \]

given by the last vertex. The functor is pre-cofibred and fibres have initial elements.

Given \( C \) let \( \text{Sd}(C) \) be the cofibred category over \( C^0 \times C \) defined by the functor \( (x,y) \mapsto \text{Hom}(x,y) \). The objects are arrows \( u: X \rightarrow Y \) and a map \( (u: X \rightarrow Y) \rightarrow (u': X' \rightarrow Y') \) is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\uparrow & & \downarrow \\
X' & \xrightarrow{u'} & Y'
\end{array}
\]

Then \( \text{Sd}(C) \) is cofibred over \( C \) (and over \( C^0 \)) with fibres having initial objects.

Suppose \( E \rightarrow C \) is cofibred and the fibres have initial objects; \( \phi_x \in E_x \). Then we define \( \phi_x \in \text{Sd}(C) \) and \( u \phi_x \rightarrow E \) as

\[
\begin{array}{ccc}
\text{Sd}(C) & \rightarrow & E \\
\downarrow & & \downarrow \\
C & \rightarrow & C
\end{array}
\]
This is a functor:

\[ \phi_{X'} \rightarrow w_i \phi_X \rightarrow (w_i \phi_{X'}) \rightarrow u_i \phi_{X'} \]

\[ \phi_X \rightarrow w_i \phi_X \rightarrow (w_i \phi_X) \rightarrow u_i \phi_{X'} \]

\[ X' \xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Y' \]

It is a co-cartesian functor (the arrow \( w \rightarrow w' \) is co-cartesian when \( w : X' \rightarrow X \)).

Reason for the notation \( \text{Sd}(C) \). I conjecture \( C \rightarrow \text{Sd}(C) \) analogous to barycentric subdivision of a simplicial complex. Hopefully it will be more suited to categories.

If \( C \) is an ordered set, then \( \text{Sd}(C) \) is the ordered set of layers \((X, Y)\), \( X \leq Y \) in \( C \) where

\[ (X', Y') \leq (X, Y) \]

means \( X \leq X' \leq Y' \leq Y \).

Examples:

\( C : \ 0 \leq 1 \)

\( \text{Sd}(C) : \ (0, 0) \leq (0, 1) \geq (1, 1) \)

\( \text{Sd}^2(C) \)
Conjecture: \((C, C') \mapsto \lim_{n} \text{Hom}(\text{Sd}^nC, C')\)

carries higs into isomorphisms. (probably need C finite).

Question: Does \(C \mapsto \text{Sd}C\) have a right adjoint \(\text{Ex}\)?

If so then

\[
\text{Ob}\{\text{Ex}(C')\} = \text{Hom}_{\text{cat}}(e, \text{Ex}(C'))
= \text{Hom}_{\text{cat}}(\text{Sd}(e), C') = \text{Hom}_{\text{cat}}(e, C')
= \text{Ob}\{C'\}
\]

and

\[
\text{Ar}\{\text{Ex}(C')\} = \text{Hom}(0 \leq 1, \text{Ex}(C'))
= \text{Hom}(\overset{\longrightarrow}{e}, C')
\]

and

\[
\text{Ar}^2\{\text{Ex}C\} = \text{Hom}(\overset{\longrightarrow}{\overset{\longrightarrow}{e}}, C')
\]

Answer: \(\text{NO}\)
Let \( f : X \to Y \) be a map of spaces (CW-complexes say). Suppose that for every finite complex \( K \) we have that the induced map of fundamental groupoids

\[
\pi \text{Hom}(K, X) \to \pi \text{Hom}(K, Y)
\]

is an equivalence of categories. Then \( f \) is a homotopy equivalence, in effect, taking \( K = pt \) we see the fundamental groupoids of \( X \) and \( Y \) are equivalent. By Whitehead's theorem, we want to show that \( \pi_k(X,x) \to \pi_k(Y,fx) \) for all \( k \) and \( x \). But if \( K \) has a basepoint \(*_{K}\), then

\[
\text{Hom}(K, Y) \to \text{Hom}(*, Y) = Y
\]

is a fibration with fibre \( \text{Hom}(K, x)(y, y) \) over \( y \in Y \). Thus we have a fibration of groupoids

\[
\pi \text{Hom}(K, Y) \to \pi Y
\]

whose fibre is a groupoid with components \( \pi_0 \text{Hom}(K, x)(y, y) \) over \( y \in Y \). Thus the hypothesis implies

\[
[(K, x), (X, x)] \to [(K, x), (Y, fx)],
\]

so done.


Let $X$ be a space, and $A, \mathcal{V}$ two subspaces. Call $\mathcal{V}$ a halo nbhd. of $A$ if \exists continuous function $\tau: X \to [0,1]$ such that $\tau(A) = 1$, $\tau(X-\mathcal{V}) = 0$.

Observe that if $X$ is normal, then by Urysohn's lemma every neighborhood of a closed set is a halo neighborhood, and conversely.

Call a sheaf of sets $F$ over $X$ soft if for any $A \subseteq X$ we have surjectivity $F(X) \to \lim_{U \supseteq A} F(U)$

where $U$ runs over the halo neighborhoods of $A$. It is enough to consider only $A$ which are closed since a halo nbhd. of $A$ and $\overline{A}$ are the same thing.

Observe that this agrees with the Godement definition when $X$ is paracompact. Indeed the inductive limit above is $F(A)$ for any closed set $A$. (Coh I, p. 151) hence the condition becomes $F(X) \to F(A)$ for all closed $A$.

Dold's principal technical result is the following, which generalizes Godement's 3.4.1 (p. 151).
Theorem: Let \( \{ U_i \} \) be a numerable covering of \( X \) (numerable means \( \exists \) refinement of form \( g^{-1}(T, U) \) where \( g \) is a locally-finite partition of \( 1\mathbb{Z} \)), and assume that \( F|U_i \) is soft for each \( i \). Then \( F \) is soft.

Examples: Let \( E \xrightarrow{f} X \) be a space over \( X \) and \( F \) the sheaf of its sections. Call \( E \) soft over \( X \) if \( F \) is soft.

Claim: If \( f \) is a fibre-homotopy-equivalence (over \( X \)), \( \exists s: X \to E \) s.t. \( fs = id_E \) and \( sf = \pi \times id_E \) then \( E \) is soft over \( X \).

Proof: Given a halo nbhd. \( U \) of \( A \subset X \) and a section \( t \) of \( f \) over \( U \) we must show that \( t \) restricted to some smaller nbhd. extends to all of \( X \). Picture

The dotted arrow gives the desired extension over \( U' \).
Converse: Assume $E \xrightarrow{f} X$ universally soft (remains soft after any base change; e.g. example above) then $f$ is a fibre homotopy equivalence over $X$.

Proof: First of all, soft $\Rightarrow F(X) \neq \emptyset$ since $F(\emptyset) = \emptyset$ pt (observe $\emptyset \subset \emptyset$ is a halo nbd.), hence $f$ has a section $s$. Now want to construct a dotted arrow in

$$
\begin{array}{ccc}
E \times I & \xrightarrow{sf + id} & E \\
\downarrow & & \downarrow f \\
E \times I & \xrightarrow{f \circ \text{pr}_1} & X
\end{array}
$$

Since $f$ is universally soft, $(f \circ \text{pr}_1)^*(-E)$ is soft over $E \times I$, hence all we need now is extend the section $s$ over $E \times I$ to a halo nbd. But this is clearly possible using constant homotopies near $0$ and $1$.

Note: We do not require that the vertical homotopy $sf \sim_X E$ preserve the section $s$. Thus $s(X)$ is not necessarily a strong deformation retract of $E$ over $X$. In good cases one might to be able to arrange this by extending the map $E \times I \cup X \times I \to E$? Here is a soft map which is not a fibration:

$$
\begin{array}{c}
\text{fiber}
\end{array}
$$
Weak covering homotopy property: We say that \( f : E \to X \) has the WCHP if given \( \alpha : K \to E \) and a homotopy \( K \times I \to X \) starting from \( f \alpha \), there is a lifting \( K \times I \to E \) whose initial position is vertically homotopic to \( \alpha \). Example:

\[
\begin{array}{c}
\begin{array}{c}
K \\
\downarrow \\
K \times I \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\to E \\
\downarrow f \\
\to X \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\to E \\
\downarrow p \\
\to B \\
\end{array}
\end{array}
\]

Lemma: Let \( f : E' \to E \) be a map of spaces over \( B \) such that there exists \( g : E' \to E \) with \( gf \sim_B \text{id} E' \). If \( E \to B \) has the WCHP, then so does \( E' \to B \).

Proof: Given

\[
\begin{array}{c}
K \times I \\
\downarrow \\
K \times I \\
\end{array}
\begin{array}{c}
\to E' \\
\downarrow f \\
\to E \\
\end{array}
\begin{array}{c}
\to B \\
\to B \\
\end{array}
\]

Consider \( H : K \times I \to E \) covering \( \beta \) as \( H \alpha \sim_B \Rightarrow gH : K \times I \to E' \) covers \( \beta \) and

\( gH \alpha \sim_B \Rightarrow \alpha \)

g.e.d.

Proposition: Let \( f : E' \to E \) be a map of spaces over \( B \) such that both \( p' : E' \to B \) and \( p : E \to B \) have the WCHP. If \( f \) is a homotopy equivalence \( \heartsuit \), then it is a fiber homotopy equivalence.
Note the maps with the WCHP are stable under composition and base change. The point is that $E \to B$ has WCHP iff the dotted arrow exists in
\[
K \times 0 \to X \\
\downarrow \quad \downarrow \\
K \times I \quad \beta \to Y
\]
provided $\beta$ is a constant homotopy in some interval $K \times [0, e]$.

Next suppose $E', E$ are spaces over $B$ with the WCHP, and let $f : E' \to E$ be a map.

\[
\begin{array}{ccc}
E' \times E & \overset{\pi_2}{\longrightarrow} & (E/B) \\
\downarrow \quad \downarrow (d_0, d_1) & & \\
E' \times_{B} E & \overset{f \times id}{\longrightarrow} & E \times_{B} E \\
\downarrow \pi_2 & & \\
E & & \\
\end{array}
\]

Since $E'/B$ has the WCHP the lower $\pi_2$, does.

Unpleasant feature: suppose $E \to B$ has the WCHP but not the CHP, i.e. $E^I \to E \times_{B} E^I$ doesn't have a section. Then as $E^I \to E \times_{B} E^I$ is a leg (both spaces have $E$ as strong def. retract), it cannot have the WCHP, or otherwise by the preceding proposition it would have a section.
The preceding proposition is proved by a covering homotopy type argument which might run as follows: provided we knew that 
\( (E/B)^I \rightarrow E \times_B E \) has the WCHP when \( E \rightarrow B \) does.

(This is alright when \( B \) is paracompact and locally contractible.)

Under this condition we may factor \( f \) in the customary way, and the second map \( g \) has the WCHP by

\[
\begin{align*}
E' \xrightarrow{i} & E' \times_E (E/B)^I \xrightarrow{g} E \\
\downarrow \text{id} & \downarrow \text{id} \xrightarrow{f \times \text{id}} E \times B \xrightarrow{E \times \text{id}} E \times E \\
\end{align*}
\]

\( \text{id} \) has WCHP

Thus if \( f \) is a homotopy equivalence, the map \( g \) will be a homotopy equivalence with the WCHP. Such a map has a section:

\[
\begin{array}{ccc}
B & \xrightarrow{s} & E' \\
\downarrow{h} & & \downarrow{w} \text{ WCHP} \\
B \times I & \xrightarrow{h} & B
\end{array}
\]

where \( h: \text{id}_B \times \text{id}_E \) is any homotopy constant near \( 0 \).

Since \( g \) has a section, it follows that \( f' : E \rightarrow E' \)
such that $f' f' \sim_B \text{id}_E$. Applying the same reasoning to $f'$ we find $f''': E' \to E'$ such that $f'' f' \sim_B \text{id}_E$.

Then one has that $f'' \sim_B f$, so $f$ is a fibre-homotopy equivalence.

**Remark:** If $X$ is a space of the homotopy type of a CW complex, then its sheaf-theoretic and singular cohomology coincide. In effect both are homotopy invariants, hence reduces to case of a CW complex, where equality follows from fact that CW complexes are paracompact and locally-contractible.

Gold shows that over paracompact contractible spaces that a WCMP space same as a space locally fibre-homotopy equivalent with a product space.
Let $I$ be an ordered set. Its realization $\text{BI} = |\text{New}(I)|$ is the ordered simplicial complex whose simplices are chains $X_0 < \ldots < X_p$ in $I$. (ordered s.s.x. = s.s.x. + ordering on vertices > each simplex is lin.ordered). $\text{Sd}(I)$ is the ordered set of layers of $I$. I want to interpret $B\text{Sd}(I)$ as a subdivision of $\text{BI}$.

**Example 1.** $I = \{0 \leq 1\}$. Then

$\text{Sd} I = \{ 0 \leq 0.1 \leq 1 \}$

so geometrically we have

$\text{BI}: \quad \begin{array}{c}
\text{Sd}(I)
\end{array}$

$\begin{array}{c}
\text{Sd}(I)
\end{array}$

**Example 2.** Suppose $C'$ is a full subcat. of $C$. Then $\text{Sd} C'$ is the full subcat. of $\text{Sd} C$ consisting of arrows $u: X \to Y$ such that $X, Y \in C'$. Thus, if $I'$ is a subcat of $I$ endowed with the induced ordering, $\text{Sd} I'$ is a sub-ordered set of $\text{Sd} I$. $\text{BI}'$ is the subcomplex of $\text{BI}$ consisting of the simplices $X_0 < \ldots < X_p$ with all $X_i$ in $I'$, $B\text{Sd} I'$ is a subcomplex of $B\text{Sd} I$.

**Example 3.** $\text{Sd}(C \times C') \rightarrow \text{Sd}(C) \times \text{Sd}(C')$. In
\textit{effect}

\[ \text{Ob} \{ \text{Sd} C \} = \text{Ar} C = \text{Hom}((0 \leq 1), C) \]

\[ \text{Ar} \{ \text{Sd} C \} = \text{Ar}_3 C = \text{Hom}(0 \leq 1 \leq 2 \leq 3 \leq C) \]

and these functors commute with products, in fact with arbitrary inverse limits, so we have

\[ \text{Sd} \left( \lim_{\longrightarrow} C \right) = \lim_{\longrightarrow} \text{Sd} C \]

\textbf{Example 4.} \( I = (0 \leq 1 \leq 2) \). This can be embedded as a sub-ordered set of \((0 \leq 1) \times (0 \leq 1) = \overline{I} \):

\[ \begin{array}{ccc}
\downarrow & < & \downarrow \\
B I & < & B \overline{I}
\end{array} \]

so \( B \text{Sd} I < B \text{Sd} \overline{I} = (B \text{Sd} (0 \leq 1))^2 \)

\textbf{Example 5.} \( I = [n] = (0 \leq 1 \leq \ldots \leq n) \) which we will embed in \([1]^n\) as the sequence

\[ (0, \ldots, 0) \leq (1, 0, \ldots, 0) \leq (1, 1, 0, \ldots) \leq \ldots \leq (1, 1, \ldots, 1) \]
Then $Bsd I$ is a subcomplex of $(sd[I])^n$.

**Example 6**: An ordered set, we have $Bsd(I)$ is a simplicial complex whose vertices are layers $x \leq y$ in $I$. Define a map

$$h_t: Bsd(I) \longrightarrow BI$$

$$h_t(x \leq y) = tx + (1-t)y \quad 0 \leq t \leq 1$$

To show this map is well-defined, we need only show that the image of the vertices of a simplex lie in a simplex. But a simplex in $sd(I)$ is of the form:

$$x_0 \leq \cdots \leq x_i \leq (x_0 \leq y_0) \leq (x_1 \leq y_1) \cdots \leq y_p$$

and

$$tx_0 + (1-t)y_0 \leq \cdots \leq tx_p + (1-t)y_p$$

all lie in the simplex $(x_0 \leq \cdots \leq y_p)$.
The preceding examples seem to establish the

**Assertion:** For any ordered set $I$, we have a map

$$h: \text{BSd}(I) \times [0, 1] \rightarrow BI$$

$$(x \leq y), t \rightarrow tx + (1-t)y$$

such that

i) for $0 < t < 1$, $h_t$ is a homeomorphism

ii) $h_1: \text{BSd}(I) \rightarrow BI$ is the map induced by the target functor $\text{Sd}(I) \rightarrow I$.

iii) $h_0$ is the map induced by source $\text{Sd}(I) \rightarrow I^o$ followed by the homeomorphism $BI^o = BI$.

If $I$ is finite, the subdivisions

$$\cdots \rightarrow \text{BSd}''(I) \xrightarrow{h_t} \text{BSd}'(I) \xrightarrow{h_t} BI$$

become arbitrarily fine for any $0 < t < 1$.

Now I want to apply the simplicial approx. thm. Suppose $I, J$ are two ordered sets, with $I$ finite, and let

$$f: BI \rightarrow BJ$$

be a map of the associated polyhedra $BJ$.

**Note:** $BJ$ has a canonical open covering - open stars of vertices $j \in J$.

For $n$ suff. large, the composed map

$$f: \text{BSd}''(I) \rightarrow BJ$$
has the property that for every vertex, the open star of every vertex is contained in the inverse image of an open star of \( \beta J \). Then we get a simplicial map

\[ N Sd^n(I) \rightarrow N J \]
July 18, 1972

The relation between what you are trying to do for categories and Kan's \( \text{Ex}^\infty \) theory:

Suppose \( C \) is a contractible category. Then I can solve the extension problem for the map

\[
\begin{align*}
\text{sd}^n \{0 \leq 1\} & \to [0 \leq 1] \\
\downarrow & \downarrow \\
\text{sd}^n \{0 \leq 1\} & \to C
\end{align*}
\]

provided I subdivide enough. Precisely, suppose I have given \( f \)

\[
\begin{align*}
\text{sd}^n \{0 \leq 1\} & \to \{0, 1\} \\
\downarrow & \downarrow \\
\text{sd}^n \{0 \leq 1\} & \to C
\end{align*}
\]

Then \( g \) exists for \( n \) sufficiently large.

**Generalization:** Suppose I have maps

\[
\begin{align*}
\text{sd}^n \{0 \leq 1\} & \to C \\
\text{sd}^n \{1 \leq 2\} & \to C \\
\text{sd}^n \{0 \leq 2\} & \to C
\end{align*}
\]

which are compatible. Then \( m \) can I enlarge \( n \) so as there exists an extension

\[
\begin{align*}
\text{sd}^n \{0 \leq 1 \leq 2\} & \to C
\end{align*}
\]
Question: Let $K$ be a finite simplicial complex, let $L$ be a subcomplex, and let $C$ be a contractible category. Given a functor $\text{Cat}(L) \to C$, does there exist a subdivision $K'$ of $K$ rel $L$ so that $f$ extends:

$$\text{Cat}(L) \xrightarrow{f} C$$

$$\cap$$

$$\text{Cat}(K') \xrightarrow{?}$$

Better questions: Given an ordered finite simplicial complex $K$, and a subcomplex $L$, we then have an inverse system of maps

$$\text{Sd}^m L \xrightarrow{f} \text{Sd}^n K$$

and we can ask if given $\text{Sd}^m L \to C$, does there exist $n > m$ and an extension

$$\text{Sd}^m L \xrightarrow{?} \text{Sd}^m L \xrightarrow{?} C$$

$$\cap$$

$$\text{Sd}^n K$$

Assume the answer to the preceding is Yes. Define a functor on ordered simplicial complexes by

$$F(K) = \lim_{m} \text{Hom}(\text{Sd}^m(K), C)$$
Then we are asking that \( F(K) \rightarrow F(L) \) if \( L \leq K \). In particular if we take
\[
\begin{align*}
K &= \Delta(n) \\
L &= \Delta(n)
\end{align*}
\]
then we see that the simplicial set
\[
\text{in } n \mapsto F(\Delta(n))
\]
is a contractible Kan complex. Now-
\[
\text{Hom} \left( \text{Sd}^m(\Delta(n)), C \right)
\]
should roughly be the same as
\[
\text{Hom} \left( \Delta(n), \text{Ex}^m(\text{New} C) \right).
\]
This suggests that I am roughly aiming for a version of \( \text{Ex}^m \) using the elementary subdivision rather than barcyclic subdivisions.

---

\textbf{Conjecture:} Let \( C \) be a contractible category. Then the simplicial set
\[
\text{in } n \mapsto \lim_m \text{Hom} \left( \text{Sd}^m([n]), C \right) = X(C)
\]
is a contractible Kan complex.
Observe that if we used barycentric subdivising, then this limit would be $\operatorname{Ex}^\infty(\operatorname{Nerve} C)$, so the conjecture would be clear.

Variations on the preceding conjecture:

1. \( C \xrightarrow{f} C' \) cofibred with contractible fibres. Then
\[
X(C) \longrightarrow X(C')
\]
is a Kan fibration with contractible fibres.

2. \( C \xrightarrow{f} C' \) cofibred such that all colimit change functors are hog's. Then
\[
X(C) \longrightarrow X(C')
\]
is a Kan fibration (with fibre \( X(C_y) \) over \( Y \) for all \( Y \in \operatorname{Ob}(C') \); observe if vertices of \( X(C) \) same as objects of \( C \)).

List all things that can be proved about $K_*(\mathbb{Z})$ using results about $K_*(\mathbb{F}_p)$ and the $J$-homomorphism.

**Claims:**

$J \{ \pi_{4d-1} \to \infty \} \to K_{4d-1} \mathbb{Z}$

cyclic of order denoted $(B_0/A_0)$.

**Proof. Diagram**

\[
\begin{array}{ccc}
B\Sigma^+ & \to & F \\
\downarrow & & \downarrow \\
BGL(\mathbb{Z})^+ & \to & BO = BGL(\mathbb{R}) \\
\downarrow (ch_{2i})_{i \geq 1} & & \downarrow \\
\prod K(\mathbb{Q}, 2i) & i \geq 1 & \\
\end{array}
\]

$F$ is the fibre of $(ch_{2i})_{i \geq 1}$. Since Chern classes of a representation of a discrete group are torsion, the dotted arrow exists. Thus we get a diagram:

\[
\pi_{4d-1} A = \pi_{4d-1} B\Sigma^+ \to \pi_{4d-1} F = \mathbb{Q}/a_0 \mathbb{Z}
\]

$a_0 = \begin{cases} 
1 \text{ even} \\
2 \text{ odd} 
\end{cases}$
I will show below that \( J \{ \pi_{q-1} SO \} \xrightarrow{e} e(\frac{q}{4}) \subset \mathbb{Z}/4\mathbb{Z} \)

so the claim will follow.

Recall the definition of the e-invariant. Given an element of \( \pi_{4q-1}^s \), represent it by a map

\[ f : S^{8k+4q-1} \longrightarrow S^{8k} \]

and let \( X \) be its mapping cone. Then

\[ 0 \longrightarrow \tilde{K}^0(S^{8k}) \xrightarrow{\alpha} \tilde{K}^0(X) \xrightarrow{\alpha} \tilde{K}^0(S^{8k+4q}) \xrightarrow{\alpha} 0 \]

so if we choose \( x \in \tilde{K}^0(X) \) mapping to the distinguished generator of \( \tilde{K}^0(S^{8k}) \), the top component of the character of \( x \)

\[ ch_{8k+4q}(x) \in H^{8k+4q}(X; \mathbb{Q}) \cong H^{8k+4q}(S^{8k+4q}; \mathbb{Q}) \]

\[ \cong \mathbb{Q} \]

is a rational number determined up to \( ch_{8k+4q}(\tilde{K}^0(S^{8k+4q})) = a_n \mathbb{Q}. \) (Observe if \( q \) is even, then real e-invariant = complex e-invariant, since \( \tilde{K}^0(S^{8k}) = K(S^{8k}) \))

Recent the preceding. Let \( BO^{8k} \longrightarrow BO \)

\[ F_{8k} \longrightarrow BO^{8k} \xrightarrow{(ch_i)_{i>2k}} \prod K(\mathbb{Q}, i) \]

with \( BO^{8k} \), \( (8k-1) \)-connected, and define
Then we have
\[
S^{8k+4i-1} \xrightarrow{f} S^{8k} \xrightarrow{\text{gen.}} X \xrightarrow{\text{ch}_{8k+4i}(x)} BO\langle S^k \rangle \xrightarrow{i>2k} \prod_{i>2k} K(Q,4i)
\]
from which we see that
\[e(f) = \text{Toda bracket of}\]
\[
S^{8k+4i-1} \xrightarrow{f} S^{8k} \xrightarrow{\text{gen.}} BO\langle S^k \rangle \xrightarrow{i>2k} \prod_{i>2k} K(Q,4i)
\]
and hence if we choose maps on the other side
\[
S^{8k+4i-1} \xrightarrow{f} S^{8k} \xrightarrow{\text{gen.}} BO\langle S^k \rangle \xrightarrow{i>2k} \prod_{i>2k} K(Q,4i)
\]
we know the arrow at the left is \(e(f)\) mod indeterminacy, so we find the following alternate description of the \(e\)-invariant.
\[e(f) = -f^*(\alpha) \in \prod_{i>2k} K(Q,4i) \hookrightarrow Q/a_\infty\mathbb{Z}\]
where \(\alpha\in \pi_{8k}(F_{8k})\) is the unique element mapped to the generator of \(\pi_{8k}(BO\langle S^k \rangle)\).
Now take $8k$-fold loop spaces

$$
\begin{array}{ccc}
\Omega^s S^8k & \rightarrow & \Omega^s B(S^2k) & \rightarrow & \prod_{i \geq 1} K(Q, 4i) \\
\downarrow & & & & \\
S^1 & \rightarrow & Z \times BO & \rightarrow & \prod_{i \geq 1} K(Q, 4i)
\end{array}
$$

Thus it follows that the various $S^8k$ induce the map

$$\lim_{s \to \infty} \Omega^s S^8k \rightarrow Z \times F$$

which covers

$$\downarrow$$

$$Z \times BO$$

and so it is now easy to see that the map on homotopy induced by $F$ in degree $4s-1$ is simply the $e$-invariant.

How about dimensions $8k, 8k+1$? According to Adams the picture is that $\pi_n^s$ contains the direct summands

![Diagram showing the direct summands](image-url)
\[ \mathcal{J} \{ \pi_{8k} \text{SO} \} \cong \mathbb{Z}/2 \quad \text{in} \quad \pi_{8k}^S \]
\[ \mathcal{J} \{ \pi_{8k+1} \text{SO} \} \oplus \langle \eta_{8k+1} \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \text{in} \quad \pi_{8k+1}^S \]
\[ \langle \eta_{8k+2} \rangle \cong \mathbb{Z}/2 \quad \text{in} \quad \pi_{8k+2}^S. \]

and that moreover \( \eta_{8k+i} \) maps non-trivially to the generator of \( \pi_{8k+i}^S \text{BO} \), \( i = 1, 2 \). Thus we have direct summands
\[ \mathbb{Z}/2 \quad \text{in} \quad K_{8k+1} \mathbb{Z} \]
\[ \mathbb{Z}/2 \quad \text{in} \quad K_{8k+2} \mathbb{Z}. \]

(\text{It should be true that the image \( \mathcal{J} \) goes to zero in} \( \pi_k \text{BO} \), because of the fibration)

\[ \text{SO} \rightarrow \text{Im} \mathcal{J} \rightarrow \text{BO} \xrightarrow{\mathbb{Z}/2-1} \text{BSO} \]
\[ \downarrow \mathcal{J} \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \text{SG} = \text{SG} \rightarrow \ast \rightarrow \text{BSG} \]

This is in fact a proof provided

\[ \text{SO} \xrightarrow{\mathcal{J}} \text{Im} \mathcal{J} \]
\[ \text{J} \quad \text{SG} \xleftarrow{\quad (\text{QSO})} \quad \text{commuted} \]
Now bring in finite fields. The diagram:

\[
\begin{array}{cccc}
B \Sigma^+_n & \longrightarrow & BGL_n \mathbb{Z} & \longrightarrow & BGL_n(\mathbb{Z}/p) \\
\downarrow \text{Braverman} & & & \downarrow \text{braverman} \\
\Sigma^*_n & \longrightarrow & BU & \quad & \\
\end{array}
\]

doesn't commute, however, it does if we restrict to a finite skeleton of \(B \Sigma^+_n\) and localize with respect to \(p^n\).

The diagram of the homotopy groups:

\[
\begin{array}{cccc}
\Sigma^+_n & \longrightarrow & BGL(\mathbb{Z}/p) & \cong & F \mathbb{P}^p \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & BU & \overset{\pi_{1-p}}{\longrightarrow} & BU \\
& & & & & \overset{\text{ch}}{\longrightarrow} \pi_{1-p} K(\mathbb{Q}, 2i) \\
& & & & & \overset{\text{ch}_i}{\longrightarrow} \pi_{1-p} K(\mathbb{Q}^+, 2i) \\
\end{array}
\]

so we get a homomorphism:

\[
\pi_{2i-1}(B \Sigma^+) \longrightarrow \pi_{2i-1} \left(F \mathbb{P}^p\right) = K_{2i-1}(\mathbb{Z}/p)
\]

with relation:

\[
\mathbb{Z}/p^{i-1} \mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}
\]

complex e-invariant
Some number theory

\[ m(2) = \text{denom} \left( B_2/4 \right) = \prod_{l \text{ prime}} m_l(2) \]

where for \( l \) odd we have

\[ m_l(t) = \nu_l(p^t - 1) \quad \text{if } p \text{ gen. } \mathbb{Z}_l^* \]

\[ = \begin{cases} 0 & \text{if } (l-1) \nmid t \\ \nu_l(2^t) + 1 & \text{if } (l-1) \mid t \end{cases} \]

and for \( l = 2 \)

\[ m_2(t) = \nu_2(3^t - 1) \]

\[ = \begin{cases} 1 & t \text{ odd} \\ \nu_2(t) + 2 & t \text{ even} \end{cases} \]

Note that the topologist's \( B_0 = B_{2n} \) in Borevich-Shaf

Examples

\( s = 1 \), \( m(2) = 2^3 \cdot 3 = 24 \), \( \frac{B_1}{4} = \frac{1}{4.6} \)

\( s = 2 \), \( m(4) = 2^4 \cdot 3 \cdot 5 \Rightarrow \frac{B_2}{8} = \frac{1}{8 \cdot 30} = \frac{1}{16 \cdot 3 \cdot 5} \)

\( s = 3 \), \( m(6) = 2^3 \cdot 3^2 \cdot 7 \Rightarrow \frac{B_3}{12} = \frac{1}{12 \cdot 42} = \frac{1}{8 \cdot 7 \cdot 7} \)

\( s = 4 \), \( m(8) = 2^5 \cdot 3 \cdot 5 \Rightarrow \frac{B_4}{16} = \frac{1}{16 \cdot 30} = \frac{1}{32 \cdot 3 \cdot 5} \)

\( s = 5 \), \( m(10) = 2^3 \cdot 3 \cdot 11 \Rightarrow \frac{B_5}{20} = \frac{1}{20 \cdot 6} = \frac{1}{8 \cdot 3 \cdot 11} \)
July 21, 1972

Observation which perhaps is important.

Let $f: C \to C'$ be cofibred, and suppose that each fibre $C_y$, $y \in C'$, is connected. Given objects

$$X', X \quad \text{in } C$$

and an arrow

$$f_{X'} \Rightarrow f_X \quad \text{in } C'$$

we would like to lift $u$ to a map from $X'$ to $X$.

Because $f$ is cofibred, we can lift $u$ to a cartesian arrow

$$X' \to u_* X',$$

which is such that

$$\text{Hom}(X', Z) \to \text{Hom}(u_* X', Z)$$

for all $Z \in C_{fx}$. Thus $u$ lifts to a map $X' \to X$ iff there is an arrow $u_* X' \to X$ in $C_{fx}$.

But we are given only that $C_{fx}$ is connected, so all we have is a chain of arrows

$$u_* X' \Rightarrow \cdots \Rightarrow X$$

in $C_{fx}$. But recall that $\text{Sp}^m[1]$
is the category

\[ \cdots \]

with \( 2^m \) arrows, and that the functor \( \text{Sd}^m [1] \to [1] \) sends all the objects except 0 to 1.

Therefore for \( m \) sufficiently large we can find a commutative diagram

\[
\text{Sd}^m \{0,1\} \xrightarrow{(x,x)} C \xrightarrow{f} C'\]

(\( f \) may be useful later to note that the dotted arrow may be chosen so that the first arrow \( f \) goes into a cocartesian arrow relative to \( f \). The point is that \( \text{Sd}^m [1] \to [1] \) is cofibred. In effect \( \text{Sd} C \to C \) is cofibred. Thus the dotted arrow is a cocartesian functor.)
July 23, 1972

On hom

Homotopy type of categories.

Up to now, I have been trying to understand the homotopy groups of a small category C, in the following way: Given a finite complex X, I want to find the set \([X, BC]\). To do this, I tried to construct a category \(T(X, C)\) such that

\[
\pi_0 T(X, C) = [X, BC]
\]

Here are some potential candidates for \(T\):

1) \(X\) compact space, then
\[
T(X, C) = \text{Tors} (X, C) = \text{Homtop} (\text{Top}X, C^\vee)
\]

2) \(X\) polyhedron
\[
T(X, C) = \lim_{K} \text{Hom} (\text{Cat} K, C)
\]

where \(K\) runs over all the admissible triangulations of \(X\).

3) \(X\) small category
\[
T(X, C) = \lim_{m} \text{Hom} (\text{Sd}^m X, C)
\]
What is going on here?

Here is an interpretation: For 3) we have

\[ T(Y, T(X, C)) = T(Y \times X, C) \]

so that

\[ [Y, B T(X, C)] = [Y \times X, B C] = [Y, B C^X]. \]

In other words, the category \( T(X, C) \) is playing the role of the function space

\[ BC^X = \text{Hom}(X, BC) = \Gamma(X \times BC/X). \]

Recall Grothendieck's \( \mathcal{T} \times S \) formalism. Suppose \( f: X \to S \) is a map and \( Z \) is over \( X \). Then Grothendieck denotes by \( \mathcal{T} \times S \) what I would write \( f_* Z \). It has the property

\[ \text{Hom}_{/S}(T, f_* Z) = \text{Hom}_{/X}(f^* T, Z) \]

For example, if \( C \) is over \( S \), then

\[ \text{Hom}_{/S}(T, f_* f^* C) = \text{Hom}_{/X}(X_S \times T, X_S \times C) = \text{Hom}_{/S}(X_S \times T, C). \]
\[
= \text{Hom}_{\mathcal{S}}\left( T, \text{Hom}_{\mathcal{S}}\left( X, C \right) \right)
\]

So

\[
f_x f^* C = \text{Hom}_{\mathcal{S}}\left( X, C \right)
\]

The picture: suppose we take seriously the philosophy that homotopy theory is to be constructed out of small categories. Over any \( X \) we consider the 2-category of cofiber categories over \( X \) with cocartesian functors as morphisms.

\[
\text{Hom}_X(Y, Z) = \text{Hom}_{\text{Cofat}/X}^{\text{cocart}}(Y, Z)
\]

Then given \( f: X \rightarrow S \) we have

\[
f^* : \text{Cofat}/S \rightarrow \text{Cofat}/X
\]

and perhaps an \( f_* \) functor which when \( S = e \) reduces to

\[
\Gamma(Z/X) = \text{Hom}_{\text{Cofat}/X}^{\text{cocart}}(X, Z).
\]

In effect one knows that a cocart. functor

\[
\begin{array}{ccc}
X \times T & \xrightarrow{f} & Z \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]
is the same as a functor

\[ T \rightarrow \Gamma(Z/X). \]

Now what you need to do is form the homotopy category of Cofcat/\( X \) by inverting the fibre-homotopy-equivalences. Then you wish to construct the derived functor

\[ Rf_x^* : \text{Ho}(\text{Cofcat}/X) \rightarrow \text{Ho}(\text{Cofcat}/S) \]

for a map \( f : X \rightarrow S \). In particular, if one takes \( f : X \rightarrow pt \), and \( C \) over \( pt \), then

\[ Rf_x^* C = T(X, C). \]

Now perhaps you might want to use a specific construction for \( Rf_x^* = \text{holim} \) such as

\[ Rf_x^*(Z) = \lim_{m} \text{Hom}(Sd^m X, Z). \]
July 26, 1972. To understand holim.

Let \( F \to S \) be a fibred category. I want to understand holim \( \frac{F}{S} \).

Example: suppose \( F \) is the fibred category in groupoids defined by a complex of abelian group functors of length 2

\[
K^*: K^0 \to K^1 \to 0.
\]

Compute \( \lim_{\to S} \frac{F}{S} \), i.e., the category of cartesian sections of \( F/S \). Now since the fibres are groupoids, every arrow is cartesian. Thus we want sections of \( F/S \) such a thing consists of

\[
\text{Ob-}S \ni y \mapsto s(y) \in \text{Ob}(F_y) = K^1(y).
\]

\[
\text{As } S \ni (u: y \to y') \mapsto t(u) \in \text{Hom}(s(y'), u^*s(y)) = K^0(y')
\]

\[
dt(u) = o(y') - u^*o(y)
\]

such that for \( y' \leftarrow y'' \xleftarrow{u} y \) we have

\[
t(vu) = t(v) + v^*t(u).
\]

Thus a section of \( F/S \) is a 1-cocycle in the complex.
\[ \mathcal{C}^0(S, K^0) \overset{\delta}{\rightarrow} \mathcal{C}^1(S, K^0) \overset{\delta}{\rightarrow} \mathcal{C}^2(S, K^0) \]

where \[ \mathcal{C}^0(S, F) = \prod_{y \in S} F(y) \]

Thus it is clear that \( \lim_{\longrightarrow} F \) is the category belonging to the complex

\[ (\mathcal{C}^i(S, K^i))^0 \overset{\delta}{\rightarrow} \cdots \]

What should the homotopy-inverse limit be? According to Kan, one wants an object \( s(y) \) over \( y \), \( y \in S \), and for every arrow \( a : y^{\prime} \rightarrow y \)

a path from \( s(y^{\prime}) \) to \( a \circ s(y) \), etc. Since the fibres of \( F \) are groupoids, it follows that every path must be an isomorphism. Thus in general we have

\[ \underset{S}{\lim} F \xrightarrow{\sim} \operatorname{holim} F \]

when \( F \rightarrow S \) is fibred in groupoids.
July 26, 1972

Dear Jack,

As I wrote you earlier, the assertion in your note that I can prove the injectivity of the map

\[ J(\pi_1^0) \subset \pi_1^s \longrightarrow K_1 \mathbb{Z} \]

is inaccurate with respect to the 2-torsion. Unfortunately, the corrections I sent are also incorrect. Since Kervaire has requested some details, I am sending the following account of what I know about the above map, in order to clear the confusion.

1. First consider the dimensions \( i = 8k, 8k+1 \), where \( J(\pi_1^0) = \mathbb{Z}/2 \). I do not know whether this group injects into \( K_1 \mathbb{Z} \), and suspect that it does not, except of course when \( k = 0 \).

However, Adams has produced elements of order 2, \( \eta_j \in \pi_j^s \), \( j = 8k+1, 8k+2 \), closely related to the image of \( J \) in the preceding dimensions, which do map non-trivially into \( K_1 \mathbb{Z} \). To see this, consider the square

\[
\begin{array}{c}
\mathcal{B}F_{\infty}^+ \\
\downarrow \delta \\
\mathcal{B}GL(\mathbb{Z})^+ \\
\end{array} \longrightarrow \begin{array}{c}
\mathcal{B}O \\
\mathcal{B}GL(\mathbb{H}) \\
\end{array}
\tag{1}
\]

induced by the various group inclusions. Passing to homotopy groups, we obtain homomorphisms \( \pi_j^s = \pi_j \mathcal{B}F_{\infty}^+ \longrightarrow K_j \mathbb{Z} \longrightarrow \pi_j \mathcal{B}O \) whose composition is the degree map for \( KO \)-theory. Since Adams has shown that the degree map carries \( \eta_j \) to the generator of \( \pi_j \mathcal{B}O = \mathbb{Z}/2 \), the image of \( \eta_j \) in \( K_j \mathbb{Z} \) is non-trivial. In fact, we have

\[ K_j \mathbb{Z} = \mathbb{Z}/2 \oplus ? \quad j = 8k+1, 8k+2 \]

I should mention that this observation appears already in one of Gersten's papers.

2. Next consider the dimension \( i = 4s-1 \), where \( J(\pi_1^0) \) is cyclic of
order $\text{denom}(B_5/4s)$. I shall prove the injectivity:

$$J(\pi_{4s-1}^s) \hookrightarrow K_{4s-1}Z$$

by showing that the Adams $e$-invariant on $\pi_{4s-1}^s$, which detects $J(\pi_{4s-1}^s)$, comes from an invariant defined on $K_{4s-1}Z$.

Following Sullivan, consider the fibration

$$\begin{array}{ccc}
F & \longrightarrow & BO \\
& \longrightarrow & \prod_{i \geq 1} K(\mathbb{Q}, 4i)
\end{array}$$

where $K(\mathbb{Q}, j)$ is an Eilenberg-MacLane space and $\text{ch}_j$ represents the $j$-th component of the Chern character. Since $B\Sigma_\infty^+$ has trivial rational cohomology, the degree map $B\Sigma_\infty^+ \longrightarrow BO$ lifts by obstruction theory, uniquely up to homotopy, to a map

$$B\Sigma_\infty^+ \longrightarrow F$$

which induces a homomorphism

$$\pi_{4s-1}^s \longrightarrow \pi_{4s-1}^s F \cong \mathbb{Q}/a_sZ$$

where $a_s$ is 1 or 2 depending on whether $s$ is even or odd.

I claim this homomorphism is the negative of the Adams $e$-invariant.

Assuming this for the moment, consider the diagram

$$\begin{array}{ccc}
B\Sigma_\infty^+ & \longrightarrow & BGL(\mathbb{Z})^+ \\
\downarrow w & & \downarrow \text{ch} \\
F & \longrightarrow & BO \\
\end{array}$$

with the map $w$ obtained from (1). Since the Chern classes of representations of discrete groups are torsion classes, the map $(\text{ch})w$ is null-homotopic, and the dotted arrow exists. The induced map from $B\Sigma_\infty^+$ to $F$ must be (2).

Thus we obtain a commutative diagram

$$\begin{array}{ccc}
\pi_{4s-1}^s & \longrightarrow & K_{4s-1}Z \\
\downarrow -e & & \\
\mathbb{Q}/a_sZ & \leftarrow & \end{array}$$

as desired.
3. To prove the claim about the e-invariant, consider the map

\[ BO(8k) \longrightarrow \bigoplus_{i \geq 1} K(Q, 8k+4i) \]

with components \( ch_{8k+4i} \), where \( BO(8k) \) is the \((8k-1)\)-connected covering of \( BO \). Denote this map briefly by \( c : BO(8k) \longrightarrow \Sigma(8k) \) and let \( F(8k) \) be its fibre. Let \( b : S^{8k} \longrightarrow BO(8k) \) represent the generator of \( \pi_{8k} BO(8k) = \pi_{8k}^BO \) provided by Bott periodicity.

Now suppose given a map \( f : S^{8k+4s-1} \longrightarrow S^{8k} \) representing an element \( \bar{f} \) of \( \pi_{4s-1}^S \). We compute the Toda bracket \( \{ c, b, f \} \) by forming the diagram

\[
\begin{array}{cccc}
S^{8k+4s-1} & \xrightarrow{f} & S^{8k} & \longrightarrow \text{Cone } f & \longrightarrow & S^{8k+4s} \\
\downarrow{u} & & \downarrow{v} & & \downarrow{x} & & \downarrow{y} \\
\Omega S(8k) & \xrightarrow{b} & F(8k) & \longrightarrow BO(8k) & \longrightarrow & E(8k)
\end{array}
\]

in which the arrows \( x, y \) and \( v, u \) can be filled in as \( bf \) and \( cb \) are null-homotopic. By definition, the Toda bracket is the element represented by \( y \) in

\[ \pi_{8k+4s} F(8k) / c_* \pi_{8k+4s} BO(8k) + f^* ch_{8k+4s} \Omega E(8k) = Q / a_s \mathbb{Z}. \]

Now Adams defines the e-invariant of \( \bar{f} \) by choosing an element \( z \) of \( \tilde{E}(\text{Cone } f) \) restricting to the generator of \( \tilde{E}(S^{8k}) \), and forming

\[ ch_{8k+4s}(z) \in H^{8k+4s}(\text{Cone } f, Q) \cong H^{8k+4s}(S^{8k+4s}, Q) \cong Q. \]

The image of this rational number in \( Q / a_s \mathbb{Z} \) is then \( e(\bar{f}) \). Clearly \( z \) and \( ch_{8k+4s}(z) \) may be identified with the maps \( x \) and \( y \) in the diagram, hence we have the formula

\[ e(\bar{f}) = \{ c, b, f \}. \]

On the other hand, from the theory of Toda brackets one knows that the map \( u \) in the diagram represents the negative of \( \{ c, b, f \} \). Thus we have the formula

\[ e(\bar{f}) = -f^*(v_k) \in \pi_{8k+4s-1} F(8k) = Q / a_s \mathbb{Z}. \]
where \( v_k = v \) is the unique element of \( \pi_{sk} F(8k) \) mapping to the generator of \( \pi_{sk} B_0(8k) \). Now by periodicity we have \( \bigwedge^{sk} F(8k) \cong \mathbb{Z} \times F \). The maps \( v_k \) fit together to induce a map

\[
\nabla : \lim_{k} \bigwedge^{sk} F(8k) \longrightarrow F
\]

which covers the degree map into \( B_0 \). Thus \( \nabla \) is the map (2). The formula (3) shows that its effect on homotopy groups is the negative of the e-invariant, which proves the claim.

4. Additional information on the image of \( J(\pi_{4s-1} 0) \) in \( K_{4s-1} \mathbb{Z} \) can be obtained from the computation of the \( K \)-groups of finite fields as follows. Let \( p \) be a prime number and \( \mathbb{F}_p \) the field with \( p \) elements, and consider the obvious homomorphisms

\[
\pi_{4s-1}^s \longrightarrow K_{4s-1} \mathbb{Z} \longrightarrow K_{4s-1} \mathbb{F}_p .
\]

I will show below that this composition is essentially the part of the complex e-invariant which is prime to \( p \). More precisely, there is a commutative diagram

\[
\pi_{4s-1}^s \longrightarrow K_{4s-1} \mathbb{F}_p \cong \mathbb{Z}/(p^{2s-1}) \mathbb{Z}
\]

\[
\xrightarrow{\theta} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}[1/p]
\]

where \( \theta \) is injective with image the unique subgroup of order \( p^{2s-1} \).

Here \( \mathbb{Q}/\mathbb{Z}[1/p] \) denotes the ring of rational numbers with powers of \( p \) in the denominator.

Assuming this, let \( \mathfrak{f} \) be an odd prime, and choose \( p \) to be a topological generator of the group \( \mathbb{Z}_\mathfrak{f}^* \) of \( \mathfrak{f} \)-adic units. According to Adams, the e-invariant is injective on \( J(\pi_{4s-1} 0) \), and the \( \mathfrak{f} \)-primary component \( J(\pi_{4s-1} 0)(\mathfrak{f}) \) is cyclic of order \( \mathfrak{f}^m \), \( n = v_\mathfrak{f}(p^{2s-1}) \), \( v_\mathfrak{f} = \mathfrak{f} \)-adic valuation. We have therefore an isomorphism

\[
J(\pi_{4s-1} 0)(\mathfrak{f}) \cong (K_{4s-1} \mathbb{F}_p)(\mathfrak{f}) .
\]
It follows that the odd part of $J(\pi_{4s-1}^0)$ is isomorphic to a direct summand of $K_{4s-1}\mathbb{Z}$.

Suppose now that $\gamma = 2$ and take $p = 3$. Using Adams work, both the source and target of the map

$$J(\pi_{4s-1}^0)_{(2)} \longrightarrow (K_{4s-1}\mathbb{F}_3')_{(2)}$$

are cyclic of order $2^n$, $n = v_2(3^{2s-1})$; and the map is essentially multiplication by $a_s$. It follows that for $s$ even, when $a_s = 1$, $J(\pi_{4s-1}^0)_{(2)}$ is isomorphic to a direct summand of $K_{4s-1}\mathbb{Z}$.

Finally, observe that the diagram (4) shows the unique element of order 2 of $J(\pi_{4s-1}^0)$, when $s$ is odd, goes to zero in $K_{4s-1}\mathbb{F}_p$ for all $p$.

Summarizing:

**Proposition:** The homomorphism $\pi_{4s-1}^s \longrightarrow K_{4s-1}\mathbb{Z}$ induces an injection of $J(\pi_{4s-1}^0)$ into $K_{4s-1}\mathbb{Z}$. For even $s$, the image of $J(\pi_{4s-1}^0)$ is a direct summand. For odd $s$, the odd-torsion part of the image is a direct summand. For odd $s$, the unique element of order 2 of the image is in the kernel of the homomorphism $K_{4s-1}\mathbb{Z} \longrightarrow K_{4s-1}\mathbb{F}_p$ for all primes $p$.

I do not know whether or not the image of $J(\pi_{4s-1}^0)_{(2)}$ is a direct summand of $K_{4s-1}\mathbb{Z}$ when $s$ is odd. The first case is $s=1$, where

$$\mathbb{Z}/24 = J(\pi_2^0) = \pi_2^2 \longrightarrow K_2^2 = H_2(\text{St}(\mathbb{Z}),\mathbb{Z})$$

Here $K_2\mathbb{F}_2 = \mathbb{Z}/8$ and the map $J(\pi_2^0) \longrightarrow K_2\mathbb{F}_2$ has a kernel of order 6.

5. It remains to construct the diagram (4). Consider the diagram
\[
\begin{align*}
F & \longrightarrow BO \xrightarrow{ch} \prod_{i \geq 1} K(q_{i,4i}) \\
\downarrow & \quad \downarrow \quad \downarrow \\
F' & \longrightarrow BU[p^{-1}] \xrightarrow{ch} \prod_{j \geq 1} K(q_{2j}) \\
\uparrow & \quad \uparrow \quad \uparrow \quad ((p^{j-1})^{-1} \text{ch}_{2j}) \\
\mathbb{F}_p & \longrightarrow BU[p^{-1}] \xrightarrow{\mathbb{F}_p} BU
\end{align*}
\]

where $F'$ and $\mathbb{F}_p$ are defined so that the rows are fibrations. Here $BU[p^{-1}]$ is the localization of $BU$ which represents the functor $K(\ast) \otimes \mathbb{Z}[p^{-1}]$. Examining the homotopy sequences of these fibrations, we obtain isomorphisms

\[
\begin{align*}
\pi_{4s-1} F & \cong \mathbb{Q}/a_s \mathbb{Z} \\
\downarrow & \quad \downarrow \\
\pi_{4s-1} F' & \cong \mathbb{Q}/\mathbb{Z}[p^{-1}] \\
\uparrow & \quad \cup \\
\pi_{4s-1} \mathbb{F}_p & \cong (p^{2s-1})^{-1} \mathbb{Z} / \mathbb{Z}
\end{align*}
\]

where the maps at the right are the obvious ones.

From the computation of the $K$-groups of a finite field, there is a homotopy equivalence

\[
\text{BGL}(\mathbb{F}_p)^+ \cong \mathbb{F}_p
\]

induced by lifting representations of finite groups over $\mathbb{F}_p$ to virtual complex representations by means of the Brauer theory. I claim that the diagram

\[
\begin{align*}
\text{BGL}(\mathbb{F}_p)^+ & \longrightarrow \text{BGL}(\mathbb{F}_p)^+ \cong \mathbb{F}_p \\
\downarrow & \quad \downarrow \\
BO & \longrightarrow BU[p^{-1}]
\end{align*}
\]

is commutative. The upper right path is obtained by lifting the obvious representation of $\sum_n$ on $\mathbb{F}_p^n$ to a virtual complex representation, while the lower right path comes from the obvious action of $\sum_n$ on $\mathbb{C}^n$. 
These two virtual representations are not the same in general. However, it is known that their characters agree on elements of \( \Sigma^I_n \) of order prime to \( p \), because both the representations \( \mathbb{W}_n^I \) and \( \mathbb{E}^n \) come from the integral representation \( \mathbb{Z}^n \). Thus the two virtual representations agree on the Sylow \( I \)-subgroups \( \Sigma^I_n \) for all primes \( I \neq p \). By a standard transfer argument, one has
\[
\left[ B \Sigma^I_n, BU[p^{-1}] \right] \xrightarrow{\mathcal{I} \neq p} \bigcap_{I \neq p} \left[ B \Sigma^I_n, BU[p^{-1}] \right].
\]
Consequently, the above diagram commutes as claimed.

Since \( B \Sigma^+_{\infty} \) has trivial rational cohomology, it follows by obstruction theory that the diagram
\[
\begin{array}{ccc}
B \Sigma^+_{\infty} & \to & B GL(F)^{+} = \mathbb{R}^P \\
\downarrow & & \downarrow \\
F & \to & F'
\end{array}
\]
is commutative, where the vertical arrow at the left is the one inducing minus the \( e \)-invariant. The desired commutative diagram (4) now results by taking homotopy groups, and using the isomorphisms (5).

This concludes the account of the map \( J(\pi_0) \to K_* \mathbb{Z} \). To the best of my knowledge, nothing more is known about \( K_* \mathbb{Z} \) beyond what this and Borel's theorem provide.

Best wishes,

Evan Miller
July 31, 1972.  On stability

Let $k$ be a field and consider $M = \text{Mod}_k(k)$. Let $C_n$ be the full subcategory of $Q(M)$ consisting of $M$ of dimension $\leq n$.

Let $f: C_{n-1} \rightarrow C_n$ be the inclusion. Then $f/V$ is equivalent to the ordered set of layers in $V$ of dimension $\leq n-1$. This clearly has a final object if $\dim V < n$, so suppose $\dim V = n$.

Let $X(V)$ be the simplicial complex whose simplices are chains $0 < W_0 < \cdots < W_p < V$ such that $W_p/W_0$ is of dim $< n$, i.e., either $W_0 > 0$ or $W_p < V$. Then $X(V)$ is clearly the suspension of the building $X(V)$.

Thus since we know:

\[ X(V) \text{ is } (n-3) \text{- connected} \]
\[ \Rightarrow X(V) \text{ is } (n-2) \text{- connected.} \]

But if $I_V$ is the ordered set equivalent to $f/V$, then $I_V$ is the ordered set of 1-simplices in $X(V)$ and we have a homotopy equivalence:

\[ \text{Cat } [X(V)] \rightarrow I_V \]

Let $I_V$ be the ordered set above which is equivalent to $f/V$, i.e., the ordered set of 1-simplices in the ordered simplicial complex $X(V)$. Then

\[ (W_0 < \cdots < W_p) \rightarrow (W_0, W_p) \]

is a homotopy equivalence.
(cofibre: \( W_0 \cdots \leq W_p \) + \( (W_0, W_p) \leq (W', W'') \) \( \mapsto \) \( W' \leq W_0 \cdots \leq W_p \leq W'' \) which is clearly the smallest simplex with ends \( W', W'' \) + which contains \( W_0 \cdots \leq W_p \). fibre is contractible has initial object.)

(General Lemma: Let \( X \) be a simplicial complex, with a [partial] ordering on vertices such that each simplex is linearly ordered, and such that any chain is a simplex provided its bottom and top form a 1-simplex. Then (i)1-simplexes in \( X \) form an ordered set \( I_X \) (ii)

\[
\begin{align*}
\text{Cat}(X) & \rightarrow I_X \\
(x_0 \cdots \leq x_p) & \mapsto (x_0, x_p)
\end{align*}
\]

is a homotopy equivalence (iii) the nerve of \( I_X \) is a subdivision of \( X \).

Thus we can conclude that \( f/V \) is \((n-1)\)-conn. for each \( V \in C_n \). And further that the h-fibre of

\[ C_{n-1} \rightarrow C_n \]

is \((n-2)\)-connected. Thus the h-fibre of

\[ C_n \rightarrow Q(M) \]

has homotopy groups beginning in dimension \( n \). (e.g. \( n=0 \), the fibre is \( Q(M) \) which begins in dim 0)
Suppose now that $A$ is a Dedekind domain with fraction field $K$. Let $M = PA$, and define again the filtration
\[ C_{n-1} \subseteq C_n \subseteq Q(M) \]
by: $C_n$ consists of $M$ of rank $\leq n$. Again if $f: C_{n-1} \rightarrow C_n$ is the inclusion, then $f/M$ is the ordered set of admissible layers in $M$ of rank $\leq n$. But, there is a 1-1 correspondence between subbundles of $M$ and subspaces of $M \otimes K$:
\[ N \subseteq M \Rightarrow M/N \text{ is in } PA \iff N = M \cap (N \otimes K) \subseteq M \otimes K \]
Therefore the fibres $f/M$ are all $(n-1)$-connected.

Suppose $A$ is the ring of integers in a number field $K$, whence it is known that the groups $\text{Gal}(A)$ have finitely generated homology in each degree. I want to try now to prove that $C_n$ has finitely generated homology in each degree. Then by the above stability considerations, we have that $Q(M)$ has f.g. homology and so, as it is an $H$-space, its homotopy groups are finitely generated.

Another way of thinking about $C_n$. Consider the fibred category over $\Delta$ whose fibre over $[p]$ is the groupoid of $p$-filtered objects.
of \( P_A \) such that \( \text{rank}(M_p) \leq n \). Call this cat. \( F_n^* \). Then we have a functor

\[ f: F_n^* \rightarrow C_n \]

and \( f/M \) is the fibrled cat. \( \Delta \) consisting of

\[ 0 < M_1 < \cdots < M_p < M \]

i.e. it is the simplicial set of

\[ M_0 < M_1 < \cdots < M_p \]

which is contractible.

Thus we can use \( F_n^* \) to calculate the homology of \( C_n \). We get usual seq. sequence

\[ E^2_{pq} = H_p(U \mapsto H_q(F_n(U), \Lambda)) \implies H_{p+q}(C_n, \Lambda) \]

But observe: \( (F_n)_\Lambda \) is the groupoid of \( \Lambda \)-filtered vector bundles

\[ 0 < M_1 < \cdots < M_p \]

with \( \text{rank } M_p \leq n \). Thus the non-degenerate part occurs with \( \nu \leq n \), and \( E^2_{pq} = 0 \), \( p > n \). But also, \( \text{rank } \Lambda \), the isom. classes of sequences \( (\ast) \) is finite (finiteness of class number), and the group of automorph. has f.g. homology. Thus \( E^2_{pq} \) is f.g., and we conclude \( H_q(C_n, \Lambda) \) is f.g.
(Checkable case: \( A \) a P.I.D. Then every projective is free, and a filtered object is determined up to isomorphism by the ranks of \( M_i/M_{i-1}, V_i \). The group of automorphisms is then an arithmetic group.

\[
\text{Conclusion: } K \text{ number field, } S \text{ finite set of places including the arch. ones, } A = \text{ ring of } S\text{-integers. Then } K_i; A \text{ is finitely generated for each } i \geq 0.
\]