March 1, 1971:

Check the grand conjecture against the Tate result for $K_2$. Thus let $C$ be a Dedekind subring of a number field $F$, and denote by $T^{(n)}$ the $n$-fold tensor product of the Tate character on $\mathbb{Z}_\ell$; we assume $\ell^{-1} \in C$. Then I expect an exact sequence

$$H^{i-1}(F, T^{(j)}) \longrightarrow \bigoplus_{v \in C} H^{i-2}(k(v), T^{(j-1)}) \longrightarrow H^i(C, T^{(j)}) \longrightarrow H^i(F, T^{(j)}) \xrightarrow{\ \delta\ }$$

where the middle map is an $i_*$. I also expect a corresponding sequence for K-groups. Moreover there should be compatibility:

$$\begin{tikzcd}
0 \arrow[r] & K_2(C) \arrow[r] \arrow[d, "c_2^{\#}"] & K_2(F) \arrow[r] \arrow[d, "c_2^{\#}"] & \bigoplus_{v \in C} K_1(k(v)) \arrow[r] \arrow[d, "\cong\ c_1^{\#}"'] & K_1(C) \arrow[d, "c_2^{\#}"] \\
0 \arrow[r] & H^2(C, T^{(2)}) \arrow[r] & H^2(F, T^{(2)}) \arrow[r, "\delta"] & \bigoplus_{v \in C} H^1(k(v), T^{(1)}) \arrow[r] & H^3(C, T^{(2)})
\end{tikzcd}$$

(The group $H^3(C, T^{(2)})$ should be zero as $\ell$ has been removed?)

Now Tate claims to have proved that

$$_{(\ell)}K_2(F) \xrightarrow{\ \sim\ } H^2(F, T^{(2)})_{\text{tors}}$$

so by five lemma, his result should be equivalent to

$$_{(\ell)}K_2(C) \xrightarrow{\ \sim\ } H^2(C, T^{(2)})_{\text{tors}}.$$

because the groups $K_*(C)$ should be finitely generated.

By my yoga there should be an Atiyah-Hirzebruch spectral sequence converging to the $\ell$-adic completion $K_*(C)_{\hat{\ell}}$ and ~~starting~~ with $H^*(C, T^{(*)})$.

| | | |
|---|---|---|
| | $H^1(C, T^{(0)})$ | |
| $\cancel{H^0(C, T^{(1)})}$ | $H^1(C, T^{(1)})$ | $H^2(C, T^{(1)})$ |
| $\cancel{H^0(C, T^{(2)})}$ | $H^1(C, T^{(2)})$ | $H^2(C, T^{(2)})$ |

Now this spectral sequence should ~~collapses~~ collapse as $C$ is of coh. dim 2 yielding isomorphisms

$$K_1(C)_{\hat{\ell}} = H^1(C, T^{(1)})$$

$$K_2(C)_{\hat{\ell}} = H^2(C, T^{(2)})$$

$$K_3(C)_{\hat{\ell}} = H^1(C, T^{(2)})$$

$$K_4(C)_{\hat{\ell}} = H^2(C, T^{(3)})$$

The first is probably OKAY because the "exact sequence"

$$0 \longrightarrow H^1(C, T^{(1)}) \longrightarrow \underset{\substack{\parallel \\ (F^{\cdot})_{\hat{\ell}}}}{H^1(F, T^{(1)})} \longrightarrow \bigoplus_{v \in C} \mathbb{Z}_\ell \longrightarrow H^2(C, T^{(1)})$$

(which has to be interpreted accurately !?) shows that $H^1(C,T^{(1)}) = (C^{\cdot})_{\hat{\ell}}$ which should coincide with $K_1 C_{\hat{\ell}}$ as $\ell$ is out of $C$. (Calvin-Moore shows that $K_2 F \twoheadrightarrow \bigoplus_{v \nmid \ell} \mu_v$, hence $K_1 C = C^{\cdot}$.)

According to Tate

$$H^1(C, T^{(2)}) \xrightarrow{\sim} H^1(F, T^{(2)})$$

has rank $r_2$ over $\mathbb{Z}_\ell$ and this fits nicely with Borel's determination of $K_3(C) \otimes \mathbb{Q}$. Finally Tate knows that

$$K_2(C)_{\hat{\ell}} = H^2(C, T^{(2)})_{tors}$$

so it seems (conjecturally) that $H^2(C,T^{(2)})$ should be finite and that

$$_{(\ell)}(K_2 F) = H^2(F, T^{(2)}).$$

---

~~Now suppose ... is the subring of F of ... integers with ... in denominators~~ Let $\bar{C}$ be the integral closure of $C$ in $F(\mu_{\ell^\infty})$. We assume $\mu_\ell \subset F$ or $\mu_4 \subset F$ if $\ell = 2$, so that the Galois group $\Gamma$ of $F(\mu_{\ell^\infty})/F$ is $\simeq \mathbb{Z}_\ell$. Set $X = H_1(\bar{C}, \mathbb{Z}_\ell)$, the Galois group of the maximal pro-$\ell$-abelian extension of $F(\mu_{\ell^\infty})$ unramified outside of $\ell$. Then

$$H^1(\overline{C}, A) = \mathrm{Hom}_{\mathbb{Z}_\ell}(X, A)$$

for any $\mathbb{Z}_\ell$-module $A$. Hence we have short exact sequences:

$$0 \longrightarrow H^1(\Gamma, H^{q-1}(\overline{C}, T^{(r)})) \longrightarrow H^q(C, T^{(r)}) \to H^q(\overline{C}, T^{(r)})^\Gamma \to 0$$

Now it should be the case that $\overline{C}$ has cohomological dimension $1$ (?), hence we hope for

$$\begin{tikzcd}
0 \arrow[r] & T^{(r)}/\Gamma \arrow[r] & H^1(C, T^{(r)}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_\Gamma(X, T^{(r)}) \arrow[r] & 0 \\
 & & {}_{(\ell)}K_{2r-1}(C) & \left\{\begin{array}{ll} \mathbb{Z}_\ell^{r_2} & \text{if } r \text{ even} \\ \mathbb{Z}_\ell^{r_1+r_2} & \text{if } r \text{ odd} \end{array}\right. \arrow[u]
\end{tikzcd}$$

like the finite field contribution

(by Borel)

$$\mathrm{Hom}(X, T^{(r)})/\Gamma \simeq H^2(C, T^{(r)}) = {}_{(\ell)}K_{2r-2}(C)$$

<u>Remarks</u>: Somehow the odd $K$ groups are trivial and don't depend on the number of points left out of $C$. Conjecture: $K_{odd}(C) \xrightarrow{\sim} K_{odd}(F)$ in dims $> 1$ ($C$ not complete).

This agrees with $H^1(C, T^{(r)}) = H^1(F, T^{(r)})$.

~~February~~ March 6, 1971 More Mather.

How Mather deloops $BG$, $G$ = orientation-preserving diffeomorphisms of $\mathbb{R}$:

<u>Bisimplicial version:</u> Let $\overline{W}(G)$ be the standard simplicial "classifying space" for $G$.

$$\overline{W}(G)_\nu = G^\nu$$

$$\begin{aligned} d_0(g_1,\dots,g_\nu) &= (g_2,\dots,g_\nu) \\ d_i(g_1,\dots,g_\nu) &= (g_1,\dots,g_i g_{i+1},\dots,g_\nu) \\ d_\nu(g_1,\dots,g_\nu) &= (g_1,\dots,g_{\nu-1}). \end{aligned}$$

Let

$$\overline{W}(G)^{(p)}_\nu \subset \overline{W}(G)^p_\nu = (G^\nu)^p$$

be the subset consisting of $(g_i^j)$ $1 \le i \le \nu$, $1 \le j \le p$ such that

$$I(g_\cdot^1) < I(g_\cdot^2) < \cdots < I(g_\cdot^p),$$

where $I((g_1,\dots,g_\nu))$ is the smallest closed interval of $\mathbb{R}$ outside of which the $g_i$ are identity.

Clearly $\overline{W}(G)^{(p)}$ is a simplicial subset of $\overline{W}(G)^p$. If we let $G(I)$ be the subgroup of $G$ consisting of the diffeos. with support in $I$, then

$$\overline{W}(G)^{(p)} = \bigcup_{I_1 < I_2 < \cdots < I_p} \overline{W}(G(I_1) \times \cdots \times G(I_p))$$

On the other hand there are homomorphisms

$$G(I) \times G(I') \longrightarrow G(I'') \qquad I < I'$$

if $I''$ is the smallest interval containing $I \cup I'$. This gives us maps

(notation: $I'' = I + I'$

$$\overline{W}(G(I_1) \times \cdots \times G(I_p)) \xrightarrow{d_j} \overline{W}(G(I_1) \times \cdots \times G(I_j + I_{j+1}) \times \cdots G(I_p))$$

which allow us to form a bisimplicial set

$$(p,q) \longmapsto \overline{W}(G)^{(p)}_q$$

with face operators

$$d^h_j (g^1_\cdot, g^2_\cdot, \dots, g^p_\cdot) = (g^1_\cdot, \dots, g^j_\cdot g^{j+1}_\cdot, \dots, g^p_\cdot)$$

$$d^v_i ((g^1_1, \dots, g^1_q), \dots (g^p_1, \dots, g^p_q)) = ((g^1_1, \dots, g^1_i g^1_{i+1}, \dots, g^1_q), \dots).$$

<u>Notation:</u> $\overline{W}(G)^{(*)}_*$ for this bisimplicial set.

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Now let $G_p$ be the group of diffeos. of $\Delta(p)$ which preserve the faces and which are the identity in a neighborhood of each vertex. Clearly $\{G_p\}$ is a simplicial group, and

$G_1 = G$ once we choose a diffeom. of $\mathbb{R}$ with $(0,1)$. Unfortunately there is no foliation of the $\Delta(p)$ - bundle over $BG_{p}$ of codimension $1$, so this doesn't seem to be very promising

Let $G_n$ be the group of diffeomorphisms of $)0,n($ which are the identity in a neighborhood of each of the points $0,1,..,n$. Let $C_n$ be the category whose objects are sets $I$ endowed with a $G_n$-equivalence class of bijections $u : I \quad )0,n($ and ~~with~~ whose morphisms are isomorphisms in the obvious sense. Thus $C_n$ is equivalent to the category with the single object $)0,n($ and the group $G_n$ for morphisms.

The face operator $d_i : C_n \quad C_{n-1}$ is defined as follows: First consider $d_1 : C_2 \quad C_1$. Choose a diffeomorphism of $)0,2($ with $)0,1($ ~~which is~~ coinciding with $y = x$ near $0$ and with $y = x - 1$ near $2$. Then given a bijection $u : I \quad )0,2($, we compose it with this chosen diffeomorphism (call it $D_1$) to get a bijection $D_1 u : I \quad )0,1($. The important thing is that $D_1$ is unique up to a canonical element of $G_1$. One defines $d_0$ and $d_2$ by deleting $)0,1)$ and $(1,2($, parametrizing the rest in the obvious way.

In general given $i$, $1 \quad i \quad n-1$, one chooses a diffeomorphism $D_i$ of $)0,n($ with $)0,n-1($ coinciding with $y = x$ for $x \quad i-1$ and near $i-1$ and coinciding with $y = x - 1$ for $x \quad i+1$ and near $i+1$. The choice of $D_i$ is unique up to composition with an element of $G_{n-1}$.

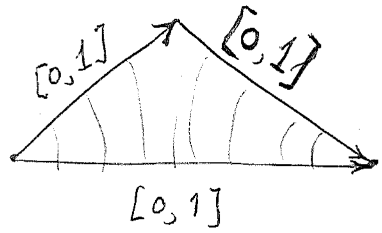
The verification that the face identities hold should be straightforward, due to essential uniqueness of the $D_i$. Degeneracies have to be added in the stupid way, as they do not seem to arise naturally geometrically.

~~February~~ March 8, 1971 Mather (again)

Let $X$ be a manifold endowed with ~~a~~ normally-oriented foliation of codimension 1. Assume that $X$ can be triangulated in such a way that over any simplex $\sigma$ the foliation is quasi-linear, i.e. after a diffeomorphism of $\sigma$ preserving-faces it is defined by a linear function on the simplex.

Choose a function $f_v$ defining the foliation ~~in~~ a nbd. of each vertex $v$ and taking the value zero at the vertex. I assume that each one simplex is transversal to the foliation, whence it obtains an orientation from that of the foliation. Now each one simplex $\sigma$ has a parameterization $\varphi : \sigma \simeq [0,1]$ unique up to an element of $G$ $\ni$ $\varphi(a) = f_{v_0}(a)$ for $a$ near $v_0$, and $\varphi(b) = f_{v_1}(b) + 1$ for $b$ near $v_1$ where $\sigma = \{v_0, v_1\}$. So choose such parameterizations $\varphi_\sigma$.

Now given a 2-simplex, one has identifications




hence if one chooses a fixed diffeomorphism of $[0,2]$ with $[0,1]$ one obtains a diffeo of $[0,1]$ with compact support as follows: One compares the function on the above simplex furnished by the foliation with the one given by the diffeom. Not like twisting situation because neither first nor last face appears

Basic geometric object consists of an $n$-simplex $\Delta(n)$
with ordered vertices $0, 1, \dots, n$
and a ~~[illegible]~~ codimension 1 foliation on $\Delta(n)$
defined by a function $\varphi : \Delta(n) \to \mathbb{R}$ conjugate by a $\Gamma$-homeo
to a linear function on $\Delta(n)$ $\ni$
~~[illegible]~~ $\varphi(0) < \varphi(1) < \cdots < \varphi(n)$.

March 10, 1971: Stasheff classification problem

Let $G$ be a topological group and let $X$ be a space. Suppose $P$ is a $G$-torsor over $X$. Then we can form the simplicial space $\mathrm{Nerve}(P,G)$:

$$P \times G^2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; P \times G \;\rightrightarrows\; P$$

over $X$; it is also the Čech complex of the map $P \to X$. We have maps of simplicial spaces

$$\begin{tikzcd} & \mathrm{Nerve}(P,G) \arrow[dl] \arrow[dr] & \\ X & & \mathrm{Nerve}(pt,G) \end{tikzcd}$$

the first of which is some sort of homotopy equivalence, and the latter has fibre $P$. Perhaps, as a topologist, I should realize these simplicial spaces, and identify them:

$$\begin{tikzcd} & P \times^G PG \arrow[dl] \arrow[dr] & \\ X & & BG \end{tikzcd}$$

Because the former map is a homotopy equivalence, we obtain a definite map $X \to BG$ in the homotopy category, which one calls the classifying map for $P$. It is associated to an equivariant map $P \to PG$.

Better diagram maybe:

$$\begin{tikzcd} P \arrow[d, dash] & P \times PG \arrow[l] \arrow[r] \arrow[d, dash] & PG \arrow[d, dash] \\ X & P \times^G PG \arrow[l] \arrow[r] & BG \end{tikzcd}$$

Important:

$$\underline{\mathrm{Hom}}_G(P, PG) = \underline{\Gamma}(X, P \times^G PG)$$

is <u>contractible.</u> ~~[illegible]~~ ($X$ "cofibrant").

Eventual problem is to make sense of this classification theorem in more general circumstances, e.g. $X$ should be a topos.

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Stasheff problem: Fix a space $F$ and consider maps $Z \to X$ which are locally fiber homotopically trivial with fibre $F$. According to Segal's (account) this means that there is a covering $\mathcal{U} = \{U\}$ of $X$ ~~[illegible]~~ and fiber homotopy equivalences for each $U$:

$$Z|U \longleftarrow U \times F.$$

I think this means that $Z \to X$ is a quasi-fibration with fibre $F$.

Given such a map $\pi : Z \to X$, let

$$P = \underline{Heq}_{/X}(X \times F, P)$$

$$M = \underline{Heq}\,(F, F)$$

where the former is the space over $X$ whose fibre at $x$ is the space of homotopy equivalences of $F$ with $Z|\{x\}$, and the latter is the space of self homotopy equivalences of $F$. It is clear that $M$ acts on $P$ on the right and that we again have a diagram of simplicial spaces

$$\begin{array}{ccccc}
P & \longleftarrow & \mathrm{Nerve}(P \times M, M) & \longrightarrow & \mathrm{Nerve}(M, M) \\
\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow \\
X & \longleftarrow & \mathrm{Nerve}(P, M) & \longrightarrow & \mathrm{Nerve}(pt, M)
\end{array}$$

It should be true that the bottom left arrow is a homotopy equivalence. The proof might consist in showing that the canonical map

$$\mathrm{Nerve}\,(P, M) \longrightarrow \mathrm{Cech}\,(P \to X)$$

is a homotopy equivalence dimension-wise. Indeed in dim. $\nu$ it is

$$\begin{array}{ccc}
P \times M^{\nu} & \longrightarrow & (P/X)^{\nu+1} \\
(p, m_1, \ldots, m_\nu) & \longmapsto & (p, pm_1, pm_1m_2, \ldots, pm_1m_2 \ldots m_\nu)
\end{array}$$

and locally $P$ is fiber-homotopically ~~…~~ equivalent to

$$\underline{Heq}_{0/X}(X\times F, X\times F) = \underline{Heq}(F,F) = M,$$

thus this last map (bottom p. 3) is locally a fibre homotopy equivalence.

Thus if what precedes is correct, we have a definite map in the homotopy category from $X$ to $\mathrm{Nerve}(pt, M)$ which "classifies" $Z$ because of the ~~…~~ quasi-fibrations:

$$\begin{tikzcd}
Z \arrow[d] & \mathrm{Nerve}(P\times M, M)\times^M F \arrow[l] \arrow[r] \arrow[d] & \mathrm{Nerve}(M,M)\times^M F \arrow[d] \\
X & \mathrm{Nerve}(P,M) \arrow[l] \arrow[r] & \mathrm{Nerve}(pt, M)
\end{tikzcd}$$

dimension-wise:

$$\begin{tikzcd}
Z \arrow[d] & P\times M^\nu \times F \arrow[l] \arrow[r] \arrow[d] & M^\nu \times F \arrow[d] \\
X & P\times M^\nu \arrow[l] \arrow[r] & M^\nu
\end{tikzcd}$$

principal bundles for a topological groupoid. 

## March 12, 1971. On Haefliger structures.

Let $\Gamma$ denote a topological ~~category~~ groupoid:

$$\Gamma \times_R \Gamma \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \Gamma \rightrightarrows R$$

(basic example: the pseudogroup of diffeomorphism of $\mathbb{R}^n$)
Given a space $X$ we consider the fibred category over Open$(X)$ assigning to $U$ the category Hom$(U, \Gamma)$:

$$\mathrm{Hom}(U, \Gamma\times_R\Gamma) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathrm{Hom}(U,\Gamma) \rightrightarrows \mathrm{Hom}(U, \mathbb{R}).$$

This is a presheaf of categories in the strict sense over $X$. Since the morphisms glue it is a pre-stack. We form the associated stack. Actually we should do this over the gross site of topological spaces. Call the resulting stack $St(\Gamma)$. Then the category $St(\Gamma)(X)$ is the category of $\Gamma$ structures on $X$. The set of isomorphism classes we denote $H^1(X, \Gamma)$.

Example: Let $\Gamma$ be the top. category defined by a topological group $G$. Then $St(\Gamma)(X)$ is the category of principal $G$-bundles over $X$.

<u>Note</u>: The stack $St(\Gamma)$ in the general case does not have the property that any two objects are locally-isomorphic.

In the example where $\Gamma$ comes from $G$ one constructs a classifying map in the following map. Given a principal $G$-bundle $P$ over $X$ one has maps of simplicial spaces

$$\begin{array}{ccccc} X & \longleftarrow & P\times G\times G & \longrightarrow & G\times G \\ \vert\vert\vert & & \vert\vert\vert & & \vert\vert\vert \\ X & \longleftarrow & P\times G & \longrightarrow & G \\ \vert\vert & & \vert\vert & & \vert\vert \\ X & \longleftarrow & P & \longrightarrow & pt \end{array}$$

In other words over $X$ we have the nerve of the category defined by $(P,G)$ which is homotopy equivalent to $X$ and this category maps to $(pt, G)$. Thus we get a situation

$$X \xleftarrow{\text{heq.}} |N(P,G)| \longrightarrow |N(pt,G)| = BG$$

Now I want to imitate this example in the case of say the pseudo group $\Gamma$.

<u>First attempt:</u> Let $\xi$ be a $\Gamma$-structure on $X$. Then there is an open covering $\mathcal{U}$ of $X$ and isomorphisms in $St(\Gamma)(U)$ between $\xi|U$ and a function $f_U : U \to R$

$$\varphi_U : \xi|U \longrightarrow f_U \qquad U \in \mathcal{U}$$

Also given $U, V \in \mathcal{U}$ we obtain a map

$$\gamma_{U,V} : U \cap V \longrightarrow \Gamma$$

realizing the isomorphism $\varphi_U \varphi_V^{-1}$:

$$f_V|U\cap V \xleftarrow{\varphi_V} \xi|U\cap V \xrightarrow{\varphi_U} f_U|U\cap V$$

(since the pre-stack on $X$ defined by $\Gamma$ is a full-subcategory of $St(\Gamma)$, the isom. $\varphi_U\varphi_V^{-1}$ is defined by a map $U\cap V \longrightarrow \Gamma$.)

For each open set $W$ of $X$ I consider the ~~functions~~ maps $f : W \longrightarrow R$ which are isomorphic in $St(\Gamma)(W)$ to $\xi|W$. In other words we consider those functions which define the structure over $W$. Now I want to know if this is a sheaf, i.e. given $W = \bigcup W_i$ and $f : W \to R$ such that $f|W_i \simeq \xi|W_i$ in some way for each $i$, does it follow that $f \simeq \xi|W$?

(Question: To what extent is a $\Gamma$-structure on $X$ the same as a subsheaf of functions from $X$ to $R$? If $\Gamma$ comes from $G$, then ~~there is~~ $R$ is a point so ~~[illegible]~~ all the $\Gamma$-structures give the same sheaf of functions. On the other hand a $q$-foliation on a ~~[illegible]~~ manifold $X$ is clearly determined by the sheaf of $C^\infty$ functions to $\mathbb{R}^q$ flat along the leaves.)

In any case denote by $F_\xi$ the sheaf of functions which locally define the $\Gamma$ structure. Then we construct a simplicial object and a map:

$$\begin{tikzcd}
F \times_R \Gamma \times_R \Gamma \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, dashed, shift right=2] & \Gamma \times_R \Gamma \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
F \times_R \Gamma \arrow[r] \arrow[d, shift left=1] \arrow[d, dashed, shift right=1] & \Gamma \arrow[d, shift left=1] \arrow[d, shift right=1] \\
F \arrow[r] \arrow[d] & R \\
X &
\end{tikzcd}$$

The problem with this is that the simplicial object is not usually **acyclic** over $X$; for example if $X = pt$ and we take constant functions. On the other hand with the differentiable pseudo-group, if the functions come from a foliation, ~~[illegible]~~ then

$$F_x \times_R \Gamma \xrightarrow{\sim} F_x \times F_x$$

$$(f, \gamma) \longmapsto (f, \gamma f)$$

and the ~~[illegible]~~ simplicial object is acyclic.

## The principal bundle associated to a $\Gamma$-structure:

Suppose given a $\Gamma$-cocycle on $X$. This means I have a covering $\mathcal{U}$ of $X$ and for each $U \in \mathcal{U}$ a map $f_U : U \longrightarrow R$ and for each pair $U \cap V$ a map

$$\gamma^U_V : U \cap V \longrightarrow \Gamma$$

such that

$$\gamma^U_V f_U = f_V \qquad \left(\begin{array}{l} \text{i.e. source } \gamma^U_V = f_U \\ \phantom{\text{i.e. }}\text{target } \gamma^U_V = f_V \end{array}\right)$$

and

$$\gamma^V_W \gamma^U_V = \gamma^U_W \qquad \text{on } U \cap V \cap W.$$

To obtain the principal bundle we glue together

$$P = \coprod_{U \in \mathcal{U}} U \times_{(f_U, s)} \Gamma \Big/ \text{equivalence relation}$$

the equivalence relation is as follows. Given $x \in U$ $\gamma \in \Gamma$ such that $f_U(x) = s(\gamma)$ let $(x, \gamma)_U$ denote the associated elt. in the disjoint union. Then

$$(x, \gamma)_U \sim (x, \gamma')_{U'}$$

iff

$$\gamma'(x)\, \gamma^U_{U'}(x) = \gamma(x)$$

$$\begin{tikzcd}
f_U(x) \arrow[rr, "\gamma^U_{U'}(x)"] \arrow[dr, "\gamma(x)"'] & & f_{U'}(x) \arrow[dl, "\gamma'(x)"] \\
& \cdot &
\end{tikzcd}$$

The transitivity condition guarantees this is an equivalence relation. The equivalence relation **implies that** $\gamma'(x)$ and $\gamma(x)$ **have the same** target, hence we have ~~the~~ maps

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & R \end{tikzcd}.$$

(Note that for a pseudo-group $s : \Gamma \longrightarrow R$ is etale hence $P$ is etale over $X$ in this case.) ~~the~~

Now the desired resolution of $X$ is

$$\begin{tikzcd} \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ P \times_R \Gamma \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ P \arrow[r] \arrow[d] & R \\ X & \end{tikzcd}$$

<u>Definition</u>: A $\Gamma$-torsor over $X$ is a space $P$ endowed with maps

$$p : P \longrightarrow X$$
$$q : P \longrightarrow R$$
$$\Delta : P \times_R \Gamma \longrightarrow P$$

which locally over $X$ is isomorphic to $P = X \times_R \Gamma$ for some $f: X \longrightarrow R$ with maps

$$pr_1 : X \times_R \Gamma \longrightarrow X$$

$$t \cdot pr_2 : X \times_R \Gamma \longrightarrow R$$

$$id \times \Delta : X \times_R \Gamma \times_R \Gamma \longrightarrow X \times_R \Gamma$$

---

<u>Remarks:</u> (1) A section $s: X \longrightarrow P$ of $p$ gives a "trivialization"

$$\begin{tikzcd} X \times_{R} \Gamma \arrow[rr, "\cong"] \arrow[dr, "s \times id"'] & & P \\ & P \times_R \Gamma \arrow[ur, "\Delta"'] & \end{tikzcd}$$

$$(X \xrightarrow{gs} R)$$

This is clear because $\Gamma$ is a groupoid (these things have only to be checked in sets).

(2) The difference of two sections is a morphism $X \longrightarrow \Gamma$. Indeed

$$P \times_X P \xleftarrow{\sim} P \times_R \Gamma$$

March 13, 1971

Review proof of classification theorem for principal $G$-bundles.

Given $P$ over $X$, we form the mixing diagram

$$\begin{tikzcd} P \arrow[d] & P \times PG \arrow[l] \arrow[r] \arrow[d] & PG \arrow[d] \\ X & P \times^G PG \arrow[l, "p_1"'] \arrow[r, "p_2"] & BG \end{tikzcd}$$

As $p_2$ is a homotopy equivalence we obtain an element of $[X, BG]$. This defines a map

$$H^1(X, G) \longrightarrow [X, BG].$$

On the other hand we have a map

$$[X, BG] \longrightarrow H^1(X, G)$$

by associating to $f$ the bundle $f^*(PG)$. (Use here that $f^*(P)$ depends only on the homotopy class of $f$, which results from the fact that $P \to X$ is a fibration.)

The composition

$$H^1(X, G) \longrightarrow [X, BG] \longrightarrow H^1(X, G)$$

is the identity by the mixing diagram. Indeed to $cl(P) \in H^1(X,G)$ we associate the homotopy class of the map $p_2 s$, where $s$ is a section of $p_2$, so we want to show that

$$\begin{array}{ccc} (p_2 s)^* \, cl(PG) & \overset{?}{=} & cl(P) \\ \| & & \\ s^* cl(P \times PG) = s^* p_1^* cl(P) & = & (p_1 s)^* cl(P). \end{array}$$

~~Apply $p_2^*$ to both sides and use that ...~~;
so done as $p_1 s = id$.

To show that the composition

$$[X, BG] \longrightarrow H^1(X,G) \longrightarrow [X, BG]$$

is the identity, we form the mixing diagram for the bundle $PG$ over $BG$.

$$\begin{tikzcd}
PG \arrow[d] \arrow[r, dashed, bend left, "\Delta"] & PG \times PG \arrow[l] \arrow[r] \arrow[d] & PG \arrow[d] \\
BG \arrow[r, dashed, bend left, "s"] & PG \times^G PG \arrow[l, "p_1"] \arrow[r, "p_2"] & BG
\end{tikzcd}$$

~~We must show that $p_2 s = id$~~ Take $s$ to be induced by the diagonal, whence $p_2 s = id$ and done.

Logic of above argument:

i) Over $BG$ there is a canonical bundle $PG$, hence there is a map

$$[X, BG] \longrightarrow H^1(X, G)$$

as $H^1(?, G)$ is a homotopy functor

ii) From the mixing diagram and contractibility of $PG$, there is a map

$$H^1(X, G) \longrightarrow [X, BG]$$

iii) The composite

$$[X, BG] \longrightarrow H^1(X, G) \longrightarrow [X, BG]$$

is the identity because given $f: X \longrightarrow BG$ it induces $P = f^*(PG)$ which comes provided with an equivariant map $P \longrightarrow PG$; hence there is a section $s: X \longrightarrow P \times^G PG$ with $p_2 s = f$; as the homotopy class of $p_2 s$ is the map ii) applied to $f$, we are done.

iv) The composite

$$H^1(X, BG) \longrightarrow [X, BG] \longrightarrow H^1(X, BG)$$

is the identity because the map $p_2 s$ constructed in the mixing diagram pulls back $PG$ to $P$.

Mixing diagram for a principal $\Gamma$-bundle:

$$\begin{array}{lccccc}
P\Gamma: & \Gamma\times_R\Gamma\times_R\Gamma & \overset{p_{23}}{\underset{1\times\Delta}{\overset{\Delta\times 1}{\rightrightarrows}}} & \Gamma\times_R\Gamma & \underset{\Delta}{\overset{p_2}{\rightrightarrows}} & \Gamma \\
\downarrow & \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle s} \\
B\Gamma: & \Gamma\times_R\Gamma & \overset{p_2}{\underset{p_1}{\overset{\Delta}{\rightrightarrows}}} & \Gamma & \underset{s}{\overset{t}{\rightrightarrows}} & R
\end{array}$$

$p_{234}$, $p_{23}$, $p_2$

$$\begin{array}{lccccc}
P\tilde{\times}_R P\Gamma: & P\times\Gamma\times\Gamma\times\Gamma & \overset{\Delta\times1\times1}{\underset{1\times1\times\Delta}{\overset{1\times\Delta\times1}{\rightrightarrows}}} & P\times\Gamma\times\Gamma & \underset{1\times\Delta}{\overset{\Delta\times1}{\rightrightarrows}} & P\times\Gamma \\
\downarrow & \downarrow{\scriptstyle p_{123}} & & \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_1} \\
(P\tilde{\times}_R P\Gamma)\times_\Gamma R: & P\times\Gamma\times\Gamma & \overset{\Delta\times1}{\underset{p_{12}}{\overset{1\times\Delta}{\rightrightarrows}}} & P\times\Gamma & \underset{p_1}{\overset{\Delta}{\rightrightarrows}} & P
\end{array}$$

This leads to a diagram of simplicial spaces

$$\begin{array}{ccccc}
P & \longleftarrow & P\tilde{\times}_R P\Gamma & \longrightarrow & P\Gamma \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{\bar{p}_1} & (P\tilde{\times}_R P\Gamma)\times_\Gamma R & \xrightarrow{\bar{p}_2} & B\Gamma = P\Gamma\times_\Gamma R
\end{array}$$

where the right horizontal maps are the projections forgetting the $P$-factors.

Consider the classification theorem now formally assuming that there is a reasonable realization functor. Then $\bar{p}_1$ is a homotopy equivalence, so choosing a homotopy inverse $s$ one obtains a map $\bar{p}_2 s : X \to B\Gamma$. This gives a map

$$H^1(X,\Gamma) \longrightarrow [X, B\Gamma]$$

and homotopic $\Gamma$-structures give rise to homotopic maps. (The proof of this will require some argument. The point is that if we have a $\Gamma$-bundle $P$ over $X \times I$, then we have

$$\begin{array}{ccc}
(\dot{P} \times_R P\Gamma) \times_\Gamma R & \longrightarrow & (P \times_R P\Gamma) \times_\Gamma R \\
\Big\downarrow \dot{\bar{p}}_1 & & \Big\downarrow \bar{p}_1 \\
X \times \dot{I} & \longrightarrow & X \times I
\end{array}$$

where the vertical maps are homotopy equivalences. It will be necessary to construct compatible homotopy inverses. Sections of $\bar{p}_1$ will be constructed using partitions of 1, so what is needed should be found in the Eilenberg-Steenrod book.) Thus we have a map

$$\Phi : H^1(X,\Gamma)/\text{hom.} \longrightarrow [X, B\Gamma].$$

On the other hand $P\Gamma$ defines an

element of $H^1(B\Gamma, \Gamma)$, hence by functorality we have a map

$$\Psi : [X, B\Gamma] \longrightarrow H^1(X, \Gamma)/\text{hom.}$$

~~The composition~~

Proof that $\Psi \Phi = id$: Start with $f : X \longrightarrow B\Gamma$. Then $\Phi(cl f) = cl(P)$ where $P = f^* P\Gamma$. Moreover there is a map $\tilde{f} : P \longrightarrow P\Gamma$ covering $f$. I claim this induces a section $s$ of $\bar{p}_1$ in the mixing diagram such that $\bar{p}_2 s = f$. As $\Psi\, cl(P) = cl(\bar{p}_2 s)$ by definition one wins. (This claim requires work due to the fact that the model $P \tilde{\times}_R P\Gamma$ is not clean somehow.)

It is necessary to digress and work out the mixing diagram more carefully.

Given principal $\Gamma$-bundles $\mathbb{P} = (X, P, p : P \to X, q : P \to R, \mu : P \times_R \Gamma \to P)$ and $\mathbb{P}' = (X', P', p' : P' \longrightarrow X', q' : P' \longrightarrow R, \mu' : P' \times_R \Gamma \to P')$ we wish to define their product which perhaps should be denoted $\mathbb{P} \times \mathbb{P}'$. For the total space take

$$P \underset{(q,q')}{\times} P' = \{ (\lambda, \lambda') \mid q(\lambda) = q(\lambda') \in R \} \qquad \left(\text{denote this } P \times_R P'\right)$$

and make $\Gamma$ act by

$$((\lambda, \lambda'), \gamma) = (\lambda\gamma, \lambda'\gamma) \qquad \text{if } q(\lambda) = q(\lambda') = s(\gamma)$$

Denote the orbit space of the action by $P \times^{\Gamma} P'$.
I want to show ~~the orbit space~~ then that the projection

$$p'' : P \times_R P' \longrightarrow P \times^{\Gamma} P'$$

together with the canonical maps

$$q'' : P \times_R P' \longrightarrow R$$

$$\mu'' : (P \times_R P') \times_R \Gamma \longrightarrow P \times_R P'$$

~~constitute~~ a principal $\Gamma$ bundle. The question is local over $X$ (also over $X'$), so we can assume

$$P = X \times_R \Gamma$$

for some $f : X \longrightarrow R$. Then

$$P \times_R P' = \left\{ (x, \gamma, \lambda') \;\middle|\; \begin{array}{l} f(x) = s(\gamma) \\ t(\gamma) = q'(\lambda') \end{array} \right\}$$

and the action is: if $q'(\lambda') = s(\gamma_1)$

$$(x, \gamma, \lambda') \cdot \gamma_1 = (x, \gamma\gamma_1, \lambda'\gamma_1)$$

(where $\gamma \cdot \gamma_1 = \Delta(\gamma, \gamma_1)$). ~~Since $\Gamma$ is ...~~ In each orbit there is a unique representative of the form $(x, id, \lambda')$ where $f(x) = s(id) = t(id) = q'(\lambda')$. Thus

$$(X \times_R \Gamma) \times^{\Gamma} P' = X \times_R P'_{q'}$$ and

$$p'' : \quad P \times_R P' \longrightarrow X \times_R P'$$
$$(x, \gamma, \lambda') \longmapsto (x, \lambda'\gamma^{-1}).$$

Next one has

$$(X \times_R P') \times_R \Gamma \xrightarrow{\sim} P \times_R P'$$
$$((x, \lambda'), \gamma_1) \longmapsto (x, \gamma_1, \lambda'\gamma_1)$$

( here $X \times_R P' \longrightarrow R$ sends $(x, \lambda') \mapsto f(x) = g'(\lambda')$ and $s(\gamma_1) = f(x)$.) because $\Gamma$ is a groupoid. This proves that we have defined a principal $\Gamma$-bundle. Thus we have the mixing diagram

$$\begin{tikzcd}
P \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & P \times_R P' \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & P' \arrow[d] \\
X & P \times^{\Gamma} P' \arrow[l, "\bar{p}_1"'] \arrow[r, "\bar{p}_2"] & X'.
\end{tikzcd}$$

(these are the two projections of $\mathbb{P} \times \mathbb{P}'$ onto the factors). Moreover if $P|U = U \times_R \Gamma$, then

$$(\bar{p}_1)^{-1} U = U \times_R P' )$$

For the classification theorem we assume that the map $g' : P' \longrightarrow R$ is a fibre homotopy equivalence, i.e. $\exists$ a section and fibre-wise retraction to the section

Then it follows that $\bar{p}_1$ is a quasi-fibration with contractible fibres (I hope), hence $\bar{p}_1$ is a homotopy equivalence. In any case $\bar{p}_1$ induces an isomorphism on cohomology by Leray.

Now the classification theorem goes through: Given $P$ over $X$, $\bar{p}_1$ has a homotopy inverse $s$ so we obtain a map $\bar{p}_2 s : X \longrightarrow X'$ such that

$$(\bar{p}_2 s)^* \, cl\, \mathbb{P}' = s^* \, cl(\mathbb{P} \times \mathbb{P}') = s^* \bar{p}_1^* \, cl(\mathbb{P})$$
$$= (\bar{p}_1 s)\, cl(\mathbb{P}) = cl(\mathbb{P})$$

the last step because $\bar{p}_1 s \sim id_X$. Thus we have defined a map

$$H^1(X, \Gamma) \longrightarrow [X, X']$$

by sending $cl(\mathbb{P})$ to the homotopy class of $\bar{p}_2 s$. Again modulo construction of $s$ this induces

$$H^1(X, \Gamma)/\text{hom.} \xrightarrow{\Phi} [X, X']$$

and we have shown that if $\Psi$ is the map

$$[X, X'] \xrightarrow{\Psi} H^1(X, \Gamma)/\text{hom.}$$

defined by $\Psi(cl f) = cl(f^* \mathbb{P}')$, then $\Psi \Phi = id$. On the other hand given $f : X \longrightarrow X'$, $\Psi(cl f) = cl(f^* \mathbb{P}')$;

setting $\mathbb{P} = f^*\mathbb{P}'$, we have a map $\mathbb{P} \longrightarrow \mathbb{P}'$ over $f$, hence a section $s$ of $\bar{P}_1$ defined by the graph such that $\bar{p}_2 s = f$; as $\Phi\Psi(\mathrm{cl}\, f) = \Phi(\mathrm{cl}\, f^*\mathbb{P}') = \mathrm{cl}(\bar{p}_2 s)$, we see $\Phi\Psi = \mathrm{id}$ also.

Things to be checked carefully using appropriate paracompactness assumptions on $X$.

i) If $g' : P' \longrightarrow R$ is a fiber-homotopy-equivalence then in the mixing diagram $\bar{p}_1$ is a homotopy equivalence

ii) Given $P$ over $X \times I$. one can find compatible homotopy inverses for the vertical arrows

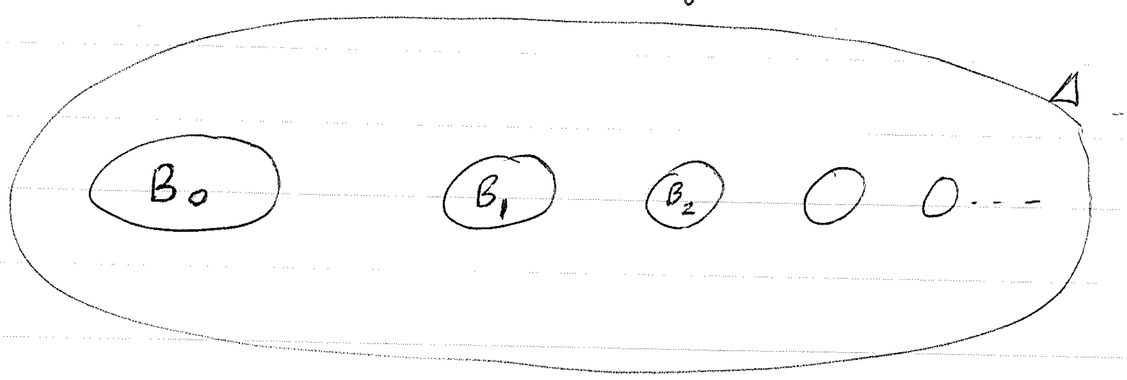
$$\begin{array}{ccc} (P|X\times\dot{I}) \times^{\Gamma} P' & \longrightarrow & P \times^{\Gamma} P' \\ \downarrow & & \downarrow \\ X \times \dot{I} & \longrightarrow & X \times I \end{array}$$

iii) The Milnor construction fulfills these requirements.

March 14, 1971

I want to understand Mather's theorem that the group of homeomorphisms of $R^n$ with compact support has no cohomology. Call this group $G$.

First I show that $G_{ab} = 0$. Let $g \in G$ have support in a ball $B_0$ and choose balls $B_1, B_2, \ldots$ which are disjoint and converge to a limit




all in some bounded region $\Delta$. Let $z \in G$ have support in $\Delta$ and carry $B_0$ onto $B_1$, $B_1$ onto $B_2$, etc. Then if $\sigma(x) = zxz^{-1}$ we have that

$\sigma(g) = zgz^{-1}$ has support in $B_1$

$\sigma^2(g)$ " " " $B_2$

and the infinite product

$$T(g) = \sigma(g)\sigma^2(g) \cdots\cdots$$

converges defining an element of $G$. Then

$$\sigma^{-1}T(g) = g\,T(g) \implies g = T(g)^{-1}z^{-1}T(g)z$$

showing that $g$ is a commutator.

(Digression: One knows that the inner auto. $g \mapsto zgz^{-1}$ acts trivially on the homology with constant coefficients of a group $G$. The homotopy operator is

$$\begin{aligned} h(g_1,\dots,g_k) = {} & (z^{-1}, zg_1z^{-1}, zg_2z^{-1}, \dots, zg_kz^{-1}) \\ & -(g_1, z^{-1}, zg_2z^{-1}, \dots, zg_kz^{-1}) \\ & \vdots \\ & +(-1)^k(g_1,\dots,g_k, z^{-1}) \end{aligned}$$

I think. It checks for $k=1$.)

Observe that the same argument shows that $\tilde{G}_{ab} = 0$ where $\tilde{G}$ = diffeos. of $\mathbb{R}$ with support contained in $(-\infty, a]$ for some $a$. Indeed given $g \in \tilde{G}$ choose $z \in \tilde{G}$ such that $z(x) = x-1$ for $x \leq a$ where $(-\infty, a]$ contains the support of $g$. Then $\sigma^k(g) = z^k g z^{-k}$ sends $x \mapsto g(x+k)-k$, and has support in $(-\infty, a-k]$ so

$$T(g) = \sigma g \cdot \sigma^2 g \cdots$$

converges in $\tilde{G}$, and $g = T(g)^{-1} z^{-1} T(g) z$. (Cleaner to note that the translation $x \mapsto x-1$ is the product of $z$ and something centralizing the subgp. with supports in $(-\infty, a]$).

<u>Question</u>: Let $G$ be a group endowed with an endomorphism $\sigma$ such that $x \mapsto x\sigma(x)^{-1}$ is bijective. If $\sigma$ acts trivially on $H^*(G)$ does it follow that $H^+(G)=0$? (true for $H^1(G)$.)

---

Mathers proof that $H_+(G)=0$ uses the algebra structure and the fact that ~~if $y \in$~~ the monoid of isom. classes of $G$ bundle over $X$ finite, then $\exists\, x \ni x+y \cong x$, hence any primitive coh. class $c$ will satisfy

$$c(x)+c(y) = c(x+y) = c(x) \Rightarrow c(y)=0$$

---

March 15, 1971 More Mather.

Given two real numbers $a \leq b$, let $G_{ab}$ be the diffeomorphisms of $R$ with support in $[a,b]$. Then we have a 2-category:

Objects: real nos. $a$
1-maps: for each pair $a, b$ there is a unique 1-arrow $a \to b$ if $a \leq b$, and none otherwise
2-maps: the two arrows from $a \to b$ to $a \to b$ are the elements of $G_{ab}$.

In other words Hom $(a,b)$ is the category associated to $G_{ab}$ if $a \leq b$ and is empty otherwise.

Using Segal's classifying space we obtain a simplicial space

$$(*) \qquad \cdots \coprod_{a \leq b \leq c} BG_{ab} \times BG_{bc} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \coprod_{a \leq b} BG_{ab} \rightrightarrows \coprod_{a} pt$$

which is a topological category with objects $a \in R$ and morphisms $BG_{ab}$ from $a$ to $b$. Hopefully the realization of this simplicial space is Mather's $B(BG)$.

Assuming this we want a map of $B(BG)$ into $B\Gamma$, that is, a $\Gamma$-structure on the realization of $(*)$. Recall that the basic map $\Sigma BG_{ab} \longrightarrow B\Gamma$ is obtained by taking the canonical foliation on the flat bundle $PG_{ab} \times_{G_{ab}} [a,b]$

and identifying $PG_{ab} \times^{G_{ab}} [a,b] \simeq BG_{ab} \times [0,1]$ using the triviality of $[0,1]$-bundles. This suggests that the map from the realization of $(*)$ to $B\Gamma$ should be obtained as follows: ~~[illegible]~~ ~~Let~~

Given real numbers ~~[illegible]~~ $a_0 \leq a_1 \leq \cdots \leq a_q$ we make ~~[illegible]~~

$$G_{a_0 a_1 \cdots a_q} = G_{a_0 a_1} \times G_{a_1 a_2} \times \cdots \times G_{a_{q-1} a_q}$$

act on $\Delta(q)$ as follows. Let $f : \Delta(q) \longrightarrow R$ be the function with

$$f(t_0, \ldots, t_q) = \sum_{i=0}^{q} t_i a_i .$$

~~[illegible]~~ Consider the case where $a_0 < a_1 < \cdots < a_q$. Then each of the slices

$$f^{-1}(x) \qquad a_0 < x < a_q$$

is an $(n-1)$-simplex and each region

$$f^{-1}\big((a_{i-1}, a_i)\big)$$

is homeomorphic to

$$f^{-1}(x) \times (a_{i-1}, a_i) \qquad a_{i-1} < x < a_i$$

in an "obvious" way. ?

$$\varphi_{a_0 \cdots a_q} : U_{a_0 \cdots a_q} \longrightarrow BG_{a_0 \cdots a_q} \qquad\qquad U_a = \rho_a^{-1}(0,1]$$

such that transitivity holds, i.e. commutativity in

$$\begin{tikzcd}
U_{a_0 \cdots a_i \cdots a_q} \arrow[d, "\varphi_{a_0 \cdots a_q}"] & \subset & U_{a_0 \cdots \hat{a}_i \cdots a_q} \arrow[d, "\varphi_{a_0 \cdots \hat{a}_i \cdots a_q}"] \\
BG_{a_0 \cdots a_q} \arrow[rr] & & BG_{a_0 \cdots \hat{a}_i \cdots a_q}
\end{tikzcd}$$

This means that we have a covering $\mathcal{U} = \{U_a ; a \in \mathbb{R}\}$ of $X$ and over each overlap $U_{ab}$, $a<b$, a principal $G_{ab}$-bundle $P_{ab}$ and for $a<b<c$ maps

$$m_{abc} : P_{ab}|U_{abc} \times P_{bc}|U_{abc} \longrightarrow P_{ac}|U_{abc}$$

equivariant for the homomorphism (canonical)

$$G_{ab} \times G_{bc} \longrightarrow G_{ac} ,$$

such that things are compatible over $U_{abcd}$ for all $a<b<c<d$.

---

Suppose given such a gadget $\{U_a, P_{ab}, m_{abc}\}$ for $X$. Then instead of the realization of ,

$$(+) \qquad \Rrightarrow \coprod_{a_0 \leq a_1} U_{a_0 a_1} \rightrightarrows \coprod_{a_0} U_{a_0}$$

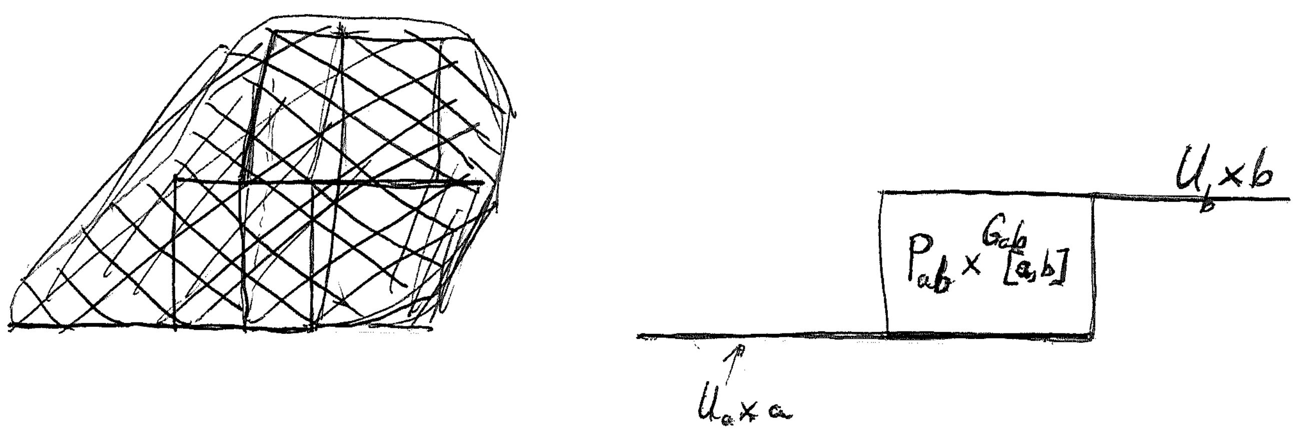
which [redacted] lies X and has partitions of 1 for its sections we can form the space

$$(++)\qquad \bigcup_q \bigcup_{a_0<\cdots<a_q} P_{a_0\cdots a_q} \times^{G_{a_0\cdots a_q}} \Delta(q)$$

which is homeomorphic to the realization of (+), once we choose compatible isos.

$$P_{a_0\cdots a_q} \times^{G_{a_0\cdots a_q}} \Delta(q) \cong U_{a_0\cdots a_q} \times \Delta(q).$$

Picture:




The point is that the realization of (++) carries a **canonical** $\Gamma$-structure.

Now in the universal case we take the space

$$(**)\qquad \bigcup_q \bigcup_{a_0<\cdots<a_q} PG_{a_0\cdots a_q} \times^{G_{a_0\cdots a_q}} \Delta(q)$$

which comes provided with a canonical gadget $\{U_a, P_{ab}, m_{abc}\}$ and moreover such that the corresponding (++) realization has a canonical section.

<u>Conclusion:</u> To prove Mather's theorem we must show how a $\Gamma$-structure over $X$ is homotopic to one produced by a gadget $\{U_a, P_{ab}, m_{abc}\}$ together with a section of the associated (**) space.

<u>March 18, 1971.</u> More Mather

Let $G_{ab}$ denotes the diffeos. of $\mathbb{R}$ with support in $[a,b]$ and

$$G_{a_0 \cdots a_q} = G_{a_0 a_1} \times \cdots \times G_{a_{q-1} a_q}$$

Letting $BG = |\overline{W}(G)|$ we have a topological category

$$\cdots\cdots \coprod_{a_0 \leq a_1 \leq a_2} BG_{a_0 a_1 a_2} \substack{\rightarrow\\\rightarrow\\\rightarrow} \coprod_{a_0 \leq a_1} BG_{a_0 a_1} \rightrightarrows \coprod_{a_0} pt$$

whose realization we wish to show is $B\Gamma$. Therefore we want to produce a $\Gamma$-structure over the realization.

The realization, which we denote $B(BG)$, consists of a union

$$\bigcup_{q \geq 0} \bigcup_{a_0 < \cdots < a_q} BG_{a_0 \cdots a_q} \times \Delta(q)$$

where the identifications are made by face operators. Thus every point $\xi$ of $B(BG)$ determines a sequence

$$\vec{a}(\xi) = (a_0 < \cdots < a_q)$$

of real numbers, a point

$$b(\xi) \in BG_{a_0 \cdots a_q} \, ,$$

and a point

$$t(\xi): \quad t_0 + \cdots + t_q = 1 \qquad 0 < t_i$$

of ~~the~~ the interior of $\Delta(q)$. As with the Milnor model for $BG$, one agrees to coallesce $\vec{a}(\xi)$ and $t(\xi)$ into a ~~linear combination~~ finite sum

$$\sum_{a\in\mathbb{R}} t_a = 1 \qquad 0\leq t_a$$

the sequence $a_0 < \cdots < a_q$ being precisely the $a$ for with $t_a > 0$. Thus every point $\xi$ of $B(BG)$ determines a ~~~~ point $t(\xi) = \{t_a(\xi), a\in\mathbb{R}\}$ $\ni$

$$\sum_{a\in\mathbb{R}} t_a(\xi) = 1 \qquad t_a(\xi)\geq 0.$$

of the simplex with vertices $\{a\in\mathbb{R}\}$; in addition it determines

$$b(\xi) \in BG_{a_0\cdots a_q}$$

where $a_0 < \cdots < a_q$ are those $a$ for which $t_a(\xi) > 0$.

The topology on $B(BG)$ we take to be the Milnor one; thus it is the coarsest such that the functions $t_a$ and ~~~~ $b$ are continuous.

To give a map from $X$ to $B(BG)$ means we give a partition of unity on $X$

$$\sum_{a\in\mathbb{R}} \rho_a = 1$$

and for each sequence ~~~~ $a_0 < \cdots < q$ a map