$\boxed{\text{points in } \mathcal{C}^\wedge = \operatorname{Pro} \mathcal{C}}$

March 20, 1971

Program: Let $\mathcal{C}$ be a small category. I want to define $\mathcal{C}$-bundles over a topos, and prove some kind of classification theorem.

Observation: Let $\mathcal{C}^\vee$ be the category of functors $F: \mathcal{C}^\circ \longrightarrow \text{Sets}$ and $h: \mathcal{C} \longrightarrow \mathcal{C}^\vee$ the canonical embedding. I claim that "points" for the topos $\mathcal{C}^\vee$ are the same thing as pro-objects in $\mathcal{C}$.

Recall that a point is a functor $\rho: \mathcal{C}^\vee \longrightarrow \text{Sets}$ which is compatible with arbitrary inductive limits and with finite projective limits. Given $F \in \mathcal{C}^\vee$ it admits a standard resolution

$$\coprod_{(X,\xi)\to(X',\xi')} h_X \rightrightarrows \coprod_{(X,\xi)\in \mathcal{C}/F} h_X \longrightarrow F$$

better:

$$\varinjlim_{(X,\xi)\in \mathcal{C}/F} h_X \xrightarrow{\sim} F$$

hence a functor $\sigma: \mathcal{C}^\vee \longrightarrow \text{Sets}$ compatible with inductive limits satisfies

$$\varinjlim_{(X,\xi)\in \mathcal{C}/F} \sigma(h_X) \xrightarrow{\sim} \sigma(F).$$

This formula shows that σ is determined by the functor $\sigma \circ h: \mathcal{C} \longrightarrow \text{Sets}$. Conversely given $\varphi: \mathcal{C} \to \text{Sets}$, define σ by

$$\sigma(F) = \varinjlim_{\mathcal{C}/F} \varphi(X).$$

~~better: $\sigma(F) = \varinjlim_{X \in \mathcal{C}} F(X) \times \varphi(X)$~~

Then $\sigma \circ h_X \simeq \varphi(X)$ because $\mathcal{C}/h_X$ has the final object (X, id_X). [redacted] Is σ compatible with inductive limits? If so, we can identify functors $\varphi : \mathcal{C} \to \mathrm{Sets}$ and $\varinjlim$-compatible functors $\sigma : \mathcal{C}^\vee \longrightarrow \mathrm{Sets}$. (Yes, see page 4)

So now given a point $\rho : \mathcal{C}^\vee \longrightarrow \mathrm{Sets}$ we know it is determined by the restriction $\rho \circ h : \mathcal{C} \longrightarrow \mathrm{Sets}$. I claim the functor $\rho \circ h$ is pro-representable. Indeed since

$$(\rho h)(Z) = \varinjlim_{(X, \xi),\ \xi \in \rho h(X)} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

[redacted] it suffices to show that the category $\mathcal{C}/\rho h$ of pairs (X, ξ), $\xi \in \rho h(X)$ is left-filtering.

a) <u>equalization</u>: Suppose given

$$X_1 \overset{a}{\underset{b}{\rightrightarrows}} X_2$$

$$\xi_1 \rightrightarrows \xi_2$$

Let k be the kernel of $h_{X_1} \rightrightarrows h_{X_2}$. Since ρ is a point it is compatible with kernels, hence

$$\rho(k) = \operatorname{Ker}\ \rho h(X_1) \rightrightarrows \rho h(X_2)$$

and there is an $\eta \in \rho(k)$, unique in fact, which goes to ξ_1. We can write k as a quotient

$$\coprod h_Z \twoheadrightarrow k$$

and since ρ is a point it is compatible with disjoint sums and surjections so

$$\coprod \rho h(Z) \twoheadrightarrow \rho(k)$$

and there is a map $h_Z \to k$ and $\zeta \in \rho h(Z)$ going to η. Thus we get

$$Z \longrightarrow X_1 \rightrightarrows X_2$$

$$\zeta \longmapsto \xi_1$$

proving the equalization condition.

b) <u>product</u>: Given $\xi_i \in \rho h(X_i)$, let $k = h_{X_1} \times h_{X_2}$ and write k as a quotient of (a sum of) representable functors. As in a) we get then a $\zeta \in \rho h(Z)$ and map $Z \to X_1$ and $Z \to X_2$ taking ζ to ξ_1 and ξ_2 respectively.

Conversely if $\varphi : \mathcal{C} \longrightarrow$ Sets is pro-representable, then write it

$$\varphi(?) = \varinjlim_{I} \operatorname{Hom}(X_i, ?) \qquad I \text{ left-filtering}$$

and set

$$\rho(F) = \varinjlim_{I} F(X_i).$$

Then ρ is clearly a point and $\rho h = \varphi$.

<u>Conclusion</u>: Points in $\mathcal{C}^{\vee}$ are the same as pro-objects in $\mathcal{C}$.

(answer to question – top page 2)
The following formula for σ shows it compatible with induction limits

$$\coprod_{Y \to X} F(X) \times \varphi(Y) \rightrightarrows \coprod_{X} F(X) \times \varphi(X) \longrightarrow \sigma(F)$$

)

Here is a more diagrammatical version of the preceding: We view $\mathcal{C}$ as

$$\mathcal{C}: \qquad \cdots\ \Gamma\times_R\Gamma \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \Gamma \rightrightarrows R$$

and an object of $\mathcal{C}^\vee$ as a set F provided with maps

$$F \longrightarrow R \qquad\qquad \Gamma\times_R F \xrightarrow{\mu} F$$

enabling us to form the category

$$\mathcal{C}/F: \qquad \cdots\ \Gamma\times_R\Gamma\times_R F \;\substack{\xrightarrow{p_{23}}\\ \xrightarrow{\Delta\times 1}\\ \xrightarrow{1\times\mu}}\; \Gamma\times_R F \;\substack{\xrightarrow{p_2}\\ \xrightarrow[\mu]{}}\; F$$

A functor from $\mathcal{C}$ to Sets we view as a set P with maps

$$P \longrightarrow R \qquad\qquad P\times_R\Gamma \longrightarrow R$$

enabling us to form the category

$$\mathcal{C}/P \qquad\qquad P\times_R\Gamma\times_R\Gamma \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; P\times_R\Gamma \rightrightarrows P.$$

Such a covariant functor P can be interpreted as a functor from $\mathcal{C}^\vee$ to Sets compatible with inductive limits (we showed this above). The value of P at F is the cokernel of the pair

$$P\times_R\Gamma\times_R F \rightrightarrows P\times_R\Gamma\ .$$

Denote this cokernel by $P\times^{\Gamma}F$.

Now let us suppose that the functor $F \mapsto P\times^{\Gamma}F$ is compatible with finite projective limits. Then I showed that $\mathcal{C}/P$ is left-filtering in essentially the following way:

a) <u>product</u>: ~~When~~ Take $F = \Gamma \times_{(s,s)} \Gamma$, that is $\underbrace{\Gamma \times \Gamma}_{R} = \{(\gamma_1,\gamma_2) \mid s(\gamma_1)=s(\gamma_2)\}$. Then

$$P\times^{\Gamma}F = \{(p,\gamma_1,\gamma_2) \mid t(p)=s(\gamma_1)=s(\gamma_2)\}\Big/$$

equivalence relation gen. by

$$(p\gamma,\gamma_1,\gamma_2) = (p,\gamma\gamma_1,\gamma\gamma_2).$$

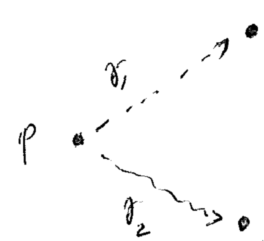
By hypothesis ($\underbrace{\Gamma \times \Gamma}_{R}$ is the product of the rep. functors Γ and Γ)

$$P\times^{\Gamma}F = (P\times^{\Gamma}\Gamma)\times(P\times^{\Gamma}\Gamma) \simeq P\times P$$

Thus we have that

$$\begin{array}{ccc} P\times_R \Gamma & \longrightarrow & P\times P \\ (p,\gamma_1,\gamma_2) & \longmapsto & (p\gamma_1, p\gamma_2) \\ t(p)=s(\gamma_1)=s(\gamma_2) & & \end{array}$$

is <u>surjective</u>. In other words we have the axiom



b) <u>equalization</u>: Take

$$F = \left\{ (\gamma, \gamma_1, \gamma_2) \;\middle|\; \begin{array}{c} t(\gamma) = s(\gamma_1) = s(\gamma_2) \\ t(\gamma_1) = t(\gamma_2) \\ \gamma\gamma_1 = \gamma\gamma_2 \end{array} \right\}$$

= fibre product

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Gamma \arrow[d, "\Delta"] \\ \Gamma \times \Gamma \times \Gamma \arrow[r] & \Gamma \times \Gamma \end{tikzcd}$$

By hypothesis the following square is cartesian

$$\begin{tikzcd} P \times^{\Gamma} F \arrow[r] \arrow[d] & P \arrow[d, "\Delta"] \\ P \times \Gamma \times \Gamma \arrow[r] & P \times P \end{tikzcd}$$

$$(p, \gamma_1, \gamma_2) \longmapsto (p\gamma_1, p\gamma_2)$$

$$t(p) = s(\gamma_1) = s(\gamma_2)$$
$$t(\gamma_1) = t(\gamma_2)$$

Thus given $p\gamma_1 = p\gamma_2$, $\exists\, (p', \gamma) \ni (\gamma, \gamma_1, \gamma_2) \in F$ and $p'\gamma = p$.

This gives the axiom:

$$\begin{tikzcd} \underset{p'}{\bullet} \arrow[r, dashed, "\gamma"] & \underset{p}{\bullet} \arrow[r, shift left, "\gamma_1"] \arrow[r, shift right, "\gamma_2"'] & \underset{p\gamma_1 = p\gamma_2}{\bullet} \end{tikzcd}$$

(note that given $F, G \in \mathcal{C}^{\vee}$ their product is $F \times_R G$.)

March 22, 1971: (Carl is 6)

Let G and H be groups. Let $\mathcal{T}_G$ be the stack over the category of spaces assigning to a space X the groupoid of G-torsors over X. Let

$$F : \mathcal{T}_G \longrightarrow \mathcal{T}_H$$

be a cartesian functor. Apply F to the torsor G over a point with right G-action and we get an H-torsor Q over a point with G acting as autos. of Q. Now given $E \in Ob\ \mathcal{T}_G(X)$ we define

$$E \times^G Q \xrightarrow{\sim} F(E)$$

as follows: There are principal G-bundle maps

$$\begin{tikzcd} E \arrow[d] & E \times G \arrow[l, "\mu"'] \arrow[r, "pr_2"] \arrow[d, "pr_1"] & G \arrow[d] \\ X & E \arrow[l] \arrow[r] & pt \end{tikzcd}$$

hence as F is cartesian there are principal H-bundle maps

$$\begin{tikzcd} F(E) \arrow[d] & F(E \times G) \arrow[l] \arrow[r] \arrow[d] & F(G) \arrow[d] \\ X & E \arrow[l] \arrow[r] & pt \end{tikzcd}$$

Thus $F(E)$ is obtained by descending $F(E\times G) = E \times F(G)$ by the map $E \longrightarrow X$. Identify $Q = F(G)$ and the descent data with the action of G, etc. and done.

So we have shown that $\overbrace{F}^{\text{a cartesian}}: \mathcal{T}_G \longrightarrow \mathcal{T}_H$ is canonically identifiable with a H-torsor endowed with an action of G. Thus we get an equivalence

$$\underline{\mathrm{Hom}}(G, H) \longrightarrow \underline{\mathrm{Homcart}}\,(\mathcal{T}_G \to \mathcal{T}_H)$$

where the former is the category of homos. from G to H under inner H-action.

$\boxed{\text{Digression}}$

Now I take G to be the group of diffeomorphisms of $\mathbb{R}$ with support in $[0,1]$. If I want to define an operation

$$\begin{tikzcd} F: \mathcal{T}_G \times \mathcal{T}_G \arrow[r] & \mathcal{T}_G \\ \mathcal{T}_{G\times G} \arrow[u, "\text{equivalence}"'] & \end{tikzcd}$$

I must therefore give a G-torsor endowed with an action of $G \times G$. So you make $G\times G$ act on

$$[0,2] = [0,1] \cup [1,2]$$

in the obvious way, and let Q be the set of diffeomorphisms

$$\varphi : [0,2] \longrightarrow [0,1]$$

such that $\quad \varphi(x) = x \quad$ for $\quad x$ near 0

$\qquad\qquad\qquad \varphi(x) = x - 1 \quad$ for $\quad x$ near 2.

This is clearly a torsor under G. Choosing such a φ we get a homomorphism $\mu : G \times G \longrightarrow G$ which is unique up to ~~any~~ inner autos. I believe I once checked that the operation F is associative, but I want to be sure now that $\boxed{\text{it is not unitary.}}$ In other words I want to see that the homomorphism composé

$$\phi : G \xrightarrow{(id,\, 1)} G \times O \xrightarrow{\mu} G$$

is not conjugate to the identity. If so

$$\phi(g) = u g u^{-1}$$

~~so~~ if $\varphi[0,1] = [0, \tfrac{1}{2}]$ say, then

$$u g u^{-1}(x) = x \qquad \text{for} \quad \tfrac{1}{2} \leqslant x \leqslant 1$$

$$g(u^{-1}(x)) = (u^{-1}x) \qquad \text{for} \quad \tfrac{1}{2} \leqslant x \leqslant 1$$

so $\quad \underline{g = id} \quad$ on $\quad u^{-1}[\tfrac{1}{2}, 1]$, which is nonsense.

The above digression seems to reinforce the idea of replacing the group of diffeos. with support in $[0,1]$ by the 2-category $\{G_{ab}\}$. The actual significance of this replacement is unclear. One feels that G_{01} bundles have a sum operation which is unitary but maybe this is so only over finite complexes. Situation analogous to stable representations for infinitely generated groups, i.e. only for finitely gen. G does

$$Hom(G, GL(R))/\!\!/GL(R)$$

has an abelian monoid structure.

Let C be a small category. By a C-torsor over a space X I mean a morphism of toposes

$$f : Sheaves(X) \longrightarrow C^{\vee}$$

i.e. a functor f^* going the other way which is $\varinjlim$ compatible and finite $\varprojlim$ compatible. Such a morphism f can be identified with a sheaf P over X endowed with

$$t : P \longrightarrow R \qquad R = Ob(C)$$

(strictly speaking a map $P \to o^*(R)$)
$o : X \to pt$

$$\mu : P \times_R \Gamma \longrightarrow P$$ satisfying usual conditions and

such that the functor

$$F \longmapsto P \times^{\Gamma} F$$

from $\mathcal{C}^{\vee}$ to Sheaves(X) commutes with finite proj. limits. This last condition is equivalent to the category ~~[illegible]~~ object

$$\mathcal{C}/P : \qquad P \times_R \Gamma \times_R \Gamma \rightrightarrows P \times_R \Gamma \rightrightarrows P$$

being left-filtering, hence a ~~[illegible]~~ ~~[illegible]~~ family of pro-objects in $\mathcal{C}$ parameterized by X in some sense. (This sort of this should be closedly connected with the limiting process of Deligne.) ∎

The next step will be to replace $\mathcal{C}$ by a 2-category and understand the corresponding torsors. ~~[illegible]~~ Here P will not be a sheaf of sets over X, but instead a sheaf of categories, i.e. a __stack__.

For simplicity ~~[illegible]~~ consider the situation ~~[illegible]~~

when $\mathcal{C}$ is a 2-category all of whose 2-arrows are isomorphism. Let R be the set of objects of $\mathcal{C}$. Let $\tilde{\mathcal{C}}$ be the stack of torsors over X for the category of 1-arrows in $\mathcal{C}$. Thus if Γ is the set of 1-arrows in $\mathcal{C}$ an object of $\tilde{\mathcal{C}}(U)$ is a $Q \longrightarrow U \times \Gamma$ endowed with action

$$Q \times_{\Gamma} \Gamma^{(2)} \longrightarrow Q$$

where $\Gamma^{(2)}$ is the set of 2-arrows in $\mathcal{C}$.

In the Mather situation Γ is the set of pairs (a,b) with $a \leq b$, and $\Gamma^{(2)} = \coprod_{a \leq b} G_{ab}$, so

$$Q = \coprod_{a \leq b} Q_{ab}$$

with Q_{ab} a torsor for G_{ab}.

Now $\mathcal{P}$ will be a stack over $X \times R$ and an object P over U will appear as a disjoint union

$$P = \coprod P_a$$

and given $Q \in \tilde{\mathcal{C}}(U)$ we have P twisted-by Q:

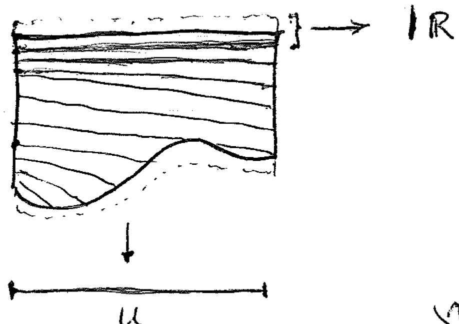
$$P \tau Q = \coprod_{a \leq b} P_a \tau Q_{ab}$$

<u>Example</u>: Suppose P_a is a torsor over U for $G_{[-\infty,a]}$. Then given a G_{ab}-torsor Q_{ab} we have

$$P_a \tau Q_{ab} = (P_a \times Q_{ab}) \times^{G_{[\infty,a]} \times G_{ab}} G_{[-\infty,b]}$$

is a $G_{[-\infty,b]}$-torsor.

<u>Main example</u>: Let δ be a Haefliger structure over X. Over U consider spaces E represented by the following picture



Thus E is a manifold over U of rel dim 1 endowed with a foliation transversal to the fibres. E comes with two sections, the bottom one defining the Haefliger structure δ and the top flat. Finally there is a function to $\mathbb{R}$ defined in a nbd. of the top section which is constant along the leaves of the foliation and a local diffeo. on each fibre at the flat section. Finally an isomorphism of two such structures is a diffeo defined in a nbd. of the region between the two sections.

March 26, 1971. Summary of problems and progress.

If C small category and $\mathcal{T}$ is a topos, we know what is a C-torsor over $\mathcal{T}$. It is an object P of $\mathcal{T}$ endowed with

$$P \longrightarrow \sigma^*(Ob\,C)$$

$$P \times_{Ob\,C} Ar\,C \longrightarrow P$$

~~satisfying~~ such that

$$C/P: \qquad P \times_{Ob} Ar \times_{Ob} Ar \rightrightarrows P \times_{Ob} Ar \rightrightarrows P$$

is left-filtering. We can form the category ~~of [illegible]~~ of C-torsors over $\mathcal{T}$, denote it $\underline{\mathrm{Tors}}(\mathcal{T}; C)$ and we can probably prove a classification theorem in Grothendieck form:

$$\underline{\mathrm{Homtop}}(\mathcal{T}, C^\vee) \longrightarrow \underline{\mathrm{Tors}}(\mathcal{T}, C)$$
$$f \longmapsto f^*(Ar\,C)$$

equivalence of categories.

~~What is significant here~~

The above is clean from Groth. viewpoint. It remains to understand from the algebraic topologist's viewpoint. Thus restrict $\mathcal{T}$ to the category of compact (Hausdorff) spaces; you want to prove that the homotopy classes of $\mathcal{C}$-torsors over X is representable say by the realization of the nerve of $\mathcal{C}$. This is interesting because it says that every homotopy class of maps $X \longrightarrow |N(\mathcal{C})|$ is obtained from a $\mathcal{C}$-torsor on X, hence it gives you representatives ~~for~~ elements of $\pi_k |N(\mathcal{C})|$.

March 30, 1971
thru April 11

C-torsors
+ initial understanding of Mather Thm

Let C be a small category and let X be a topological space. We have defined a C-torsor over X. It is a sheaf P [redacted] of sets over X endowed with a map $P \longrightarrow X \times (\mathrm{Ob}\, C)$ (= const. sheaf on X) and a map

$$P \times_{\mathrm{Ob}\, C} \mathrm{Ar}\, C \longrightarrow P$$

making each stalk P_x a pro-representable covariant functor from C to (Sets). C-torsors form a fibred category over X, in fact, a stack over X. Denote by [redacted] $\underline{\mathrm{Tors}}(X, C)$.

[redacted]

Let BC be the realization of the nerve of C, or the Milnor model. The conjecture is that [redacted] for X nice, e.g. a finite complex, the categories

$$\underline{\underline{\pi}}\ \underline{\mathrm{Tors}}(X, C) \qquad \underline{\underline{\pi}}\ \underline{\mathrm{Hom}}(X, BC)$$

are canonically equivalent. On the right is the fundamental groupoid of the space of maps from X to BC and on the left [redacted] the groupoid associated to the category $\underline{\mathrm{Tors}}(X, C)$.

We have rigged the definition of C-torsor over X so that the [redacted] functor

$$\underline{\mathrm{Hom\,top}}\,([redacted]\ \mathrm{Top}(X), C^\vee) \longrightarrow \underline{\mathrm{Tors}}\,(X, C)$$

defined by the canonical $\mathcal{C}$-torsor $\mathrm{Ar}\,\mathcal{C}$ in $\mathcal{C}^\vee$ is an equivalence of categories. Now it is clear that to any morphism of topoi $\mathcal{C}_1^\vee \longrightarrow \mathcal{C}_2^\vee$, or what amounts to the same thing essentially, a functor $\mathcal{C}_1 \longrightarrow \mathrm{Pro}\,\mathcal{C}_2$, there is associated a functor

$$\underline{\mathrm{Tors}}(X, \mathcal{C}_1) \longrightarrow \underline{\mathrm{Tors}}(X, \mathcal{C}_2).$$

Here is a formula: Let P be a $\mathcal{C}_1$-torsor over X and let Q be a $\mathcal{C}_2$-torsor in $\mathcal{C}_1^\vee$, so that $\mathcal{C}_2$ acts to the right of Q and $\mathcal{C}_1$ acts to the left. Then we can form

$$P \times^{\mathcal{C}_1} Q = \mathrm{Coker}\left\{ P \times_{\mathrm{Ob}\,\mathcal{C}_1} \mathrm{Ar}\,\mathcal{C}_1 \times_{\mathrm{Ob}\,\mathcal{C}_1} Q \rightrightarrows P \times_{\mathrm{Ob}\,\mathcal{C}_1} Q \right\}$$

and this is a $\mathcal{C}_2$-torsor over X. (If we are given $u: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, then $Q = \mathrm{Ob}\,\mathcal{C}_1 \times_{\mathrm{Ob}\,\mathcal{C}_2} \mathrm{Ar}\,\mathcal{C}_2$ and in general an element of Q should be thought of as an arrow [redacted] from an object of [redacted] $\mathcal{C}_1$ to an object of $\mathcal{C}_2$.)

Example: Let M be an abelian monoid and let $M \longrightarrow G$ be the universal map to an (abelian) group. Then Q in this case is G and the functor

$$\underline{\mathrm{Tors}}(X, M) \longrightarrow \underline{\mathrm{Tors}}(X, G)$$

is given by sending P to

$$P \times^M G = \mathrm{Cokernel}\{ P \times M \times G \rightrightarrows P \times G \}$$

$$= \varinjlim P$$

the limit being taken [redacted] over the category [redacted] with objects M and morphisms $M \times M$. In this example Q is an ind-object of $\mathcal{C}_1$.

March 31, 1971

Suppose $u : C_1 \longrightarrow C_2$ is a functor. We want to understand when the restriction relative to u of a C_2-torsor is a C_1-torsor. First of all observe that a ~~Osm~~ sheaf P over X endowed with an action of C on the right is a kind of covariant functor on C. Hence there are adjoint maps

$$\begin{array}{ccc} & P \mapsto P \times^{C_1} (\mathrm{Ar}\, C_2) & \\ (\text{right } C_1\text{-spaces}/X) & \rightleftarrows & (\text{right } C_2\text{-spaces}/X) \\ & P' \times_{\mathrm{Ob}\, C_2} \mathrm{Ob}\, C_1 \longleftarrow\!\!\shortmid P' & \end{array}$$

It would be better to replace u by a functor $u^* : C_1^\vee \longleftarrow C_2^\vee$ compatible with inductive limits. Such a thing is furnished by a Q with left C_1 and right C_2 action by the formula

$$u^* F = Q \times^{C_2} F = \mathrm{Coker}\left\{ Q \times_{\mathrm{Ob}\, C_2} \mathrm{Ar}\, C_2 \times_{\mathrm{Ob}\, C_2} F \rightrightarrows Q \times_{\mathrm{Ob}\, C_2} F \right\}$$

The other functor $u_* : C_1^\vee \longrightarrow C_2^\vee$ is given by

(see below) $$\xcancel{u_* G = \mathrm{Ker}\left\{ G \times_{\mathrm{Ob}\, C_1} Q \rightrightarrows G \times_{\mathrm{Ob}\, C_1} \mathrm{Ar}\, C_1 \times_{\mathrm{Ob}\, C_1} Q \right\}}$$

These are adjoint functors and constitute of morphism of topoi when u^* commutes with finite proj. limits, i.e. when

Q is a $\mathcal{C}_2$-torsor over $\mathcal{C}_1^\vee$.

One defines the action of u on covariant functors

$$\underline{\mathrm{Hom}}(\mathcal{C}_1, \mathrm{Sets}) \underset{u_+}{\overset{u^+}{\rightleftarrows}} \underline{\mathrm{Hom}}(\mathcal{C}_2, \mathrm{Sets})$$

by

$$u^+(P) = P \times^{\mathcal{C}_1} Q = \mathrm{Coker}\{P \times \mathrm{Ar}\,\mathcal{C}_1 \times Q \rightrightarrows P \times Q\}$$

(see below) $\cancel{u_+(P') = \mathrm{Ker}\{Q \times P' \rightrightarrows Q \times \mathrm{Ar}\,\mathcal{C}_2 \times P'\}}$.

We know that u^+ carries $\mathrm{Pro}\,\mathcal{C}_1$ to $\mathrm{Pro}\,\mathcal{C}_2$ provided (u^*, u_*) is a morphism of topoi. We want now to understand when u_+ carries $\mathrm{Pro}\,\mathcal{C}_2$ to $\mathrm{Pro}\,\mathcal{C}_1$. ~~As u_+ commutes with filtered inductive limits, it is enough to know that u_+ of a representable functor is pro-representable.~~

(Correct formulas:

$$u_* G = \mathrm{Ker}\{\mathrm{Hom}_{/\mathrm{Ob}\,\mathcal{C}_1}(Q, G) \rightrightarrows \mathrm{Hom}_{/\mathrm{Ob}\,\mathcal{C}_1}(\mathrm{Ar}\,\mathcal{C}_1 \times_{\mathrm{Ob}\,\mathcal{C}_1} Q, G)\}$$

$$u_+(P') = \mathrm{Ker}\{\mathrm{Hom}_{/\mathrm{Ob}\,\mathcal{C}_2}(Q, P') \rightrightarrows \mathrm{Hom}_{/\mathrm{Ob}\,\mathcal{C}_2}(Q \times_{\mathrm{Ob}\,\mathcal{C}_2} \mathrm{Ar}\,\mathcal{C}_2, P')\}.)$$

Consider the case of $u : \mathcal{C}_1 \to \mathcal{C}_2$ where

$$Q = \coprod_{X,Y} \operatorname{Hom}(uX, Y) = Ob\,\mathcal{C}_1 \underset{Ob\,\mathcal{C}_2}{\times} Ar\,\mathcal{C}_2$$

and

$$u^*F = F \circ u$$

$$u_+ P' = P' \circ u \qquad\qquad P' \times_{Ob\,\mathcal{C}_2} Ob\,\mathcal{C}_1$$

In this case it is clear that u_+ commutes with filtered inductive limits, so to see that $u_+(\operatorname{Pro}\mathcal{C}_2) \subset \operatorname{Pro}\mathcal{C}_1$ it is enough to know that u_+ carries representable functors to pro-representable ones. Thus given $Y \in \mathcal{C}_2$ we want that the covariant functor

$$(u_+ h^Y)(X) = \operatorname{Hom}_{\mathcal{C}_2}(Y, uX)$$

is pro-representable. This ~~means~~ means

i) given $Y \xrightarrow{\xi} uX_1$ equalized by $X_1 \rightrightarrows X_2$, there is a $X \to X_1$ equalizing the pair and $\eta : Y \to uX$ yielding ξ.

ii) given $\xi_1 : Y \to uX_1$ and $\xi_2 : Y \to uX_2$, there exist $Y \to uX$ and $X \to X_1$, $X \to X_2$ yielding ξ_1 and ξ_2.

Assume this holds for each $Y \in Ob\,\mathcal{C}_2$. Then the functor $u_! : \mathcal{C}_1^\vee \to \mathcal{C}_2^\vee$ given by

$$(u_! G)(Y) = \varinjlim_{Y \to u(X)} G(X)$$

is exact. As it is automatically compatible with arbitrary inductive limits, this means that $(u_!, u^*)$ is a morphism of topoi. Hence we have a diagram of topoi

$$\begin{tikzcd} \mathcal{C}_1^{\vee} \arrow[r, shift right, "u"'] & \mathcal{C}_2^{\vee} \arrow[l, shift right, "v"'] \end{tikzcd}$$

The way to think of this is in terms of the torsors and the way they move.

I want now to check that the map on points produced by v is precisely u_+. Enough to check for ev_Y.

$$ev_Y(u_! G) = (u_! G)(Y) = \varinjlim_{Y \to u(X)} G(X)$$

On the other hand if we take the pro-object Y in $\mathcal{C}_2$ which gives rise to the functor h^Y, and pull this back to $u_+ h^Y = h^Y \circ u$ $\quad ((h^Y u)(X) = \mathrm{Hom}(Y, uX))$ then the corresp. pro-object in $\mathcal{C}_1$ is

$$\varinjlim_{Y \to uX} X$$

so we get the same point as above.

<u>Definition</u>: $u: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is <u>strongly</u> <u>cofinal</u> if the following equivalent conditions are satisfied.

i) $(u_!, u^*)$ is a morphism of topoi from $\mathcal{C}_2^\vee \longrightarrow \mathcal{C}_1^\vee$

ii) ~~[illegible]~~ composition with u carries $\text{Pro}\, \mathcal{C}_2$ into $\text{Pro}\, \mathcal{C}_1$

ii)' for every Y in $\mathcal{C}_2$, $X \longmapsto \text{Hom}_{\mathcal{C}_2}(Y, uX)$ is a pro-object in $\mathcal{C}_1$.

<u>Remark</u>: If u is strongly cofinal, then

$$\begin{array}{ccc} u^*: \; H^q(\mathcal{C}_2, F) & \xrightarrow{\sim} & H^q(\mathcal{C}_1, u^*F) \\ \| & & \| \\ R^q \varprojlim_{\mathcal{C}_2} (F) & & R^q \varprojlim_{\mathcal{C}_1} (u^*F) \end{array}$$

because u^* is exact and it preserves injectives, since it ~~[illegible]~~ is the direct image for a morphism of topoi.

Finally I note that

$$\text{Pro}\, \mathcal{C}_1 \underset{u_+}{\overset{u^+}{\rightleftarrows}} \text{Pro}\, \mathcal{C}_2$$

are adjoint functors, hence

$$\underline{\text{Tors}}(X, \mathcal{C}_1) \underset{u_+}{\overset{u^+}{\rightleftarrows}} \underline{\text{Tors}}(X, \mathcal{C}_2)$$

will also be so that the homotopy theory of these torsors will be equivalent. For future reference note that there is a canonical arrow

$$u^+ u_+ P' \longrightarrow P' \qquad \text{i.e.}$$

$$\left(P' \times_{Ob\,C_2} Ob\,C_1\right) \times^{C_1} Ar\,C_2 \longrightarrow P'$$

Now return to the situation of an abelian monoid M and the universal arrow to an abelian group G. Then the strongly cofinal condition amounts to showing that G as a right M-set is a pro-object in the M-category, i.e. that the category with objects G and arrows $G \times M$ is left-filtering:

i) If $gm_1 = gm_2$, then $\exists\, m \ni mm_1 = mm_2$ and $gm^{-1} . m = g$.

ii) Given g_1, g_2, $\exists\, g$ and $m_i \ni gm_i = g_i$. (write $g_i = m^{-1} m_i$ and take $g = m^{-1}$).

Exactness:

Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of sheaves of groups over X. Then there are functors

$$\underline{\mathrm{Tors}}(X, G') \longrightarrow \underline{\mathrm{Tors}}(X, G) \longrightarrow \underline{\mathrm{Tors}}(X, G'').$$

Somehow from these functors we obtain an exact sequence

$$\begin{array}{l} 1 \to H^0(X,G') \longrightarrow H^0(X,G) \longrightarrow H^0(X,G'') \\ \xrightarrow{\delta} H^1(X,G') \longrightarrow H^1(X,G) \longrightarrow H^1(X,G'') \end{array}$$

(and further using the Giraud H^2).

First consider the case where G' is a subgroup of G and G'' is the homogeneous space G/G'. Then we wish to understand the functor

$$\begin{array}{ccc} \underline{\mathrm{Tors}}(X, G') & \longrightarrow & \underline{\mathrm{Tors}}(X, G) \\ P' & \longmapsto & P' \times^{G'} G \end{array}$$

Given P' such that $P' \times^{G'} G$ is trivial, better - given P' and a G'-map $u : P' \longrightarrow G$ (same as an isom. of $P' \times^{G'} G \xrightarrow{\sim} G$), one obtain a section of G'' by passage to quotient

$$X = P'/G' \xrightarrow{u/G'} G/G' = G''.$$

Conversely given $s: X \longrightarrow G''$ one forms cartesian square

$$\begin{array}{ccc} P & \longrightarrow & G \\ \downarrow & & \downarrow \\ X & \longrightarrow & G'' \end{array}$$

(pull-back of canonical G'-torsor over G''). What we have shown therefore is the equivalence of the following two categories:

i) objects = pairs (P', φ) where P' is a G'-torsor over X and $\varphi: P' \longrightarrow G$ is a G'-map. morphism from (P_1', φ_1) to (P_2', φ_2) = map $P_1' \to P_2'$ rendering evident diagram commutative. There is at most one such map and it is an isomorphism.

ii) objects = sections $X \longrightarrow G''$, morphism = identities.

We can also define morphisms in i) to be comm. diagrams

$$\begin{array}{ccc} P_1' & \xrightarrow{\varphi_1} & G \\ {\scriptstyle \text{morph of } G'\text{-tors}} \downarrow & & \downarrow {\scriptstyle \text{morph of } G\text{-tors.}} \\ P_2' & \longrightarrow & G \end{array}$$

(the right vertical arrow is given by a left multiplication) and in ii) by defining a morphism between sections to be something produced by a left multiplication.

This raises the following question: Is the

sheaf of categories (G'', G) a stack? No. The associated stack should be the category of triples (P', φ, P) where P' is a G'-torsor, P is a G-torsor and $\varphi : P' \to P$ is a G'-map. Equivalently the category of pairs (P, s) where P is a G-torsor and $s \in H^0(X, P/G')$.
or the cat of G'-torsors
So the situation is that

$$\begin{tikzcd}
\{s \in H^0 G''\} \arrow[r, "\text{fibre}", "\text{over trivial torsor}"'] & \{(P', \varphi, P)\} \arrow[r] \arrow[d, "\text{equiv.}"] & \underline{\mathrm{Tors}}(X, G) \\
& \underline{\mathrm{Tors}}(X, G') \arrow[ur] &
\end{tikzcd}$$

(Remark: A morphism of groupoids $\mathcal{G}_1 \xrightarrow{f} \mathcal{G}_2$ is fibrant iff

$$\mathrm{Ar}\, \mathcal{G}_1 \twoheadrightarrow \mathrm{Ar}\, \mathcal{G}_2 \times_{\mathrm{Ob}\, \mathcal{G}_2} \mathrm{Ob}\, \mathcal{G}_1 .$$

If this is the case, and $a \in \mathrm{Ob}\, \mathcal{G}_1$ is a basepoint, then

$$0 \to \pi_1(\text{Fib over } fa) \to \pi_1(\mathcal{G}_1, a) \to \pi_1(\mathcal{G}_2, fa) \to \pi_0(\text{Fib. over } fa) \to \pi_0 \mathcal{G}_1 \to \pi_0 \mathcal{G}_2$$

is exact by usual argument.)

Next the case where G' is normal in G. Then we consider the functor

$$\underline{\mathrm{Tors}}(X, G) \longrightarrow \underline{\mathrm{Tors}}(X, G'')$$
$$P \longmapsto P/G'$$

The fibre over the trivial torsor is the category ~~[illegible]~~ of pairs (P, φ) where P is a G-torsor and $\varphi : P \to G''$ is a G-map. This category is equivalent to the category of G'-torsors by the maps

$$P' \longmapsto P' \times^{G'} G \quad + \text{ map } \varphi(p', g) = gG'$$

$$(P, \varphi) \longmapsto \varphi^{-1}\{1\}.$$

One slight problem with this approach is that we need ~~[illegible]~~ ultimately an argument identifying $\underline{\text{Tors}}(X, G'')$ with the quotient of $\underline{\text{Tors}}(X, G)$ by G'. Thus perhaps one should start with $\underline{\text{Tors}}(X, G)$ and define new morphisms between objects:

$$\overline{\text{Hom}}(P, Q) = \text{Hom}_{G''}(P/G', Q/G')$$

Then morphisms glue, ~~[illegible]~~ so one can form the associated stack which can be identified with $\underline{\text{Tors}}(X, G'')$ because:

$$\underline{\text{Tors}}(X, G)/G' \longrightarrow \underline{\text{Tors}}(X, G'')$$

is fully faithful and "locally-surjective".