March 1, 1972

Infinitesimal form of corcycle:

Consider first the problem of integrating a vector field \( \mathbf{v} = a(x) \frac{d}{dx} \) on \( \mathbb{R} \) (with compact support, i.e., \( a(x) = 0 \) for \( |x| \) large) to a flow \( t \mapsto F_t : \mathbb{R} \to \mathbb{R} \).

Thus for \( x \) fixed, the tangent vector to the path \( t \mapsto F_t(x) \) should be the vector field \( \mathbf{v} \) at \( F_t(x) \):

\[
\frac{d}{dt} f(F_t(x)) = a(F_t(x)) \frac{d}{dx} (F_t(x))
\]

for all functions \( f \) on the line. Taking \( f = \text{id} \)

\[
\begin{cases}
\frac{d}{dt} F_t(x) = a(F_t(x)) \\
F_0(x) = x
\end{cases}
\]

This differential equation determines \( F_t \). (Check that \( F \) is a flow: \( F_{t+s} = F_t \circ F_s \).)

But

\[
\frac{d}{dt} F_{t+s}(x) = \frac{d}{dt} F_{t+s}(x) = a(F_{t+s}(x))
\]

and

\[
\frac{d}{dt} F_t(F_0(x)) = a(F_t(F_0(x)))
\]

so have two solutions of the initial value problem.
\[
\frac{d}{dt} u_t = a(u)
\]

\[
u_0 = F_0(x)
\]

hence they must be equal.

So let's now solve the DE to the second order

\[
F_t(x) = x + t \left. \frac{d}{dt} F_t(x) \right|_{t=0} + \frac{t^2}{2} \left. \frac{d^2}{dt^2} F_t(x) \right|_{t=0}
\]

\[
\frac{d^2}{dt^2} F_t(x) = \frac{d}{dt} \left( a(F_t(x)) \right) = a'(F_t(x)) \frac{dF_t(x)}{dt}
\]

\[
= a'(F_t(x)) a(F_t(x)) = \left( a a' \right)(F_t(x))
\]

Thus

\[
F_t(x) = x + t a(x) + \frac{t^2}{2} (a a')(x) + O(t^3)
\]

\[
= e^{ta(x)} x + \sum_{n=0}^{\infty} \frac{t^n}{n!} (a^{(n)})(a(x))
\]

Now consider the cocycle on page 17 with

\[
g_{vu}(x) = x + t a(x) + \frac{t^2}{2} (a a')(x) + O(t^3)
\]

\[
g_{wu}(x) = x + t b(x) + \frac{t^2}{2} (b b')(x) + O(t^3)
\]

and determine the leading terms as \( t \to 0 \).

\[
\log \left( \frac{1+u}{u} \right) = \frac{u - \frac{u^2}{2} + \frac{u^3}{3}}{u} = 1 - \frac{u}{2} + O(u^2)
\]
\[ g'_{\nu}(x) = 1 + t a'(x) + \frac{t^2}{2} (a a')'(x) + \ldots \]

\[ g''_{\nu}(x) = 1 + t b'(x) + \frac{t^2}{2} (b b')'(x) + \ldots \]

\[ g''_{\nu}(x) = t a''(x) + \frac{t^2}{2} (a a'')'(x) + \ldots \]

\[ g''_{\nu}(x) = t b''(x) + \frac{t^2}{2} (b b'')'(x) + \ldots \]

\[
\log \frac{g'_{\nu}}{g'_{\nu} - 1} = 1 - \frac{1}{2} [t a'] + O(t^2)
\]

\[
\log \frac{g''_{\nu}}{g''_{\nu} - 1} = 1 - \frac{1}{2} [t b'] + O(t^2)
\]

\[ g'_{\nu} - g'_{\nu} = t (b' - a') + O(t^2) \]

Thus, the leading term of the expression in brackets at bottom of page 17 is

\[
\frac{-\frac{1}{2} t a' + \frac{1}{2} t b'}{t (b' - a')} = \frac{1}{2}
\]

and

\[
\begin{vmatrix}
  g'_{\nu} - 1 & g''_{\nu} - 1 \\
  g''_{\nu} & g''_{\nu}
\end{vmatrix} = \begin{vmatrix}
  t a' & t b' \\
  t a'' & t b''
\end{vmatrix}
\]

\[ = t^2 (a'b'' - a''b') + O(t^3) \]
Thus we get the infinitesimal cocycle
\[
\lambda\left(\frac{a}{\text{d}x}, \frac{b}{\text{d}x}\right) = \frac{1}{2} \int_{-\infty}^{\infty} (a' b'' - a'' b') \, dx
\]

\[
= \int_{-\infty}^{\infty} a' b'' \, dx
\]

Check this is a cocycle on the Lie algebra of vector fields on $\mathbb{R}$ with compact support. This means it satisfies the Jacobi identity
\[
\lambda([X, Y], Z) + \lambda([Y, Z], X) + \lambda([Z, X], Y) = 0
\]

$X = \frac{a}{\text{d}x}$, $Y = \frac{b}{\text{d}x}$, $Z = \frac{c}{\text{d}x}$

\[
[X, Y] = (ab' - ba') \frac{d}{dx}
\]

\[
\lambda([X, Y], Z) = \int (ab' - ba') c'' \, dx = \int (ab'' c'' - ba'' c) \, dx
\]

\[
\lambda([Y, Z], X) = \int (bc'' a'' - ab'' c') \, dx
\]

\[
\lambda([Z, X], Y) = \int (ca'' b'' - a'' c' b) \, dx
\]

so it is indeed a 2-cocycle.

(This is the Heisenberg-Fuchs 2-cocycle for vector fields on $S^1$.
They claim that $\int \frac{a a' a''}{b b' b''} \, dx$ is a 3-cocycle.)
March 2, 1972

I want to understand the Lie algebra extension defined by this cocycle. Thus if \( \mathfrak{g} \) is the Lie algebra of vector fields on \( \mathbb{R} \) with compact support, the extension is \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \) with bracket

\[
[X + \alpha, Y + \beta] = [X, Y] + \lambda(X, Y).
\]

Hence to realize \( \tilde{\mathfrak{g}} \) concretely, what we want to do is associate to each \( X \in \mathfrak{g} \) an operator \( A(X) \) satisfying the commutation relations

\[
[A(X), A(Y)] = A([X, Y]) + \lambda(X, Y)
\]

\( \lambda(X, Y) \) being viewed as a scalar operator.

Idea: Let \( M \) be a symplectic manifold, \( \omega \) the canonical closed non-degenerate 2-form. Then

\[
\Theta(X)\omega = 0 \quad \iff \quad d(i(X)\omega) = 0.
\]

so there is a 1-1 correspondence between Hamiltonian vector fields on \( M \) and closed 1-forms. In particular to each \( f \in C^\infty(M) \) belongs \( X_f : i(X_f)\omega = df \).

The Poisson bracket of two functions \( f, g \) is defined by

\[
\{f, g\} = X_f g = i(X_f)d g = i(X_f)i(X_g)\omega.
\]

Then

\[
d\{f, g\} = \Theta(X_f)dg = \Theta(X_f)i(X_g)\omega = i([X_f, X_g])\omega.
\]
so that \( f \mapsto X_f \) is a Lie homomorphism. Thus we get a Lie algebra extension

\[
0 \to \mathbb{R} \to \left\{ \text{functions under Poisson bracket} \right\} \to \left\{ \text{Hamiltonian vector fields} \right\} \to 0
\]

(These are sheaves of Lie algebras.) I might hope to induce the extension \( \tilde{\omega} \) above by making \( \tilde{\omega} \) act as Hamiltonian vector fields on some symplectic manifold. The obvious candidate, \( M = \) cotangent bundle of \( \mathbb{R} \) doesn't work. In effect if

\[
\tilde{\omega} = \sum \alpha(x) \frac{\partial}{\partial x}
\]

is a vector field on \( \mathbb{R} \), then the induced vector field \( \tilde{\omega} \) on the cotangent bundle \( \mathbb{T}^* \) can be shown to be

\[
\tilde{\omega} = X_{\tilde{\omega}} - \left( \sum \alpha(x) \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial \alpha} \right) dp dq
\]

the Hamiltonian vector field \( X_{\tilde{\omega}} \) where \( \tilde{\omega} : M \to \mathbb{R} \) sends \( \tilde{\omega} \) to \( \alpha \). Formulas:

- Canonical 1-form on \( M \): \( \omega = y \, dx \)
- 2-form on \( M \): \( \omega = dy \wedge dx \)

\[
\tilde{\omega} = \alpha \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \left( y \alpha \right) \frac{\partial}{\partial y}
\]

\[
i(\tilde{\omega}) \omega = \alpha dy + y \alpha' dx = d(\alpha y)
\]

Thus in this example \( \tilde{\omega} \) lifts to the Lie
Further possibilities:

\( \mathbb{R} \rightarrow \text{functions} \rightarrow \text{Hamilton v. f.} \)

Non-trivial on the formal level (formal power series at a point)? Probably otherwise the extension would be trivial for a canonical reason. In fact Hamiltonian vector fields is probably a perfect Lie algebra and so this extension would have to split canonically if it split.

In fact there is a local section. Assign to \( \mathfrak{v} \) the unique \( f \) with \( X_f = v \) such that \( f(0) = 0 \). Then the cocycle is going to be given by

\[
0, \omega \mapsto [\tilde{v}, \tilde{w}] - [v, w].
\]
Review of Kostant's theory: Let a Lie algebra $g$ act on a symplectic manifold $(M,\omega)$. Then to each $x \in g$ we have a Hamiltonian vector field $\nu_x$ on $M$. Exact sequence of Lie algs.

$$0 \to \mathbb{R} \to \text{functions on } M \to \text{Hamilt. v.f.} \to H^1(M, \mathbb{R}) \to 0$$

where $H^1(M, \mathbb{R})$ is abelian. In effect if $X,Y$ are Hamiltonian v.f., we can locally solve for $f,g$:

$$i(X)\omega = df \quad i(Y)\omega = dg$$

and then $\{f,g\}$ is a well-defined global function on $X$ such that

$$i([X,Y])\omega = d\{f,g\}.$$ 

Thus the last map above vanishes on brackets. Now suppose then that $H^1(g, \mathbb{R}) = 0$, i.e. that $g = [g, g]$. Then $\nu_x = X_f$, where $f_x$ is a function unique up to constants. If in addition $H^2(g, \mathbb{R}) = 0$, then the central extension obtained by pull-back

$$0 \to \mathbb{R} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

$$0 \to \mathbb{R} \to \text{functions} \to \text{exact Ham. v.f.} \to 0$$

will be trivial, hence we obtain a Lie homomorphism

$$\tilde{\mathfrak{g}} \to \text{functions on } M \text{ under } \{,\}$$

By duality this gives us an equivariant map.
Conclude: If \( H'(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0 \), then there exists a canonical map \( M \rightarrow \mathfrak{g}^* \) for any symplectic \( \mathfrak{g} \)-manifold.

In particular any homogeneous symplectic manifold must cover an orbit of \( \mathfrak{g} \) in \( \mathfrak{g}^* \) in a canonical way. But more is true in this case: Thus suppose \( M \) homogeneous and that \( m \in M \), and let \( \mathfrak{g}_m \) be the stabilizer, i.e., \( x \in \mathfrak{g}_m \Leftrightarrow \mathfrak{v}_x(m) = 0 \).

Let \( \lambda \) be the elt of \( \mathfrak{g}^* \) determined by \( m \), i.e.,

\[
\lambda(x) = f_x(m)
\]

and \( \mathfrak{g}_\lambda \) the stabilizer of \( \lambda \):

\[
x \in \mathfrak{g}_\lambda \Leftrightarrow \lambda([x,y]) = 0 \quad \forall y \in \mathfrak{g}.
\]

Now:

\[
\lambda([x,y]) = f_{[x,y]}(m) = \{ f_x, f_y \}(m) = (i(v_x)i(v_y)\omega)(m)
\]

and by assumption \( M \) is homogeneous so \( \mathfrak{g}_\lambda \rightarrow \mathcal{T}_M(m) \) is surjective. Thus if \( x \in \mathfrak{g}_\lambda \), we have \( i(v_x)\omega(m) = 0 \), so \( \mathfrak{v}_x(m) = 0 \) and \( x \in \mathfrak{g}_m \). So \( \mathfrak{g}_\lambda \subset \mathfrak{g}_m \), and as the other inclusion is clear, we have \( \mathfrak{g}_m = \mathfrak{g}_\lambda \). Thus \( \mathfrak{g}/\mathfrak{g}_\lambda \rightarrow \mathcal{T}_M(m) \) and so one has that \( M \) is a covering space of the orbit of \( \lambda \).
The idea behind reviewing Kostant theory was to construct the representation mentioned on page 23 by the following scheme. Thus suppose of perfect and form the universal central extension \( G' \) of \( G \) by \( H_2(G) \).

Given a 2-cocycle \( \omega \) on \( G \), it determines an element of \( (G_{\text{uni}})^* \), on which \( G \) operates, so we can look at the orbit \( O \), which is a symplectic homogeneous \( G \)-manifold. Quantizing \( O \) as in Kostant's theory should lead to the desired representation.

Example. Take \( G \) to be abelian and suppose the 2-cocycle \( \omega \) is a non-degenerate bilinear form. Then \( O = G \) acting as translations and quantization here means we construct the Heisenberg representation:

\[
[A(x), A(y)] = \lambda(x,y) \cdot \text{id}
\]

This roughly signifies that to produce a representation when \( G \) = vector fields on \( R \) with compact support, we will need an infinite dimensional \( O \), i.e., second quantization?
March 3, 1972:

Consider $\text{SL}_2 \mathbb{R}$ as a discrete group. We propose to define an element of $H^2(\text{SL}_2 \mathbb{R}, \mathbb{R})$. Let $\mathbb{Z}$ be the upper half plane, $\{z \in \mathbb{C} | \text{Im } z > 0\}$. Then $\text{SL}_2 \mathbb{R}$ acts continuously on $\mathbb{Z}$ by

$$(a \ b) (z) = \frac{az + b}{cz + d}$$

and the stabilizer of $i$ is

$$
\begin{pmatrix}
  a & b \\
  -b & a
\end{pmatrix}
$$

which is $\text{SO}_2$. Thus,

$$Z = \text{SL}_2(\mathbb{R})/\text{SO}_2.$$

$Z$ is the symmetric space belonging to $\text{SL}_2(\mathbb{R})$; $\text{SO}_2$ is the maximal compact subgroup. Since $Z$ is contractible, $\text{SO}_2 \rightarrow \text{SL}_2 \mathbb{R}$ is a homotopy equivalence, hence the universal covering of $\text{SL}_2 \mathbb{R}$ as a top. group is contractible.

$$0 \rightarrow Z \rightarrow \hat{\text{SL}_2 \mathbb{R}}^{\text{top}} \rightarrow \text{SL}_2 \mathbb{R} \rightarrow 1$$

Let $\omega$ denote an invariant volume form on $Z$, e.g.,

$$\omega = \frac{dx \, dy}{y^2} = \frac{i \, dz \, d\bar{z}}{y^2}$$
(Proof of invariance)

\[
(a \ b)^* \omega = \frac{d(\frac{az+b}{cz+d}) \cdot \overline{d(\frac{az+b}{cz+d})}}{\text{Im} \left( \frac{az+b}{cz+d} \right)^2}
\]

\[
\text{Im} \left( \frac{az+b}{cz+d} \right) = \frac{1}{2i} \left( \frac{az+b}{cz+d} - \frac{a\overline{z} + b}{c\overline{z} + d} \right)
\]

\[
= \frac{1}{2i} \frac{1}{|cz+d|^2} \left[ ac\overline{z}^2 + azd + bc\overline{z} + bd - a\overline{z} - bc - ad \overline{z} - bd \right]
\]

\[
= (ad-bc) \frac{\text{Im}(z)}{|cz+d|^2}
\]

\[
d(\frac{az+b}{cz+d}) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} \, dz = (ad-bc) \frac{dz}{(cz+d)^2}
\]

\[
(a \ b)^* \omega = \frac{(ad-bc)^2}{(ad-bc)^2} \frac{\frac{i}{2} dz \overline{dz}}{|cz+d|^4} = \frac{dx \, dy}{y^2} = \omega
\]

Observe that \( \omega \) is in fact invariant under \((GL_2^+ \mathbb{R})^+\) (it signifies det > 0) so that UHP is preserved.

Now to define an element of \( H^2(\text{SL}_2^+ \mathbb{R}, \mathbb{R}) \) it will suffice to define \( \omega \) in class in \( H^2(X) \) for any principal \( \text{SL}_2 \mathbb{R} \) bundle over a manifold \( X \). But form the fibre bundle

\[
E = P \times \text{SL}_2 \mathbb{R} Z \longrightarrow X
\]
with contractible fibres \( \cong Z \). On this fibre bundle we can define a closed 2-form \( \tilde{\omega} \) by pulling \( \omega \) up to \( P \times Z \) via the projection and then descending. More precisely, if \( U \) is an open set of \( X \) over which \( P \) is trivial, then \( E|U \cong U \times Z \) has form \( \tilde{\omega}_U = p_2^*(\omega) \), \( p_2 : U \times Z \to Z \). The transition functions being constant map \( U \cap V \to SL_2(\mathbb{R}) \) we have \( \tilde{\omega}_U = \tilde{\omega}_V \) on \( U \cap V \), because \( \omega \) is invariant. Pulling \( \tilde{\omega} \) back by a section of \( E \) gives a well-defined element of \( H^2_{\text{DR}}(X) \).

To show this element is non-trivial, let \( \Gamma \) be a discrete subgroup of \( SL_2(\mathbb{R}) \) which is torsion-free and has compact quotient, so that the quotient manifold \( \Gamma \backslash Z \) exists and is compact. Then take \( X = \Gamma \backslash Z \)

\[
P = Z \times SL_2(\mathbb{R}) \quad \quad \quad P = Z \times \Gamma \backslash Z
\]

\[
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow
\]

\[
X = \Gamma \backslash Z \quad \quad \quad X = \Gamma \backslash Z
\]

Observe that \( \tilde{\omega} \) descends to a closed 2-form on \( \Gamma \backslash Z \) and that in this case, \( \tilde{\omega} \) on \( Z \times \Gamma \backslash Z \) is the pull-back of \( \omega \) by the map \( Z \times \Gamma \backslash Z \to \Gamma \backslash Z \). The diagonal section \( \Gamma \backslash Z \to Z \times \Gamma \backslash Z \), and so we see that the form we get on \( X \) is just \( \tilde{\omega} \), which is the induced volume form on \( \Gamma \backslash Z \). In particular since \( \Gamma \backslash Z \) is compact this cohomology class is non-trivial.
Construction of such $\Gamma$: $\Gamma \mid Z$ must be a closed Riemann surface (of genus $> 1$, as its universal covering is $Z$.). Conversely, given a closed Riemann surface $X$ of genus $> 1$, its universal covering is analytically isomorphic to $Z$. Since $\text{PSL}_2(\mathbb{R})$ is the group of analytic isom. of $Z$, it follows that we have a homom. $\pi_1 X \rightarrow \text{PSL}_2(\mathbb{R})$, well-defined up to inner autos., which is injective. Thus:

Discrete torsion-free subgroups of $\text{PSL}_2(\mathbb{R})$ with compact quotient are the same thing as uniformized Riemann surfaces.

Now, to lift $\pi_1 X$ up into $\text{SL}_2(\mathbb{R})$ is possible when an obstruction in $H^2(X, \mathbb{Z}/2)$ vanishes (maybe same as putting a spinor structure on $X$?) in any case by Poincare duality we can kill such a class by passing to any double covering, so passing to any subgroup of $\pi_1 X$ of index 2 we get a $\Gamma \subset \text{PSL}_2(\mathbb{R})$:

Observe that $Z$, being a Riemann surface of constant negative curvature, its volume form is a neg. constant times its Gauss-Bonnet form, so consequently for any $\Gamma \subset \text{SL}_2(\mathbb{R})$ as above

$$\int \omega = \text{(constant)} \chi(\Gamma \mid Z)$$

where $\chi$ indicates that the class in $H^2(\text{PSL}_2(\mathbb{R}), \mathbb{R})$ comes from an integral, probably the class of the universal covering extension.
Action of $\text{PSL}_2 \mathbb{R}$ on $\mathbb{P}_1(\mathbb{R}) = S^1$. Identify $\mathbb{P}_1(\mathbb{R})$ with $S^1$ the unit circle by sending a line in the plane to the angle it makes with the $x$-axis.

\[ i.e. \quad x = \tan \left( \frac{\theta}{2} \right), \quad -\pi < \theta < \pi \]

if the line contains $(1, x)$. Another way of viewing this transformation:

\[ z \rightarrow \frac{1 + iz}{1 - iz} \]

maps $\mathbb{H} \rightarrow$ holomorphically onto $|z| < 1$ and the real axis onto the circle:

\[ \frac{1 + ix}{1 - ix} = e^{i\theta} \]

Also stereographic projection:

Recall that $\mathfrak{sl}_2$ has basis

\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

satisfying:

\[ [H, X] = 2X \]
\[ [H, Y] = -2Y \]
\[ [X, Y] = H \]
\[
\exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}
\]

acts on \( P_1(\mathbb{R}) \) by \( \exp(tH)(x) = e^{2tx} \).

So, the vector field on \( P_1(\mathbb{R}) \) is

\[
f \mapsto \frac{d}{dt} (e^{2tx}) \bigg|_{t=0} = f(x) 2x = \langle 2x \frac{d}{dx}, df \rangle(x)
\]

i.e.

\[
\vec{v}_H = 2x \frac{d}{dx}
\]

which becomes

\[
2 \tan \left( \frac{\Theta}{2} \right) (\frac{d}{d\Theta})^{-1} \frac{d}{d\Theta} = \frac{2 \tan \left( \frac{\Theta}{2} \right)}{\frac{1}{2} \sec^2 \left( \frac{\Theta}{2} \right)} \frac{d}{d\Theta} = \frac{4 \sin \Theta \cos \Theta}{2 \cos^2 \Theta} \frac{d}{d\Theta}
\]

\[
\vec{v}_H = 2 \sin \Theta \frac{d}{d\Theta}
\]

\[
(\exp(tX)(x) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}(x) = x + t \frac{d}{dx}
\]

\[
\vec{v}_X(x) = \frac{d}{dx} = 2 \cos^2 \Theta \frac{d}{d\Theta}
\]

\[
\vec{v}_X = (\cos \Theta + 1) \frac{d}{d\Theta}
\]

(check \( v_x \) on \( P_1 \) vanishes at \( x = \infty \) which is \( \Theta = \pi \)).

\[
(\exp(tY)(x) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}(x) = \frac{x}{tx+1} = \frac{1}{\frac{x}{x} + t}
\]

\[
\vec{v}_Y = \frac{d}{dt}(\frac{x}{tx+1}) \bigg|_{t=0} \cdot \frac{d}{dx} = -x^2 \frac{d}{dx} = -2 \tan \frac{\theta}{2} \cos \frac{\theta}{2} . \frac{d}{d\theta}
\]
\[ v_y = (\cos \theta - 1) \frac{d}{d\theta} \]

Unfortunately signs are off, because

\[
[v_x, v_y] = \left[ \frac{d}{dx}, -x^2 \frac{d}{dx} \right] = -2x \frac{d}{dx}
\]

\[
[v_N, v_x] = \left[ 2x \frac{d}{dx}, \frac{d}{dx} \right] = -2 \frac{d}{dx}.
\]

The reason for this comes from general incompatibility of conventions adopted at the beginning. Thus we want the Lie group \( G \) to act on \( X \) to the left and we want to define vector field \( v_H \) asso. to \( Hg \) by formula

\[
\frac{d}{dt} f(e^{th} x) \bigg|_{t=0} = (Hf)(x)
\]

so that we have "Taylor formula"

\[
f(e^{th} x) = \sum_n t^n (H^n f)(x) \overset{\text{def}}{=} (e^{th} f)(x).
\]

Unfortunately this will force us to set

\[
(g f)(x) = f(g x)
\]

making \( G \) act to the right on functions. So the only consistent thing to do (from the category viewpoint)
is to define 
\[(Hf)(x) = \frac{d}{dt}(e^{tH}f)(x)\big|_{t=0} = \frac{d}{dt}f(e^{-tH}x)\big|_{t=0}\]
and accept 
\[f(e^{-tH}x) = \sum \frac{t^n}{n!} (H^n f)(x)\]
which is an ugly version of Taylor's formulas. Thus on \(R\) we have to accept the formula 
\[e^{-tD}x = x + t\]
so that 
\[(e^{tD}f)(x) = f(x+t)\).

This seems unpleasant, but there is a real problem: 
We can't preserve all of:
(i) bracket of vector fields = commutator of inf. flows
(ii) bracket of vector fields = commutator of operators on the functions.
(iii) formula 
\[\frac{d}{dt}f(e^{tH}x)\big|_{t=0} = (Hf)(x)\]

Perhaps the thing to do is to define 
\[(\nabla_H f)(x) = -\frac{d}{dt}f(e^{tH}x)\big|_{t=0}\]

Thus we have 
\[g \mapsto (gf)(x) = f(g^{-1}x)\]
\[H \mapsto (Hf)(x) = -\nabla_H f(x)\]
Then \( v \) will be a Lie homomorphism. So in the present situation we must put

\[
\begin{align*}
v_H &= -2 \sin \theta \frac{d}{d\theta} \\
v_X &= -(1 + \cos \theta) \frac{d}{d\theta} \\
v_Y &= -(\cos \theta - 1) \frac{d}{d\theta}
\end{align*}
\]

A slightly better basis is perhaps \( \left( \frac{d}{d\theta}, \cos \theta \frac{d}{d\theta}, \sin \theta \frac{d}{d\theta} \right) \)

---

Ideas for future work: Construct explicitly the representations of the universal covering of \( \text{PSL}_2(\mathbb{R}) \) and try to see if these can be extended to the group of orientable diffeomorphisms of \( S^1 \).

Explicit realization of \( \text{PSL}_2(\mathbb{R}) \)? (means orientation preserving)

Borel subgroup \( B \) of \( \text{SDiff}(S^1) \) might be subgroups fixing \( x = \infty \). Observe have character \( B \to \mathbb{R}^+ \) obtained from \( \theta \) derivative at \( x = \infty \). Can one induce characters of \( B \) up to representations of \( \text{SDiff}(S^1) \)? Borel decomposition of \( \text{SDiff}(S^1) \) and the construction of its universal central extension by Moore-Matusmoto methods.

(Thurston point out that \( \text{SDiff}(S^1) \) has 2 real classes of dim 2— the Euler class and the interesting (Godbillon-Vey) class. Thus even if we construct \( \text{PSL}_2(\mathbb{R}) \) and extend to \( \text{SDiff}(S^1) \), we don't get the interesting class.)
March 3, 1971. Milnor model for BG

A review of general nonsense, which should be well understood:

**Milnor model for BG:** Let $G$ be a group (sans topologie pour fixer les idees). The principal bundle over Milnor's BG is the infinite join

$$
\bigcup_{n} G \times \mathbb{N}
$$

whose points are linear combinations $\sum_{i=0}^{\infty} t_i g_i$ where $t_i \geq 0$, $\sum t_i = 1$, and almost all the $t_i$ are zero. Thus the Milnor PG is a simplicial complex with vertices $\mathbb{N} \times G$ and where a subset of vertices

$$(g_1, g_2, \ldots, g_m)$$

forms a simplex iff the $g_i$ are distinct.

The quotient of this by the action of $G$ freely in the simplices of PG, the quotient simplicial complex. One sees the notion of BG.

The Milnor BG is the quotient of this by the action of $G$. Unfortunately the simplicial complex structure of PG does not induce one on BG, because the $G$-orbit of a sequence $(g_1, \ldots, g_m)$ in $G^m$
is not determined by the m-tuples of G-orbits of the vertices. Nevertheless, we ought to be able to describe it as the realization of the singular complex of a category, so legal claims.

Fact: Let $K$ be a simplicial complex endowed with an ordering on its vertices such that each simplex is linearly ordered. Then to $K$ is associated a semi-simplicial set $\Gamma(K)$, its singular complex:

$$\Gamma(K)_p = \text{Hom} \left( \Delta(p), K \right) \text{ ordering-preserving}$$

This is also equivalently $\Gamma(K)$ is the nerve of $K$ viewed as a category. Then the canonical map

$$|\Gamma(K)| \to |K|$$

is a homeomorphism. (This is clear set-theoretically because a point in $|\Gamma(K)|$ is of the form $\sum_{i=0}^p t_i k_i$, all $t_i > 0$, $(k_0, \ldots, k_p)$ a non-deg. simplex of $K$. The same is true for the geometric realization of $|K|$).

Apply this fact to $PG$ whose vertices $\mathbb{N} \times G$ are ordered via the natural order on $\mathbb{N}$. Thus $PG$ is the realization of the nerve of the category whose objects are pairs $(i, g)$ and where

$$\text{Hom}((i, g), (i', g')) = \begin{cases} \varphi & i > i' \text{ or } i = i' \text{ and } g \neq g' \\ \{ \text{id}_g \} & i = i' \text{ and } g = g' \\ \emptyset & i < i' \end{cases}$$
Since $G$ acts freely on the objects, hence also the arrows of this category, we see that $BG$ is the realization of the category whose set of objects is $N$ and

$$\text{Hom}(i, i') = \begin{cases} \emptyset & \text{if } i \neq i' \\ \{id\} & \text{if } i = i' \\ G & \text{if } i < i'\end{cases}.$$

(Must check this against Segal’s paper eventually.)

As a check we observe that geometrically we get in the realization of the nerve of this category one $g$-simplex for each collection

$$g_0 < g_1 < \ldots < g_g$$

and that the same is true for the Milnor $BG$, namely to the simplex $\left( (l_0, g_0), \ldots, (l_g, g_g) \right)$ in $\mathbb{P}G$ goes the simplex

$$l_0 < l_1 < \ldots < l_g$$

Now this construction makes sense for a monoid $M$, hence we have a Milnor $BM$. Moreover there is a canonical map

$$BM \to B\mathbb{P}N = \bigcup_{n \geq 0} \Delta(n)$$
obtained by mapping \( N \) to 1. Let now \( X \) be a compact space, and let \( f: X \to BM \) be a map. The map \( \text{pf}: X \to BN \) is the same thing as a family of out functions \( \sigma_i : X \to [0,1] \) \( i \geq 0 \), almost all zero such that

\[
\sum_{i=0}^{\infty} \sigma_i = 1
\]

Such a partition of unity determines an open covering \( U_i \) of \( X \) by \( U_i = f^{-1}(0,1) \). If \( \sigma \) is a finite subset of \( N \) such that

\[
U_\sigma = \bigcap_{i \in \sigma} U_i \neq \emptyset
\]

then for \( x \) in \( U_\sigma \), \( f(x) \) is a point of \( BM \) of the form

\[
i_0 < i_1 < \ldots < i_8
\]

and \( m^x_{i-1,i} \) is a locally constant function of \( x \in U_\sigma \), because the simplices over \( U_\sigma \) are topologically disjoint.

Let \( x \in X \) and let \( \sigma = \{i_0, \ldots, i_8\} \) be the subset of \( N \) such that \( i \in \sigma \iff \sigma_i(x) > 0 \). Then \( f(x) \) is a point of \( BM \) of the form

\[
\begin{align*}
1_0 < l_1 < l_2 < \ldots < l_8
\end{align*}
\]
with \( m_{i-j} \in M \). If \( i, j \in \sigma \), let \( m_{ij}(x) \) be the product of the various \( m \)'s between \( i \) and \( j \).

I claim then that \( m_{ij} : U_i \cap U_j \rightarrow M \) is a locally constant function. The only thing to show is that if \( x \) specializes to a point \( y \) in a such a way that certain of the \( f_k(x) \), \( k \in \sigma - \{ij\} \) go to zero, if \( f_i(y) \) and \( f_j(y) > 0 \), then \( m_{ij}(x) \rightarrow m_{ij}(y) \). But this is clear from the topology on \( BM \).

**Conclusion:** A map \( f : X \rightarrow BM \) is the same as a partition of unity

\[
\sum_{i \geq 0} f_i(x) = 1 \quad f_i : X \rightarrow [0, 1]
\]

Togethet with locally constant maps

\( m_{ij} : U_i \cap U_j \rightarrow M \quad i \neq j \)

satisfying the cocycle condition

\( m_{ij} m_{jk} = m_{ik} \) on \( U_i \cap U_j \cap U_k \) if \( i < j < k \)

(Here \( U_i = f_i^{-1}(0, 1) \).)
As a check suppose given this data, and try to construct \( f \). Then given \( x \) define \( f(x) \) in the way you must, namely if \( \sigma = \{ i_0, \ldots, i_k \} \) are the indices \( \sigma \ni \forall i \in \mathcal{A} \), then

\[
\hat{f}(x) = \underbrace{i_0 < i_1 < \cdots < i_k}_{\text{point of the simplex}} \quad \text{with coordinates} \ (f_i(x)).
\]

Now you want to check the continuity of \( f \), which somehow seems messy (?).
Example 1: Let $K$ be a simplicial complex and let $K'$ be its barycentric subdivision. A simplex of $K'$ is a sequence of simplices $\sigma_i \prec \cdots \prec \sigma_m$ in $K$. Thus $K'$ has a natural ordering and its vertices form a category, namely, the category of simplices of $K$ with inclusion maps.

$|K|$ has a natural covering by stars of vertices ($= \text{open sets } U_v$ of points whose $v$th coordinate is $> 0$). Lubkin forms the family of finite intersections, thus obtaining the open stars of simplices ($= \text{open sets } U_\sigma$ consisting of the points whose coordinates at each $v$ in $\sigma$ are $> 0$). Then the category of these open sets is the same as the category of simplices of $K$, $\text{Cat}(K)$.

Let Nerve Cat($K$) be the nerve of $\text{Cat}(K)$; it is the semi-simplicial set where $i$-simplices are chains of proper inclusions of length $i+1$. Thus an $i$-simplex of Nerve Cat($K$) is the same as an $i$-simplex of $K'$. Consequently

$$\text{real(Nerve Cat}(K)\text{)} = |K'|$$

Summary: Given a simplicial complex $K$, it determines a category, namely, its ordered set of simplices, and the nerve real of the nerve of this category is the real of $K'$. 
Example 2: Given a category $C$, denote by $\mathbb{N} \cdot C$ the category whose objects are pairs $(n, X)$ with $n \in \mathbb{N}$, $X \in \text{Ob } C$, in which the morphisms are given by

$$\text{Hom}(n, X; n', X') = \begin{cases} \emptyset & \text{if } n > n' \\ \{\text{id}\} & \text{if } n = n' \text{ and } X = X' \\ \text{Hom}(X, X') & \text{if } n < n', \end{cases}$$

with evident composition. Then there are two obvious functors

$$\mathbb{N} \cdot C \xleftarrow{N} \mathbb{N} \xrightarrow{C}$$

inducing maps of $\text{Reel}(\text{Nerve } ?)$. The functor $\mathbb{N} \cdot C \rightarrow C$ is surely going to induce a homotopy equivalence on nerve realizations.

How $\text{Reel}(\text{Nerve } C)$ looks: A typical point might be written

$$t_0 X_0, t_1 f_1, t_1 X_1, t_2 f_2, \ldots, t_b X_b$$

with $t_i > 0$, $\sum t_i = 1$.

This belongs to the simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \ldots \rightarrow X_b$$

One makes identifications as follows.
faces: If \( t_i \to 0 \) one deletes \( t_i X_i \) and composes \( f_{i-1,i} \) and \( f_{i,i+1} \).

degeneracies: If \( f_{i,i+1} = \text{id} \) one deletes \( X_{i+1} \) and adds \( t_i \) and \( t_{i+1} \).

The space \( \text{Real}(\text{Nerve}(N,C)) \) has points corresponding to sequences

\[
\pi_0 X_0 \xrightarrow{f_{i_0,i_1}} \pi_1 X_1 \xrightarrow{f_{i_1,i_2}} \cdots \xrightarrow{f_{i_{g-1},i_g}} \pi_g X_g
\]

where \( i_0 < i_1 < \cdots < i_g \) are in \( \mathbb{N} \) and \( t_i > 0 \), \( \sum t_i = 1 \). A map of a space \( X \) into \( \text{Real}(\text{Nerve}(N,C)) \) is therefore the same as a partition of unity

\[
\sum_{i \in \mathbb{N}} \rho_i = 1
\]

and for each \( U_i = \rho_i^{-1}(0,1] \) a continuous map \( \pi_i : U_i \to \partial_b C \) and for each \( i < j \) a continuous map \( f_{ij} : U_i \cap U_j \to \partial_a C \) such that the cocycle condition holds.

Problem: Is there a reasonable way to think about maps of \( X \) into \( \text{Real}(\text{Nerve}(C)) \)?
March 4, 1972: Group completion theorem.

Let \( M \) be a topological monoid, to fix the ideas, I want to understand its "group-completion" \( \Omega BM \).

The basic construction: Let \( M \) act on \( M \times M \) by the rule \((m_1, m_2) m = (m_1 m, m_2 m)\). Thus \( M \times M \) is a right \( M \) space and so we obtain a topological category \((M \times M/\Delta M)\) whose nerve \( Nw-(M \times M/\Delta M) \) is the simplicial space:

\[
(M \times M) \times M \times M \xrightarrow{\mathbf{e} = pr_1} (M \times M) \times M \xrightarrow{\mathbf{s} = \text{act}_M} M \times M
\]

This will be the candidate for \( \Omega BM \). To obtain a map \( p \) of \( Nw-(M \times M/\Delta M) \) to \( \Omega BM \), we produce a "fibre" space over \( BM \) with \( Nw-(M \times M/\Delta M) \) as fibre whose total space is contractible.

Let \( M \) act to the left on \( M \times M \) by the rule \( m(m_1, m_2) = (mm_1, m_2) \). This commutes with the right action, hence \( M \) acts on \( Nw-(M \times M/\Delta M) \) and we can form a simplicial topological category \((M \mid Nw-(M \times M/\Delta M))\) whose nerve \( Nw-(M \mid Nw-(M \times M/\Delta M)) \) will be a bisimplicial space

\[
(p, q) \mapsto M^p \times (M \times M) \times M^q
\]

It is clear that the "vertical augmentation"

\[
Nw-(M \mid Nw-(M \times M/\Delta M)) \rightarrow Nw-(M \setminus e)
\]
has fibres as \( \text{New}(M \times M/\Delta M) \).

To obtain contractibility consider the map
\[
\text{New}(M \setminus \text{New}(M \times M/\Delta M)) \rightarrow \text{New}(M/M)
\]
\[m, m_1, m_2, m' \mapsto m_2, m'
\]

more precisely the map
\[
M^p \times (M \times M) \times M^q \rightarrow M \times M^q
\]
given by projection on the last two factors. The fibres of this map are as \( \text{New}(M/M) \) which is contractible and the base is contractible, so the "total" space is contractible.

We now get from the above considerations a "map"
\[
\text{New}(M \times M/\Delta M) \rightarrow \Omega BM.
\]

For this to be a homotopy equivalence, it is necessary and sufficient that the action of any element of \( M \) produce a homotopy self-equivalence of \( \text{New}(M \times M/\Delta M) \).
March 5, 1971

Integrating Classifying Toposes.

Let $T$ be a topos, let $S$ be an object of $T$, and let $G$ be a group in $T/S$. By $BG$, I mean the topos $(T/S)_G$ consisting of objects $M$ of $T$ over $S$ with an action of $G$:

$$G \times_S M \rightarrow M$$

By $\Gamma(S, BG)$ I mean the stack over $T$ obtained by "integrating" $BG$ over $S$. Thus $BG$ is a pre-stack over $T/S$; to each $U \rightarrow S$ we have the category associated to the group $H^0_k(U, G)$. One enlarges this in the standard way to a stack; to each $U \rightarrow S$ one associates the category of $G_U$-torsors. Now one takes the direct image of this stack relative to the map $S \rightarrow e$; one obtains the stack associating to $U$ in $T$, the category of $G_{B \times U}$-torsors over $S \times U$. This last stack is $\Gamma(S, BG)$.

It is clear that there is a canonical map

$$B \Gamma(S, G) \rightarrow \Gamma(S, BG)$$

obtained by viewing the group $\Gamma(S, G)$
in $T$ as a prestack.

It is clear that this map is the full subcategory consisting of the trivial torsor.
March 5, 1972:

I want now to understand the homotopy type of $\text{New}(M \times M/\Delta M)$ in the case of $K$-theory. Thus $M$ will now be replaced by the category of finitely generated $R$-modules and their isomorphisms.

For $\text{New}(M \times M/\Delta M)$ I take the category $C$ cofibred over $\Delta^0$ whose fibre $C_n$ is the category $\text{fin}^\perp_{n+2} = (\text{fin}^\perp_2) \times \text{fin}^\perp_n$. An arrow from $(V_0^\pm, V_1, \ldots, V_n)$ to $(W_0^\pm, W_1, \ldots, W_m)$ lying over a monotone map $[m] \to [n]$ is a collection of isomorphisms

$$W_j^\pm \cong \bigoplus_{1 \leq i \leq n} V_i^\pm$$

$(V_i^+ = V_i^- \quad \text{for} \quad i \geq 0)$.

Thus the source operator $C_1 \to C_0$ (coarse change with the last vertex $[0] \to [1]$) is

$$(V_0^\pm, V_1) \mapsto V_0^\pm \oplus V_1$$

while the target operator $\varphi(0) = 0$ is

$$(V_0^\pm, V_1) \mapsto V_0^\pm$$

Now an important thing to note is that the source operator is faithful, hence the (pseudo-)simplicial category

$$\cdots \implies C_2 \implies C_1 \implies C_0$$

is essentially the nerve of a category object in $\text{Cat}(\text{Cat})$ with $\text{Ob} C = C_0$ and $\text{Arr} C = C_1$, etale over $\text{Ob} C$. Then I know a nice topos of sheaves to consider.

More precisely, let $C_1$ denote the cofibred category
over \( C_0 = \text{Ax} \) defined by the functor

\[
F(E^\pm) = \{ \text{splittings } E^\pm \cong F^\pm \oplus p^\pm \text{ together with } \text{ an ism } p^+ \cong p^- \}.
\]

(a splitting is simply a projection operator. Thus I ask for projection operators \( \pi^\pm \) together with an isom. \( \text{Im } \pi^+ \cong \text{Im } \pi^- \).

Thus an object of \( C_1 \) consists of a pair \( E^+_1, E^-_1 \) together with splittings

\[
E^\pm = \text{Im } \pi^\pm \oplus \ker \pi^\pm
\]

and an isomorphism \( \alpha : \text{Im } \pi^+ = \text{Im } \pi^- \). Define

\[
\mathbf{s} : C_1 \longrightarrow C_0
\]

to be the structural map, and

\[
t : C_1 \longrightarrow C_0
\]

with

\[
t(E^\pm, \pi^\pm, \alpha) = (\ker \pi^+).
\]

Finally it is clear how to define composition

\[
C_1 \times C_0 \longrightarrow C_1.
\]

Thus we have a category object in \( \text{Cat} \) whose nerve is \( \text{equivalent to the pseudo-simplicial category } \mathbf{C} \times \text{ defined above.}

Now intuition from Mather’s theorem tells me that it is natural to consider the category of local
Coefficient systems on $C_\ast$. Such a thing consists of a sheaf $F_0$ over $C_0$ together with an action of $C_1$. Now $F_0$ will be a discrete fibred and cofibred category over $C_0$, and we must lift $F_0$ to $C_1$ via some

in order that it be etale over $C_1$. Think of $F_0$ as a bifibred category over $C_0$ with discrete fibres. From the sheaf theory it is natural to ask for a right action of $C_1$

\[(x) \quad F_0 \times_{C_0} C_1 \longrightarrow F_0.\]

This means given $E^\pm = V^\pm \oplus P$, we want a map

\[(\ast) \quad F(V^\pm) \longrightarrow F(E^\pm).\]

More precisely, $F_0 \times_{C_0} C_1$ has for its objects $(f, V^\pm \oplus P^\pm \in E^\pm)$ with $f \in F(V^\pm)$ and morphisms are isomorphisms. Thus the action $(x)$ being a functor means that $(\ast)$ is equivariant for $\text{Aut}(V^\pm)$. The identity axiom for a functor implies that the map $(\ast)$ must reduce to the given action of $\text{Aut}(V^\pm)$ on $F(V^\pm)$.

Conclude that the natural category of sheaves on $C_\ast$ is the topos of covariant functors on the category with objects pairs $V^\pm$ and morphisms

$V^\pm \longrightarrow E^\pm$

to consist of two complemented inclusions $V^\pm \oplus P^\pm \longrightarrow E^\pm$. 

together with an isomorphism \( p^+ \cong p^- \).

So now let \( B \) be this category. I want to show that it has the good properties. The first thing is to show it gives rise to an \( H \)-space. But we have direct sum

\[
\begin{align*}
\begin{array}{c}
V^+, W^+ \\
\downarrow \\
\uparrow \\
V'^+, W'^+ \\
\uparrow \\
\downarrow
\end{array}
\end{align*}
\]

which is associative and commutative up to isomorphism. Moreover \( (0,0) \) behaves as a unit for this operation. Thus the realization of \( B \) is an \( H \)-space. In fact, it an invertible \( H \)-space because given any object \( V^+ \) of \( B \), its inverse is \( (V^-, V^+) \). That's because

\[
(V^-, V^+) \oplus (V^+, V^-) \Rightarrow (V^- \oplus V^+, V^+ \oplus V^-) \leftarrow (0,0).
\]

Why \( B = \Omega \text{BBA} \):

By \( \text{BBA} \) we mean the analogue of \( \text{Ner}\{M\} \), i.e., the (pseudo)-simplicial category in \( \text{B} \). Using the evident \( A \)-action on \( \text{B} \) we can form a pseudo-simplicial category \( \text{BBA} \). An important point is again the fact that the map

\[
\begin{align*}
\begin{array}{c}
V \oplus (V^+) = (V \oplus V^+, V^-)
\end{array}
\end{align*}
\]
Why $B = \Omega BBA$. Let $A$ act on $B$ by the rule $V(V^+, V^-) = (V \odot V^+, V^-)$ and form the pseudo-simplicial category $\text{Nerve}(A \backslash B)$:

$$\Rightarrow A \times B \Rightarrow B.$$  

Observe that the action map $A \times B \to B$ is "stale" i.e. $A \times B$ is equivalent to the $\text{category}$ $\mathcal{D}$ over $B$ defined by the functor $F(V^+, V^-) = \{\text{splittings of } V^+\}$.

(The way to check these things is to note that an object of $A^n \times B$ is a collection $(V_1, \cdots, V_n, V^\pm)$ and the first $n$-vertex map

$$(V_1, \cdots, V_n, V^\pm) \mapsto (V_1 \oplus \cdots \oplus V_n \oplus V^+, V^-)$$

allows one to identify $A^n \times B$ with an object of $B$ together with an $(n+1)$-fold splitting of the first space. It follows then that $\text{Nerve}(A \backslash B)$ is homotopy equivalent to the category with objects $V^\pm$ and in which a map $V^\pm \to W^\pm$ consists of splittings

$$V^+ \oplus P \oplus Q^+ \Rightarrow W^+$$

$$V^- \oplus Q^- \Rightarrow W^-$$

together with an isomorphism $Q^+ = Q^-$. To show the last category, call it $L$, is contractible, since
A acts invertibly on \( B \).

\[ \text{New}(A \setminus B) \to \text{New}(A) \] is a quasi-fibration with fibres \( \sim B \), so this will establish \( \exists \text{New}(A) \sim B \).

To show \( L \) contractible, project: \( V^\pm \to V^- \). This provides a functor from \( L \) to the category \( J \) of complements inclusions (reduced version of \( B(A/A) \)). Then

\[ L \to J \]

is cofibred. The fiber over \( V^- \) is the category \( J \)

\( (V^+ \oplus \Delta V^+ \sim V^+) \), hence is contractible as it has an initial element. Thus \( L \) is contractible.

Now I have to understand the homology of \( B \).

The idea will be to consider the functor

\[ \text{New}(A \ast A/A) \to B \]

given by last vertex:

\[ (V_0, V_1, \ldots, V_n) \mapsto V_0^+ \oplus \Delta V_1 \oplus \cdots \oplus \Delta V_n \]

This functor is a homotopy equivalence as mentioned before. Why?

This is the nerve of a category in \( B^\sim \).
Claim: Let $C$ be a topological category with etale source maps, and $C^\wedge$ the associated category of sheaves. Then for computing cohomology in $C^\wedge$, I have found useful the resolution

$$\text{Ar}_2 C \implies \text{Ar}_1 C \rightarrowtail \text{Ob}_C \quad (= \text{nerve of a category})$$

where $\text{Ar}_1 C$ acts on the right. This is the nerve of a category $\mathcal{T}$ in $C^\wedge$ with

$$\text{Ob}(\mathcal{T}) = \text{Ar}_1 C$$

$$\text{Ar}(\mathcal{T}) = \text{Ar}_2 C$$

etc. The classifying topos of $(C^\wedge, \mathcal{T})$ is $C^\wedge$ itself.

In effect we already know that $C^\wedge / \text{Ar}_1 C \simeq (\text{Ob}_C)^\wedge$

Thus a diagram

\[
\begin{array}{ccc}
\implies & F \times \text{Ar}_2 C & \rightarrowtail \ F \\
& \downarrow \text{Ar}_1 C & \\
\implies & \text{Ar}_2 C & \rightarrowtail \text{Ar}_1 C
\end{array}
\]

in $C^\wedge$ will be equivalent to a diagram

\[
\begin{array}{ccc}
\implies & F' \times \text{Ob}_C & \rightarrowtail \ F' \\
& \downarrow \text{Ob}_C & \\
\implies & \text{Ob}_C & \rightarrowtail \text{Ob}_C
\end{array}
\]

so it's pretty clear. (The origin of this question arose because I thought source etale top. categories had
to be treated differently from categories in topos.}
March 6, 1972

Mumford's conjecture again

Let $V$ be a representation of a group $G$, $G$ being discrete. Assume $G$ perfect and no mod $p$ cohomology where $V$ is of characteristic $p$. Now consider the bigraded ring

$$H^*(G, SV)$$

Think of this as $H^*(X, O_X)$, where $X$ is a ringed topos of char $p$. Thus it has Steenrod operations with $a$ Bockstein operation of degree 1, and $F^0$ operation induced by Frobenius.
March 6, 1972.

Let $R$ be a ring, and $J = J(R)$ the category of finitely generated projective $R$-modules with complemented injections for morphisms (i.e., a map $P \to P'$ in $J(R)$ consists of a pair of $R$-module maps $\varepsilon : P \to P'$, $\pi : P' \to P$ such that $\pi \varepsilon = \text{id}_P$). I propose to determine the category $\text{Ind}(J(R))$.

So let $I$ be a filtering category and

\[ I \longrightarrow J \]

\[ \begin{array}{c}
    i \\
    \lor
\end{array} \quad \begin{array}{c}
    P_i \\
    \lor
\end{array} \]

a functor. Let

\[ P = \varinjlim P_i \]

and for each $i$ let the maps

\[ P_i \leftrightarrow P \]

\[ \begin{array}{c}
    \pi_i \\
    \lor
\end{array} \quad \begin{array}{c}
    \varepsilon_i
\end{array} \]

be defined by taking the limit over the category $i \downarrow I$ of the maps

\[ P_i \leftrightarrow P_j \]

\[ \begin{array}{c}
    \pi_i \\
    \lor
\end{array} \quad \begin{array}{c}
    \varepsilon_j
\end{array} \]

Then $\pi_i \varepsilon_i = \text{id}_P$, so $E_i = \varepsilon_i \pi_i$ is an idempotent in $\text{End}(P)$. Suppose now that $u : i \to j$. Then we clearly have

\[ P \leftrightarrow P_i \leftrightarrow P_j \leftrightarrow P \]

\[ \begin{array}{c}
    \pi_j \\
    \lor
\end{array} \quad \begin{array}{c}
    \varepsilon_i \\
    \lor
\end{array} \quad \begin{array}{c}
    \pi_i \\
    \lor
\end{array} \]

Finally, note that
\[ E_i E_j = \varepsilon_i \pi_i \varepsilon_j \pi_j = \varepsilon_i \pi_i \pi_j \varepsilon_j \pi_j = E_i \]

\[ E_j E_i = \varepsilon_j \pi_j \varepsilon_i \pi_i = \varepsilon_j \pi_j \varepsilon_j \varepsilon_i \pi_i = E_i \]

so that \( E_i \leq E_j \) in the usual sense of projectors.

Thus

\[ i \mapsto E_i \]

is a map from \( I \) to the ordered set of projectors in \( P \).

Its image \( \bar{I} \) will be a directed set and the functor \( I \to \bar{I} \) will be cofinal.

Therefore to any ind-object in \( I \), we can associate an \( R \)-module \( P \) together with a set \( E \) of projectors in \( P \) satisfying

i) \( E \) directed

ii) \( \forall p \in P, \exists E \) with \( p \in \text{Im}(E) \)

iii) \( \forall E \in E, \text{Im}(E) \) is a fin. proj. \( R \)-module.

The functor represented by the ind-object is

\[ Q \mapsto \{ Q \xleftarrow{\pi} P \mid i \pi \leq \text{same member of } E \} \]

**Proposition:** \( \text{Ind}(I) \) is equivalent to the following category:

**Objects:** An \( R \)-module \( P \) endowed with a set \( E \) of projectors which is directed, exhaustive, and hereditary, and such that \( \forall E \in E, \text{Im}(E) \) is fin. gen. proj.

**Arrows:** \((P, E) \xrightarrow{} (P', E')\) consists of \( p \xleftarrow{\pi} P \)

such that \( E \mapsto \varepsilon E \pi \) carries \( E \) into \( E' \).
Proof: Let $I'$ be the category just described. We have a functor $\text{Ind}(I) \to I'$, and similarly a functor in the opposite direction. Observe that compositions are the same.

Example to show that we can have $P \supset P'$ but not $E = E'$. Take $R$ to be a field to simplify and consider $E'$ to be all projection operators on $V = k^n$. The point is that given $E_1', E_2'$ we can find a subspace of finite codimension $Q \subset \ker E_1' \cap \ker E_2'$ and such that $Q \cap [\text{Im}(E_1') + \text{Im}(E_2')] = 0$. Then extending the sum of the images to a complement for $Q$ and letting $E'$ be the resulting projector we have

\[
\begin{align*}
\ker E' \subset \ker E_1' & \quad \implies \quad E_2' \subset E_1' \\
\text{Im} E' \supset \text{Im} E_1' & \quad \implies \quad E_2' \subset E_1'.
\end{align*}
\]

Thus the set of all projectors works, so we can take $E$ to be the projectors on subspaces corresponding to finite $S \subset N$.

Question: Is $P$ necessarily projective?
Let $S$ denote the monoid of isomorphism classes of fin. gen. proj. $R$-modules. Let $(S/S)$ be the category obtained by letting $S$ act on itself by addition. Then we have an evident functor

\[ f: \mathcal{S} \rightarrow (S/S) \]

which sends $P$ to its iso. class $\text{cl}(P) \in S$, and a morphism $P \xrightarrow{\tau} P'$ to the morphism $(\text{cl}(P), \text{cl}(P'))$.

**Proposition:** $f$ is acyclic, i.e. for all $F: (S/S) \rightarrow \mathsf{Ab}$ we have

\[ H_\bullet(\mathcal{S}, f^*F) \cong H_\bullet(S/S, F). \]

**Proof.** It suffices to show that for each object $s$ in $S/S$ the category of arrows $s \rightarrow \text{cl}(P)$ is contractible, i.e. the category of pairs $(P,t)$ with $s+t=\text{cl}(P)$ is contractible. But observe we have a functor

\[ (P,t), (P',t') \mapsto (P \oplus P', s+t+t') \]

and natural transfs.

So the contractibility follows from.

**Lemma:** Let $C$ be a category with a functor

\[ C \times C \rightarrow C \]

\[ (X,Y) \mapsto X+Y \]

together with natural transfs.

\[ X \xrightarrow{i_1} X+Y \xrightarrow{i_2} Y. \]

If $C$ is non-empty it is contractible.
Proof. The functor \( X \mapsto X_0 \) is joined to the functor \( X \mapsto X + X_0 \) by the natural transf. \( i_2 \).

Similar the latter is joined to the identity by \( i_1 \).

Better to observe simply that if \( \text{cl} (P_0) = 0 \) then there are natural transformations

\[
\begin{array}{ccc}
(p, t) & \xrightarrow{\sim} & (p, t) \\
\downarrow & & \downarrow \\
(P_0, 0) & \longrightarrow & (P \oplus P_0, t+0)
\end{array}
\]

Joining the identity to the constant functor.

Corollary: \( H^* (\mathcal{A}, f^* F) = \lim_{\longrightarrow} \lim_{\leftarrow} F(s) \) \( * > 0 \)

Proof: The category \( S/S \) is filtering, hence the inductive limit functor is exact.

Proposition Let \( B \) denote the category of pairs \((V^+, V^-)\) with morphisms \( f: (V^+, V^-) \rightarrow (W^+, W^-)\) to be a complemented inclusion together with an isomorphism \( \text{Ker} (\pi^+) \cong \text{Ker} (\pi^-) \). Then

\[
H^*_x (B, \mathbb{Z}) \cong H^*_x (\mathcal{A}) \left[ \pi_0 \mathcal{A} \pi_0^{-1} \right]
\]
Proof: The projection $B \xrightarrow{p} I$ is cofibred with fibre over $V$ equivalent to $\mathbb{A}^n$, hence we have a spectral sequence

$$H_* (B, \mathbb{Z}) \otimes H_* (\mathcal{S}, \mathbb{Z}) = E^2$$

$$L_p^* (\mathcal{S} \otimes \mathbb{Z}) = H_*(\mathcal{S}, \mathbb{Z})$$

Thus $L_p^* (\mathcal{S})$ is the functor on $I$ which sends $V$ to $H_*(\mathcal{S}) = \bigoplus_{s \in S} H_*(\text{Aut}(P_s))$

and which sends a map $V \xrightarrow{\phi} V'$ into multiplication by $\text{cl}(Ker \phi) \in \pi_0 \mathcal{S}$. Applying the preceding we have $E^2_{pq} = 0$ for $q > 0$, and the spectral sequence degenerates yielding the desired result.
Idea: Go back to the category $J$ of pairs $(V^+, V^-)$ and assume that there exists a stable range function, namely a function $n(i)$ so that

$$H^i_J(GL_n) \to H^i_J(GL_{n+1})$$

for all $j \leq i$, $n(i) \leq n$. Then I see by considering the projection $(V^+, V^-) \to V^+$ that the component of $J$ with $\dim V^- = \dim V^+ + m$ will clearly have the right homotopy type in a range. The point is that the local coefficient system

$$V^+ \to H^*_J(GL_{\dim V^+ + m}, L)$$

is locally constant, hence its homology over $\mathbb{Q}$ is trivial as $J$ is contractible. Now the next point is to use the equivalence of the components.

Scheme for new proof of computation of $H_\ast (J)$: Idea is to consider projection $(V^+, V^-) \to V^+$ giving spectral sequence

$$E_2^{p,q} = \lim_{\to} (V^{-} \to H_\ast (\text{Aut} V^{-})) \Rightarrow H_\ast (J)$$

Now multiply by $S = \text{Ioo classes} \otimes$ and localize. It doesn't affect $H_\ast (J)$ and it makes the local coeff. system invertible. Then the spectral sequence degenerates by contractibility of $J$. 
Thurston described Eilenberg–MacLane cohomology in the following way. Let $G$ be a Lie group, say. Then an $E$-$M$ class for $G$ with coeff. $R$ “is a class for the underlying discrete group $\pi_1 G$ which varies continuously”. For example, given a family of homomorphisms

$$\varphi_0 : \Gamma \longrightarrow G$$

parametrized by $s \in S$, then $\varphi_0^*(\alpha) \in H^*(\Gamma, R)$ should be continuous in $s$ for any $E$-$M$ class $\alpha$ of $G$.

We can view the family as a map

$$S \times \Gamma \longrightarrow S \times G$$

of top. groups over $S$, where $S \times \Gamma$ is discrete over $S$.

One should also allow the group $\Gamma$ to vary over $S$. (?)

We can consider a smooth map $X \longrightarrow S$ and a principal $G$-bundle $P \longrightarrow X$ which is stratified with respect to $S$. Then if we give a section $\varepsilon$ of $X/S$ and a trivialization $\pi^*P \cong X \times G$ of $P$ over the section, we obtain a family of homomorphisms

$$\varphi_0 : \pi_1(X_0, \varepsilon(s)) \longrightarrow G$$

as described above. Thus an $E$-$M$ class $\alpha$ of $G$ should
give a section of the DR cohomology of \( X/S \):

\[
\delta \mapsto \pi_{0}^{*}(\delta) \in H^{*}(\pi_{1}(X_{0}, \varepsilon(0)), \mathbb{R}) \rightarrow H^{*}(X_{0}, \mathbb{R})
\]

**Question:** Can we identify an EM class \( \alpha \) of \( G \) with a characteristic class for stratified principal \( G \)-bundles over foliated manifolds, that is, a char. class \( \Theta \) which assigns to a foliated manifold \((X, S \subseteq T_{X})\) and a principal \( G \)-bundle \( P \) stratified with \( S \) a class

\[
\Theta(P, X, S) \in H^{*}(X, [2(x)]) \land s^{*}
\]

where

\[
\delta^{*} \rightarrow \pi^{*} \rightarrow \pi^{*} \rightarrow \pi^{*}
\]

**Example:** \( G \) Lie group, say connected, \( Z = G/K \) its associated symmetric space. Then given a principal \( G \)-bundle \( P \rightarrow X \) stratified with respect to a foliation \( S \), we form \( P \times GZ = P/K \). Let \( \omega \) be a left invariant differential form on \( Z \); the complex of these is \((\Lambda^{*}(g/k))^{*} \rightarrow (\Lambda^{*}(g))^{*}\). In fact one has an embedding

\[
(\Lambda^{*}(g/k))^{*} \rightarrow (\Lambda^{*}(g))^{*} \quad \text{(left back via} \ k \rightarrow g/k)\]

with image the forms \( \omega + i(k)\omega = 0 \). Therefore choose a \( G \)-invariant connection on \( P \) extending the given \( S \)-connection. This is possible because there exists such an invariant connection on \( S \) as a section of an affine bundle over \( X \). More precisely, consider the fiber above...
the differential being induced by the embedding

\[(\Lambda^i(g/k)^*)^K \hookrightarrow \Lambda^i g^* \]  

(morph $G \to G/K$).

Then if $\pi : P \times G \mathcal{Z} \rightarrow X$ is the projection, we can associate to $\omega$ a form along the fibres of $\pi$ obtaining a map of complexes

\[(\Lambda^i(g/k)^*)^K \rightarrow \Gamma(P \times G \mathcal{Z}, \Lambda^i T^*_\pi)\]

Now, the point is that because $P$ is stratified with respect to the foliation $\mathcal{S}$, this map lifts to a map

\[(\star) \quad (\Lambda^i(g/k)^*)^K \rightarrow \Gamma(P \times G \mathcal{Z}, \Lambda^i U^*)\]

where $U \subset T_{P \times G \mathcal{Z}}$ is the subbundle spanned by the lift of $\mathcal{S}$ provided by the $S$-connection, and $T^*_\pi$, that is

\[0 \rightarrow U \rightarrow T_{P \times G \mathcal{Z}} \rightarrow \pi^* Q \rightarrow 0\]

\[0 \rightarrow T^*_\pi \rightarrow U \rightarrow \pi^* S \rightarrow 0\]

(connection provides a splitting of latter sequence, hence a lifting of evs: $\Lambda^i T^*_\pi \rightarrow \Lambda^i U^*$) Another version: locally $\mathcal{F}$ quotient $X$ by foliation, and $P$ comes from $F$ over $X$.

Then we have

\[(\Lambda^i(g/k)^*)^K \rightarrow \Gamma(P \times G \mathcal{Z}, \Lambda^i T^*_\pi) \rightarrow \Gamma(P \times G \mathcal{Z}, \Lambda^i U^*)\]

and being canonical this gives a global map $(\star)$.

Now given a section $s : X \rightarrow P \times G \mathcal{Z}$ we have a map

\[\Gamma(P \times G \mathcal{Z}, \Lambda^i U^*) \rightarrow \Gamma(X, \Lambda^i S^*)\]
pull-back of forms. We see therefore that to any reduction of $P$ to $K$ we have associated a map of complexes

\[ [\Lambda^\ast (g/k)^\ast]^K \longrightarrow \Gamma (X, \Lambda^\ast s^\ast) \]

hence a well-defined map in cohomology (by usual argument)

\[ H^\ast (g, k) \longrightarrow H^\ast (X, \Lambda^\ast s^\ast) \].

I'd like a converse to this example. Suppose we have a characteristic class for $G$-torsors stratified with respect to foliations. Do we get a class in $H^\ast (g, k)$?

Simpler description: Given $P, X, S, \alpha : X \to P/K$ as above. Given a tangent vector $v \in S(x)$ it can be lifted in two ways, via $\alpha$ and via the connection. The difference then is tangent to the fibre, so we get

\[ S(x) \longrightarrow \mathfrak{t}_F \alpha(x) \]

where $g : P \to P/K$. This gives the map $(+)$ above, I guess.
Example: Do back to the case of $\pi: Y \to X$ foliated $R$-bundle flat at the ends. Then if we choose a fibre coordinate $z: y \to R$ such that $z = f_u$ at the "ends" of $Y$, we have a way of mapping tangent vectors on $X$ into vector fields with compact support on $R$, hence we can pull back the Schottky--Veech cocycles. In the notation used before (except $z$ replaces $t$)

$$\omega = dz + \sum a_i dx_i$$

Then $\frac{\partial}{\partial x_i}$ lifts to

$$-a_i \frac{\partial}{\partial z} + \frac{\partial}{\partial x_i} \quad \text{via foliation}$$

$$\frac{\partial}{\partial x_i} \quad \text{via } z$$

and the difference is

$$\left( \frac{\partial}{\partial x_i} \right)_x \to a_i(x, z) \frac{\partial}{\partial z} \quad \text{vector field on } \mathbb{R}$$

so the $\mathfrak{z}$-form is

$$\left( \frac{\partial}{\partial x_i} \right)_x \left( \frac{\partial}{\partial x_j} \right)_x \to \lambda \left( a_i^{x_0} \frac{\partial}{\partial z}, a_j^{x_0} \frac{\partial}{\partial z} \right)$$

$$\sum_{i=1}^{\infty} \frac{\partial a_i^{x_0}}{\partial z} \frac{\partial a_i^{x_0}}{\partial z} (x, z) \, dz$$
i.e. we get our previous formula

\[ \pi_x^\ast(\theta \cdot d\theta) = \sum \int_{i_1}^{\infty} \frac{\partial^2 a_i(x, z)}{\partial z^2} \frac{\partial q_z}{\partial z}(x, z) \, dz \, dx_i \, dx_j \]

(up to sign).
March 11, 1972.

Fix a space \( J \). Consider the following topological category \( \mathcal{J}(J) \). Its objects are pairs of finite sets \( s^+ \rightarrow J \) \( s^- \rightarrow J \) over \( J \). An arrow \( (s^+, s^-) \rightarrow (T^+, T^-) \) consists of a pair of injections

\[
\begin{align*}
s^+ & \rightarrow T^+ \\
s^- & \rightarrow T^- 
\end{align*}
\]
over \( J \) together with an isomorphism \( T^+ - s^+ \cong T^- - s^- \) over \( J \).

\[
\text{Ob } \mathcal{J}(J) = \bigsqcup_{s^+, s^- \text{ finite sets}} J^{s^+} \times J^{s^-}
\]

\[
\text{Ar } \mathcal{J}(J) = \bigsqcup_{s^+ \rightarrow T^+, s^- \rightarrow T^-} J^{T^+} \times_{J^{T^+ - s^+} = J^{T^- - s^-}} J^{T^-}
\]

Note that the target map is not etale. Thus the simplicial category

\[
\Rightarrow (M \times M) \times M \Rightarrow M \times M
\]

\[
M = \bigsqcup_n (X^n, \Sigma^n)
\]
is not of the type you studied before i.e.

\[
(\times^p \times \times^q) \times X^r \Rightarrow X^{p+r \times q} \times \times^r
\]
is not etale.

Next suppose \( \mathcal{J} \) is a space with basepoint. Consider pairs of finite sets \( s^+, s^- \) plus a map \( s^+ \rightarrow \mathcal{J} \). An arrow \( (s^+, s^-, \eta) \rightarrow (T^+, T^-, \eta) \) consists of
\[
\begin{align*}
    s^+ & \rightarrow T^+ \quad \text{over } \mathcal{J} \\
    s^- & \rightarrow T^- \\
    T^+ - s^+ & \rightarrow T^- - s^-
\end{align*}
\]

such that \( \eta(T^+ - s^+) = \bullet \) the basepoint.

\[
\begin{align*}
\text{Ob} &= \frac{\prod_{s^+, s^-} \mathcal{J}^s^+}{s^+, s^-} \\
\text{Ar} &= \frac{\prod_{s^+ \rightarrow T^+, s^- \rightarrow T^-} \mathcal{J}^s^+}{s^+ \rightarrow T^+, s^- \rightarrow T^-}
\end{align*}
\]

Thus source is etale here. \( S \rightarrow \mathcal{J}^S \) functor covariant \( \mathcal{J} \) for \( S \rightarrow \mathcal{T} \) if we use the basepoint. This shouldn't work either.

Concluded: In order to work \( \prod_{n} \mathcal{P} \Sigma_{n} \times \Sigma_{n} \mathcal{J}^n \) into your setup it is necessary to make the diagonal maps \( \mathcal{J} \rightarrow \mathcal{J}^k \) etale.
Lang's theorem again

Let $\sigma$ be an endomorphism of a group $G$. Given a $G$-torsor $P$ let

$$\tau_x P = P \times^G \mathcal{S}_G$$

and consider the category of $(P, \alpha)$ where $\alpha: \tau_x P \to P$, i.e. $\alpha: P \to P$ satisfies $\alpha(pg) = \alpha(p) \sigma(g)$. Define

$$(*) \quad (G^\text{-torsor}) \quad \longrightarrow \quad \text{cat. of } (P, \alpha)$$

$$Q \quad \longmapsto \quad Q \times^G \mathcal{S}_G \quad \text{with} \quad \alpha(qg) = q\sigma(g).$$

Claim: $(*)$ is fully faithful.

It is an equivalence of categories $\iff$

$$G / G^\sigma \longrightarrow G$$

$$g G^\sigma \longmapsto g \sigma^{-1}(g).$$

Proof. $\Theta: Q \times^G \mathcal{S}_G \longrightarrow Q' \times^G \mathcal{S}_G$ choose $g \in Q$, and let $\Theta(g) = g'g_0$. Then

$$g'g_0 = \alpha'(g_0) = \Theta \alpha(g) = \Theta g = g' \sigma g_0$$

so $g_0 = g_0$ and $g'g_0 \in Q'$, so fully faithful.

Given $(P, \alpha)$, choose $p \in P$, whence $\alpha(p) = pg$; let

$$g = g_0 \sigma^{-1}$$

whence $\alpha(pg) = pg \sigma g = pg_1$ and hence $P = Q \times^G \mathcal{S}_G$ where $Q = \{ p \mid \alpha(p) = p \}$. Thus equivalence if $G / G^\sigma = G$.

Conversely, given $g_0 \in G$ and define $\alpha: G \to G$ by

$$\alpha(g) = g_0 \sigma(g).$$

If an equivalence, $g_1$ with $\alpha(g_1) = g_1$, i.e.

$$g_0 = g_1 \sigma^{-1}(g_1).$$
March 12, 1972.

\[ \text{stable splitting theorem} \]

A category of f.g. proj. \( R \)-modules and iso.

\[ \text{cat. of pairs } (V^+, V^-) \text{ an arrow } (V^+, V^-) \to (W^+, W^-) \]

consists of split injections \((i^+, i^-): V^+ \to W^+\) together with \(Q^+ \to Q^-\).

Fix \( V \) in \( A \). Let \( A_V \) be the category of injections \( V \to E \) with \( E/V \) in \( A \), arrows being isoms.

Under \( V \). Then we have a direct sum operation

\[ E, E' \to E + E' \]

which is associative, commutative, and unital.

\[ \text{is not an additive category} \]

Note that the category of injections \( V \to E \) is not an additive category. Filtrated over additive category \( F_R \).

Form \( A_V \) in analogy with \( J \). Its objects are the same as those of \( A_V \). A morphism \( E \to E' \) in \( A_V \) consists of an injection \( i: E \to E' \) under \( V \) together with a submodule \( Q \) of \( E' \) containing \( V \) such that

\[ E + Q \to E'. \]

Composition is clear.

\[ \text{Claim that} \]

\[ V \to J \]

essentially is cofibrant. Since the fibre over \( W \) is \( \text{Ext} (W, V) \).
What can be done in the above is to define \( E' \) with \( E \) defined to be objects of \( V \) and arrows of \( E \).

\( E' \) is not compatible with \( E \), and \( E' \) is not compatible with \( E_2 \).

Then, \( E \) and \( E_2 \) are not compatible with \( E' \), and \( E_2 \).

This means we have an isomorphism:

\[
\begin{array}{ccc}
E' & \cong & E_2 \\
\downarrow & & \downarrow \\
E & \cong & E_2
\end{array}
\]

Then \( E \) given is compatible with \( E' \).

Thus, we have an isomorphism:

\[
\begin{array}{ccc}
E' & \cong & E \circ V \\
\downarrow & & \downarrow \\
E' & \cong & E_2 \circ V
\end{array}
\]

This is a graphoid (in fact a Picard category), as we must show every arrow in \( V \) is commutative.
is coprime over $I$.

The point: Suppose $G$ acts on $E$. Then $G$ acts on $E \oplus E$. The point is that stability $\Rightarrow$ invertibility. What does this amount to?

Consider characteristic classes of representations in $A_Y$. Then any stable char. class extends to the Grothendieck group of representations, in which we have the identity $E \cong E$. The point is that stability $\Rightarrow$ invertibility.

Suppose we have a stable class. My idea is to send $E \mapsto E$ from $A_Y \rightarrow A$. Then stable classes in $A$ will induce stable classes for $A_Y$. The point is that if $G$ acts on $E$, then $G$ acts on $E \oplus QY$ which is isomorphic as a $G$-module to $E \oplus (QY)$.

Thus the value of a stable class on $E$ and on $E \oplus QY$ will be the same.

Question: Can you see geometrically why stability implies invertibility? Thus in the preceding we first want to understand why in $J_Y$ we will have a homotopy

$$\begin{array}{c}
\text{Aut}(E) \\
\downarrow \\
\text{Aut}(E) \\
\end{array} \xrightarrow{\gamma} \begin{array}{c}
J_Y \\
\end{array}$$

This is easy. We have arrows in $J_Y$

$$(E,F) \rightarrow (E \oplus E, F \oplus E) \cong (E \oplus E, F \oplus E) \leftrightarrow (E, F).$$
Thus if $G$ acts on $E$, then we have the functor

$$G \rightarrow \mathcal{F}_V$$

and the natural transformation

$$e \rightarrow (E, V)$$

$$(g, id) \rightarrow (g \cdot E, V)$$

Thus if $G$ acts on $E$, we have the following diagram of $G$-objects in $\mathcal{F}_V$

$$(E, F) \xrightarrow{\text{in} \cdot \text{in}} (E \cdot E, F \cdot F) \xrightarrow{(\text{in} \cdot \text{in})} (E, F)$$

where $G$ acts trivially on $F$. Thus the functors from $G$ to $\mathcal{F}_V$ given by the $G$-objects $(E, F)$, $(E, F)$ are related by natural transformations.

This argument is the geometric reason why $E$ and $E$ become equivalent in $\mathcal{F}_V$. Now to see why $E$ and $E$ become equivalent in $\mathcal{G}$.

Cohomologically what seems to happen is that we compute

$$H_x(\mathcal{F}_V) = \lim_{E} H_x(\text{Aut}(E))$$

In other words: for each extension $E$ we have its group...
of autos and for each map $E \to E'$ in $\mathcal{V}$ we have a well-defined homomorphism

$$\text{Aut}(E) \to \text{Aut}(E')$$

except that more is true, namely the functor

$$E \to H_*(\text{Aut}(E))$$

depends only on the isomorphism class of $E$, and the arrow

$$H_*(\text{Aut}(E)) \to H_*(\text{Aut}(E'))$$

depends only on the isomorphism class of the complement. Thus we have a filtered inductive limit.
March 13, 1972

Suppose over $I$ we form the cofibred category belonging to the functor $V \rightarrow \text{Aut}(V)$ to sets. Thus we consider pairs $(V, \Theta)$ where $\Theta \in \text{Aut}(V)$ and an arrow $(V, \Theta) \rightarrow (V', \Theta')$ is a split injection $V \rightarrow V'$ such that $\Theta \rightarrow \Theta'$. To the pair $(V, \Theta)$, associate the diagram

\[
\begin{array}{ccc}
(0,0) & \xrightarrow{(\delta, \alpha, \Theta)} & (V, V) \\
\downarrow & & \downarrow \\
(0,0) & \xrightarrow{(\delta, \alpha, \text{id})} & (V, V) \\
& & \leftarrow \begin{array}{c} \longrightarrow \atop \downarrow \text{i} \end{array} V' \\
& & V \\
\end{array}
\]

in $J$. We see that if $(V, \Theta) \rightarrow (V', \Theta')$ is an arrow, then

\[
\begin{array}{ccc}
(0,0) & \xrightarrow{\Theta} & (V, V) \\
\downarrow & & \downarrow \\
(0,0) & \xrightarrow{\text{id}} & (V, V) \\
\end{array}
\]

commutes. It therefore is clear that the category of $(V, \Theta)$ is equivalent to the category of paths in $I$ of the form

\[
\begin{array}{ccc}
(0,0) & \xrightarrow{\Theta} & (V, V) \\
\downarrow & & \downarrow \\
(0,0) & \xrightarrow{\text{id}} & (V, V) \\
\end{array}
\]
Question: I have already seen that the category $\mathcal{A}$ of objects of $\mathcal{I}$ under a given object is contractible, because of the existence of an initial object. But suppose you give several objects $A_1, \ldots, A_n$ and consider the category of $n$-tuples $(V, v_1, \ldots, v_n)$ where $v_i : A_i \to V$. Is this category contractible? For example consider the category of diagrams

\[ A_1 \rightarrow B \]
\[ A_2 \]

Is this contractible? NO

\[ \text{Heuristic argument: We know } \mathcal{I} \text{ is contractible, hence so must be the space of paths joining } A_1 \text{ to } A_2 \text{ in the realization of the nerve of } \mathcal{I}. \text{ But because of the relative sums, such paths should be replaceable by 2-stage paths.} \]

\[ A_1 \xrightarrow{a} B \xleftarrow{b} A_2 \]

\[ A_1 \xleftarrow{a} \xrightarrow{b} A_2 \]

Why this doesn't work: Assign to $A_1 \xrightarrow{a} B$ the kernel of the map $A_1 + A_2 \to B$, and observe the kernel does not change for maps $B \to B'$ as they are injective. So the category is not connected.
Given $(V^+, V^-)$ in $I$ we consider the category of arrows

$$(V^+, V^-) \rightarrow (W^+, W^-) \rightarrow (0, 0)$$

i.e. of stable isomorphisms of $V^+$ and $V^-$. Then $\text{Aut}(V^+, V^-) = \text{Aut}(V^+) \times \text{Aut}(V^-)$ acts on this category. Observe that the category is not connected. In effect

$$V^+ \leftarrow W^+ \leftarrow$$

$$V^- \leftarrow W^-$$

so we can intersect $V^+$ and $V^-$, and the dimension of the intersection $I$ at least

$$0 \rightarrow I \rightarrow V^+ \rightarrow V^+ + V^- \rightarrow W$$

is an invariant.

**Question:** Can we modify the above category so that its components are $K_1$ at least in the stable range?

Recall that power operations lead to maps

\[ H^2_i(X) \xrightarrow{\text{ext.}} H^2_{2ni}(X^n) \xrightarrow{\text{A}} H^{2ni}_{\Sigma_n}(X) \]

I shall be interested in power operations in mod p cohomology. Begin with the external power operation

\[ H^{2i}(X) \xrightarrow{\text{ext.}} H^{2ni}_{\Sigma_n}(X^n) \]

\[ H^{2i+1}(X) \xrightarrow{\text{ext.}} H^{n(2i+1)}_{\Sigma_n}(X^n, \mathbb{F}_p) \]

I don't know how to think of this yet, so from now on treat \( p = 2 \). Then we have power operation (external)

\[ P_i^{\text{ext}}: H^i(X) \xrightarrow{\text{ext.}} H^i_{\Sigma_n}(X^n) \]

with the following properties

\[ P_i^{\text{ext}}(x+y) = \sum_{i+j=n} \text{ind } \Sigma_i \times \Sigma_j (P_i^{\text{ext}} \otimes P_j^{\text{ext}}) \]

\[ P_i^{\text{ext}}(xy) = P_i^{\text{ext}} P_j^{\text{ext}} x P_i^{\text{ext}} y \]

\[ P_i^{\text{ext}}(0) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases} \]
Here if $H \subset G$ is a subgroup of finite index, and if $X$ is any $G$-space, then

$$\text{Ind}_H^G(X) \longrightarrow H_\ast(X)$$

is the trace map for the covering

$$\begin{array}{rcl}
\mathcal{P}_H \times B/H X & \longrightarrow & \mathcal{P}_G \times B/G X \\
\downarrow \text{cont.} & & \downarrow \\
B H & \longrightarrow & B G.
\end{array}$$

It is natural for maps of $G$-spaces, so

$$\Delta^\ast \text{Ind}_{\Sigma_i \times \Sigma_j} (P_i \otimes P_j) = \text{Ind}_{\Sigma_n} (P_i \otimes P_j)$$

where

$$P_i x = \Delta^\ast \text{Ind} P_i^\ast(x)$$

is the internal Steenrod operation.

Now I want to arrange the family of $P_n$ in a coherent way. So let me consider the functor

$$F(X) = \left\{ (\alpha_n) \in \prod_{n \geq 0} H_n^*(X^n) \mid \forall i+j = n \quad \Sigma_n \alpha_n = \alpha_i \otimes \alpha_j \right\} \quad \alpha_0 = 1$$

**Proposition:** $F(X)$ is a ring with

$$(\alpha + \beta)_n = \Sigma \text{Ind} \Sigma_i \times \Sigma_j \alpha_i \otimes \beta_j \quad \alpha_n = \alpha_i \otimes \beta_n$$
Proof: Must show that addition is well-defined. Need to know
\[ \text{res} \frac{\sum_n \text{ind} \Sigma a \times \Sigma b}{\Sigma_i \times \Sigma_j} \]

For this use the double coset formula
\[ \text{res}_K \text{ind}_G H = \sum_K \text{ind}_K H \]

Now, \( \sum_n / \Sigma_a \times \Sigma_b = \text{subsets of } \{1, \ldots, n\} \) of order \( a \times (a, b) \) shuffles.

Thus given \( A \leq S \) of order \( a \) there is a nice shuffle permutation \( \sigma_A \) sending \( \{b, \ldots, i\} \) to \( A \) in order.

Now we want to look at the orbits of \( \sum_i \times \Sigma_j \) on \( \sum_n / \Sigma_a \times \Sigma_b \). So what's important is \( A \cap \{b, \ldots, i\} \) so given a double coset we get a decomposing

\[ a = a' + a'' \]
\[ b = b' + b'' \]
\[ a' + b' = i \]
\[ a'' + b'' = j \]

\[ A \]

The double cosets are thus in 1-1
correspondence with $0 \leq a' \leq a$. For our coset representative take the element $\tau_a$, interchanging $a_1, a_2$ sending

\[
[a_1+1, \ldots, a_n] \mapsto [a_1+1, \ldots, a_1+b, \ldots, a_n]
\]

\[
[a_1, \ldots, a_n+b] \mapsto [a_1, \ldots, a_1+b, \ldots, a_n+b]
\]

and fixing the rest. The stabilizer of $A$ is clearly

\[
\Sigma_a' \times \Sigma_a'' \times \Sigma_b' \times \Sigma_b'' \rightarrow \Sigma_a \times \Sigma_b
\]

and so it is clear that

\[
\sum_{i}^{n} \Sigma_i \times \Sigma_j \times \Sigma_i \times \Sigma_j
\]

where

\[
\begin{align*}
    a + b &= n \\
i + j &= n
\end{align*}
\]

Starting with $\alpha_a \boxtimes \alpha_b$

\[
\sum_{i}^{n} \text{res} \text{ind} \Sigma_i \times \Sigma_j \times \Sigma_i \times \Sigma_j = \sum_{0 \leq a' \leq a} \text{ind} \Sigma_i \times \Sigma_j \times \Sigma_i \times \Sigma_j \times \Sigma_i \times \Sigma_j \times \Sigma_i \times \Sigma_j
\]

\[
\begin{align*}
    a &= a' + b' \\
b &= b'' + b'' \\
i &= a'' + b''
\end{align*}
\]
so addition is well-defined.

Associativity and commutativity of addition clear. The fact that $\alpha$ exists is clear by recursion

$$(\alpha + \beta)_n = \alpha_n + \text{ind } \alpha_i \beta_{n-1} + \ldots + \beta_n.$$

Distributivity clear from

$$[F(\alpha + \beta)]_n = \sum_{i+j=n} \text{ind } \Sigma_i x_\Sigma_j (a_i \beta_j) = \sum_{i+j=n} \text{ind } \Sigma_i x_\Sigma_j (\text{res } \Sigma_i x_\Sigma_j \alpha_i).$$

Variants. Let $R$ be a graded anti-comm. $F$-algebra, and consider

$$F(R) = \left\{ (\alpha_n) \in \prod_{n \geq 0} H^0(B\Sigma_n, R) \mid \forall i + j = n \quad \alpha_0 = 1 \quad \text{res } \Sigma_i x_\Sigma_j \alpha_n = \alpha_i \otimes \alpha_j \right\}$$

This is also a ring. But such an $\alpha$ is the same as a ring homomorphism

$$\Gamma = \bigoplus_{n \geq 0} H^*_x(B\Sigma_n) \rightarrow R.$$

Thus

$$F(R) = \text{Hom}_{\text{range}}(\Gamma, R)$$

is a ring so $F$ is an affine ring scheme.
Question: I also have an interpretation of \( F(R) \) as \( \exp \) characteristic classes for representations of groups on finite sets with coefficients in \( R \). The operation on such \( \exp \) classes

\[
(\Theta, \Theta')(E) = \Theta(E) \Theta'(E)
\]
corresponds to the multiplication in \( F(R) \). What is the interpretation of the addition of \( F(R) \)?

The point seems to be this. Given an \( n \)-fold covering \( E \rightarrow X \), consider the induced covering of \( i \)-fold subsets of \( E \), call it \( Y_i \rightarrow X \). Then we can form \( \Theta(E_i) \Theta'(E_i) \) and then

\[
(\Theta + \Theta')(E) = \sum_{i \geq 0} [\Theta(E_i) \Theta'(E_i)]
\]

Proof that \( (\Theta + \Theta') \) is an exponential characteristic class. Let \( A_i(E) \) for the \( i \)-fold subsets of the fibres of \( E \). Then

\[
A_i(E + F) = \frac{1}{i!} \sum_{j=0}^{i} A_j(E) \times A_{i-j}(F)
\]
March 15, 1972

I have seen already the usefulness of étale toposological categories \( C \) in which the source map is étale. Consider now the analogue in which the space \( \text{Ob} \, C \) is replaced by the topos of \( G \)-sets.

Thus \( \text{Ob} \, C \) is the topos of \( G \)-sets, and \( \text{Ar} \, C \) is the topos of \( G \)-sets over a \( G \)-set \( S \)

\[
\text{Ar} \, C = (\text{G-set})_S
\]

and the source map is the localisation arrow

\[
(\text{G-set})_S \longrightarrow (\text{G-set})
\]

The target map must be a similar morphism of topos, hence must be given by a \( G \)-torsor \( P \) over \( S \) in the category of \( G \)-sets. Thus \( P \) is a \((G \times G)\)-set which is free for the action of the second factor.

The composition arrow will be given by a map

\[
P \times G P \longrightarrow P
\]

of \( G \times G \) sets, which is associative in the evident sense. The point is that once \( S \) is given, we can consider sheaves over \( \text{Ob} \, C \) on which \( \text{Ar} \, C \) acts

\[
F \downarrow \text{Ob} \, C
\]

\[
F \times \text{Ar} \, C \longrightarrow F
\]

\[
\text{Ob} \, C
\]

i.e. we give a map

\[
\pi^* F \longrightarrow s^* F
\]

\[
P \times G F \longrightarrow (P/6) \times F
\]
Thus the action will be a map \( P \times G \rightarrow F \) of \( G \)-sets. 

Associativity will be expressed in the obvious way. Next the identity, which must be a section of \( S \), i.e. a fixed point \( \text{fix} S \), must be expressed by giving a trivialization of the torsor \( P \) over \( e \), i.e., we give a point \( e \in P \). Then it is more or less clear that the identity axiom may be expressed as saying that \( e \) is an identity for the multiplication of \( P \).

Thus \( P \) is an associative monoid. The group \( G \) acts to the left and right of \( P \), freely on the right. I claim that \( g \mapsto ge = eg \) is an inj. homomorphism from \( G \) to \( P \) and that the left and right actions are just multiplication with respect to the monoid structure of \( P \). Indeed let \( \mu : P \times P \rightarrow P \) be the mult. of \( P \) with \( \mu(x, eg) = xg \).

(Note that the homom. \( g \mapsto g' \) defined by \( ge = eg \) is required to be the identity of \( G \), so \( ge = eg \).)

Then

\[
\mu(x, eg) = \mu(xg, e) = xg \\
\mu(e, gx) = \mu(g, x) = gx
\]

so the claim is clear.

Summarizing...
Proposition: Let \( G \) be a group and let \( \mathcal{M} \) be a monoid containing \( G \) such that right mult. \( p \cdot g \mapsto pg \) is a free \( G \)-action. Then

\[
\begin{align*}
\text{Ob } C &= (G\text{-sets}) \\
\text{Ar } C &= (G\text{-sets})/\langle E/G \rangle \\
F &= (G\text{-sets})/P/G \\
\end{align*}
\]

\[
\begin{align*}
s^*F &= P/G \times F \\
t^*F &= P \times F \\
\end{align*}
\]

constitutes the analogue of an étale topological category with \( \text{Ob } C = G\text{-sets} \). Every such is obtained in this way.

Examples:

1) Support \( G \) is a subgroup of a group \( P \). Then we have to consider \( P \) as a \( G \)-torsor (right) in the category of \( G \)-sets (left action). Such a \( P \) is determined up to isomorphism by

\[
[0,1] \to [0,t] \subset [0,1]
\]

where the first diffeomorphism = \( x \mapsto x^t \) near \( x = 0 \). Then
$G$ is a subgroup of $P$. Moreover $G$ acts freely on $P$ to the right.

2) Suppose $B$ is a subgroup of a group $G$. Then with the notation change $(G \to B, P \to G)$, we can consider $G$ as a $B \times B$-set $P$. Then we consider the $B$-set $S = G/B$ and break it into $B$-orbits:

$$G/B = \bigsqcup_{w \in W} BwB/B$$

where $w$ runs over double coset representatives. For each $w$, we have stabilizers of coset $wB$.

$$BwBw^{-1},$$

and two injections of it into $B$. Thus one way of describing $P$ as a $B \times B$-set is by choosing representatives $\{wB, w_0w\}$ for the $B$-orbits on $G/B$ and using the stabilizers and the homomorphism $BwBw^{-1} \to B$, defined by the $B$-torus over the point $wB$. To reconstruct $G$, I also need what data?

$$G \times_B G \longrightarrow G$$

$$\bigsqcup_{w_1w_2} Bw_1B \times_B Bw_2B$$

$$\bigsqcup_w Bw_1Bw_2B$$

$$G = \bigsqcup_w B \times_{wBw^{-1}} B$$
Example: Consider the following category. Its objects are pairs $(S, V)$ where $S$ is a finite linearly ordered set, $V$ is a vector space over $k$. An arrow $(S, V) \rightarrow (S', V')$ consists of a monotone injection $S \rightarrow S'$, a split injection $V \oplus Q \rightarrow V'$, and an isomorphism $S' \cong S$ with a basis of $Q$. We consider the full subcategory with $\dim V - \text{card } S = m$. It is clear the category is equivalent to the full subcategory with objects $(\mathbb{N}, k^{m+n})$.

It is the cofibrant category belonging to the functor

$$\langle n \rangle = \{1, \ldots, n\} \rightarrow \mathbb{GL}_{m+n}(k) = \text{Aut}(k^{m} \oplus \langle n \rangle)$$

where given a map $\langle n \rangle \rightarrow \langle n' \rangle$ one considers the induced map $\mathbb{GL}_{m+n} \rightarrow \mathbb{GL}_{m+n'}$, doing the appropriate thing on the last coordinates.

(Should be careful. Clear that fibre over $\langle n \rangle$ equivalent to $\text{Aut}(k^{m+n})$ with the understanding that this be $\phi$ if $m < n$. Now given $k^{m+n}, k^{m+n'}$

$$\langle n \rangle \rightarrow \langle n' \rangle$$

we get a split injection $k^{m+n} \oplus Q \rightarrow k^{m+n'}$ together with an isomorphism of the complement of $u$ with a basis for $Q$. When $m \geq 0$ there is a canonical such arrow working on the last $\oplus n$-coordinates, so the arrow $(\langle n \rangle, k^{m+n}) \rightarrow (\langle n' \rangle, k^{m+n'})$ is uniquely expressible.
as the product of the canonical arrow and an auto.
of $k^{m+n}$. When $m < 0$, there is no canonical such arrow, however we could look at all subspaces of $k^n$ of codim $-m$ if we wanted to.

Anyway let's worry only about $m > 0$. Then we have

$$\begin{align*}
GL_m & \rightarrow GL_{m+1} \rightarrow GL_{m+2} \Rightarrow GL_{m+3} \\
0 & \rightarrow 1 \Rightarrow 2 \Rightarrow 3
\end{align*}$$

What is the fundamental group of the category $GL_m \rightarrow GL_{m+1} \rightarrow GL_{m+2}$?

Assuming $m$ is in a stable range, the homology of the fibres will be constant over the base which is the category $0 \rightarrow 1 \Rightarrow 2$

which has an initial object, hence is contractible.

**Proof.** Let $i \rightarrow G_i$ be a functor from $I$ to groups. If $I$ is connected, then the fundamental group $\pi_1(G, i_0)$ is the limit $\varprojlim_{i \in I} G_i$ over $I$. Assume $I$ has an initial object $i_0$. Then $\pi_1(G, i_0)$ is isomorphic to the inductive limit of the groups $G_i$. The limit inductive being taken in the category of groups.
Proof. Let $F$ be a local coefficient system on $Y$. Then for each $i$, $F(i)$ is a $G_i$ set, and for each $i \to i'$, $F(i) : F(i') \to F(i)$ is compatible with the map $G_i \to G_{i'}$ induced by $u$.

Suppose $S$ is a $\lim_i G_i$ set. Then set $F(i) = S$ for all $i$, and let $G_i$ act on $F(i)$ as it should. Then we get a local coefficient system on $Y$ with $F(u) = \text{id}s$ for all arrows $u$ in $I$. This defines a homomorphism

$$\pi_1(S, i_0) \to \lim_i G_i$$

in general.

Now in general we have a map

$$\lim_{i_0 \to i} G_i \to \pi_1(S, i_0)$$

for any object $i_0$. This assigns to $F$ the set

$$\lim_{i_0 \to i} F(i) \leftarrow F(i_0).$$

The composition

$$\lim_{i_0 \to i} G_i \to \pi_1(S, i_0) \to \lim_i G_i$$

is evidently the obvious one, so it is the identity when $i_0$ is an initial object. Finally it is clear that any arrow $u$ in $Y$ comes from an element of $G_i$ for some $i$, $\square$.ED.
So we see that it is necessary to consider the inductive limit of

\[ \text{Gl}_m \rightarrow \text{Gl}_{m+1} \rightarrow \text{Gl}_{m+2} \]

\[ \circ \rightarrow 1 \rightarrow 2 \]

We must identify the two images of an element of \( \text{Gl}_{m+1} \) in \( \text{Gl}_{m+2} \). The two embeddings are conjugate by the matrix

\[ \begin{pmatrix} I_m & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \]

whose centralizer is small, consequently lots of commutators \( \varepsilon x^{-1} y^{-1} \) become zero in the inductive limit. Now with \( m \geq 1 \), we have to divide out by the normal subgroup of \( \text{SL}_3(\mathbb{Z}) \) generated by the difference of a transposition and any conjugate. Thus we have to kill all of \( \text{SL}_3(\mathbb{Z}) \). So with \( m \geq 1 \) the fundamental group probably is \( \text{Gl}_{m+2} \) modulo the normal subgroup generated by elementary matrices.
Consider the local field situation $K, A, m, k$. If $V$ is a finite-dimensional vector space over $K$, let $X(V)$ be its building. Thus $X(V)$ is the simplicial complex whose $i$-simplices are chains of lattices $L_0 < \cdots < L_i$ such that $m_i L_i < L_0$. I know that $X(V)$ is contractible, hence the cohomology of $\text{Aut}(V)$ should be accessible through the stabilizers of the simplices of $X(V)$.

What I want to do is consider the complex of chains on $X$:

$$0 \rightarrow C_{n-1}(X) \rightarrow \cdots \rightarrow C_1(X) \rightarrow C_0(X) \rightarrow \mathbb{Z}$$

$$C_i(X) = \mathbb{Z}[X_i]$$

where $X_i$ is the set of $i$-simplices. Given a simplex $\sigma = L_0 < \cdots < L_i$, let

$$b(\sigma) = (b_0, b_1, \ldots, b_i)$$

$$b_0 = \dim_k L_0 / mL_i$$

$$b_j = \dim L_j / L_{j-1}, \quad 1 \leq j \leq i$$

so that $b(\sigma)$ is a sequence satisfying

$$\begin{cases} b_0 > 0, & b_i > 0, \ldots, b_i > 0 \\ \sum_{j=0}^{i} b_j = n. \end{cases}$$
It is easy to see that \( b(\sigma) = b(\sigma') \iff \sigma \text{ and } \sigma' \) are conjugate under \( G = \text{Aut}(V) \). Thus the \( G \)-orbits on \( X_i \) are in 1-1 correspondence with sequences \((\star)\), so

\[
X_i = \bigsqcup \frac{G}{G_\sigma} \quad \sigma \in S_i
\]

where \( S_i \) is a set of representatives for the orbits, so

\[
C_i(X) = \bigoplus \frac{\mathbb{Z}[G] \otimes \mathbb{Z}[G_\sigma]}{\mathbb{Z}[G_\sigma]}.
\]

Now if I regard \( G \) as a discrete group, then since the complex of chains \( C_i(X) \) is a resolution of \( \mathbb{Z} \), I have a spectral sequence

\[
E^1_{p \star} = H_\star(G, C_p(X)) \Rightarrow H_\star(G, \mathbb{Z})
\]

\[
\bigoplus_{\sigma \in S_i} H_\star(G_\sigma, \mathbb{Z})
\]

and similarly one in cohomology, using the cochain complex. I recall having trouble with the topological situation.
Consider $G$ as a topological category.

(More invariantly, consider the top. cat. whose objects are f.d. v.s. $/K^\text{dis}$ (discrete top.) and whose morphisms are isomorphisms, $\text{Isom}(V,W)$ being endowed with its customary topology.) We then can consider $(G,X)$ as a cofibred and fibre category over $G$. Specifically, the category whose objects are pairs $(V,\sigma)$ where $\sigma$ is a simplex of $X(V)$, and where a morphism $(V,\sigma) \to (W,\tau)$ consists of an isom $\alpha : V \cong W$ such that $\alpha(\sigma) \subseteq \tau$.

Now consider the morphism

$$(G,X) \xrightarrow{f} G.$$

The fibres are contractible. Here I have an idea that $f$ is an essential morphism of toposi when one considers gross toposi.

On the other hand we have a functor

$$(G,X) \xrightarrow{\Delta_{\leq n}^+} \{\text{full subcat. of } \Delta \text{ consisting of } [1], [2], \ldots, [n], \text{ and injective monotone maps.}\}

This functor is fibred. So we should be able to conclude that cohomology $H^*(G, F_\epsilon)$ (E-M coh.) is abutment of a spectral sequence with

$$E^P_1 = \prod_{\sigma \in \Delta} H^0(G_{\sigma}, F_\epsilon).$$
Alternative approach. \( X \) simplicial set (allow degenerate simplices). \( G \) acts on \( X \), so can form \((X, G)\) simplicial category, and \( \text{New}(X, G) \) which is a bi simplicial space. Now given a top. abelian group \( A \), we obtain a double complex of abelian groups:

\[
C^*(\text{New}(X, G), A) = \text{Map}(\square G^b \times X_p , A).
\]

Now for fixed \( p \) we have the cochains with values in \( A \) of the top. category \((G, X_p)\) which is equivalent to \( \Pi_{G^b} (G_p, e) \), so we get the desired spectral sequence:

\[
E^1_{p,q} = \bigoplus_{\sigma \in S_p^+} H_q^b(G^\sigma, A) \Rightarrow H^{p+q}(C^*(\text{New}(X, G), A))
\]

In the other direction:

\[
\text{Map}(G^b \times X_p, A) = \prod_{X_p} \text{Map}(G^b, A)
\]

so

\[
C^*(\text{New}(X, G), A) = C(X, C(\otimes \text{New-G, A}))
\]

so we get contractibility.

Conclude: The Eilenberg-MacLane cohomology of \( \text{GL}_n(K) \) with coefficients in any top. abelian group \( A \) can be computed reduced to that of the stabilizers of the simplices of the building.
Applications:

1. Take $A = \mathbb{Q}_p$. According to Lazard

$$\lim_{U \to} H^*(U, \mathbb{Q}_p) = H^*(\mathcal{O}_f, \mathbb{Q}_p)$$

where $U$ runs over the open subsets of $\text{GL}_n(A)$. The point now is that if $U$ is normal in $V$, then

$$H^*(U, \mathbb{Q}_p)^{\text{iso}} \cong H^*(V, \mathbb{Q}_p)$$

so taking the limit over $U$ and using the fact that $\text{GL}_n K$ acts trivially on $H^*(\mathcal{O}_f, \mathbb{Q}_p)$ we see that for all $\sigma$ in the building

$$H^*(G_\sigma, \mathbb{Q}_p) \xrightarrow{\sim} H^*(\mathcal{O}_f, \mathbb{Q}_p).$$

Consequently, the $E_1^*$ term is the chains on the orbit space $X/G$. Thus in the $SL_n$ case done as this orbit is a simplex. In the case of $\text{GL}_n K$, I have to consider the simplicial set whose $i$-simplices are sequences

$$b_0, b_1, \ldots, b_i \geq 0$$

with $\Sigma b_i = n$ where the faces and deg. ops.

$$d^j_j(b_0, \ldots, b_i) = \begin{cases} (b_0 + b_i, b_2, \ldots, b_i) & \text{if } j = 0 \\ (b_0, \ldots, b_{i-1} + b_i) & \text{if } j = i-1 \\ (b_0 + b_i, b_1, \ldots, b_{i-1}) & \text{if } j = i \end{cases}$$

and
To compute the homology as follows. Let \( R = \mathbb{Z}[T] \) be the group ring of \( \mathbb{Z}_n \). Then we start with the standard resolution

\[
\rightarrow \oplus R \rightarrow \rightarrow \oplus R
\]

\[
d_j (b_0 \otimes \cdots \otimes b_n) = b_0 \otimes b_j b_{j+1} \otimes \cdots \otimes b_n, \quad 0 \leq j \leq n
\]

of \( R \) as an \( \oplus R \)-module, (let \( [b_0, \ldots, b_n] = b_0 \otimes b_n \) and \( \text{res} [b_0, \ldots, b_n] = [rb_0, \ldots, b_n] \).

The complex we are looking at is obtained by tensoring with \( R \) over \( \oplus \), because

\[
\text{res} (\oplus R \otimes \oplus \otimes R) \oplus \oplus = \oplus R \otimes R
\]

\[
\otimes [b_0, \ldots, b_{n+1}] \rightarrow b_0 \times b_{n+1} \otimes [b_1, \ldots, b_n]
\]

\[
\downarrow d_n \downarrow d_n
\]

\[
\otimes [b_0, \ldots, b_n b_{n+1}] \rightarrow b_0 \times b_n b_{n+1} \otimes [b_1, \ldots, b_{n-1}]
\]

Okay. Thus the homology of the complex under consideration is
Thus it seems that $X/G \simeq S^1$; (have to take the degree $n$ part of the preceding, and this contributes for $n \geq 1$.)

Conclude: E-M cohomology of $GL_n(K)$, $(K:Q_p) \to \infty$

with coefficients in $Q_p$ is

$$H^*(S^1, Q_p) \otimes H^*(O_n, Q_p)$$

$$\bigotimes_{i=1}^{2n-1} K[i]$$

(2) Take $A = F_l$, where $l$ prime to $p$. Here we know that $H^*(G_0, F_l)$

is finite-dimensional in each dimension, in fact, if $b_0, b_1, \ldots, b_n$.

$$H^*(G_0) = \bigotimes_{i=0}^{n} H^*(GL_{b_i}(k))$$

if $b_0, b_1, \ldots, b_n$. It will be preferable to work with homology defined as the dual of cohomology.

Then

$$E^1_{p*} = \bigoplus_{\sigma \in S_p} H^*_\sigma(G_0) = \bigoplus_{(b_0, \ldots, b_n)} \bigotimes_{i=0}^{n} H^*(GL_{b_i}(k))$$

$$\bigotimes_{b_i} \Sigma b_i = n$$

$$= R \otimes R^p \text{ where } R = \bigoplus_{n \geq 0} H^*_n(GL_n(k))$$
Again it should be possible to identify
\[ E^{2}_{p+1} = \text{Tor}_{p}^{R \otimes R}(R, R) \]

But I know
\[ R = \mathbb{E}[\varepsilon, \xi_1, \xi_2, \ldots] \otimes \Lambda[\eta_1, \eta_2, \ldots] \]

Thus
\[ \text{Tor}_{p}^{R \otimes R}(R, R) = \text{Tor}_{R_1 \otimes R_1}(R_1, R_1) \otimes \text{Tor}_{R_2 \otimes R_2}(R_2, R_2) \]
\[ = \mathbb{P}[\varepsilon, \xi_1, \xi_2, \ldots] \otimes \Lambda[\eta_1, \eta_2, \ldots] \]
\[ \otimes \Lambda[\bar{\varepsilon}, \bar{\xi}_1, \bar{\xi}_2, \ldots] \otimes \Gamma[\bar{\eta}_1, \bar{\eta}_2, \ldots] \]
March 19, 1972

Problem: To prove the equivalence of the following two categories:

(i) \( B(M \times M, \Delta M) \) where \( M = B(\text{f.g. proj } R\text{-modules + iso.}) \). Its objects are pairs \( (V^+, V^-) \) of f.g. proj. \( R\)-modules. An arrow \( (V^+, V^-) \rightarrow (W^+, W^-) \) consists of a pair of split injections and an isomorphism of the complements.

(ii) Simple version of \( B(M \times N, N) \) where \( N = B(\text{finite sets}) \). Call it \( \Gamma \). Its objects are pairs \( (V, S) \) where \( V \) is a f.g. module and \( S \) is a finite set. An arrow consists of a split injection \( V \rightarrow V' \), injection \( S \rightarrow S' \), and isomorphism \( \Gamma \).

Idea of a solution. Let \( M \times N \) act on \( M \times M \times N \) by

\[
(w, (v^+, v^-, s)) \cdot ((u^+, u^-, s'), (w^+, w^-, t)) = ((v^+ + u^+ + w^+, v^- + u^- + w^-, s' + t))
\]

and call the result \( B(M \setminus M \times M \times N / N) \). Then we have projections

\[
\begin{array}{ccc}
B(M \setminus M \times M \times N / N) & \twoheadrightarrow & B(M \setminus M) \\
\downarrow & & \downarrow \\
B(M \setminus N) & \longrightarrow & B(N, N)
\end{array}
\]

and the point is that the projections are quasi-fibrations with contractible base.
so I wish to consider the category $J^1$, whose objects are triples $(V^+, V^-, S)$ and in which an arrow $(V^+, V^-, S) \rightarrow (W^+, W^-, S')$ consists of

\[
\begin{align*}
V^+ & \rightarrow W^+ \\
V^- & \leftarrow W^- \\
S & \rightarrow S'
\end{align*}
\]

\[\text{Ker } \pi^- \leftarrow \text{Ker } \pi^+ \oplus R[S-S']\]

(intuitively \(W^+ = V^+ + Q, \ W^- = V^- + Q + R[T], \ S' = S + T\))

Consider now the projection \((V^+, V^-, S) \rightarrow S\) from \(J^1\) to \(J_0\) (finite sets and injections). The fibre over \(S\) consists of pairs \((W^+, W^-)\) with usual maps, hence the fibre is equivalent to \(J\). Now given \(S \rightarrow S'\) and an object \((V^+, V^-)\) over \(S\) I want to consider the arrows

\[
(V^+, V^-, S) \xrightarrow{\alpha} (W^+, W^-, S')
\]

lying over \(S \rightarrow S'\). Such an \(\alpha\) consists of

\[
\begin{align*}
V^+ & \rightarrow W^+ \leftarrow Q^+ \\
V^- & \rightarrow W^- \leftarrow Q^-
\end{align*}
\]

\[Q^- \leftarrow Q^+ \oplus R[S'-S].\]

Thus if I consider \((V^+, V^- \oplus R[S'-S], S')\) and canonical arrow \((V^+, V^-, S') \rightarrow \) \(J\) unique map in \(J\)

\[
(V^+, V^- \oplus R[S'-S]) \rightarrow (W^+, W^-)
\]

yielding \(\alpha\). Thus \(J_1 \rightarrow J_0\) is fibred,
and further, the cobase change functors
\[(V^+, V^-; S) \rightarrow (V^+, V^- \oplus R[S'; S])\]
are homotopy equivalences as we have seen before. Thus, since \(I_0\) is contractible, the inclusion of a fibre \(f : \mathcal{F} \rightarrow \mathcal{F}_1\)
is a homotopy equivalence. Here I use

**Lemma:** Let \(E \rightarrow B\) be cofibrad and suppose the cobase change functors between the fibres are homotopy equivalences (e.g. if bifibrad), if \(B\) is contractible, then for any \(b\), \(E_b \rightarrow E\) is a homotopy equivalence.

In the situation just considered, one can contract \(\mathcal{F}_1\) to the fibre \(\mathcal{F}\) over the initial object \(\phi\) of \(I_0\) as follows. Start with the identity functor
\[(V^+, V^-; S) \rightarrow (V^+, V^-; S)\]
and the functors
\[(V^+, V^-; S) \rightarrow (R[S] \oplus V^+, R[S] \oplus V^-; S)\]
\[(V^+, V^-; S) \rightarrow (R[S] \oplus V^+, V^-, \phi)\].
The vertical arrows are natural transformations giving the deformation.

On the other hand, we have the projection from \(\mathcal{F}_1\) to \(I\) sending \((V^+, V^-; S)\) to \(V^+\). This should again be cofibrad associated to the functor
Thus $f_1$ is cofibred over $I$ with fibres $f_0$, $A$ acting by sum on the first factor. Now I know that the action is invertible so again by the lemma it should follow that the inclusion of a fibre $f_0 \hookrightarrow f_1$ is a homotopy equivalence.
Consider the category of vector bundles over $P_1$ over a field $k$. According to Grothendieck–Hilbert any such bundle is a direct sum of the line bundles $\mathcal{O}(n)$. How this may be proved? One associates to $\mathcal{E}$ the graded module

$$\Gamma_*(\mathcal{E}) = \bigoplus_n \Gamma(P_1, \mathcal{O}(n))$$

over $\bigoplus_n \Gamma(P_1, \mathcal{O}(n)) = k[T_0, T_1]$. To prove $\Gamma_*(\mathcal{E})$ is a free $\Gamma_*(\mathcal{O})$-module, proceed as follows. Choose a trivialization of $\mathcal{E}$ iff $T_0 = 0$, which is possible as $k[z]$ for $z = T_1/T_0$, is a P.I.D. Thus we know that

$$M = \lim_{\rightarrow n} \Gamma(P_1, \mathcal{O}(n))$$

is a free $k[z]$-module. Now $\mathcal{E}$ is completely determined by $M$ and by the stalk at $z = \infty$ which is a $\mathcal{O}_\infty$-lattice in

$$L \subseteq k(z) \otimes_{k[z]} M$$

where $\mathcal{O}_\infty = k[z^{-1}]$ localized at ideal $z^{-1}k[z^{-1}]$. Moreover

$$\Gamma(P_1, \mathcal{E}) = L \cap M$$

$$\Gamma(P_1, \mathcal{O}(n)) = z^n L \cap M$$

with $T_0$ acting as the inclusion of $z^n L \cap M$ in
\( z^{n+1} \mathcal{L} \mathcal{M} \) and \( \mathcal{T}_1 \) as multiplication by \( z \). Now \( \mathcal{T}_0 \) is regular as far as \( \Gamma^* (\mathcal{E}) \) is concerned so what we must show is that \( z \) is injective on

\[
\frac{\mathcal{F}_x (\mathcal{E}) / \mathcal{T}_0 \Gamma^* (\mathcal{E})}{\oplus \frac{z^n \mathcal{L} \mathcal{M}}{z^{n-1} \mathcal{L} \mathcal{M}}} \quad \text{for } n \in \mathbb{Z}.
\]

But if \( \omega \) is \( z^n \mathcal{L} \mathcal{M} \) and \( z \omega \in z^n \mathcal{L} \mathcal{M} \), then

\[
z \omega = z^n \mathcal{L} \quad \text{for } \mathcal{L}
\]

so \( \omega = z^{n-1} \mathcal{L} \in z^{n-1} \mathcal{L} \mathcal{M} \)

which proves what we want.

Let \( \mathcal{E} \) be a vector bundle on \( \mathcal{P}_1 \) as above. Then we know

\[
\mathcal{E} = \oplus \mathcal{O}(a_i) \quad r = \text{rank } \mathcal{E}
\]

where \( a_1 > a_2 > \ldots > a_k \). I claim there is a canonical filtration \( \mathcal{F} \) on \( \mathcal{E} \).

\[
\mathcal{F} \mathcal{E} = \oplus \mathcal{O}(a_i) \quad \text{decreasing filtration}
\]

Thus we can filter the category \( \mathcal{A} \) by saying \( \mathcal{E} \in \mathcal{F}^p \mathcal{A} \)

\[
\Leftrightarrow \mathcal{E} \text{ is a direct sum of } \mathcal{O}(n) \text{ with } \mathcal{O}(n) \text{ for } n \geq p
\]

\[\mathcal{F} \Rightarrow \mathcal{E} \text{ has a largest subobject } \mathcal{F}_p \mathcal{E} \]
which is in $F_{pA}$. Can characterize $F_{pE}$ as being generated by the images of all the maps from $O(p)$ to $E$.

Now by construction we know that the filtration

$$F_{p+1}E \subset F_p E \subset \cdots$$

splits. Therefore let us consider the direct sum $K$-theory of $A$.

$$K_*(A, +) = \bigoplus$$

Thus we can describe any $E$ up to isomorphism by the numbers

$$\nu_p = \dim \frac{F_p E}{F_{p+1} E}$$

i.e. the number of times the indecomposable $O(p)$ occurs in $E$. The direct sum Grothendieck group is

$$K_0(a, +) = \mathbb{Z}[X, X^{-1}] \quad X = c(O(1)).$$

Next consider the $K$-theory. Given a repn. of $G$ on $E$ it must preserve the filtration which splits forgetting the action. Thus for any invertible repn. class for repn. in $A$ we have

$$\Theta(E) = T \Theta(gpE).$$

by the stable splitting theorem. Therefore it is clear (use $\text{Hom}(A(p), O(p)) \cong \mathbb{K}$) that the direct sum $K$-theory
associated to $A$ is the direct sum of copies of the $K$-theory of $k$, one for each $\phi(p)$, $p \in \mathbb{Z}$.

I want to understand the $K$-theory of $k[\mathbb{Z}]$. So I consider the category of finitely generated projective $k[\mathbb{Z}]$-modules and their isomorphisms, and later the direct sum operation.

I associate the building $\mathcal{B}$ which is a $\mathbb{Z}$-graded complex.

Let $K = k(\mathbb{Z})$, $A = k[\mathbb{Z}^{-1}] M$, $M = \mathbb{Z}^{-1} k[\mathbb{Z}^{-1}]$. To a $\text{f.g. proj. } k[\mathbb{Z}]$-module $M$, I associate the building $X(M)$ of $A$-lattices in $K \otimes_{k[\mathbb{Z}]} M$. Thus $X(M)$ is a simplicial complex whose vertices represent the extension of $M$ to a vector bundle on $\mathbb{P}^1$. To a lattice $L$ we have $E_L$ as on page 1.

$X = X(M)$ being contractible it furnishes a spectral sequence relating the cohomology

$$E^2_{pq} = \mathbb{C}^p(X/\Gamma, \mathcal{H}^q) \Rightarrow H^*_{\Gamma}$$

of $\Gamma$ and its stabilizers. The first thing I want to do is determine the orbit of $\Gamma$ on the various simplices and their stabilizers.

Claim: Two lattices $L, L'$ are $\Gamma$-conjugate $\iff$ the sheaves $E_L$ and $E_{L'}$ are isomorphic. Moreover
$\Gamma_L$ is the group of automorphisms of $E_L$.

Proof: This is clear because the map $E \to E'$ determined by what it does at $1$ and may be identified with a map $M = \Gamma(A^1, E) \to M' = \Gamma(A^1, E')$ carrying $L$ into $L'$.

It might be better to note that the category of $L$ in $\mathcal{X}(M)$ with morphisms $M \to L'$ is equivalent to the category of $E$ on $\mathbb{P}^1$ of rank $r$ with morphisms $E \to E'$ as injections which are views on $A^1$.

I should review now my earlier ideas on the homotopy axioms.

Given $M$ as above, the idea was to consider $L$ which are sufficiently positive so that $V = L \cap M$ is an "involutive" $k$-subspace of $M$, i.e., $V$ generates $M$ and $z^{-1}V = \{m \in M \mid zm \in V\} \subset V$, whence it follows that

$$0 \to k[z] \otimes z^{-1}V \xrightarrow{\sim} k[z] \otimes V \to M \to 0$$

is exact. Observe that if $V \subset V'$ are both involutive, then

$$z : V'/V \xrightarrow{\sim} V' + zV'/V + zV$$

This should roughly mean that $E_L$ is of filtration $\geq 0$ i.e. contains copies of $0(n)$, $n \geq 0$. 
March 22, 1972  (Carl is 17 today).

The situation: the field. We are interested in K-theory of $k[z]$ and of the projective line $\mathbb{P}^1$ over $k$.

$K = k(z)$ function field

$A =$ valuation ring at $z = \infty$

$= k[z^{-1}]_p, \quad y = z^{-1}k[z^{-1}]_p$.

Then a vector bundle $E$ over $\mathbb{P}^1$ is the same as a free f.t. $k[z]$-module $M$ together with an $A$-lattice $L$ in $K \otimes k[z] M$. In fact

$M = \Gamma(A^1, E)$

$L = E_{\infty}$ stalk at $\infty$.

(The correspondence $E \leftrightarrow (M, L, L \otimes K \otimes M)$ is essentially a special case of the one used by Artin to describe sheaves in the situation

$Y \subseteq X \leftarrow U$

$E$

$E \leftrightarrow (j^*E, i^*E, \overset{i^*E}{\longrightarrow} i^* j^* i^*E)$

Remarks:

The square

$\begin{array}{ccc}
\text{Vect}(\mathbb{P}^1) & \longrightarrow & \text{Vect}(A) \\
\downarrow & & \downarrow \\
\text{Vect}(A^1) & \longrightarrow & \text{Vect}(K)
\end{array}$

is 2-cartesian,
and, at least conjecturally, it gives rise to a Mayer-Vietoris sequence. Somehow therefore the arrows $\to$ are "transversal". A basic problem is to define a suitable notion of transversality.

---

Idea for proving $K_*(k) \cong K_*(k[z])$.

The point is to show surjectivity. Thus suppose a free f.t. $k[z]$-module $M$. We then consider the extensions $E = (M, L)$ of $M$ to a vector bundle over $\mathbb{P}$ such that $E \geq 0$, meaning that it is generated by its global sections (hence that it is isomorphic to $\oplus O(n)$ with $n_i \geq 0$). To such an $E$ we associate the pair of $k$-modules of f.t.

$$(\Gamma(E), \Gamma(E(-1)))$$

If $E \leq E'$, (i.e. $L \leq L'$) then from

$$0 \to E \to E' \to F \to 0$$

support at $\infty$

we obtain exact sequence,

$$0 \to \Gamma(E) \to \Gamma(E') \to \Gamma(F) \to 0$$

$$0 \to \Gamma(E(-1)) \to \Gamma(E'(-1)) \to \Gamma(F(-1)) \to 0$$

because $E \geq 0 \Rightarrow H^1(E) = H^1(E(-1)) = 0$.

But multiplication by $t_i$ and $t_0 = 0$ at $\infty$ defines an isomorphism.
\[ \Gamma F \xrightarrow{\sim} \Gamma F(-1). \]

Thus to the inclusion \( E \rightarrow E' \) belongs an inclusion of pairs\( (\Gamma F, \Gamma F(-1)) \rightarrow (\Gamma F', \Gamma F'(-1)) \) together with a trivialisation of the cokernel. It's clear this is compatible with compositions of injections.
March 23, 1972

Conjecture: Let $k$ be a finite field of characteristic $p$, and let $X$ be the building of proper subspaces of a f.d. $V$ over $k$, $G = \text{Aut}(V)$. Then

$$H^*_G(X; F_p) \cong H^*_G(\text{pt}; F_p)$$

Meaning: We know that there is a spectral sequence

$$E^2_{rs} = \begin{array}{c}
H^r(G, H^s(X)) \\ \Rightarrow H^{r+s}_G(X)
\end{array}$$

(mod $p$ coefficients) and that

$$H^0(X) = H^0_p$$

$$H^{-1}(X) = \text{Hom}(\text{Steinberg representation}, F_p)$$

$$H^i(X) = 0 \quad i \neq 0, r-1.$$

Thus the conjecture asserts that

$$H^*(G, (\text{Steinberg} \otimes F_p)) = 0$$

Note that if $B$ is a Borel subgroup

Recall that if $P$ is a Sylow-$p$ subgroup of $G$, then
\[ \text{(*) } \quad \text{res}_{\mathbb{P}}(\text{Steinberg}) = \mathbb{Z}[P] \]

hence

\[ H^*(G, \text{Hom}(\text{Steinberg}, \mathbb{F}_p)) \] \[ \downarrow \text{res} \]

\[ H^*(P, \text{Map}(P, \mathbb{F}_p)) = \begin{cases} 0 & x > 0 \\ \mathbb{F}_p & x = 0. \end{cases} \]

So what is required to prove the conjecture is to show that the \( \mathbb{F}_p \) is not there, for example that it is not fixed by the Borel subgroup.

**Example:** \( r = 2 \). Here one knows that \( P \) is abelian with normalizer \( B \), so

\[ H^*(G, M) \cong H^*(B, M) = H^*(P, M) \]

for all \( G \)-modules \( M \) which are \( p \)-primary. Thus

\[ H^*_G(\mathbb{F}_p X) = H^*_G(G/B) = H^*_B(B) \]

The conjecture is clear.

To prove the conjecture, suppose we can take the following improvement of (*)
March 24, 1972

Let \( G = GL_n(k) \), \( k \) a finite field of characteristic \( p \). Let \( B \) be the standard Borel subgroup:

![Diagram]

The dots denote the simple roots associated to \( B \). There are

\[
    t = (t_i) \mapsto t_{i+1}/t_i
\]

for \( 1 \leq i < n \).

Better notation: Let \( \sigma \) denote a simplex of the building of \( G \). Thus \( \sigma \) is a flag

\[
    0 < V_1 < \cdots < V_p < k^n
\]

in \( k^n \). Let \( P_\sigma \) denote the corresponding parabolic subgroup. One knows then that every subgroup \( P_\sigma < K < G \) is of the form \( K = P_\tau \)

for a unique \( \tau < \sigma \). \( (G = P_\phi) \).

Conjecture: Let \( \mathbb{Z}[G] \) be any parabolic subgroup of \( G \).

Conjecture: Let \( M \) be any \( \mathbb{Z}[\phi^{-1} G] \)-module. Then for any \( \sigma \) (including \( \phi \)), the complex
\[ \oplus H_\times(P_\sigma, M) \longrightarrow \oplus H_\times(P_\tau, M) \longrightarrow H_\times(P_\sigma, M) \longrightarrow 0 \]

\[ \sigma \preceq \tau \preceq \sigma \text{ and } |\tau - \sigma| = 2 \]

\[ \text{is acyclic. (Here } S \text{ is a fixed max. flag containing } \sigma). \]

**Remarks:**

1. Maybe \( M \) should be a complex of modules.
2. The geometric significance of the above complex: Given \( \sigma \) consider all conjugate flags and all possible refinements of this. This forms a simplicial category whose \( \mathbf{P} \text{-simplices consists of flags} \]

\[ W_i < V_i < \cdots W_i < V_q < \cdots W_p \]

3. Induction. Given \( P_\sigma \supset P_\tau = B \), all subflags between \( P_\tau \) and \( B \) contain the radical (unipotent) \( N_\sigma \) of \( P_\sigma \), and hence

\[ H_\times(P_\tau, M) = H_\times(P_\tau / N_\sigma, \mathbf{Z} \otimes_{\mathbf{Z}[N_\tau]} M) \]
Example: rank 1. \( G = GL_2(k) \) acts on \( G/B = \mathbb{P}^1 \). Since \( B \) contains a Sylow \( p \)-subgroups, one have

\[
H^*(G,M) \longrightarrow H^*(B,M) \longrightarrow H^*_B(G/B,M)
\]

for any \( \mathbb{Z}/p^{\ell-1}[G] \) module. But there are two orbits

\[
G\bar{B} = eB \cup B \cdot B
\]

for \( B \) on \( G/B \) and the second orbit is free. Thus we has immediately that the second orbit doesn't count for positive dimensional cohomology, i.e.

\[
H^+(G,M) \longrightarrow H^+(B,M).
\]

On the dimension zero all one knows from the exact sequence is that

\[
H^0(G,M) = \{ m \in H^0(B,M) \mid \sigma m = m \}.
\]

But this shows the conjecture is wrong because

\[
M = \text{Map}(G,A)
\]

where \( A \) a \( \mathbb{Z}/p^{\ell-1} \)-module

\[
H^0(G,M) = \text{Map}(B\backslash G/G, A) \cong A,
\]

\[
H^0(B,M) = \text{Map}(B\backslash G/B, A) \cong A \oplus A
\]

The problem was with the generalized Brauer theorem which will not be valid for \( G \)-modules.
Thus if the Sylow $p$-subgroup $P$ of $G$ is abelian
\[ H^*(G, A) \xrightarrow{i_*} H^*(P, A)^N = H^*(N, A) \]
\[ N = \text{Norm}_G(P), \]
for any trivial $\mathbb{Z}[p^{-1}, G]$-module $A$, but not for non-trivial ones. In effect, given
\[ P \times P^{-1} \xrightarrow{i_x \times j_x} P \]

one knows as $P$ is abelian that $\exists y \in N$ carrying $i_x$ into $j_x$, but one needs to worry about the effect of $y$ on $A$. Review the argument: $P$ being abelian, $P$ and $xP^{-1}$ are both Sylow subgroups of the centralizer of $P \times P^{-1}$ in $G$, so $\exists z \in \text{centralizer of } P \times P^{-1}$ such that $zPz^{-1} = xP^{-1}$. Then $y = z^{-1}x$ normalizes $P$ and $y(i_x) y^{-1} = j_x$. So to make things work with a non-trivial $G$ modules it is necessary to know that $z$ can be chosen to centralize $M$.

---

To modify the conjecture by putting
\[ M = E_p. \]

Example of rank 1. Let $S < G$ be the complement of a single element. Thus $P_0$ is the stabilizer of a flag with a single jump of 1 dimension
\[ P_0 : \begin{array}{c}
V_0 < \cdots < V^p < V^{p+2} < \cdots < V^h = V
\end{array} \]
Is \( H^*(B) \to H^*(P) \) again there are two double cosets
\[
P_0 = B \cup BsB
\]
and
\[
B \cap sBs = N \quad \text{the unipotent part of } P_0.
\]

Choose
\[
V^p \subset W \subset \Sigma V^{p+2}
\]

and let \( B \) stabilize \( W^* \), and \( b\) is interchanged \( W \) and \( W' \). The question is whether the homomorphism
\[
\begin{array}{c}
N \to B \\
\text{incl.} & \to \\
\text{conj. by } x
\end{array}
\]

induce the same homomorphism from \( H^*(B) \) to \( H^*(N) \).

Example: In dimension 3
\[
\left( \begin{array}{ccc} 1 & 0 & x \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{array} \right) \quad \left( \begin{array}{ccc} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & \delta \end{array} \right)
\]
\[
N \to x \to sx^0 \to B
\]

This is certainly not conjugate in \( B \) to the standard embedding of \( N \) in \( B \), and it clearly does not induce the identity on \( H^1 \) when \( k = \mathbb{F}_2 \).
Thus the conjecture should be modified to

Conjecture: Let $G = \text{Aut}(V)$ and let $\mathcal{F}$ be a maximal flag in $V$, that is, a top-dimensional simplex of the building. Here $V = k^n$, $k$ finite of char. $p$. Then with coefficients in $\mathbb{F}_p$, the complex

\[
\begin{array}{cccccc}
\oplus H_*(P_\tau) & \rightarrow & \oplus H_*(P_\tau) & \rightarrow & H_*(G) & \rightarrow 0 \\
\end{array}
\]

is exact.

Let $\tilde{C}$ be the complex of chains on the suspension of $X$:

\[
o \rightarrow C_n(X) \rightarrow \cdots \rightarrow C_1(X) \rightarrow C_0(X) \rightarrow F_p \rightarrow 0
\]

where $X$ is the building. The homology of this complex is concentrated in degree $n$ and is the Steinberg representation of $G \otimes \mathbb{F}_p$. I claim

\[
H_*(G, \tilde{C}_p) = \bigoplus_{\substack{\tau \in \mathcal{F} \otimes \mathbb{F}_p \\ \text{card} \tau = p}} H_*(P_\tau).
\]

In effect

\[
\tilde{C}_p(X) = C_{p-1}(X) = F_p \left[ \bigoplus_{\substack{\tau \in \mathcal{F} \otimes \mathbb{F}_p \\ \text{card} \tau = p}} G/P_\tau \right]
\]

so this is clear. The point is that each simplex in the building is conjugate to a unique $\tau \in \mathcal{F}$. Pursuing
this, one sees that the differential in $(\circ)$ is induced by the differential in $E_1$.

Thus I see that $(\circ)$ is the $E_1$ term of the spectral sequence

$$E_1^{pq} = H_q(G, \tilde{C}_p) \Rightarrow H_{p+q}(G, \text{Steinberg} \otimes F_p)$$

so what the conjecture signifies is that the $E^2$ term is zero. I am going to try to prove this by induction on $n$, having seen already that it is true for $n = 2$.

Let $L$ be a line in $V$ and let $P$ be the parabolic subgroup normalizing $L$. Then

$$H_\ast(G, M) \xrightarrow{\text{ind}} H_\ast(P, M) \xrightarrow{\text{res}} H_\ast(G, M)$$

is multiplicative by $[G; P] = \text{no. of lines} = \frac{q^n - 1}{q - 1} = 1 \quad (p)$, so that if we knew triviality over $P$, we would have it over $G$. So we wish to consider the complex

$$P \mapsto H_\ast(P, \tilde{C}_p).$$

Review now how one inductively determines the homotopy type of $X$. Let $H_L$ be the set of hyperplanes in $V$ complementary to $L$, and $Y \subset X$ the subcomplex consisting of all vertices not in $H_L$. Then $Y$ is contractible ($W \mapsto W + L$ retracts $Y$ into the closure of the vertex $L$.) Observe that this
retraction gives an equivariant for $P$ map

$$Y \times \Delta(1) \longrightarrow \text{Closed star}_L$$

and that $C$ is a functor from simplicial complexes to chain complexes transforming simplicial homotopies to chain homotopies. Thus $Y$ is $P$-equivariantly contractible, whence the complex

$$p \mapsto H_p(P, C(Y))$$

will contract for $H_p(P, T_0)$ for $p=0$. A better way of putting it is to say that the complex $\tilde{C}(Y)$, defined in analogy with $\tilde{C}(X)$, is equivariantly homotopic to zero. Thus

$$p \mapsto H_p(P, \tilde{C}(Y))$$

is acyclic for all $\underline{q}$.

But we have clearly an exact sequence

$$(\star) \quad 0 \longrightarrow \tilde{C}(Y) \longrightarrow \tilde{C}(X) \longrightarrow \tilde{C}(X, Y)[1] \longrightarrow 0.$$

Claim

$$C(X, Y) = \bigoplus_{W \in H_L} \tilde{C}(X_W).$$

In effect $C_p(X, Y)$ has as basis all $p+1$ simplices $\sigma$ containing some $W$, necessarily unique, so if

$$0 < V_1 < \cdots < V_p < W,$$
determines a simplex of some \( X_W \) or possibly the empty simplex.

Claim for each \( g, P \)

\[(*) \quad 0 \to H_q(P, \tilde{C}_p(Y)) \to H_q(P, \tilde{C}_p(X)) \to H_q(P, \tilde{C}(X, Y)) \to 0\]

is exact, because as a \( P \)-module \( \tilde{C}_p(X) \) is the free abelian group generated by a \( P \)-set, while \( \tilde{C}_p(Y) \) is generated by a \( P \)-subset. Now choosing a \( W \) and letting \( P_W \) be its stabilizer, we have

\[H_q(P, \tilde{C}_p(X, Y)) = H_q(P_W, \tilde{C}_p(X_W)).\]

Now \( P_W = k^* \times \text{Aut}(W) \), \( k^* \) acting trivially. By induction

\[H_x(p \mapsto H_q(\text{Aut}(W), \tilde{C}_p(X_W))) = 0 \quad \text{(assuming } n \geq 3)\]

so we see using the long exact sequence in homology associated to \((*)\) (considered as an exact seq of cox, \( p \) varying, \( q \) fixed) that

\[H_x(p \mapsto H_q(P, \tilde{C}_p(X))) = 0\]

concluding the induction. \( \Box \). Conjecture on page 6 is proved.

Remark: Observe for \( n = 1 \) that the conjecture is not true. Thus the complex is \( \mathbb{H}_X(G) = F_p \) in degree \( p = 0 \). 

Note that

\[H^*_x(B, F_p \otimes \text{Steinberg}) = F_p \quad n \geq 2\]

\[H^*_x(G, F_p \otimes \text{Steinberg}) = 0 \quad n \geq 2\]
follows from the preceding proof.

Questions

1. Can the preceding be generalized to any Tits system with finite Weyl group?

Suppose given \( G, B, N, S \). Don't see the analogue of the line.

2. \( H^*(G, \text{Steinberg} \otimes \mathbb{F}_l) = ? \quad l \neq p \).

In mod \( l \) cohomology the complex \( H^*_k(G, \mathcal{C}_\ast(x)) \) takes the form

\[
\longrightarrow \bigoplus_{0 \leq i < j < n} H^*_k(GL_i \times GL_{j \cdot i} \times GL_n) \longrightarrow \bigoplus_{0 \leq i \leq n} H^*_k(GL_i \times GL_{n-i}) \longrightarrow H^*_k(GL_n) \longrightarrow 0
\]

Recall that if \( R \) is an augmented algebra over \( k \), then \( T(R[1]) \) with differential defined to be the

\[
d[e_1, \ldots, e_n] = \sum_{i=1}^{n-1} (-1)^i [e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n]
\]

is the bar resolution for calculating \( \text{Tor}_R^k(k, k) \).

Do not confuse the map \( H^*_k(M) = \bigoplus_{n \geq 0} H^*_k(GL_n) \rightarrow k \), sending \( GL_n \) to \( 0 \) for \( n > 0 \), with the map induced by the map \( M \rightarrow \text{pt.} \).
Recall the exact sequence model for BM:

\[
\begin{align*}
\ldots & \longrightarrow \bigoplus_{a+b=n} \Pi_{a,b>0} B G_{a,b} \longrightarrow \bigoplus_{a>0} \Pi_{a>0} B G_a \longrightarrow \text{pt} \\
& \longrightarrow \bigoplus_{a+b=n} \Pi_{a+b>0} B G_{a+b} \longrightarrow B G_n \longrightarrow \text{pt}
\end{align*}
\]

If we filter this by calling \(B G_{a_1, a_2, \ldots, a_p}\) filtration \(n\) if \(a_1 + a_2 + \ldots + a_p \leq n\), then the \(n\)-th graded complex is

It appears that the normalized homology is the complex

\[
\begin{align*}
\longrightarrow & \bigoplus_{a+b=n} H_\ast(G_{a,b}) \longrightarrow H_\ast(G_n) \longrightarrow 0
\end{align*}
\]

which is the thing obtained before, i.e., the complex we showed was acyclic for \(n \geq 2\). Thus it appears that

\[\text{Filt}_n(x)\] is acyclic mod \(p\)

except for the trivial copy of \(BZ = S^1\).
March 25, 1972

K local field of characteristic p. To understand the mod p cohomology of $GL_2(K)$, which hopefully is nice.

I consider the building $X(K^2)$ with its natural $G = GL_2(K)$ action. This gives rise to a category cofibred over $\Delta^0$ whose fibre over $[\mathfrak{p}]$ is the category of filtered $A$-modules $L_0 < L_1 < \cdots < L_p$,

where each $L_i$ is free of rank 2 and $\pi^i L_p \subset L_0$, and their isomorphisms. The non-degenerate objects are as follows:

$p = 0$ $L_0$ free $A$-module of rank 2 $\cong GL_2 A$

$p = 1$ $L_0 < L_1$ with $\dim L_1/L_0 = 1$ $\cong$ Iwahori subgroup $(\star \star)$

$p = 1$ $L_0 < L_1$ with $L_0 = \pi^1 L_1$ $\cong GL_2 A$

$p = 2$ $L_0 < L_1 < L_2$ $\cong$ Iwahori subgroup.

Let $U = GL_2 A$, $U' = Iwahori subgroup$. Then the category, or properly its non-degenerate part takes the form:

\[
\begin{array}{ccc}
U' & \xrightarrow{d_1 = \text{id}} & U \\
\downarrow{d_0 = \text{id}} & & \downarrow{d_0 = \text{id}} \\
U' & \xrightarrow{d_2 = \text{conj}} & U
\end{array}
\]
To understand this a bit better, let us compute these homomorphisms. The idea is that we take for the last vertex the lattice $Ae_1 + Ae_2$ and identify the stabilizers of this with matrices.

- brisk point lattice $\Lambda = Ae_1 + Ae_2$, stabilizer $U = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$
- codim 1 lattice $\Lambda' = Ae_1 + A\pi e_2$, stabilizer $U' = \begin{pmatrix} * & * \\ \pi & 0 \end{pmatrix}$

Then all the faces by the last are inclusions on the stabilizers. Next we want

$d_1$ from $\Lambda' \leq \Lambda$ to $\Lambda'$

$d_2$ from $\pi \Lambda \leq \Lambda' \leq \Lambda$ to $\pi \Lambda \leq \Lambda'$

Thus we have to choose an isomorphism

$$\varphi: \Lambda \xrightarrow{\sim} \Lambda'$$

$$\Lambda' \xrightarrow{\sim} \pi \Lambda$$

So

$$\varphi(e_1) = \pi e_2$$

$$\varphi(e_2) = e_1$$

Then want the induced map

$$\begin{array}{c}
\text{Aut}(\Lambda' \leq \Lambda) = U' \\
\rightarrow \\
\text{Aut}(\Lambda') \\
\rightarrow \\
\text{Aut}(\Lambda)
\end{array}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \pi^{-1} \\ 0 \end{pmatrix}(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix})(\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}) =$$

$$\begin{pmatrix} 0 & \pi^{-1} \\ 1 & 0 \end{pmatrix}(\begin{pmatrix} \beta \pi & \alpha \\ \delta \pi & \gamma \end{pmatrix}) = \begin{pmatrix} \delta \pi & \pi^{-1} \gamma \\ \pi \beta \pi & \alpha \end{pmatrix}$$
Conclude that the category realizing $\text{GL}_2(K)$ is

where

\[ U = \text{GL}_2 A \]

\[ U' = \begin{pmatrix} \ast & \ast \\ \equiv 0 & \ast \end{pmatrix} \quad \text{subgroup of } U \]

\[ i : U' \longrightarrow U \quad \text{inclusion} \]

\[ h \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} \delta \pi^{-1} \gamma \\ \pi \beta \alpha \end{pmatrix} \quad \text{h order 2} \]
March 26, 1972

model for $GL_2(k)$:

$$
\begin{array}{ccc}
\left( \begin{smallmatrix} * & * \\
* & \end{smallmatrix} \right) & \rightarrow & GL_2(k) \\
\downarrow & & \\
GL_1(k) & (?)
\end{array}
$$

If $k$ is finite then the vertical map is finite. The vertical map induces isoms on mod $l$ cohomology and $\mathbb{Q}$ cohomology, $l \neq \text{char}(k)$. Thus in these case $GL_2(k) \rightarrow (?)$ induces isos on cohomology. When $k$ is finite of characteristic $p$, the horizontal arrow induces isoms on mod $p$ coh. Thus (?) has trivial mod $p$ homology.

To generalize to higher dimensions, the idea I think is to look for what one might call $Filt_n \cdot BGL(k)^+$

which should be generated by vector spaces of dimension $\leq n$. Thus

$$Filt_n(BU) = Bu_n$$

and

$$H^*(Filt_n(BU), Filt_{n-1}(BU)) = H^*(Bu_n, Bu_{n-1})$$

$$= \tilde{H}^*(Mu_n)$$
From my earlier work on stability I have a good idea as to what $H_\ast^n_{\text{BGL}(k^\ast)}$ might be. It should be $H_\ast^n(\text{GL}_n, k, \mathbb{Z}(X(k^n)))$ where $X(k^n)$ is the unimodular vector complex of $k^n$.

Problem with the preceding model: A d.v.r. in $K$, $[K:Q_p] < \infty$ where $\mu_p \subset A$. Then if I try the preceding to modify $\text{GL}_2(A)$, I get the wrong spectrum.

Try instead

$$
\begin{array}{ccc}
\text{GL}_1(A) & \longrightarrow & \text{GL}_2(A) \\
\downarrow & & \downarrow \\
\text{GL}_1(A)^2 & \longrightarrow & (?) \\
\end{array}
$$

I know that

$$
\begin{array}{ccc}
\text{Spec } H_\ast^n(\text{GL}_1(A)) & \longrightarrow & \text{Spec } H_\ast^n(\text{GL}_2(A)) \\
\downarrow & & \downarrow \\
\text{Spec } H_\ast^n(\text{GL}_1(A)^2) & \longrightarrow & \text{Spec } H_\ast^n(?)
\end{array}
$$

will be a pushout diagram. The point is that any elementary abelian $p$-subgroup of $\text{GL}_2(A)$ comes from $\text{GL}_1(A)$ and the different ones of rank 2 in $\text{GL}_1(A)$ all get identified in $(\text{GL}_1(A))^2$. 
March 29, 1972

Compactification of the
Buildings

A d.v.r. quotient field $K$. Note: The building of a $K$-vector space $V$ is the same as for the $K$-vector space $V = K \otimes K$, so we will suppose $A$ complete.

I propose to compactify the building $\text{Im}(V)$.

Let $L_\alpha$ be a directed system of lattices. Assuming the residue field $k$ is finite we can, for each interval $L \subset L'$ in the building, arrange, by selecting a suitable subsystem, that

\[ L + L_\alpha \cap L' = (L + L_\alpha) \cap L' \]

stabilizes. Call it $E_{LL'}$, and note that

\[ L_0 + E_{L_0L'} = E_{LL'} \quad L_0 \subset L \subset L' \]
\[ L' \cap E_{LL''} = E_{L' \cap L''} \quad L \subset L' \cap L'' \]

Set

\[ E_{L'} = \lim_{L \subset L'} E_{LL'} = \bigcap_{L \subset L'} E_{LL'} \]

and note that by completeness of $A$ (pass to limit in (1))

\[ L + E_{L'} = E_{LL'} \quad (1') \]

Also by passage to limit in (2)

\[ L' \cap E_{L''} = E_{L'} \quad (2') \]

\[ L' \cap L'' \]
Now set

\[ E = \bigcup L' E_L' \]

so that

\[ E_L' = L' \cap E \]

and so

\[ E_{LL'} = L + L' \cap E = (L + E) \cap L' \quad L \leq L' \]

Thus have proved.

\[ \text{Lemma: Given any directed system of submodules } E_{\alpha} \text{ of } V, \text{ there exists a subsystem which converges to a submodule } E \text{ of } V \text{ in the sense that } \forall L \leq L' \]

\[ L + E \cap L' = (L + E) \cap L' \]

\[ = L + E_{\alpha} \cap L' \]

for all sufficiently large \( \alpha \).

What this amounts to is that

\[ (A\text{-submodules of } V) \overrightarrow{\sim} \lim_{[LL']} \left( A\text{-submodules of } L/L \right) \]

and this must be due to the fact that \( A \) is compact.
So we consider the set of $A$ submodules of $V$ as the vertices of the compactification. A simplex will be a chain $E_0 < E_1 < \cdots < E_n$ of submodules such that $\pi E_n < E_0$. It will be necessary to add a topology.

Note that for any $E$ we have a canonical exact sequence

$$0 \rightarrow E_{\text{div}} \rightarrow E \rightarrow \hat{E} \rightarrow 0$$

$$\hat{E} = \bigcap E/nE$$

where $E = A^r$ and $E_{\text{div}} = K^s$. Thus given a chain $E_0 < E_1 < \cdots < E_n$ we have $E_0 \supset \pi E_n \supset (E_n)_{\text{div}}$ and so the simplex comes from a simplex of the building $X/E_{\text{div}}$. Thus without topology

$$Y(V) = \bigcap_{w \in W \subset V} X(N/W)$$

where $E_{\text{div}} \subset K$

Picture for $\dim V = 2$:
Theorem: Define a topology on the realization of \( Y(V) \) as follows. A point of the realization \( Y(V) \) is a pair \((\sigma, z)\) where \( \sigma \) is a simplex and \( z \in \sigma \) is a point in the geometric simplex with vertices \( \sigma \), i.e. if \( \sigma = (E_0, \ldots, E_n) \), then \( z = \sum t_i E_i \).

It's clear what is meant by a nbd of \( \sigma \) - we give an interval \( L < L' \) in \( V \) and consider all simplices \( \tau \) whose image in the interval is the same as \( \sigma \). Example: suppose \( \tau = L_0 < \ldots < L_m \) is in the interior. Then take \( L = L_0 \) and \( L' = L_m \). If then \( E_0 < \ldots < E_n \) is in this neighborhood, we must have

\[
\{ \pi L_0 + E_i \cap L_m \} = \{ L_i \} \]

\[
\pi L_0 + E_0 \cap L_m = L_0 \Rightarrow E_0 \cap \pi^{-1} L_m = L_0
\]

(by Makayama's lemma) \( \Rightarrow \) \( L_0 \subset E_0 \) and \( E_0 \cap \pi^{-1} L_0 = L_0 \)

that is, \( L_0 \in \pi^{-1} (V_{L_0}) \).

Example: Let \( E \cong A^s \) in \( V \). Choose \( L \) so that \( \pi L \cap E \subset \pi E \). Then if \( E' \) is in the \( L, L' \) neighborhood, we have

\[
L + E' \cap L' = L + E \cap L'
= L + E
\]

\[
\frac{E' \cap L'}{E' \cap L} = \frac{L + E' \cap L'}{E} = \frac{L + E}{E} = \frac{E}{L \cap E}
\]
The topology is then defined by saying that a nbd of \((x, z)\) consists of \((y, w)\) where \(y\) is in an \((L, L')\) nbd of \(x\) and where \(w\) is in the image of a nbd of \(z\).

**Conjecture:** 1) The above definition makes sense and makes \(|Y(V)|\) into a compact space.
2) \(|Y(V)|\) is homeomorphic to the suspension of the Borel–Serre compactification of the building associated to \(SL(V)\), \(V\) being given a volume.

The argument on page 4 uses

**Lemma:** If \(L\) is a lattice in \(V\), \(E\) a submodule, and if
\[
\pi L + \pi^{-1} L \cap E = L
\]
them \(E = L\).

**Proof:** Nakayama => \(\pi^{-1} L \cap E = L\), so \(L \subseteq E\). But \(\pi^{-1} L / L\) is the socle of \(V / L\), so \(E / L \cap \pi^{-1} L / L = 0 \Rightarrow E / L = 0\).

**Corollary:** Assume \(E\) generates \(V\), so that we can find an \(L\) in \(E\) such that \(L \rightarrow \hat{E}\).
Let \(E'\) be in the \((\pi L, \pi^{-1} L)\) nbd of \(E\), i.e.
\[
\pi L + E' \cap \pi^{-1} L = \pi L + E \cap \pi^{-1} L
\]
Then \(E' \subseteq E\) and \(\hat{E'} \rightarrow E\).
Why is the Bruhat-Tits building of $V$ contractible? Recall that if $X(V)$ is a Tits building and $Y(V)$ theirs there is a map

$$X(V) \xrightarrow{f} Y(V)$$

Vertices of $Y(V)$ are homothety classes of $L \subset V$, i.e., $L = xL'$ for $x \in K^*$. A subset of $Y(V)$ is a simplex if it is the image of a simplex of $X(V)$. Thus the map $f$ is simplicial and $Y(V)$ is contractible provided all of the fibres are. But a typical fibre looks like

![Diagram](image)

So it is clear all the fibres are contractible.

It is also clear from this, granted the Borel-Serre theorem, that the cohomology with compact supports of $X$ is the suspension of that of $Y$. 
April 29, 1972

Compactification of the building

$A, K, k, m$ usual d.v.r. situation.
$V$ a $K$-module f.t. $X(V)$ its building. Then

$$X(V) = \bigcup_n X(\pi^{-n}L/\pi^n L)$$

where $L$ is a fixed lattice in $V$ and $X(\pi^{-n}L/\pi^n L)$ is the sub-complex of $X(V)$ whose vertices lie between $\pi^n L$ and $\pi^{-n} L$. More generally have $X(L_i/L_0)$ when $L_0 < L_1$ are lattices in $V$.

Suppose the layer $L_0 < L_1$ is contained in the layer $L_0 < L_1$, i.e.,

$$L_0 < L_0 < L_1 < L_1$$

Then there is a retraction

$$X(L_1/L_0) \longrightarrow X(L_1'/L_0')$$

$$L \longmapsto (L \cap L_1') + L_0 = (L + L_0) \cap L_1'$$

Better, this is a retraction of $X(V)$ to $X(L_1'/L_0')$.

Using these retractions we obtain an inverse system of simplicial complexes

$$\longrightarrow X(\pi^{-n}L/\pi^n L) \longrightarrow X(\pi^{-n+1}L/\pi^{n-1} L) \longrightarrow \ldots$$

and we can take the inverse limit

$$\overline{X}(V) = \lim_{n} X(\pi^{-n}L/\pi^n L).$$
Note that when $k$ is finite, $X(V)$ is a compact space, since $X(π^{-n}Λ/π^{n+1}Λ)$ is a finite simplicial complex.

Fix $n$ and let $L$ be a member of $X(π^{-n}Λ/π^{n+1}Λ)$ but not $X(π^{-n}Λ/π^nΛ)$. Then there are two cases—either $L \not\in π^{-n}Λ$ or $L \not\in π^nΛ$ and both can occur simultaneously, e.g. $L = \langle π^{-1}e_1, π^{n+1}e_2 \rangle$ where $Λ = \langle e_1, e_2 \rangle$. But if $L$ is immediately joinable to the subcomplex $X(π^nΛ/π^{n+1}Λ)$, suppose $[L_0, L_0]$ is a one simplex; then either

1. $L_0 < L < Λ_0 \Rightarrow π^{-n}Λ < L < π^{n+1}Λ$
2. $πL_0 < L < Λ_0 \Rightarrow π^{n+1}Λ < L < π^{-n}Λ$

and these cases can't occur at the same time. Perhaps this can be used to compute the cohomology of $X(V)$ with supports in $X(π^{-n}Λ/π^nΛ)$?

I want to identify the space $X(V)$. Suppose that $x = (x_n) \in X(V)$ with $x_n \in X(π^{-n}Λ/π^nΛ)$. Then the dimension of the open simplex containing $x_n$ is a bounded monotone function of $n$, hence it stabilizes: $d = d_n$ for $n$ large. Then

$$x_n = \sum_{i=0}^{d_n} t_i L_{i, n} \quad L_{0,n} < L_{1,n} < \ldots < L_{d,n}$$

for uniquely determined $L_{i,n}$ in the layer $π^nΛ < π^{n+1}Λ$. Assuming $A, K$ complete, we know that

$$\lim_{n} L_{i,n} = E_i$$

where $E_i$ is an $A$-submodule of $V$, and that
\( E_0 < \cdots < E_d \quad \pi E_d \subset E_0. \)

Thus we see that set-theoretically, \( \overline{X}(V) \) is identifiable with the simplicial complex whose simplices are sequences of the form \((*)\). Thus set-theoretically at least

\[
\overline{X}(V) = \bigcup_{o = w_0 c w_1 c V} X\left(W_1/W_0\right)
\]

by our previous work (uses the exact sequence

\[0 \to E_{\text{div}} \to E \to \hat{E} \to 0.\)
March 31, 1972

Why the last vertex functor

\[ \text{New}(C) \xrightarrow{f} C \]

\[ (x_1 \leftarrow \cdots \leftarrow x_0) \quad \longmapsto \quad x_0 \]

is a homotopy equivalence. In general, given a functor \( f : C \to C' \), consider

\[ (x, y) \quad \longmapsto \quad \text{Hom}_{C'}(y, f(x)) \]

\[ C \times (C')^0 \quad \longmapsto \quad \text{Sets} \]

and form the category \( M_f \) cofibred over \( C \times (C')^0 \) associated to this functor. Thus, we have

\[ M_f \quad (x, y, \xi : y \to f(x)) \]

\[ \xymatrix{ C \ar[r]^{s} \ar[dr]^{p_1} & M_f \ar[dl]^{p_2} \\ & C' } \]

where \( p_1 \) is cofibred, and \( p_2 \) is fibred. In addition, define a section of \( p_1 \)

\[ C \xrightarrow{s} M_f \]

\[ x \quad (x, f(x), \text{id} : f(x) \to f(x)) \]

such that \( p_2 \circ s = f \). Since

\[ \text{Hom}_{M_f}(x, y, \xi : y \to f(x)), (u, f(u), \text{id} : f(u) \to f(u)) = \text{Hom}_{C}(x, u) \]

it follows we have adjoint functors.
so that $\text{pr}_1$ and $s$ are homotopy equivalences. Thus to prove $f$ is a h.e., it suffices to show that each of the categories $y/C$ consisting of $(x, f: y \to f(x))$ is contractible.

Now return to the last vertex functor:

$$\text{New}(C) \xrightarrow{f} C$$

Then given $y$ in $C$, we consider the category of arrows $y \to f(x)$, i.e., the simplicial set whose $p$-simplices are:

$$y \to x_0 \to \cdots \to x_p.$$  

Anyway, we still need to show contractibility of this.

The idea will be to consider the functors

$$(y \to x_0 \to \cdots \to x_p) \xrightarrow{d_0 \text{ nat. transf.}} (y \to x_0 \to \cdots \to x_p)$$

$$(y \to x_0 \to \cdots \to x_p) \xrightarrow{d_p} (y \to y \to x_0 \to \cdots \to x_p)$$

which actually do provide the desired contraction.
So now let $M$ be the category of coh. sheaves on a noetherian scheme $X$ and their isomorphisms. Let $\mathcal{R}$ be the cofibred category over $\Delta$ whose fibre at $p$ is the category of filtered objects of $M$ of length $p$ up to isom.

$$0 \subset M_1 \subset \cdots \subset M_p.$$ 

Let $I$ be the category with $\text{Ob}(I) = \text{Ob}(M)$, in which an arrow $M' \to M$ is an isom. of $M'$ with a subquotient of $M$, i.e.:

$$F \subset M' \to \downarrow \quad M.$$

Then we obtain a functor

$$R \xrightarrow{f} I$$

$$(0 \subset M_1 \subset \cdots \subset M_p) \longmapsto M_p$$

which I want to show is a homotopy equivalence. So I fix an object $V$ of $I$ and consider the category of arrows $y \to f(x)$. Thus I wish to consider diagrams

$$F \subset V \downarrow \quad 0 \subset M_1 \subset \cdots \subset M_p$$

as the objects of the category. A morphism in $R$
From $0 \leq M_1 \leq \cdots \leq M_p$ to $0 \leq N_1 \leq \cdots \leq N_q$ consists of $\varphi : [p] \to [q]$ and an isomorphism

$$\xi : M_{\varphi(p)}/M_{\varphi(0)} \to \tilde{N}_q$$

which induces:

$$M_{\varphi(p)}/M_{\varphi(0)} \to \tilde{N}_i$$

for $0 \leq i \leq q$.

Now we are given $F \in V$ and $E \in V$

so if $(\varphi, \xi)$ carries the former to the latter then

$$E \hookrightarrow F$$

must be cartesian

and

$$E \to M_{\varphi(p)} \to \tilde{N}_q$$

must be the given map from $E \to \tilde{N}_q$. Observe that $\xi$ is uniquely determined. So what I am trying to say is that this $y \to f(x)$ category is cofibred over $\Delta^0$ with discrete fibres, in fact the category is equivalent to the category belonging to the simplicial set whose $p$-simplices are chains of subobjects

$$M_0 \leq M_1 \leq \cdots \leq M_p \leq V$$

This is the nerve of the category of all $M \to V$ which is a category with a final object and hence is contractible.
To be very careful, given
\[ F \subseteq V \]
\[ 0 \subset M_1 \subset \ldots \subset M_p \]
and it into
\[ 0 \times_{M_p} F \subset M_1 \times_{M_p} F \subset \ldots \subset M_p \times_{M_p} F \subset V. \]

This is a functor of cofibred cats \( \mathcal{D}^\circ \). In the opposite direction send
\[ M_0 \subset M_1 \subset \ldots \subset M_p \subset V \]
\[ 0 = M_p \subset V \]
\[ 0 \subset M_1 / M_0 \subset \ldots \subset M_p / M_0. \]

(The point: The fibre category of \( y \rightarrow f(x) \) is cofibred over \( R \) hence also over \( \mathcal{D}^\circ \). And there is a unique isomorphism from any diagram (*) to any other. Thus the fibre category is a simplicial set.) I have proved:

**Proposition:** \( R = \text{the cat cof/} \mathcal{D}^\circ \text{ of} \)
\[ 0 \subset M_1 \subset \ldots \subset M_p \]
\[ S = \text{cat of } M \text{ where } M^0 \rightarrow M' \text{ is } F \subset M \]
\[ f : R \rightarrow S \]
\[ (0 \subset M_1 \subset \ldots \subset M_p) \mapsto M_p \]
is a homotopy equivalence.