April 1, 1971

Let $M' \longrightarrow M$ be a homomorphism of monoids. Assume M with the right action is an M'-torsor over $M'' = M/M'$. ~~[illegible]~~ (Everything takes place in a topos T with final object X, say.) Then to a section $s: X \longrightarrow M''$ we can associate the pull-back of M via s:

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & M \arrow[d] \\ X \arrow[r] & M'' \end{tikzcd}$$

It is a M'-torsor P' endowed with an M'-map $\varphi: P' \longrightarrow M$. Conversely given (P', φ) we can associate the section $X \longrightarrow M''$ ~~[illegible]~~ which is the map induced by φ on the M'-orbit spaces. Thus we get adjoint functors:

$$\left\{ \begin{array}{l} s: X \to M'' \\ \text{morphisms} = id \end{array} \right\} \begin{array}{c} (P'/M' \xrightarrow{\sim} M/M') \longleftarrow\!\!\!\mapsto (P', \varphi) \\ \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- \\ -\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\longrightarrow \\ (X \to M'') \longmapsto (X \times_{M''} M) \end{array} \left\{ \begin{array}{l} (P', \varphi) \quad P' \text{ an } M'\text{-torsor} \\ \qquad\qquad \varphi: P' \to M \text{ an } M'\text{-map} \\ \text{morphisms } P'_1 \to P'_2 \text{ compatible} \\ \text{with } \varphi_1 \text{ and } \varphi_2 \end{array} \right\}$$

We have therefore a map

$$H^0(X, M'') \xrightarrow{\delta} H^1(X, M') \quad \left(\overset{\text{defn.}}{\equiv} \pi_0 \text{ of } \underline{Tors}(X, M') \right)$$

and we now wish to see when its image is the kernel of the canonical map $H^1(X, M') \longrightarrow H^1(X, M)$.

One expects this to be so only when M'' is the fibre of $BM' \longrightarrow BM$, which implies that M acts on M'' as autos. since it acts through $\pi_1(BM)$. So assume that M acts on M'' as autos.

Then given a map $P_1 \longrightarrow P$ of M-torsors, the induced map $P_1/M' \longrightarrow P/M'$ is an isomorphism. Indeed from the point of view of pro-objects, the map $P_1 \to P$ is a filtered inductive limit of maps $M \xrightarrow{\text{left mult by } m} M$ which on passage to the quotient yield the isom.

$$M'' \underset{\cong}{\xrightarrow{m\cdot}} M''.$$

This implies that given

$$\begin{tikzcd} & P_1 \arrow[d, "\text{map of } M\text{-tors.}"] \\ P' \arrow[r, "M'\text{-map}"'] & P \end{tikzcd}$$

then the section $P'/M' \longrightarrow P/M'$ lifts to P_1/M', hence there is a diagram

$$\begin{tikzcd} & X \times_{P_1/M'} P_1 \arrow[r] \arrow[d] & P_1 \arrow[d] \\ P' \arrow[r] & X \times_{P/M'} P \arrow[r] & P. \end{tikzcd}$$

Thus given an M'-torsor P' and a path in $\underline{\text{Tors}}(X, M)$

from $P' \times^{M'} M$ to the trivial torsor $X \times M$, the path can be lifted to a ~~map of~~ path from P' to the torsor belonging to a map $X \longrightarrow M''$. This proves exactness of

$$H^0(X, M'') \xrightarrow{\delta} H^1(X, M') \longrightarrow H^1(X, M).$$

Actually what we have is the following diagram of categories.

$$\begin{tikzcd}
\{\text{cat. of } (P', P, \varphi : P' \xrightarrow[M']{} P)\} \arrow[d, shift left] \arrow[dr, shift left] & \\
\underline{\mathrm{Tors}}(X, M') \arrow[u, shift left, "P' \mapsto P' \times^{M'} M"] & \{\text{cat. of } (P, s : X \to P/M')\} \arrow[ul, shift left] \arrow[d] \\
& \underline{\mathrm{Tors}}(X, M)
\end{tikzcd}$$

cofibred cat. belonging to covariant functor $P \mapsto H^0(X, P/M')$

By the hypothesis that M acts as autos. of M'' it is also fibred

associates to $s : X \to P/M'$ the map

$$X \times_{\underset{s}{\to} P/M'} P \xrightarrow{pr_2} P$$

obviously right adjoint

(Note to define δ in general one only needs an M'-torsor over M''.)

Back to Mather's theorem and an understanding of the generalization to stacks. We let θ the group of diffeos of $(0,1)$ with compact support and let $\Theta(X)$ be the category of θ-torsors on X. There is a functor

$$\Theta(X) \times \Theta(X) \longrightarrow \Theta(X)$$

unique up to canonical isomorphism which is also associative up to a canonical isomorphism, but not unitary.

Now let $\mathcal{P}$ be a stack over the category of open sets of X endowed with an action of Θ, that is, a cartesian functor

$$\mu : \mathcal{P} \times \Theta \longrightarrow \mathcal{P}$$

(X taken over $Op(X)$)

together with an associativity isomorphism

$$\begin{tikzcd}
\mathcal{P} \times \Theta \times \Theta \arrow[r, "id \times \mu"] \arrow[d, "\mu \times id"'] & \mathcal{P} \times \Theta \arrow[d, "\mu"] \\
\mathcal{P} \times \Theta \arrow[r, "\mu"'] \arrow[ur, phantom, "\nearrow\!\!\!\nearrow"] & \mathcal{P}
\end{tikzcd}$$

(reasonable to have isom. as Θ, hence $\mathcal{P}$, are groupoids)

compatible with the associativity isomorphism in Θ. The left-filtering condition must now be made explicit.

(i)

$$\mathcal{P} \times \Theta \times \Theta \longrightarrow \mathcal{P} \times \mathcal{P}$$

$$(p, m_1, m_2) \longmapsto (pm_1, pm_2)$$

locally essentially surjective

This means that given $p_1, p_2 \in \mathcal{P}(U)$ there is a covering of U by $\{V\}$ such that $p_i | V$ is isomorphic to $p m_i$.

(ii) Given p, m_1, m_2 over U and an isomorphism of pm_1 and pm_2, then by refining U one can find p' and m, such that $m m_1 \simeq m m_2$ and $p'm \simeq p$ and such that the isomorphisms

$$\begin{array}{ccccccc} (p'm)m_1 & \overset{\text{assoc.}}{\simeq} & p'(mm_1) & \simeq & p'(mm_2) & \overset{\text{assoc.}}{\simeq} & (p'm)m_2 \\ \wr\| & & & & & & \wr\| \\ pm_1 & & & & & & pm_2 \end{array}$$

coincide with the given isom. of pm_1 and pm_2.

(hopefully the above is the good definition of Θ-torsor over X.)

Now we make these Θ-torsors into a 2-category. A 1-arrow from $\mathcal{P}$ to $\mathcal{P}'$ is a cartesian functor

$$F : \mathcal{P} \longrightarrow \mathcal{P}'$$

together with an isomorphism of functors

$$\begin{tikzcd} \mathcal{P} \times \Theta \arrow[r] \arrow[d] & \mathcal{P}' \times \Theta \arrow[d] \\ \mathcal{P} \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \mathcal{P}' \end{tikzcd} \qquad F(p.m) \cong F(p).m$$

which is compatible with the associativity isom:

$$\begin{tikzcd} F(p.(m_1 m_2)) \arrow[r, "\simeq", "F(\mathrm{assoc})"'] \arrow[d, "\wr"] & F(pm_1 . m_2) \arrow[r, "\simeq"] & F(pm_1).m_2 \arrow[dl, "\wr"] \\ F(p).(m_1 m_2) \arrow[r, "\simeq"', "\mathrm{assoc.}"] & (F(p)m_1)\, m_2 & \end{tikzcd}$$

A 2-arrow is a natural transformation $F \Longrightarrow F'$ such that

$$\begin{tikzcd} F(pm) \arrow[r, "\simeq"] \arrow[d, "\wr"] & F(p).m \arrow[d, "\wr"] \\ F'(pm) \arrow[r, "\simeq"] & F'(p).m \end{tikzcd}$$

commutes.

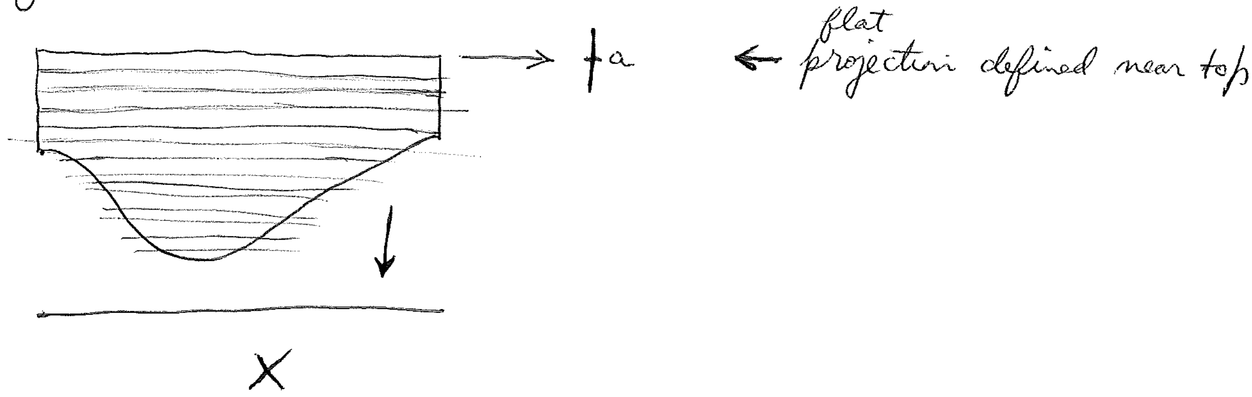
Thus we obtain a 2-category $\underline{\underline{\mathrm{Tors}}}(X, \Theta)$.

<u>Example</u>: Suppose given a Haefliger structure over X, i.e. a Γ-torsor, where Γ is the pseudogroup of orientation-preserving diffeos. of $\mathbb{R}$. Want to describe associated Θ-torsor.

<u>Idea</u>: Recall that Γ is the topological category ~~whose~~ with objects $\mathbb{R}$ and morphisms the etale space Γ over $\mathbb{R}$ of germs of diffeos between points of $\mathbb{R}$.

Fix a real number a and define the topological groupoid Q_a to have for objects the real numbers $x \leq a$ ~~and~~ in which a morphism from x to x' is a germ of diffeo. from $[x, a]$ to $[x', a]$ which is the identity near a.

These morphisms form an étale space Q_a over $R_{\leq a}$. It is clear (more or less) that a $\mathbb{Q}_a$-torsor over X is a foliated space looking so



We take $a=0$ and abbreviate to $\mathbb{Q}$. It is clear that $\textcircled{H}$ acts on $\mathbb{Q}$:

$$\mathbb{Q} \times \textcircled{H} \longrightarrow \mathbb{Q}$$

and that there is a functor

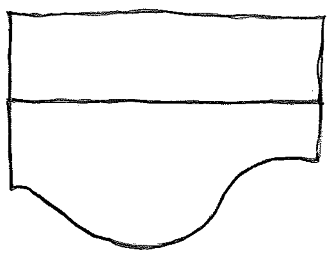
$$\mathbb{Q} \longrightarrow \Gamma$$

compatible with $\textcircled{H}$ action, if $\textcircled{H}$ acts trivially on Γ. Now the conjectures are that one has "homotopy equivalences"

$$\Gamma \longleftarrow (\mathbb{Q}, \textcircled{H}) \longrightarrow (pt, \textcircled{H})$$

the first because $\mathbb{Q}$ is a $\textcircled{H}$-torsor over Γ and the second because $\mathbb{Q}$ is contractible.

I want to check the categorical aspects of the preceding example now. First of all it is necessary to understand what is meant by the action of Θ on $\mathbb{Q}$. From the pictures this is clear, namely given a $\mathbb{Q}$-torsor and a Θ-torsor one stacks one on top of the other



Thus $\mathbb{Q}$ is a stack with an assoc. operation of Θ.

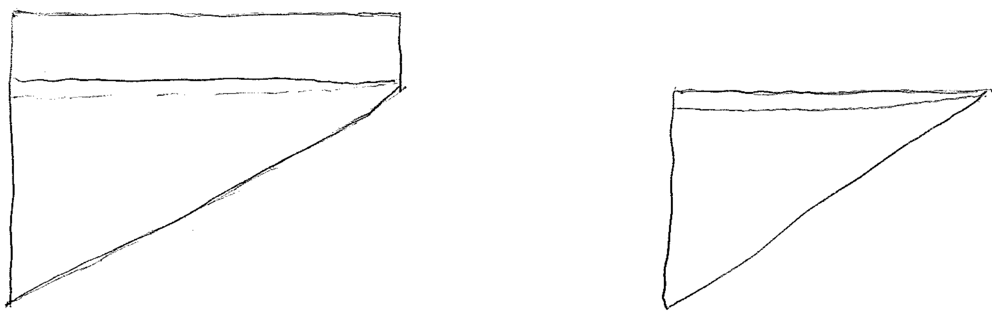
From the category viewpoint $\mathbb{Q}$ is a topological groupoid whose objects are $\mathbb{R}_{<0}$ and whose space of morphisms is the etale space over $\mathbb{R}_{<0}$ whose stalk at x consists of germs of diffeos. from $[x,0]$ to some other interval $[y,0]$. There is an obvious functor of top. groupoids $\mathbb{Q} \to \Gamma$. On the other hand Θ has been described as a groupoid with one object I and with an associative operation. For each x, choose a diffeo. germ

$$\gamma_x : [x,1] \xrightarrow{\sim} [x,0]$$

such that

$$\gamma_x(t) = t-1 \quad \text{for} \quad t \text{ near } 1.$$

and do this continuously as x ranges over $\mathbb{R}_{<0}$



This is possible and is unique up to a unique section of $\mathbb{Q}$ over $\mathbb{R}_{<0}$. Using γ one gets a functor

$$\mathbb{Q} \times \Theta \longrightarrow \mathbb{Q}$$

which is associative just as Θ is. This functor sends (x, I) to x and a morphism $g \in Q_{xy}$, $g : [x,0] \longrightarrow [y,0]$, and a $g \in G_{01}$ into the arrow

$$\begin{tikzcd}
{[x,0]} \arrow[d] & {[x,1] = [x,0] \cup [0,1]} \arrow[l, "\gamma_x"'] \arrow[d, "g \cup g"] \\
{[y,0]} & {[y,1] = [y,0] \cup [0,1]} \arrow[l]
\end{tikzcd}$$

Next to understand what it means precisely for the quotient of this action to be Γ.

Remark: Let a group G act to the right on a set S. Then we have a topological groupoid

$$
\begin{tikzcd}
S \times G \times G \arrow[r, shift left=2, "\mu \times 1"] \arrow[r, "1 \times \mu" description] \arrow[r, shift right=2, "pr_{12}"'] & S \times G \arrow[r, shift left=1, "\mu"] \arrow[r, shift right=1, "pr_2"'] & S
\end{tikzcd}
$$

and the associated stack of torsors assigns to a space X the category of G-torsors P over X endowed with an equivariant map $t: P \longrightarrow S$ (strictly $X \times S$). If G acts freely on S, then

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] & S \arrow[d] \\
X \arrow[r] & S/G
\end{tikzcd}
$$

is cartesian, so

$$P \xrightarrow{\sim} X \times_{(S/G)} S$$

and the stack is the discrete stack represented by S/G.

The **conjecture** therefore is that $\mathbb{Q}$ is a $\textcircled{H}$-torsor over Γ. What this means is that the filtering conditions are satisfied locally. Thus first we have a functor

$$\mathbb{Q}(X) \xrightarrow{\ f\ } \Gamma(X)$$

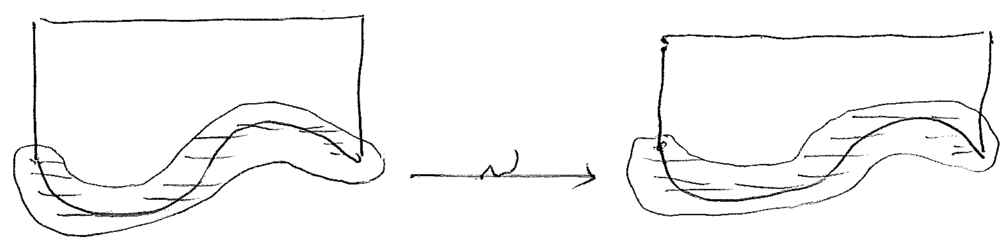
and an isomorphism $f(q \cdot m) \simeq f(q)$ if $q \in Ob\,\mathbb{Q}(X)$

and $m \in \mathrm{Ob}\, \Theta(X)$. Locally f is surjective: Given a Π-structure over X, locally it is isomorphic to an object in the image of f.

Π is the quotient of Q by Θ-action:

$$\begin{array}{ccc} Q \times \Theta \times \Theta & \longrightarrow & Q \overset{2}{\times}_{\Pi} Q \\ (q, m_1, m_2) & & q.m_1 \quad q.m_2 \end{array}$$

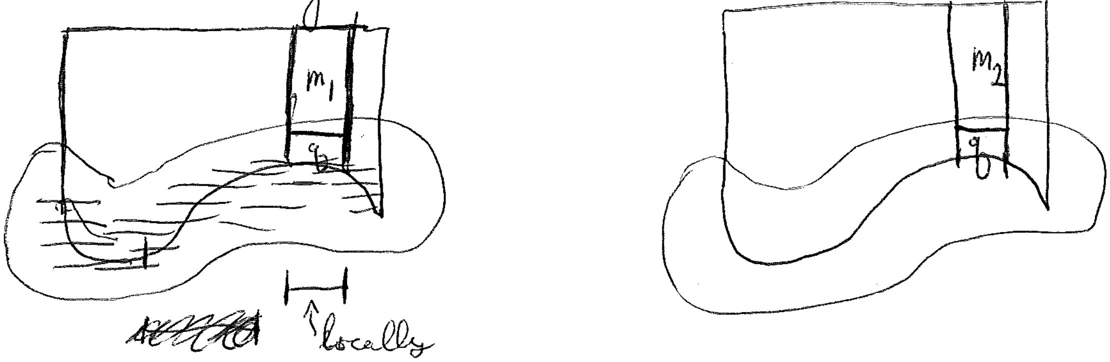
locally essentially surjective. This means that given Q-structures q_1 and q_2 over X and an isomorphism $f(q_1) \simeq f(q_2)$



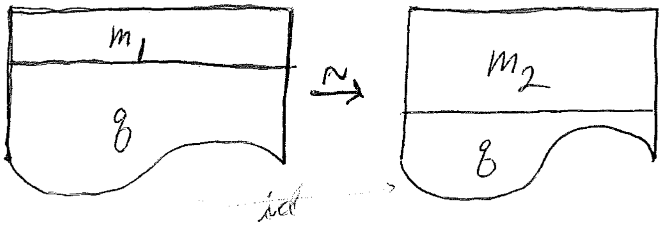
then locally there is a q and Θ-structures m_1 and m_2 and isomorphisms $q.m_1 \simeq q_1$ and $q.m_2 \simeq q_2$ such that the composite isom.

$$f(q_1) \simeq f(q.m_1) \overset{\text{can.}}{\simeq} f(q) \overset{\text{can.}}{\simeq} f(q.m_2) \simeq f(q_2)$$

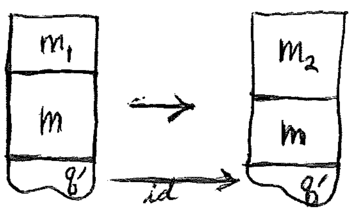
agrees with the given one.



Finally we must check the equalization condition, namely given a g and two Θ-structures m_1, m_2 and an isom. $g\cdot m_1 \simeq g\cdot m_2$



such that $f(g) \overset{can.}{\simeq} f(g.m_1) \simeq f(g\cdot m_2) \overset{can}{\simeq} f(g)$ is the identity. Then locally the situation looks



Thus we can find an isom. $g'.m \simeq g$ and an isom. $u: m.m_1 \simeq m.m_2$ such that the given isom. is the composition

$$g.m_1 \simeq (g'.m).m_1 \overset{assoc}{\simeq} g'.mm_1 \overset{id.u}{\simeq} g'.mm_2 \simeq (g'm).m_2 \simeq g.m_2.$$

Similarly one checks for when m_1 is the identity, i.e. given $g \simeq g.m$.

Thus it seems that our earlier guess at what Θ-torsors ought to be is essentially correct except that the base of such a torsor should be a stack.

April, 1971

<u>Outline</u>: homology of categories

$\mathcal{C}$ small category

$$\mathcal{C}^\wedge = \underline{\mathrm{Hom}}\,(\mathcal{C}^\circ, \text{Sets})$$
$$\mathcal{C}^\vee = \underline{\mathrm{Hom}}\,(\mathcal{C}, \text{Sets}).$$

$f: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ functor, have adjoint functors

$$\begin{tikzcd} \mathcal{C}_1^\vee \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=3, "f_*"'] & \mathcal{C}_2^\vee \arrow[l, "f^*"'] \end{tikzcd}$$

$$(f^*G)(x) = G(f(x))$$
$$(f_!F)(y) = \varinjlim_{f(x)\to y} F(x)$$
$$(f_*F)(y) = \varprojlim_{y\to f(x)} F(x)$$

Let the category of arrows $f(x) \longrightarrow y$ with y fixed (precisely, pairs (x, ξ) with $x \in \mathrm{Ob}\,\mathcal{C}_1$ and $\xi: f(x) \to y$) be denoted

$$\mathcal{C}_1/y \text{ rel } f$$

objects of $\mathcal{C}_1$ over y relative to f. Abbreviate to $\mathcal{C}_1/y$. Similarly denote the category of arrows $y \to f(x)$ with y fixed by

$$y \text{ rel } f \backslash \mathcal{C}_1 \quad \text{or simply} \quad y\backslash\mathcal{C}_1 .$$

If $f : \mathcal{C} \longrightarrow pt$, then

$$f_! F = \varinjlim F$$
$$f_* F = \varprojlim F .$$

Given $f : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, $g : \mathcal{C}_2 \longrightarrow \mathcal{C}_3$ one has canonical isomorphism

$$g_! f_! \simeq (gf)_!$$
$$g_* f_* \simeq (gf)_*$$

In particular taking $\mathcal{C}_3$ to be a point

$$\varinjlim_{\mathcal{C}_2} \circ f_! \simeq \varinjlim_{\mathcal{C}_1}$$

and similarly for inverse limits.

If $\mathcal{C}_1 \xrightarrow{u} \mathcal{C}_2$ is left adjoint to $v : \mathcal{C}_2 \to \mathcal{C}_1$, then

$$u_! = v^*$$
$$v_* = u^*$$

so have adjoint functors

$$\begin{tikzcd} \mathcal{C}_1^{\wedge} \arrow[r, shift left=4.5, leftarrow, "v_!"] \arrow[r, shift left=1.5, "u_! = v^*"] \arrow[r, shift right=1.5, leftarrow, "u^* = v_*"] \arrow[r, shift right=4.5, "u_*"'] & \mathcal{C}_2^{\wedge} \end{tikzcd}$$

?

(reversal of order: $(u, v) \longmapsto (v^*, u^*)$).

We are interested in the case when $f : \mathcal{C}_1 \to \mathcal{C}_2$ induces an isom.

$$\varinjlim_{\mathcal{C}_2} F \xleftarrow{\sim} \varinjlim_{\mathcal{C}_1} f^*F$$

for all $F \in \mathcal{C}_2^\wedge$, in which case say that f is final. Since

$$\mathrm{Hom}_{pt}(\pi_{1!} f^*F, S) \simeq \mathrm{Hom}_{\mathcal{C}_2}(F, f_* \pi_1^* S)$$

$$\mathrm{Hom}_{pt}(\pi_{2!} F, S) \simeq \mathrm{Hom}_{\mathcal{C}_2}(F, \pi_2^* S)$$

one sees that $\quad f$ is final $\iff \pi_2^* S \xrightarrow{\sim} f_* \pi_1^* S$ for all sets S.

Assume that $f : \mathcal{C}_1 \to \mathcal{C}_2$ is cofibred. Claim that

$$(f_! F)(y) = \varinjlim_{\mathcal{C}_{1,y}} F$$

where $\mathcal{C}_{1,y}$ is the fibre over y. Indeed the former is taken over $\mathcal{C}_1/y$ hence it suffices show the inclusion functor

$$v : \mathcal{C}_{1,y} \longrightarrow \mathcal{C}_1/y$$

$$z \longmapsto (f(z) \xrightarrow{id} y)$$

is final. But

$$\mathrm{Hom}_{\mathcal{C}_1/y}\left(f(x) \xrightarrow{\xi} y,\ f(z) \xrightarrow{id} y\right) = \mathrm{Hom}_{\mathcal{C}_1}(x, z)_\xi$$

$$= \mathrm{Hom}_{\mathcal{C}_{1,y}}\left(\xi_* x, z\right)$$

so v has for left adjoint the functor

$$u : \mathcal{C}_1/y \longrightarrow \mathcal{C}_{1,y}$$

$$f(x) \xrightarrow{\xi} y \longmapsto \xi_* x$$

Then

$$\varinjlim_{\mathcal{C}_1/y} F = \varinjlim_{\mathcal{C}_{1,y}} u_! F$$

$$= \varinjlim_{\mathcal{C}_{1,y}} v^* F$$

showing finality of v. (This argument to be inserted after adjoint functors. In general given $f : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, the canonical map

$$\varinjlim_{\mathcal{C}_1} f^* F \xrightarrow{\sim} \varinjlim_{\mathcal{C}_2} F$$

is an isomorphism for all $F \in \mathcal{C}_2^\wedge$ provided f admits a left adjoint.)

From now on we work with abelian functors

$$\mathcal{C}^{\wedge}_{ab} = \mathrm{Hom}(\mathcal{C}, ab).$$

Given a map $f: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, we have adjoint functors

$$\mathcal{C}^{\wedge}_{1,ab} \underset{f_*}{\overset{f_!}{\underset{\longleftarrow f^*}{\longrightarrow}}} \mathcal{C}^{\wedge}_{2,ab}$$

given by similar formulas but where the limits are taken in the category of abelian groups. Note that f^*, f_* are the restrictions of the corresponding functors ~~[illegible]~~ for set-valued functors, but not necessary $f_!$.

The ~~[illegible]~~ derived functors of $\varinjlim$ and $\varprojlim$ will be denoted

$$L_q \varinjlim_{\mathcal{C}} F = H_q(\mathcal{C}, F)$$

$$R^q \varprojlim_{\mathcal{C}} F = H^q(\mathcal{C}, F).$$

~~[illegible]~~ Given $f: \mathcal{C}_1 \to \mathcal{C}_2$, $f_!$ carries projectives to projectives since it has an exact right adjoint; similar f_* carries injectives to injectives. One knows then that there are spectral sequences

$$E^2_{pq} = H_p(\mathcal{C}_2, L_q f_!(F)) \Longrightarrow H_{p+q}(\mathcal{C}_1, F)$$

$$E_2^{pq} = H^p(\mathcal{C}_2, R^q f_*(F)) \Longrightarrow H^{p+q}(\mathcal{C}_2, F).$$

<u>Proposition:</u>

$$L_q f_!(F)(y) = H_q(\mathcal{C}_1/y\,; F)$$

$$R^q f_*(F)(y) = H^q(y\backslash\mathcal{C}_1\,; F)$$

<u>Proof</u>: We prove the first. True for $q=0$, hence both sides being homological functors of F, general nonsense shows one has only to prove the effaceability of the right side. Enough to show that for the functor

$$i : \mathcal{C}_1/y \longrightarrow \mathcal{C}_1$$

$$f(x)\to y \longmapsto x$$

i^* carries projectives to projectives, or what comes to the same thing, that i_* is exact. But if $G \in (\mathcal{C}_1/y)^{\wedge}_{ab}$, then

$$(i_* G)(z) = \varprojlim G(f(x)\to y)$$

where the limit is taken over the category of triples (x, ξ, η) where $\xi : z \to x$ is a map in $\mathcal{C}_1$ and $\eta : f(x) \to y$ is a map in $\mathcal{C}_2$. This category is the disjoint union of categories Γ_ζ as ζ runs over $\mathrm{Hom}_{\mathcal{C}_2}(f(z), y)$; Γ_ζ is the category of triples (x, ξ, η) with $\eta\cdot f(\xi) = \zeta$. Γ_ζ has the initial object (z, id, ξ), hence

$$(i_* G)(z) = \prod_{\xi \in \mathrm{Hom}_{C_2}(f(z), y)} G(f(z) \xrightarrow{\xi} y)$$

showing that i_* is exact. q.e.d.

<u>Cor</u>: If f is cofibred, then

$$L_q f_!(F)(y) = H_q(\mathcal{C}_{1,y}; F)$$

<u>Proof</u>: We have already established that the map $\mathcal{C}_{1,y} \xrightarrow{v} \mathcal{C}_1/y$ has a left adjoint u. Thus we want to know

<u>Lemma</u>: If $f: \mathcal{C}_1 \to \mathcal{C}_2$ has a left adjoint, then

$$H_q(\mathcal{C}_1, f^*F) \xrightarrow{\sim} H_q(\mathcal{C}_2, F)$$

for all q and $F \in \hat{\mathcal{C}}_{2,ab}$.

As we've established this for $q = 0$, and both sides are homological functors, we must prove the left side is effaceable. This will follow if f^* preserves projectives, equivalently if f_* is exact. But if u is a left adjoint for f, then $f_* = u^*$ is exact.

Relation with (semi-) simplicial sets:

Δ = category of ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$; simplicial objects = functors from Δ°. Let X be a simplicial set and think of it in terms of the category Δ°/X. Then

$$(\Delta^\circ/X)^\vee = \text{simplicials sets} / X$$

Δ°/X is the category one draws: $X_2 \rightrightarrows X_1 \rightrightarrows X_0$

and given $f: X \longrightarrow Y$ we have

$$(\Delta^\circ/X)^\vee \underset{f_*}{\overset{f_!}{\underset{\xleftarrow{f^*}}{\rightrightarrows}}} (\Delta^\circ/Y)^\vee$$

given by

$$f_!(F \xrightarrow{u} X) = F \xrightarrow{fu} Y$$

$$f^*(G \longrightarrow Y) = X \times_Y G \longrightarrow X$$

$$(f_* F)(y) = \varprojlim_{y \to f(x)} F(x) = H^\circ(X_y, F|X_y)$$

$(f_! F)(y) = \coprod_{f(x)=y} F(x)$ as f is cofibred

where X_y is the simplicial set

$$\Delta(q) \underset{y \to Y}{\times} X .$$

More generally for $F \in (\Delta^\circ/X)^\vee_{ab}$ we have

$$R^q f_*(F)(y) = H^q(X_y, F|X_y)$$

hence the spectral sequence

$$E_2^{pq} = H^p(\Delta^\circ/Y, R^q f_* F) \Longrightarrow H^{p+q}(\Delta^\circ/X, F)$$

is the simplicial Leray spectral sequence. Since

$$f_{!,ab}(F)(y) = \bigoplus_{f(x)=y} F(x) .$$

the functor $f_{!,ab}$ is exact and the spectral sequence in homology collapses yielding

$$H_q(\Delta^\circ/X, F) = \check{H}_q\Big(\nu \longmapsto \bigoplus_{x \in X_\nu} F(x)\Big)$$

if $Y = \Delta(0)$ final object, showing simplicial homology of X coincides with left-derived functors of $\varinjlim$ over Δ°/X. ~~Thus the homology spectral sequence is the~~ [illegible] (To establish this one needs to know that $\check{H}_q = L_q \varinjlim_{\Delta^\circ}$ which follows from the fact they coincide in dim 0, and $\check{H}_q$ is effaceable (using Dold-Puppe thm. Details appear in appendix to Gabriel-Zisman.)

Dual results are obtained for $(\Delta^\circ/X)^\wedge$ which is the Deligne topos of simplicial sheaves on the simplicial space X.

~~Given $f: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, then f_* is exact~~

~~If (i)~~

We have taken the homological point of view (which is convenient if you want to have a Kunneth theorem for the product of two categories.) Cohomologically one associates the topos $\mathcal{C}^\wedge$ ~~to~~ $\mathcal{C}$ and considers $f_!\ \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ as a comorphism of (chaotic) sites. Then (f^*, f_*)

$$\begin{tikzcd} \mathcal{C}_1^\wedge \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=3, "f_*"'] & \mathcal{C}_2^\wedge \arrow[l, "f^*" description] \end{tikzcd}$$

constitute a morphism of topoi, and $(f_!, f^*)$ do only if $f_!$ is exact. Sufficient conditions:

i) f has a left adjoint u, whence

$$(f_!F)(y) = \varinjlim_{y \to f(x)} F(x) = \varinjlim_{u(y)\to x} F(x) = (u^*F)(y)$$

is exact.

ii) $\mathcal{C}_1$ closed under finite projective limits and f preserves these, whence ~~representables~~ the category of arrows $y \to f(x)$ is cofiltering.

(Note that ii) has no analogue for the homology pictures, only i): f_* is exact if f has a left adjoint u:

$$(f_*F)(y) = \varprojlim_{y\to f(x)} F(x) = \varprojlim_{u(y)\to x} F(x) = (u^*F)(y).\ )$$

<u>April 3, 1971</u>: (Classifying C-torsors question.)

Let C be the category belonging to a partially ordered set (at most one morphism between objects and only isos. are the identity.) Let P be a C-torsor over X. Then the maps of sheaves

$$P \longrightarrow X \times S$$

is injective because P_x being a pro-representable functor, it assigns at most one element to each object. (Pro C is easy to identify: it is equivalent to the family of subsets I of S such that (i) $s \leq s'$, $s \in I \Rightarrow s' \in I$ (ii) $s', s'' \in I \Rightarrow \exists\, s \in I$ $s \leq s'$, $s \leq s''$. There is a one morphism $I \to I'$ precisely when $I \supset I'$. A representable functor is an I of the form $I_{s_0} = \{ s \mid s_0 \leq s \}$ for some s_0. Note the embedding $s \mapsto I_s$ is covariant with these formulas.) It is clear then that the category of C-torsors over X is thus the category opposed to the ordered set of subsheaves of the constant sheaf $X \times S$ which are co-hereditary and locally co-filtering.

For each $s \in S$, P_s is an open subspace of X. We thus have a functor

$$C \longrightarrow Op(X)$$

because if $s \leq s'$ then $P_s \subset P_{s'}$. This family is not arbitrary; in effect for each x the set of s with P_s containing x is a co-hereditary co-filtering subset of S. ~~[illegible]~~

~~This if we fix X there is a universal such torsor over X, namely take the ordered set $Op(X)$ and for each $U \subset X$ take $\tilde{P}_U$ to be the sheaf "U". Then given a torsor $P \subset X \times S$ with S partially ordered, there is a~~ ~~[illegible]~~ ~~morphism~~

$$\cancel{S \longrightarrow Op(X)}$$

cofiltering coheredi tary subsets

So now if we consider the function $f: X \longrightarrow Pro(S)$ which assigns to x the subset $P_x \subset S$ we have a function to ~~[illegible]~~ a partially ordered set which is semi-continuous in the sense that as you specialize you jump ~~[illegible]~~ upward, i.e.

~~[illegible]~~

(*) $\{x \mid f(x) \leq s\}$ is open for $s \in S$.

Thus a C-torsor which is representable on each fibre is (the same as) a semi-continuous function from X to S. ~~[illegible]~~
A general C-torsor is a ~~[illegible]~~ function from X to $Pro(S)$, which is semi-continuous in the sense that (*) holds for $\forall s \in S$.

(Digression: In defining filtering category one should add that the set of objects of the category is non-empty. Indeed this becomes clear if one recalls that the first axiom morally means that for every finite subset of objects, there is another object dominated by each member of the subset. Thus in defining torsor, this means that P is non-empty over each point of X when $\mathcal{C}$ is non-empty.)

Next we compute the torsor associated to a map of X to the Milnor BC. This map furnishes us with a covering $\{U_i\}$ of X indexed by $\mathbb{N}$ and for each i and object

$$O_i : U_i \longrightarrow S$$

such that for $i < j$ we have $O_i \le O_j$ on $U_i \cap U_j$. In addition $U_i = \rho_i^{-1}(0,1]$ for a partition of 1 on X; in particular each x belongs to only finitely many U_i. Given x let $i_0 < \dots < i_g$ be the i for which $x \in U_i$ and set

$$f(x) = O_{i_0}(x) \in S.$$

Then $f : X \longrightarrow S$. I claim f is semi-continuous. Indeed if x, i_0 are so, and $y \in U_{i_0}$ then $f(y) = O_{j_0}(y) \le O_{i_0}(y) = f(x)$ for y close enough to x so that O_{i_0} is constant.

Now suppose we have a C-cocycle as used by Graeme Segal. Thus we have a covering $\{U_i\}_{i\in I}$ (I arbitrary) and for each finite $\sigma \subset I$ a map

$$o_\sigma : U_\sigma \longrightarrow S$$

such that if $\sigma \supset \sigma'$ then $o_\sigma \le o_{\sigma'}$ on $U_\sigma \subseteq U_{\sigma'}$. Define a pro-object by

$$f(x) = \{ o_\sigma(x) \mid U_\sigma \ni x \}$$

Again f is semi-continuous.

Let $f : X \longrightarrow S$ be semi-continuous describing the torsor

$$P = \coprod_{s \in S} U_s$$

where

$$U_s = \{x \mid f(x) \le s\}.$$

Then $s \le s' \Rightarrow U_s \subset U_{s'}$ and so

$$x \in U_s \Longrightarrow U_{f(x)} \subset U_s$$

showing that $U_{f(x)}$ is the smallest member of the covering $\{U_s\}$ containing x. Thus ~~it should be so that $U_s \subset U_{s'} \Leftrightarrow s \le s'$, i.e. $S \to \mathrm{Open}(X)$ is an ordered subset, then~~ we have a Lubkin covering of X.

April 4, 1971

Let $f: X \longrightarrow S$ be semi-continuous, [redacted] i.e. $U_s = \{x \mid f(x) \leqslant s\}$ is open for each $s \in S$. Then $s \leqslant s' \Longrightarrow U_s \leqslant U_{s'}$. So

$$f(x) \leqslant f(y) \Longrightarrow U_{f(x)} \leqslant U_{f(y)}$$

Conversely

$$U_{f(x)} \leqslant U_{f(y)} \Longrightarrow x \in U_{f(y)} \Longrightarrow f(x) \leqslant f(y).$$

[redacted] Let L be the image of f and endow L with the partial ordering induced from S. The above implications show that L is [redacted] isomorphic as an ordered set with the family of open sets of the form $U_{f(x)}$.

<u>Definition:</u> A [redacted] Lubkin covering of X is [redacted] a [redacted] subset $L \subset \text{Open}(X)$ such that i) for every x ~~there~~ the set of $U \in L$ containing x has a least member U_x, ii) every $U \in L$ is of the form U_x for some x.

Given a Lubkin covering L one has a map

$$\begin{aligned} X &\xrightarrow{\theta} L \\ x &\longmapsto U_x \end{aligned}$$

which is surjective and semi-continuous since

$$\{y \mid U_y \subset U_x\} = \{y \mid y \in U_x\} = U_x.$$

Conversely given a semi-continuous map $f: X \to S$ it factors uniquely

$$X \xrightarrow{p} L \xrightarrow{i} S$$

where i is an embedding of ordered sets (fully faithful functor) and L is a Lubkin covering and p is the canonical map.

One might also think of a Lubkin covering L as a stratification of X with order scheme L in some sense.

It is clear that Lubkin coverings form a partially ordered set. Indeed $\{U'_x\}$ refines $\{U_x\}$ iff $U'_x \subset U_x$ for all x, or equivalently if $\exists$ comm. triangle

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & L' \arrow[d, "u"] \\ & L \end{tikzcd}$$

with u a map of ordered sets. ($u(U') =$ least U containing U'.) Also Lubkin coverings are closed under finite intersection: the intersection of L and L' is the image of $f: X \to L \times L'$.

Now we try to handle general C-torsors, where C is the category associated to a partially-ordered set S. According to previous calculations such a torsor may be identified with a map

$$f : X \longrightarrow \mathrm{Pro}(S)$$

such that for each $s \in S$

$$U_s = \{x \mid f(x) \leq s\}$$

is open. Perhaps a better way of putting it is that we have a ~~[illegible]~~ morphism

$$S \xrightarrow{U} \mathrm{Open}(X) \qquad s \mapsto U_s$$

of ordered sets such that for each x

$$\{s \mid U_s \ni x\}$$

is a pro-object (left-filtering subset of S). Let g be the composition:

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "g"] \\
\mathrm{Pro}(S) \arrow[r, "U"'] & \mathrm{Pro}(\mathrm{Open}\,X)
\end{tikzcd}$$

Then

$$f(x) \leq f(y) \overset{U \text{ morphism}}{\Longrightarrow} U_{f(x)} \leq U_{f(y)} \overset{\text{defn of } g}{\Longleftrightarrow} g(x) \leq g(y).$$

On the other hand suppose $U_{f(x)} \leq U_{f(y)}$

i.e. for every s with $f(y) \leq s$, there exists a t such that $f(x) \leq t$ and $U_t \subset U_s$. Then $x \in U_t \subset U_s$, hence $f(x) \leq s$ for every s such that $f(y) \leq s$. Thus $f(x) \leq f(y)$.

(Digression: Alternative definition of Lubkin covering: It is a map $X \xrightarrow{f} \mathrm{Open}(X)$ such that

$$x \in f(y) \iff f(x) \subset f(y).\text{)}$$

I claim the map $g : X \longrightarrow \mathrm{Pro}(\mathrm{Open}(X))$ above has the following properties:

i) semi-continuity: Given U open, the set

$$\tilde{U} = \{x \mid g(x) \leq U\} \text{ is open}$$

ii) $\tilde{\tilde{U}} = \tilde{U}$.

Proof: $\tilde{U} = \{x \mid \exists s \quad x \in U_s \subset U\}$. But $U_s \subset U \Rightarrow U_s \subset \tilde{U}$; in effect $y \in U_s \subset U \Rightarrow y \in \tilde{U}$. Therefore $\tilde{U}$ is open because if $x \in \tilde{U}$, then $\exists s \ni x \in U_s \subset U$ and $x \in U_s \subset \tilde{U}$. On the other hand $x \in \tilde{U} \Rightarrow \exists s,\ x \in U_s \subset U \Rightarrow \exists s,\ x \in U_s \subset \tilde{U} \Rightarrow x \in \tilde{\tilde{U}}$.

~~Suppose given g satisfying i) and ii)~~

Start again:

Let S be a partially ordered set and let P be a torsor over X for the category defined by $\mathcal{C}$. Thus

$$P = \coprod_{s \in S} U_s$$

where U_s is open in X and $s \mapsto U_s$ is a morphism

$$S \longrightarrow \mathrm{Open}(X)$$

of ordered sets such that $\forall x \in X$

$$\{s \mid x \in U_s\} \subset S$$

is a pro-object in $\mathcal{C}$ (a left-filtering, left-hereditary subset). Thus we get a map

$$g : X \longrightarrow \mathrm{Pro\,Open}(X) = \left\{ \begin{matrix} \text{filters of open subsets} \\ \text{of } X \end{matrix} \right\}$$

associating to x the filter $\{U_s \mid x \in U_s\}$.

For any $U \in \mathrm{Open}(X)$, set

$$g^* U = \{x \mid \exists s \quad x \in U_s \subset U\}.$$

and note that $U_s \subset U \iff U_s \subset g^* U$ for any s.

~~[illegible]~~ Then g^*U is open, hence (i) g is semi-continuous as

$$g^*U = \{x \mid g(x) \leq U\}.$$

Also (ii) $g^* g^* U = g^* U$

$$\left(g^* g^* U = \bigcup_{U_s \subset g^* U} U_s = \bigcup_{U_s \subset U} U_s = g^* U \right).$$

(iii) $U \subset V \Rightarrow g^*U \subset g^*V$

$g^*(U \cap V) = g^*U \cap g^*V$

Proof: First evident as $x \in g^*U \Rightarrow \exists s \quad x \in U_s \subset U \Rightarrow \exists s \quad x \in U_s \subset V \Rightarrow x \in g^*V$. Hence $g^*(U \cap V) \subset g^*U \cap g^*V$ is clear. Given $x \in g^*U \cap g^*V \Rightarrow \exists s, s' \quad x \in U_s \subset U, x \in U_{s'} \subset V$; $\Rightarrow \exists t \leq s, s' \quad x \in U_t \Rightarrow \exists t \quad x \in U_t \subset U \cap V \Rightarrow x \in g^*(U \cap V)$. Thus $g^*(U \cap V) = g^*U \cap g^*V$.

(iv) the map g is determined by the operator g^* by

$$g(x) \leq U \iff x \in g^*U.$$

Conversely suppose given an operator $\varphi : \mathrm{Open}(X) \longrightarrow \mathrm{Open}(X)$ morphism of ordered sets such that ~~[illegible]~~

$\alpha)$ $\quad \varphi(U) \subset U$

$\quad\quad \varphi\varphi(U) = \varphi(U)$

$\beta)$ $\quad \varphi(U \cap V) = \varphi(U) \cap \varphi(V)$.

Set

$$g(x) = \{ U \mid x \in \varphi(U) \}.$$

Then by $\beta)$ $\quad x \in \varphi(U) \cap \varphi(V) = \varphi(U \cap V) \Rightarrow U \cap V \geq g(x)$ and $g(x) \leq U \subset V \Longrightarrow x \in \varphi(U) \subset \varphi(V) \Longrightarrow g(x) \leq V$. Thus $g(x)$ is a filter and

$$g(x) \leq U \Longleftrightarrow x \in \varphi(U)$$

showing that $\quad \varphi(U) = g^* U$.

Conclude that a map $g: X \longrightarrow \mathrm{Pro}(\mathrm{Open}\, X)$ satisfying the conditions (i) and (ii) on p. 35 is the same as a functor $\varphi: \mathrm{Open}(X) \longrightarrow \mathrm{Open}(X)$ satisfying $\alpha)$ and $\beta)$.

(Thus to a torsor $\coprod_s U_s$ we have associated an operator φ such that $\varphi(U) = U$ iff U is a union of a subset of the U_s. In some sense φ carries the torsor. When the torsor is fibrewise-representable then for each x $\exists$ least U_x $\ni$ $\varphi(U_x) = U_x$ and this assignment $x \mapsto U_x$ is a Lubkin covering L such that the torsor can be obtained by a functor $L \to S$.)

Actually it appears we are examining the wrong problem. Instead of fixing X and trying to classify S-torsors for various posets S, we should fix S and try to classify S-torsors over various spaces X. In this case the solution is trivial because a S-torsor is the "same" as a continuous map

$$X \xrightarrow{f} Pro(S) \overset{\text{defn}}{=} \{\text{subsets of } S \ \substack{\text{cofiltering +} \\ \text{cohereditary}}\}.$$

where $Pro(S)$ is endowed with the topology generated by the sets

$$U_s = \{z \in Pro(S) \mid z \leq s\}$$

as s runs over S. More precisely every torsor $P \hookrightarrow X \times S$ determines a map

$$\begin{aligned} X &\longrightarrow Pro(S) \\ x &\longmapsto \{s \mid (x,s) \in P\}. \end{aligned}$$

and the category of torsors is equivalent to the ordered set of semi-continuous maps.

Now the problem is to determine whether the category of C-torsors is "cofinally" equivalent to a category of maps from X to BC.

April 5, 1971.

I want to investigate the problem of when the representable $\mathcal{C}$-torsors are dense homotopically in the $\mathcal{C}$-torsors. If ~~[illegible]~~ we consider torsors over a small category X, then we are considering the functor

$$\underline{\mathrm{Hom}}(X, \mathcal{C}) \xrightarrow{\ \alpha\ } \underline{\mathrm{Hom}}(X, \mathrm{Pro}\,\mathcal{C}).$$

What we want to do is show this arrow ~~[illegible]~~ is strongly cofinal (p. 8).
~~[illegible]~~

Special case. Assume $\mathcal{C}$ closed under finite projective limits so that a covariant functor $\mathcal{C} \longrightarrow$ Sets is pro-representable iff it is left exact. Then given $g : X \longrightarrow \mathrm{Pro}\,\mathcal{C}$, the functor of $f \in \underline{\mathrm{Hom}}(X, \mathcal{C})$

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}(X, \mathrm{Pro}\,\mathcal{C})}\big(g, \alpha(f)\big) = \mathrm{Ker}\Big\{\prod_{X} \mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(g(X), f(X)) \rightrightarrows \prod_{X \to Y} \mathrm{Hom}(g(X), f(Y))\Big\}$$

is clearly left exact, hence is pro-representable. This means that α_+ of a representable functor is pro-representables, hence α is strongly cofinal.

~~Example: Take $\mathcal{C}$ to be a category in which every endomorphism is an isomorphism e.g. category of finite sets with surjective maps for morphisms. Take~~

April 6, 1971

Δ is the category of ordered sets of the form $[n] = \{0, .., n\}$ with $n \geq 0$. Then $\underline{\mathrm{Hom}}(\Delta^\circ, \text{Sets})$, the category of simplicial sets, contains $(\mathrm{Pro}\,\Delta^\circ)^\circ$ which we now wish to determine.

Thus let X be a pro-object in Δ°. Let $x \in X_n$ be a non-degenerate simplex, and let $\varphi, \psi : \Delta(p) \rightrightarrows \Delta(n)$ be two simplicial operators such that $\varphi^* x = \psi^* x$. By the equalizer condition $x = \rho^* y$ where $\rho : \Delta(n) \to \Delta(m)$ equalizes φ, ψ. As x is non-degenerate ρ is injective, hence $\varphi = \psi$. Thus distinct simplicial operators applied to a non-degenerate simplex have distinct results.

Let $x \in X_p$ and $y \in X_q$ be non-degenerate. By the first filtering condition $\exists$ $\varphi : \Delta(p) \to \Delta(n)$ $\psi : \Delta(q) \to \Delta(n)$ and $z \in X_n$ such that $\varphi^* z = x$, $\psi^* z = y$. By composing with a surjective map $\Delta(n) \to \Delta(n')$, we can suppose z non-degenerate. Note φ, ψ are injective as x, y are non-degenerate. (We can suppose that $\{0, .., n\} = \varphi\{0, .., p\} \cup \psi\{0, .., q\}$, i.e. that $\{\text{vertices } z\} = \{\text{vert } x\} \cup \{\text{vert } y\}$.)

Now take $p = q$ and suppose x and y have the same set of vertices. Then z has $(p+1)$ vertices and since it is non-degenerate its $n+1$ vertices are distinct, hence $n = p$ and $z = x = y$. ~~Thus X_p and ... by vertices~~

~~If $v, v' \in X_0$ call $v < v'$ if $\exists\, x \in X_1$ with vertices v, v' in that order.~~ ~~let $v < v'$ if ...~~

~~and $v \neq v'$. If $v < v'$ and $v' < v''$, then there is a 2-simplex~~

Thus each non-degenerate simplex is determined by the set of its vertices and the set of these vertices has a natural linear ordering. If x is a non-degenerate p-simplex with vertices $v_0, \ldots, v_p$ in that order we write $x = (v_0, \ldots, v_p)$. The same notation make sense for degenerate simplices.

Given $v, v' \in X_0$ say $v \leq v'$ if $(v, v') \in X_1$. If $v < v'$ (i.e. $\leq$ and $\neq$) and then $v' < v$ is impossible because to have both of the simplices (v, v') and (v', v) would contradict fact that a simplex is determined by the set of its vertices. Finally if $v < v'$ and $v' < v''$, then $v < v''$ as we know there is a simplex (v, v', v'').

Thus X_0 is a linearly ordered set and X_n is the set of sequences $(v_0, \ldots, v_n)$ with $v_0 \leq \cdots \leq v_n$.

<u>Conclude</u>: $(\mathrm{Pro}\, \Delta^\circ)^\circ = \mathrm{Ind}\, \Delta$ is the category of linearly ordered sets.

Conclude C-small $\not\Rightarrow$ Pro C small.

Fundamental question: Is the $\mathcal{C}$-torsor approach you have been using going to work? Intuitively you want (to know that) a functor $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ which is a homotopy equivalence **induces** a homotopy equivalence

$$\underline{\mathrm{Tors}}(X, \mathcal{C}_1) \longrightarrow \underline{\mathrm{Tors}}(X, \mathcal{C}_2)$$

for a nice space X. Up to now you know this only if $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is strongly **cofinal**. **Somehow** you still have to understand a homotopy equivalence like <u>subdivision</u>

$$\Delta / \mathrm{Nerv}(\mathcal{C}) \longrightarrow \mathcal{C} .$$

Columbia talk, ~~March~~ April 7, 1971

(I)

Recall about K-theory.

X finite cx.

$\cancel{K_i(X)}$

1) $$K^0(X) = [X, \mathbb{Z}\times BU] = \pi_0\, \underline{\mathrm{Hom}}\,(X, \mathbb{Z}\times BU)$$

$$K_i(X) = K^{-i}(X) = \pi_i\, \underline{\mathrm{Hom}}\,(X, \mathbb{Z}\times BU)$$

$i \geq 0$.

and Postnikov system of BU, or skeleton filtration of X, yield a spectral sequence

2) $$E_2^{pq} = H^p\left(X, \begin{cases} \mathbb{Z} & -q = 2i \\ 0 & -q = 2i+1 \end{cases}\right) \Longrightarrow K^{p+q}(X).$$

3)
$$\begin{array}{c|cccc}
q\uparrow & & & & \to p \\
\hline
 & H^0(X,\mathbb{Z}) & H^1 & H^2 & \cdots \\
 & 0 & 0 & 0 & \cdots \\
 & H^0 & H^1 & H^2 & \cdots \\
 & 0 & 0 & 0 & \cdots
\end{array}$$

4) Also $$c_i : K^0(X) \longrightarrow H^{2i}(X, \underset{K^{-2i}(pt)}{\mathbb{Z}})$$

i-th Chern class.

II

1) S scheme ℓ prime no invertible $/S$

2) $T_\ell = T_\ell(\mathbb{G}_m)$ Tate module

repn. of $\pi_1(S)$ on $\mathbb{Z}_\ell$

(i.e. a character $\pi_1(S) \longrightarrow \mathbb{Z}_\ell^*$)

constructed from $\{\mu_{\ell^\nu}\}$.

3) $T_\ell^{(n)} = T_\ell^{\otimes n}$.

4) If $E \longrightarrow S$ is a v.b. its Chern classes lie in

$$c_i(E) \in H^{2i}_{et}(S, T_\ell^{(i)})$$

5) R ring I have constructed a space $BGL(R)^+$ from $\{GL_n(R)\}$ $\ni$

$$K_i(R) = \pi_i\{BGL(R)^+\} \qquad i \geq 1$$

form a natural generalization of Bass $K_1(R)$ and Milnor $K_2(R)$.

6) Defn $BGL(R)^+_\ell$ = ℓ-adic completion of $BGL(R)^+$ (Artin–Mazur)

7) Then

(A) $K_i(R)$ fin. gen. ab. gps. $\Longrightarrow$ $\pi_i\left(BGL(R)^+_\ell\right) = K_i(R) \otimes \mathbb{Z}_\ell$

$i \geq 1$.

III)

1) Conjecture: $\exists$ spectral sequence (like (Atyiah Hirzebruch)

$$E_2^{pq} = H^p_{et}\left(\mathrm{Spec}\, R, \begin{cases} T_\ell^{(i)} & -q = 2i \\ 0 & -q = 2i+1 \end{cases}\right)$$

$$\overset{?}{\Longrightarrow} \quad \begin{cases} & p+q \geq 0 \\ \pi_{-(p+q)}\left(BGL(R)^+_\ell\right) & p+q < 0 \end{cases}$$

2) Picture:

$$\begin{array}{r|llll}
q & & & & p \\
q=-1 & 0 & 0 & 0 & \\
q=-2 & H^0(T^{(1)}) & H^1(T^{(1)}) & H^2(T^{(1)}) & \\
 & 0 & 0 & 0 & \\
q=-4 & H^0(T^{(2)}) & H^1(T^{(2)}) & H^2(T^{(2)}) & \\
 & 0 & 0 & 0 & \\
 & \vdots & & &
\end{array}$$

3) For ~~...~~ rings one first looks at one has

$$H^0(T^{(n)}) = 0 \qquad n \neq 0$$

(B) $$H^p(T^{(n)}) = 0 \qquad p > 2 \qquad \underline{\ell \text{ odd}}$$

so spectral sequence degenerates.

yielding formulas

$$K_{2i-2}(R)\otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^2(\operatorname{Spec} R, T_\ell^{(i)}) \qquad i\geq 2$$

$$K_{2i-1}(R)\otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^1(\operatorname{Spec} R, T_\ell^{(i)}) \qquad i\geq 1$$

assuming (hyp. A) B & conjecture

IV) Examples:

1) finite field (formulas OKAY).

$$K_{ev}(\mathbb{F}_q) = 0$$

$$K_{2i-1}(\mathbb{F}_q)_\ell \simeq \mathbb{Z}_\ell/(q^i-1)\mathbb{Z}_\ell$$

$$= T_\ell^{(i)}/\operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$$

$$\simeq H^1(\mathbb{F}_q, T_\ell^{(i)})$$

1) finite field. k order q

$$H^t(Sp\,k, T_\ell^{(i)}) = H^t(\hat{\mathbb{Z}}, T_\ell^{(i)})$$

$$\varphi \qquad q^{-i}$$

$= 0 \qquad t \geq 2$

$= 0 \qquad t=0,\ i \neq 0$

$\doteq \mathbb{Z}_\ell/(q^i-1)\mathbb{Z}_\ell. \qquad t=1,\ i\neq 0.$

$K_{2i}(k) = 0 \qquad i > 0$

$K_{2i-1}(k) \simeq \mathbb{Z}/(q^i-1)\mathbb{Z} \qquad$ "

2) affine curve over a finite field k.

to simplify suppose we have a complete non-singular curve C over a finite field with a

3) rational point ∞ and let $\boxed{R = \Gamma(C - \{\infty\}, \mathcal{O}_C)}$ be the coordinate ring of the affine curve.

4)
$$\begin{tikzcd} R \arrow[r] & \bar{R} \\ k \arrow[u] \arrow[r, "\mathbb{Z}"'] & \bar{k} \arrow[u] \end{tikzcd}$$

abbreviate $H^q_{et}(\mathrm{Spec}\, R, T_\ell^{(i)})$ to $H^q(R, T^{(i)})$

5) $$E_2^{pq} = H^p(\hat{\mathbb{Z}}, H^q(\bar{R}, T^{(i)})) \Longrightarrow H^{p+q}(R, T^{(i)})$$

6) $$0 \to H^{q-1}(\bar{R}, T^{(i)})_{\hat{\mathbb{Z}}} \to H^q(R, T^{(i)}) \to H^q(\bar{R}, T^{(i)})^{\hat{\mathbb{Z}}} \to 0$$

One knows

7) $$H^q(\bar{R}, T^{(i)}) = 0 \qquad q \geq 2$$
$$H^0(\bar{R}, T^{(i)}) = T^{(i)}$$
$$H^1(\bar{R}, T^{(i)}) = \mathrm{Homcont}(T_\ell(J), T^{(i)})$$

where $T_\ell(J)$ is the Tate module of the Jacobian of the curve $\overline{C} = \overline{k} \times_k C$.

$$0 \to \boxed{T^{(i)}_{\hat{\mathbb{Z}}} \longrightarrow H^1(R, T^{(i)})} \longrightarrow \operatorname{Homcont}(T_\ell(J), T^{(i)})^{\hat{\mathbb{Z}}} \to 0$$

$$\operatorname{Homcont}(T_\ell(J), T^{(i)})^{\hat{\mathbb{Z}}} = 0$$

as Frob. on $T_\ell(J)$ has eigenvalues abs. value $\sqrt{q} \neq q^{-i}$

$$\boxed{\operatorname{Homcont}(T_\ell(J), T^{(i)})_{\hat{\mathbb{Z}}} \xrightarrow{\sim} H^2(R, T^{(i)}).}$$

~~Thus~~

$$\cong \leftarrow \text{(some computation I don't recall)}$$

$$\left(J_{\ell^\infty} \otimes T^{(i-1)}\right)^{\hat{\mathbb{Z}}}$$

~~has order~~ $\det\left(1 - q^i \,\mathrm{Frob.}\right)$

$$= (1-q^i)(1-q^{i+1})\,\zeta(-i).$$

~~[illegible]~~ We rewrite this in terms of $K_*(C)$ assuming these have been defined & $\exists$ exact sequence

$$\longrightarrow K_i(C) \longrightarrow K_i(R) \xrightarrow{\partial} K_{i-1}(k_\infty) \longrightarrow K_{i-1}(C).$$

and one obtains

$$K_{2i-1}(C) = \underset{\left(\substack{\text{rest. to a}\\ \text{point of } C}\right)}{K_{2i-1}(k)} \oplus \underset{(\chi)}{K_{2i-1}(k)}$$

$$K_{2i}(C) \simeq \cancel{\text{subgroup of } J(\bar{k})} \; J(i)^{Gal(\bar{k}/k)}$$

$$= \cancel{\text{subgroup of } J(\bar{k})} \; \{\lambda \in J(\bar{k}) \mid (1-q^i F)\lambda = 0\}.$$

mod p-torsion $\qquad i>0.$

$$\text{order}\left(K_{2i}(C)/p\text{-torsion}\right) = \left|(1-q^i)(1-q^{i+1})\,\zeta_C(-i)\right|$$

V) Integers in number fields

$[k : \mathbb{Q}] < \infty$

$R = \overline{\mathbb{Z}[l^{-1}]}$ in k

~~If $\exists$ extension of Bass–Tate sequence~~

$$\cancel{\to K_i(R) \to K_i(k) \to \coprod_{\mathfrak{p} \nmid l} K_{i-1}(R/\mathfrak{p}) \to}$$

~~One expects that~~

$$\cancel{K_i(R) \otimes \mathbb{Z}_l \cong \operatorname{Ker}\left\{ K_i(k)^{\otimes \mathbb{Z}_l} \xrightarrow{\text{tame}} \coprod_{\mathfrak{p}} K_{i-1}(R/\mathfrak{p})^{\otimes \mathbb{Z}_l} \right\}}$$

$$\begin{array}{ccc} k & \longrightarrow & K \\ \cup & & \uparrow \\ R & \longrightarrow & R[\mu_{l^\infty}] = \overline{R} \end{array}$$

assume $\mu_l \subset k$, l odd.

Galois extension $\subset 1 + l\mathbb{Z}_l$

Γ - extension

$$X = H_1(\bar{R}) = \text{Galois group of max. abelian } l\text{-extension of } K \text{ unramified outside of } l.$$

$$H^1(\bar{R}, T_l^{(i)}) = \mathrm{Homcont}(X, T_l^{(i)})$$

$$0 \to T^{(i)}{}_\Gamma \longrightarrow H^1(R, T_l^{(i)}) \longrightarrow \mathrm{Homcont}(X, T^{(i)})^\Gamma \to 0$$

$$\mathrm{Homcont}(X, T^{(i)})_\Gamma \cong H^2(R, T_l^{(i)})$$

$$0 \to T^{(i)}{}_\Gamma \longrightarrow K_{2i-1}(R)^{\otimes \mathbb{Z}_l} \longrightarrow \mathrm{Homcont}(X, T^{(i)})^\Gamma \to 0$$

$$\mathrm{Homcont}(X, T^{(i)})_\Gamma \cong K_{2i-2}(R) \otimes \mathbb{Z}_l$$

tested this for $i = 1$
seems to agree with Tate for $i = 2$
assuming $K_2(R) \hookrightarrow K_2(k)$.

finally Borel has shown that rank $K_{odd}(R) = r_2$

$r_2 = [k : \mathbb{Q}]/2$. while Iwasawa has shown $\mathrm{Homcont}(X, T^{(i)})^\Gamma \supset \mathbb{Z}_l^{r_2}$

ℓ prime number

S scheme $\ni$ ℓ unit / S

S_{et} etale homotopy type of S

$GL_{n,S}$ group scheme / S

~~$(BGL_{n,S})_{et}$ etale~~

~~$BGL_{n,S}$ classifying topos~~ (etale top.)

$(BGL_{n,S})_{et}$ homotopy type of the classifying topos of $GL_{n,S}$.

~~canonical map~~

$(BGL_S)_{et}$ limit as $n \to \infty$

$$\begin{array}{c} (BGL_S)_{et} \\ \downarrow p \\ S_{et} \end{array}$$

canonical map.

$$S[\mu_{\ell^\infty}] = S \times_{\mathrm{Spec}\, \mathbb{Z}[\frac{1}{\ell}]} \mathrm{Spec}\, \mathbb{Z}[\mu_{\ell^\infty}, \ell^{-1}]$$

$S[\mu_{\ell^\infty}] \longrightarrow S$ Galois extension with group $\mathbb{Z}_\ell^*$

BU_ℓ ℓ adic completion of BU

Known (essentially) that

$$\boxed{(BGL_S)_{et,\ell} \simeq S[\mu_{\ell^\infty}] \times^{\mathbb{Z}_\ell^*} (BU_\ell)}$$

where $\mathbb{Z}_\ell^*$ acts on BU_ℓ via Adams operation

$$\begin{array}{ccc} (BGL_S)_{et,\ell} & \simeq & S[\mu_{\ell^\infty}] \times^{\mathbb{Z}_\ell^*} (BU_\ell) \\ \downarrow & & \\ S_{et,\ell} & & \end{array}$$

$\exists$ canonical map

$$B\,\Gamma(S, GL_n) \longrightarrow \underline{\text{Sections}} \left\{ \begin{array}{c} (BGL_{n,S})_{et} \\ \downarrow \\ S_{et} \end{array} \right\}$$

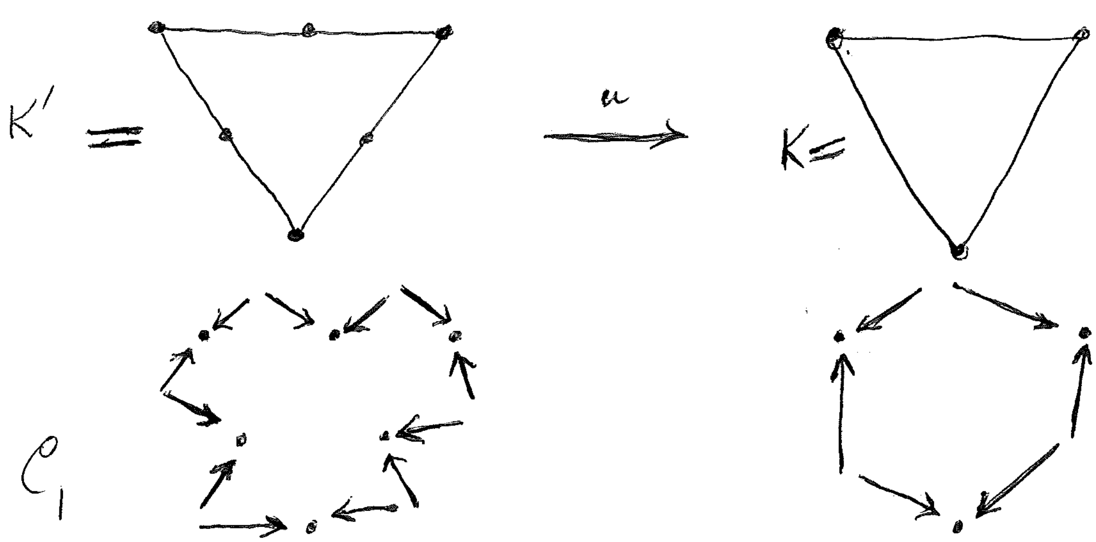
and I have constructed $\ulcorner$ for S affine

April 11, 1971:

False Hope: We know that when a functor $u: \mathcal{C}_1 \to \mathcal{C}_2$ is strongly cofinal, then there are adjoint functors

$$\underline{\text{Tors}}(X, \mathcal{C}_1) \rightleftarrows \underline{\text{Tors}}(X, \mathcal{C}_2).$$

The hope was that by inverting the strongly cofinal maps in Cat one obtains the homotopy category. But consider the case where C_2 is the category of simplices of a finite simplicial complex K and C_1 is the category of simplices of a subdivision K'. Then $u: C_1 \to C_2$ is a homotopy equivalence but it is not possible to connect $\mathcal{C}_1$ to C_2 by a chain of strongly cofinal functors. Indeed it would then be possible to map $\text{Pro}\, C_2 = C_2$ to $\text{Pro}\, C_1 = C_1$ so that the map is homotopy inverse to u. But this isn't possible:



We are interested in functors $u : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ such that

$$(*) \qquad H^q(\mathcal{C}_2, G) \xrightarrow{\sim} H^q(\mathcal{C}_1, u^* G)$$

for all $G : (\underline{\underline{\pi}}\mathcal{C}_2)^{\circ} \longrightarrow$ (Sets $q=0$, resp. Groups $q \leq 1$, resp. Ab. $\forall q$), ~~[illegible]~~ as such a functor by definition is a homotopy equivalence. Here are two criteria guaranteeing this, in fact more generally guaranteeing that $(*)$ is an isom. for G defined on $\mathcal{C}_2^{\circ}$.

1) $u_!$ exact (commutes with fin. proj. limits): Then $u_! e_1 = e_2$ and u^* preserves injectives so

$$\begin{aligned} H^q(\mathcal{C}_2, G) &= H^q \, \mathrm{Hom}_{\mathcal{C}_2}(\overset{u_! e_1 \,\Vert}{e_2}, I^{\bullet}) \\ &= H^q \, \mathrm{Hom}_{\mathcal{C}_1}(e_1, u^* I^{\bullet}) \\ &= H^q(\mathcal{C}_1, u^* G) \end{aligned}$$

2) $$R^p u_* \cdot u^* = \begin{cases} id & p = 0 \\ 0 & p > 0 \end{cases}$$

Then Hochschild-Serre for u^* yields isom

$$H^p(\mathcal{C}_2, G) \simeq H^p(\mathcal{C}_2, u_* u^* G) \cong H^p(\mathcal{C}_1, u^* G)$$

Review old approach.

acyclic maps

Prop: 1-1 corresp between $f: X \to ?$ acyclic and perf normal subgroups of $\pi_1 X$.

$BGL(A) \longrightarrow BGL(A)^+$! acyclic map $\ni$ $\pi_1 BGL(A)^+ = GL(A)/E(A)$.

universal property: $X \in \{\text{finite complexes}\}$, then

$$\begin{array}{ccc} [X, BGL(A)] & \longrightarrow & [X, BGL(A)^+] \\ \| & & \\ \pi_0 \underline{\mathrm{Rep}}(\underline{\pi}_1 X, A)/\text{stable iso.} & & \end{array}$$

universal to a functor of the form $[X, B]$, $\underline{\pi}_1 B$ no perfect subgroups.

$$\boxed{\begin{array}{l} R(X;A) = \text{Grothendieck group of } A\text{-bundles over } X. \\ R(X;A) \longrightarrow [X, K_0 A \times BGL(A)^+] \quad \text{has a certain univ. prop.} \\ \gamma\text{-filtration, } \lambda\text{-operations, weights.} \end{array}}$$

important to define weights, λ operations

Thm. loc. nilp. of γ-filt., ~~[illegible]~~ existence weights.

conjectures on stability.

April 12, 1971

Exact sequences in algebraic K-theory problem:

First recall what I have to show in order to prove Mather's theorem. I recall that for $a \in \mathbb{R}$ we let Q_a be the topological groupoid with objects $R_a = \{x < a\}$ and with morphisms from x to x' a germ of diffeo from $[x, a]$ to $[x', a]$ which is the identity near a. The set of morphisms Q_a is made into an etale space over R_a via the source map. There is an obvious functor

$$Q_a \times G_{ab} \longrightarrow Q_b$$

where G_{ab} denotes the category with one object $[a,b]$ and with autos. the diffeos with support in $[a,b]$.

Now denote by $W_p(Q_a) = Q_a \times_{R_a} \cdots \times_{R_a} Q_a$ p-times the space in degree p of the nerve of Q_a. Then we have a bisimplicial space with an augmentation to $B\Pi$

$$
\begin{array}{ccccc}
\coprod\limits_{a_0 \le a_1 \le a_2} W_2(Q_{a_0} \times G_{a_0 a_1} \times G_{a_1 a_2}) & \Rrightarrow & \coprod\limits_{a_0 \le a_1 \le a_2} Q_{a_0} \times G_{a_0 a_1} \times G_{a_1 a_2} & \rightrightarrows & \coprod\limits_{a_0 \le a_1 \le a_2} R_{a_0} \\
\downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow \\
\coprod\limits_{a_0 \le a_1} W_2(Q_{a_0} \times G_{a_0 a_1}) & \Rrightarrow & \coprod\limits_{a_0 \le a_1} Q_{a_0} \times G_{a_0 a_1} & \rightrightarrows & \coprod\limits_{a_0 \le a_1} R_{a_0} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\coprod\limits_{a_0} W_2(Q_{a_0}) & \Rrightarrow & \coprod\limits_{a_0} Q_{a_0} & \rightrightarrows & \coprod\limits_{a_0} R_{a_0} \\
\vdots & & \vdots & & \vdots \\
\Gamma \times_R \Gamma & = & \Gamma & \text{——} & R
\end{array}
$$

Now what you have to prove is

(i) (realization of bisimplicial space) $\longrightarrow B\Gamma$ is a weq because of cofinality

(ii) (real. of bisimp space) $\longrightarrow$ BBG is a weq because BQ_a is contractible for each a.

It might be simpler to realize horizontally first

$$
\coprod_{a_0 \le a_1 \le a_2} BQ_{a_0} \times BG_{a_0 a_1} \times BG_{a_1 a_2} \Rrightarrow \coprod_{a_0 \le a_1} BQ_{a_0} \times BG_{a_0 a_1} \rightrightarrows \coprod_{a_0} BQ_{a_0}
$$

$$
\coprod_{a_0} BQ_{a_0} \dashrightarrow B\Gamma
$$

The way to say things is this: We have a simplicial object in Cat top

$$\cdots \coprod_{a_0 \leq a_1} Q_{a_0} \times G_{a_0 a_1} \rightrightarrows \coprod_{a_0} Q_{a_0} \dashrightarrow \Gamma$$

with an augmentation to Γ. Then we want to know when this map is a "homotopy equivalence."

Next consider the situation of the Dedekind domain A with quotient field K. Here the analogue of Γ will be a simplicial category or more intrinsically a fibred category over Δ^0. Thus given $n \geq 0$ consider f.g. K-modules E endowed with a filtration of length n:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

and let $\mathcal{C}_n(K)$ be the category of these. Now given a monotone map $\varphi : [n] \longrightarrow [m]$ we define a functor

$$\mathcal{C}_m(K) \longrightarrow \mathcal{C}_n(K)$$

up to canonical isomorphism by

$$\varphi(0 \subset E_1 \cdots \subset E_m) = 0 \subset E_{\varphi(1)}/E_{\varphi(0)} \subset \cdots \subset E_{\varphi(n)}/E_{\varphi(0)}.$$

This gives us a fibred category over Δ

<u>Definition:</u> A category $\mathcal{C}$ with an associative unitary operation is a fibred category Δ° such that for each n the functor

$$\mathcal{C}_n \longrightarrow \mathcal{C}_1^{\,n}$$

given by the faces $(i-1,i) \subset (0,\dots,n)$ is an equivalence of categories. (for $n=0$, one means that $\mathcal{C}_n$ is equivalent to the final category.)

The analogue of the simplicial object $Q \times G \cdots$ etc. is the fibred category over $\Delta \times \Delta$ such that over $[p]\times[q]$ we have the category of f.t. A-modules E with two filtrations

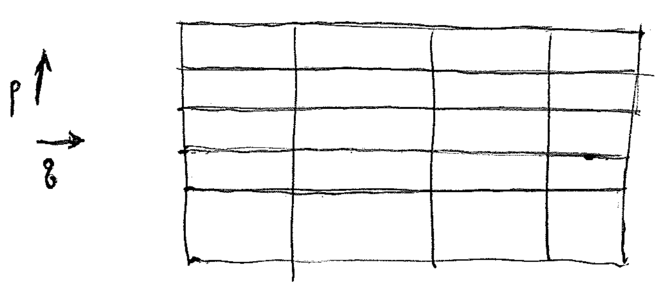
$$0 = F'_{-1}E \subset F'_0 E \subset \cdots \subset F'_p E = E$$

$$0 = F''_0 E \subset \cdots \cdots \subset F''_q E = E$$

such that

$$F'_i E / F'_{i-1} E \qquad 1 \leq i \leq p$$

is of finite length. Picture:



Next we have a cartesian functor

$$\begin{tikzcd} Q \arrow[r, "p"] & \Gamma(K) \end{tikzcd}$$

$$\begin{tikzcd} \Delta \times \Delta \arrow[r, "pr_2"'] & \Delta \end{tikzcd}$$

which sends $\{E, F_i', F_j'' \quad -1 \le i \le p,\ 0 \le j \le q\}$ into $\{E \otimes_A K,\ F_j'' E \otimes_A K,\ 0 \le j \le q\}$. It will be necessary to prove that p induces a homotopy equivalence on the realizations.

Next we have a ~~[illegible]~~ cartesian functor

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d] & B\Theta \arrow[dl] \\ \Delta \times \Delta & \end{tikzcd}$$

coming from killing F_0'. The fibre of f appears to be $\Gamma(A)$. This might not be too hard: if one pulls back $P\Theta$ by f, the result should contract to $\Gamma(A)$.

The above exhausts the geometry. We have now to prove that

(i) p induces a heq: $|Q| \longrightarrow |\Gamma(K)|$

(ii) $|f| : |Q| \longrightarrow |B\Theta|$ has fibre $|\Gamma(A)|$

(iii) $|B\Theta| = B|\Gamma(\text{fin. length})|$

(ii) and (iii) involve probably the same thing.

April 14, 1971

Fibred topos: $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ is intuitively a covariant functor from $\mathcal{B}$ to topoi. It is a functor both fibred and cofibred with topoi for fibres so that the base and cobase change associated to a morphism in $\mathcal{B}$ constitute a morphism of topoi. Thus given $f: b \longrightarrow b'$ in $\mathcal{B}$ we have a morphism of topoi

$$\tilde{f} : \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$$

constituted by the pair

$$\mathcal{E}_b \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{E}_{b'}$$

Think of $\mathcal{B}$ as the category of spaces and $\mathcal{E}_b$ as the sheaves on b.

Make the Deligne construction, or what is the same, the gross topos construction. One considers contravariant functors on $\mathcal{E}$ which are sheaves on each fibre, hence for each b give an object F_b of $\mathcal{E}_b$ and for $f: b \to b'$ a map $f^* F_{b'} \to F_b$ satisfying compatibility.

Example: Start with a pseudo-functor $\mathcal{B} \longrightarrow Cat$, $b \longmapsto \mathcal{C}_b$ and form the associated cofibred category $\pi: \mathcal{C} \to \mathcal{B}$. Then $b \mapsto \mathcal{C}_b^{\wedge}$ (contra) is a pseudo-functor $\mathcal{B} \longrightarrow$ Topoi. A gross sheaf then is a functor $F_b : \mathcal{C}_b^{\circ} \longrightarrow$ Sets for each b plus maps for each $\varphi: b \to b'$

$$F_{b'} \circ \tilde{\varphi} \longrightarrow F_b .$$

Now one defines morphisms in $\mathcal{C}$ by

$$\mathrm{Hom}_{\varphi}(x, y) = \mathrm{Hom}_{\mathcal{C}_{b'}}(\tilde{\varphi} x, y) \qquad x \in \mathcal{C}_b, y \in \mathcal{C}_{b'}$$

hence one has a map

$$F_{b'}(y) \longrightarrow F_{b'}(\tilde{\varphi} x) \longrightarrow F_b(x)$$

associated to a map $x \to y$ in $\mathcal{C}$ over φ. Thus a gross-sheaf is the same as a functor $F: \mathcal{C}^{\circ} \longrightarrow$ Sets. So we see that the gross topos for a cofibred category $\mathcal{C} \to \mathcal{B}$ is the topos $\mathcal{C}^{\wedge}$.

Possible version of Grothendieck's theory of glueing topoi. Start with a "base" site $\mathcal{S}$ and a stack $\mathcal{E} \longrightarrow \mathcal{S}$ such that each $\mathcal{E}_s$ is a topos and each base change is a morphism of topoi. One then tries to enlarge $\mathcal{E}$ to $\tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{S}}$ where $\tilde{\mathcal{S}}$ is the topos of sheaves on the site $\mathcal{S}$. The intuitive picture is the following: think of $\mathcal{S}$ as the open sets of a space S which are small relative to some sieve and $\mathcal{E}_U$, $U \in \mathcal{S}$ as the topos of sheaves on some space X_U. Then one is trying to construct the space $X \longrightarrow S$. In this example one obtains the sheaves on X as the category of cartesian sections of $\mathcal{E} \to \mathcal{S}$.

However suppose $\mathcal{S} = \Delta^\circ$ and that $\mathcal{C} \to \Delta^\circ$ is the cofibred category belonging to a simplicial set X so $\mathcal{C} = (\Delta/X)^\circ$. Then $\mathcal{E}_n = \text{Sets}/X_n$ and $\mathcal{E}$ consists of families $F_n \to X_n$ with maps $\tilde{\varphi}^* F_m \to F_n$ for $\tilde{\varphi} : X_n \longrightarrow X_m$. Thus $\mathcal{E}_n = \text{Sets}/X_n$ and given $\tilde{\varphi} : X_n \to X_m$ we have base change $\tilde{\varphi}^* F_m = X_n \times_{X_m} F_m$ for morphism of topos. A cartesian section means that $\tilde{\varphi}^* F_m \xrightarrow{\sim} F_n$ for all arrows $\tilde{\varphi}$ in Δ°. Thus the category of cartesian sections is equivalent to the sheaves on the fundamental groupoid of X.

General situation. Given a site $\mathcal{S}$, a stack $\pi : \mathcal{E} \to \mathcal{S}$ such that fibres $\mathcal{E}_s$ are topoi and base changes are inverse images functors for maps of topoi, let $S \in \text{Ob}\, \tilde{\mathcal{S}}$. Then we can consider the categories $\mathcal{S}/S$ consisting of $U \to S$ with $U \in \text{Ob}\, \mathcal{S}$ and $\mathcal{E}/S$ consisting of pairs $(E, \pi E \to S)$ so that $\mathcal{E}/S \longrightarrow \mathcal{S}/S$ is a fibred category. Set

$$\tilde{\mathcal{E}}_S = \varprojlim_{\mathcal{S}/S} \mathcal{E}/S$$

the category of cartesian sections. Intuitively an object of this is a compatible family E_α of sheaves over U for each $U \xrightarrow{\alpha} S$ and each $U \in \mathcal{S}$. Its clear that if S is representable by U, then $\tilde{\mathcal{E}}_S$ is equivalent to $\mathcal{E}_U$. One needs to show that $\tilde{\mathcal{E}}_S$ is a topos, but this follows by Giraud's criterion since the base change functors commute with arb. $\varinjlim$'s and finite $\varprojlim$'s.

Examples:

1.) Let a group G act on a space X. Take $\mathcal{S}$ to be the category (with one object s) defined by G and $\mathcal{E}_s$ to be the topos of sheaves on X with map $g : \mathcal{E}_s \to \mathcal{E}_s$ to be the morphism of topoi associated to $g : X \to X$. Then the category of cartesian sections consists of sheaves F on X with isomorphisms $g^*F \xrightarrow{\sim} F$ for each $g \in G$, hence this is the category of G-sheaves.

2.) Let $i \mapsto X_i$ be an inductive system of spaces indexed by a directed set I and set $X = \varinjlim X_i$. Then there is an evident functor

$$\mathrm{Top}(X) \longrightarrow \varprojlim_{I} \mathrm{Top}(X_i)$$

associating to a sheaf F on X its inverse images $\{F|X_i\}$. I claim this is an equivalence of categories. Indeed to show fully faithful take an etale space $F \to X$ and let $F_i = X_i \times_X F$; it suffices to show that the canonical map

$$\varinjlim F_i \to F$$

is a homeomorphism. It is bijective and given a subset A of F such that $A_i = X_i \times_X A$ is open in F_i for each i, we want to know A is open in F. But A open $\Leftrightarrow$ $s^{-1}A$ open for all (s, U), $s \in \Gamma(U, F)$; $X_i \times_X s^{-1}A = X_i \times_X (U \times_F A) = U_i \times_{F_i} A_i$ open in U_i as A_i is open in F_i. This proves the functor is fully faithful. To prove essentially surjective, suppose given a compatible family of sheaves $F_i \to X_i$ and set $F = \varinjlim_i F_i$. Then

$$X_j \times_X F = \varinjlim_i X_j \times_{X_i} F_i = F_j$$

where we have used that I is directed. Define a section $s: U \to F$ to be good if it is the limit of compatible sections $s_i : U_i \to F_i$. Then where two sections coincide is an open subset of X so one gets an etale space topology on F which, clearly by above, coincides with the inductive limit topology of the F_i.

<u>Critical questions</u>: Given $\mathcal{E} \to \mathcal{S}$ situation we think of $\tilde{\mathcal{E}}_e$ as the inductive limit of the topoi $\mathcal{E}_s$. Then when is this limit topos the "correct" limit from the viewpoint of cohomology?

April 24, 1971: K-theory

Let $\mathcal{A}$ be an additive category. Given a monotone map $\varphi : m \longrightarrow n$ we have a functor

$$\varphi^* : \mathcal{A}^n \longrightarrow \mathcal{A}^m$$

$$\varphi^*(X_1, \ldots, X_n)_i = X_{\varphi(i-1)+1} \oplus \cdots \oplus X_{\varphi(i)} .$$

This depends on the choice of $\oplus$ in $\mathcal{A}$, so ~~more intrinsic description~~ φ is only defined up to canonical isomorphism. The way to do this correctly is to note that the direct sum in $\mathcal{A}$ is defined by a universal property

$$\mathrm{Hom}(X_1 \oplus X_2, Y) \cong \mathrm{Hom}(X_1, Y) \times \mathrm{Hom}(X_2, Y)$$

and hence

$$\mathrm{Hom}\big(\varphi^*(X_1, \ldots, X_n), (Y_1, \ldots, Y_m)\big) \simeq \prod_{j=1}^{n} \mathrm{Hom}(X_j, Y_{\psi(j)})$$

where ψ is the adjoint monotone function, i.e.

$$\psi(j) = i \iff \varphi(i-1) < j \leq \varphi(i)$$

or

$$\mathrm{Hom}_{[n]}(j, \varphi(i)) = \mathrm{Hom}_{[m]}(\psi(j), i)$$

$$\simeq \prod_{i=1}^{m} \prod_{\varphi(i-1) < j \leq \varphi(i)} \mathrm{Hom}(X_j, Y_i)$$

This product is intrinsic, hence we obtain a natural cofibred category over Δ^0 whose fibre over n is $\mathcal{A}^n$. (The above construction works for any category with finite direct sums. For a category with finite **products** a similar construction yields a fibred category over Δ.)

Now when forming K-theory one ignores the maps in $\mathcal{A}$ except the isomorphisms. Then $n \longmapsto \mathcal{A}^n$ is a cofibred category over Δ^0 with groupoids for fibres. So we obtain a cofibred category $\mathcal{C} \longrightarrow \Delta^0$ such that

$$\mathcal{C}_n \xrightarrow{\simeq} (\mathcal{C}_1)^n$$

where $\mathcal{C}_1$ is the groupoid of objects and isomorphisms in $\mathcal{A}$. The space belonging to the category $\mathcal{C}$ will be the first space in Segal's spectrum – denoted $\mathrm{Seg}^{(1)}\mathcal{A}$.

$\mathrm{Seg}^{(\nu)}\mathcal{A}$ will be the space associated to the cofibred category $\mathcal{C}^{(\nu)} \longrightarrow (\Delta^0)^\nu$ whose fibres over $(q_1, \ldots, q_\nu)$ are families

$$\{ X_{i_1 \cdots i_\nu} \mid 1 \le i_j \le q_j \quad 1 \le j \le \nu \}$$

and isomorphisms. The category $\mathcal{C}^{(\nu)}$ has a distinguished object, namely, the empty family over $(0, 0, \ldots, 0)$. Hence $\mathrm{Seg}^{(\nu)}\mathcal{A}$ has a basepoint.

I now want to understand why

$$\Omega\, \mathrm{Seg}^{(\nu)} \mathcal{A} \sim \mathrm{Seg}^{(\nu-1)} \mathcal{A}$$

for $\nu \geq 2$. Segal proves this using quasi-fibration theory. Now the point is to proceed inductively. Thus $\mathrm{Seg}^{(\nu)} \mathcal{A}$ is a category over $(\Delta^{\circ})^{\nu}$ whose fibres have a composition operation and $\mathrm{Seg}^{(\nu+1)} \mathcal{A}$ is the classifying category for this.

General setup: Given $\underbrace{\mathcal{C} \longrightarrow S}_{\text{a cofibred cat}}$ and assume that there is a composition operation on the fibres.+ This enables me to form a cartesian functor

$$(*) \qquad \mathcal{B} \longrightarrow \Delta^{\circ} \times S$$

of cofibred categories over Δ° whose fibre over q is

$$\mathcal{C}/_S^{q} \longrightarrow S$$

Assume that $\exists$ [redacted] an object in S [redacted] giving rise to a homotopy equivalence $pt \longrightarrow S$. Then $\mathcal{B}$ has a basepoint and it seems reasonable to expect that when $\mathcal{C}$ is connected then $|\mathcal{C}|$ and $\Omega|\mathcal{B}|$ are homotopy equivalent.

+ compatible with cobase change operation

(*) should be cofibred

General setup: Start with $C \longrightarrow S$ and assume this cofibred, the fibres have a composition operation, and the operation is preserved by cobase change. Then I can form a cofibred category $\mathcal{B} \longrightarrow \Delta^{\circ}$ whose fibre over q is $(C/S)^{q}$. In fact $\mathcal{B}$ is $\overset{\text{already}}{}$ cofibred over $\Delta^{\circ} \times S$. ("already" put in because composition of two ~~[illegible]~~ fibrant functors is fibrant.) Because $\mathcal{B}$ is cofibred over Δ° I get the spectral sequence

$$E_2^{pq} = \check{H}^{p}\big(\nu \mapsto R^{q} \varprojlim_{(C/S)^{\nu}} F\big) \Longrightarrow R^{p+q} \varprojlim_{\mathcal{B}} F \,.$$

$$E_2^{pq} = \check{H}^{p}\big(\nu \longmapsto H^{q}((C/S)^{\nu}, F)\big) \Longrightarrow H^{p+q}(\mathcal{B}, F).$$

Now I need some kind of Kunneth formula for evaluating the cohomology of $(C/S)^{\nu}$. Note that if

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "f'"] \arrow[d, "p'"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "f"'] & \mathcal{B} \end{tikzcd}$$

is cartesian in Cat and p is cofibrant, then so is p'. Thus we have the base change theorem

$$
\begin{aligned}
f^* R^q p_*(F)(B') &= R^q p_*(F)(fB') \\
&= H^q(\mathcal{E}_{fB'}, F/\mathcal{E}_{fB'}) \\
&= H^q(\mathcal{E}'_{B'}, f'^* F/\mathcal{E}'_{B'}) \\
&= R^q p'_*(f'^* F)(B').
\end{aligned}
$$

~~The corresponding situation in homology yields a Kunneth formula. Thus if $f: \mathcal{C}_1 \to S$ and $g: \mathcal{C}_2 \to S$ are cofibred we consider the Leray spectral sequence in homology for the projection $\mathcal{C}_1 \times_S \mathcal{C}_2 \to \mathcal{C}_1$:~~

~~$E^2_{pq} = H_p(\mathcal{C}_1, f^* H_q$~~

Now consider the cartesian square

$$
\begin{tikzcd}
\mathcal{C}_1 \times_S \mathcal{C}_2 \arrow[r, "f'"] \arrow[d, "g'"] & \mathcal{C}_2 \arrow[d, "g"] \\
\mathcal{C}_1 \arrow[r, "f"] & S
\end{tikzcd}
$$

where f and g are cofibrant. In order to obtain a Kuaneth formula without properness restrictions, we must work with homology. Recall from page 2: using covariant functors on our categories to sets we have

$$
(f_! F)(s) = \varinjlim_{fx \to s} F(x) = \varinjlim_{\mathcal{C}_{1,s}} F/\mathcal{C}_{1,s}.
$$

Thus

$$f^* \circ \underline{\underline{L}} g_! = \underline{\underline{L}} g'_! \circ f'^*$$

and

$$\underline{\underline{L}} f_! \left(\underline{\underline{L}} g'_! (\Lambda) \right) = \underline{\underline{L}} f_! \left(\underline{\underline{L}} g'_! (f'^* \Lambda) \right)$$

$$= \underline{\underline{L}} f_! \left(f^* \left(\underline{\underline{L}} g_! (\Lambda) \right) \right)$$

But

$$\left(L_q f_! (f^* F) \right)(s) = L_q \varinjlim_{\mathcal{C}_{1,s}} (f^* F)$$

$$= \left(L_q \varinjlim_{\mathcal{C}_{1,s}} \Lambda \right) \otimes F(s) \qquad \text{if } F \text{ free.}$$

so

$$\underline{\underline{L}} f_! (f^* F) = \underline{\underline{L}} f_! (\Lambda) \otimes F$$

showing that

$$\underline{\underline{L}} (fg')_! \Lambda = \underline{\underline{L}} f_! \, \underline{\underline{L}} g'_! \, \Lambda = \underline{\underline{L}} f_! (\Lambda) \otimes \underline{\underline{L}} g_! (\Lambda).$$

In other words to compute the homology of $\mathcal{C}_1 \times_S \mathcal{C}_2$ when $\mathcal{C}_1$ and $\mathcal{C}_2$ are both cofibred over S we are reduced to computing over S the homology of a tensor product

$$H_*(S, F \otimes_\Lambda G) \overset{?}{=} H_*(S, F) \otimes_\Lambda H_*(S, G)$$

(say if Λ is a field.)

One know (Eilenberg-Zilber) that this holds for $S = \Delta^{\circ}$. Question: When is the canonical map

$$\mathbb{L} \varinjlim_{S} \left(F \overset{\mathbb{L}}{\otimes} G \right) \longrightarrow \mathbb{L} \varinjlim_{S} F \overset{\mathbb{L}}{\otimes} \mathbb{L} \varinjlim_{S} G$$

in the derived category of abelian groups an isomorphism? It seems reasonable that the right side coincides with

$$\mathbb{L} \varinjlim_{S \times S} F \overset{\mathbb{L}}{\boxtimes} G$$

hence the sought for property of S will hold if the diagonal

$$\Delta : S \longrightarrow S \times S$$

is acyclic:

$$\Delta_* : H_*(S, \Delta^* F) \xrightarrow{\sim} H_*(S \times S, F).$$

~~Both of these amount to knowing that~~

$$\cancel{\mathbb{L}\Delta_!(\mathbb{Z}) = \mathbb{Z}}$$

~~since~~ $\cancel{H_*(S, \Delta^* F) = H_*(S \times S, \mathbb{L}\Delta_!(\Delta^* F))}$

April 26, 1971:

Let G be a group. Consider the fibred category in groupoids over the category of spaces consisting of principal G-bundles. It is a stack. The category is equivalent to the category $\mathcal{P}$ of principal G-spaces P. I want to understand to what extent BG is the "homotopy-theoretic" limit:

$$\varinjlim_{P \in \mathbf{P}} P/G = BG.$$

Observe that PG is a "homotopy" final object of $\mathcal{P}$.

SGAA gives us a **yoga** for understanding direct limits. Thus we wish to consider the topos consisting of contravariant functors F on $\mathcal{P}$ ~~which~~ such that

$$U \underset{\text{open}}{\subset} P/G \longmapsto F(P/U)$$

is a sheaf on P/G for each P in $\mathbf{P}$. Call these $\mathcal{P}$-sheaves. Thus for each principal G-bundle $P \to X$ we have a sheaf F^P over X and for each map

$$\begin{array}{ccc} P' & \xrightarrow{\varphi} & P \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\tilde{\varphi}} & X \end{array}$$

we have a homomorphism

$$\tilde{\varphi}^* F^P \longrightarrow F^{P'}$$

(The analogy with crystalline cohomology is clear.)

Now we consider the analogues of crystals, i.e. cartesian functors from $\mathcal{P}$ to the stack of sheaves. Locally every principal bundle is trivial, hence such a crystal is determined by the G-set F^G; G [redacted] denoting the canonical G-torsor over a point. Hence the category of crystals is equivalent to the category of G-sets as it should be.

The cohomology of the constant $\mathcal{P}$-sheaf A should fit into a spectral sequence

$$E_2^{pq} = R^p \varprojlim_{\mathcal{P}} \left(H^q(P/G, A) \right) \Longrightarrow H^{p+q}(\mathcal{P}; A)$$

and this degenerates because up to homotopy $\mathcal{P}$ has a final object PG, so we should have

$$H^*(BG, A) \simeq H^*(\mathcal{P}; A).$$

<u>April 28, 1971</u>

Let S be a category and work with covariant functors $S \longrightarrow \mathcal{Ab}$. Under what conditions can we expect an Eilenberg-Zilber theorem:

$$H_*(S, \Delta^* F) \simeq H_*(S \times S, F)$$

where $\Delta : S \longrightarrow S \times S$ is the diagonal and $F: S \times S \longrightarrow \mathcal{Ab}$. The usual E-Z thm says this is true for $S = \Delta^0$.

First consider a general functor $f: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$. Then there are canonical morphisms

$$\mathbb{L} f_! \circ f^* \longrightarrow id$$

$$id \longrightarrow \mathbb{R} f_* \circ f^*$$

where

$$\mathbb{L} f_! : D^-(\mathcal{C}_1) \longrightarrow D^-(\mathcal{C}_2)$$

$$\mathbb{R} f_* : D^+(\mathcal{C}_1) \longrightarrow D^+(\mathcal{C}_2)$$

are derived functors. In effect if $F_\cdot \in D^-(\mathcal{C}_2)$ and $u: P_\cdot \longrightarrow f^* F_\cdot$ is a projective resolution, then have

$$\begin{array}{ccccc} \mathbb{L} f_!(P_\cdot) & \xrightarrow[\cong]{can} & f_!(P_\cdot) & \xrightarrow{u^\#} & F_\cdot \\ {\scriptstyle u_*}\downarrow{\scriptstyle \cong} & & & & \\ \mathbb{L} f_!(f^* F) & & & & \end{array}$$

Suppose now that $G \in D^-(\mathcal{C}_2)$ and $F \in D^+(\mathcal{C}_2)$.
Then have

$$\begin{tikzcd}
\mathrm{Hom}_{D(\mathcal{C}_2)}(\mathbb{L}f_!(f^*G), F) & & \mathrm{Hom}_{D(\mathcal{C}_1)}(f^*G, f^*F) \arrow[ll, "\sim"'] \arrow[d, "\simeq"] \\
\mathrm{Hom}_{D(\mathcal{C}_2)}(G, F) \arrow[u, dashed] \arrow[urr, dashed] \arrow[rr, dotted] & & \mathrm{Hom}_{D(\mathcal{C}_2)}(G, \mathbb{R}f_*(f^*F))
\end{tikzcd}$$

showing the equivalence of:

i) $\mathbb{L}f_! \circ f^* \xrightarrow{\sim} id$ on $D^-(\mathcal{C}_2)$

ii) $id \xrightarrow{\sim} \mathbb{R}f_* \circ f^*$ on $D^+(\mathcal{C}_2)$

iii) $\mathrm{Hom}_{D(\mathcal{C}_2)}(G, F) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{C}_1)}(f^*G, f^*F)$

on $D^-(\mathcal{C}_2) \times D^+(\mathcal{C}_2)$

Such a map f should be termed <u>universally acyclic</u> because of the geometric terminology.

Taking $G = \mathbb{Z}$, the constant functor with value $\mathbb{Z}$, one sees the equivalence of

i) $\mathbb{L}f_!(\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$

ii) $H^*(\mathcal{C}_2, F) \xrightarrow{\sim} H^*(\mathcal{C}_1, f^*F)$ all F in $D^+(\mathcal{C}_2)$

I am going to be interested in an analogue for homology. Let $\pi_i : \mathcal{C}_i \longrightarrow pt$ be the canonical maps. By devissage it ~~its~~ clear that (i) implies

$$\mathbb{L}f_!(\pi_1^* K) \xrightarrow{\sim} \pi_2^* K$$

for all $K \in D^-(\mathbb{Z})$. Claim that following are equiv.

i) $\mathbb{R}f_*(\pi_1^* K) \xleftarrow{\sim} \pi_2^* K \qquad K \in D^+(\mathbb{Z})$

ii) $H_*(\mathcal{C}_1, f^*F) \xrightarrow{\sim} H_*(\mathcal{C}_2, F) \qquad F \in D^-(\mathcal{C}_2)$

Indeed if $I \in D^+(\mathbb{Z})$ ~~is an injective complex of abelian groups~~

$$\begin{tikzcd}
\mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{L}\pi_{1!} f^*F, I) \arrow[r, equal] & \mathrm{Hom}_{D(\mathcal{C}_1)}(f^*F, \pi_1^* I) \arrow[d, equal] \\
\mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{L}\pi_{2!} F, I) \arrow[u, "\text{map ii)}"'] \arrow[dr, equal] & \mathrm{Hom}_{D(\mathcal{C}_2)}(F, \mathbb{R}f_*(\pi_1^* I)) \\
 & \mathrm{Hom}_{D(\mathcal{C}_2)}(F, \pi_2^* I) \arrow[u, "\text{map i)}"']
\end{tikzcd}$$

.

~~So this gives a criterion means to prove ii): For each object y of $\mathcal{C}_2$ the category of~~

Now (sauf erreur) i) is equivalent to

iii) for every y in $\mathcal{C}_2$, the category of arrows $y \longrightarrow f(x)$ is acyclic, i.e. has homology $\mathbb{Z}[0]$.

<u>Example 1:</u> Let $S = \Delta^\circ$ and $f = \Delta : S \longrightarrow S \times S$. Then for each $p, q \in S \times S$ we must consider the homology of the category of arrows

$$\begin{array}{l} p \searrow \\ \quad\;\; ? \\ q \nearrow \end{array} \qquad \text{in } \Delta^\circ$$

i.e. the category of simplices in the simplicial set $\Delta(p) \times \Delta(q)$. This category has the same homology as $\Delta(p) \times \Delta(q)$, namely $\mathbb{Z}[0]$, so the criterion iii) holds and so

$$H_*(\Delta^\circ, \Delta^* F) = H_*(\Delta^\circ \times \Delta^\circ, F)$$

for an bisimplicial abelian group F.

<u>Example 2:</u> Take S to be the category of simplices in a simplicial set N. (N nerve of a linearly ordered set is the example I have in mind) Then for each pair of simplices σ, τ in N I wish to consider the category of arrows

$$\begin{array}{l} \sigma \searrow \\ \quad\;\; ? \\ \tau \nearrow \end{array}$$

Unfortunately σ, τ may have no vertices in common, whence this category is void and the criterion iii) does not hold.

<u>Example 3:</u> As in Ex 1, one sees $\Delta^\circ \xrightarrow{\Delta_n} (\Delta^\circ)^n$ satisfies i) ii) iii) on page 3. This means that for any map $(\Delta^\circ)^p \xrightarrow{f} (\Delta^\circ)^q$ associated to a map $\{1, \ldots, q\} \to \{1, \ldots, p\}$ we have

$$H_*((\Delta^\circ)^p, f^* F) \xrightarrow{\sim} H_*((\Delta^\circ)^q, F)$$

because both are isomorphic to $H_*(\Delta^0, \Delta^*_\delta F)$. Consequently each of the categories $S = (\Delta^0)^n$ have the property that $S \longrightarrow S \times S$ satisfies the conditions.

Terminology: f is <u>homologically acyclic</u> (resp. coh. acyclic) if

$$H_*(C_1, f^*F) \xrightarrow{\sim} H_*(C_2, F)$$

$$\left(\text{resp. } H^*(C_2, F) \xrightarrow{\sim} H^*(C_1, f^*F)\right).$$

[redacted] equivalently if $\forall y \in C_2$ the category of arrows $y \longrightarrow f(x)$ (resp. [redacted] $f(x) \longrightarrow y$) is acyclic, equivalently $\mathbb{R}f_*(\mathbb{Z}) = \mathbb{Z}$ (resp. $\mathbb{L}f_!(\mathbb{Z}) = \mathbb{Z}$).