March 1, 1971

Check the grand conjecture against the Tate result for $K_2$. Thus let $C$ be a Dedekind subring of a number field $F$, and denote by $T^{(n)}$ the $n$-fold tensor product of the Tate character on $\mathbb{Q}_p$; we assume $C^{-1} \subseteq C$. Then I expect an exact sequence

$$H^i(F, T^{(g)}) \rightarrow \bigoplus_{v \in C} H^{i-2}(k(w), T^{(g)}) \rightarrow H^i(C, T^{(g)}) \rightarrow H^i(F, T^{(g)})$$

where the middle map is an $i_*$ I also expect a corresponding sequence for $K$-groups. Moreover there should be compatibility:

$$0 \rightarrow K_2(C) \rightarrow K_2(F) \rightarrow \bigoplus_{v \in C} K_1(k(v)) \rightarrow K_1(C)$$

$$0 \rightarrow H^2(C, T^{(a)}) \rightarrow H^2(F, T^{(a)}) \rightarrow \bigoplus_{v \in C} H^1(k(w), T^{(a)}) \rightarrow H^3(C, T^{(a)})$$

(The group $H^3(C, T^{(a)})$ should be zero all have been removed?)

Now Tate claims to have proved that

$$K_2(F) \sim H^2(F, T^{(2)})_{\text{tors}}$$

so by five lemma, his result should be equivalent to

$$K_2(C) \sim H^2(C, T^{(2)})_{\text{tors}}.$$
By my yoga there should be an Atiyah-Hirzebruch spectral sequence converging to the l-adic completion $K_*(\mathcal{C})$ and starting with $\text{H}^i(\mathcal{C}, \mathcal{T}^{(l)})$.

<table>
<thead>
<tr>
<th>$\text{H}^0(\mathcal{C}, \mathcal{T}^{(l)})$</th>
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<th>$\text{H}^2(\mathcal{C}, \mathcal{T}^{(l)})$</th>
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</table>

Now this spectral sequence collapses as $\mathcal{C}$ is of coh. dim 2, yielding isomorphisms

$K_1(\mathcal{C}) = \text{H}^1(\mathcal{C}, \mathcal{T}^{(l)})$

$K_2(\mathcal{C}) = \text{H}^2(\mathcal{C}, \mathcal{T}^{(l)})$

$K_3(\mathcal{C}) = \text{H}^4(\mathcal{C}, \mathcal{T}^{(l)})$

$K_4(\mathcal{C}) = \text{H}^2(\mathcal{C}, \mathcal{T}^{(l)})$

The first is probably OK because of the exact sequence

$$0 \longrightarrow \text{H}^1(\mathcal{C}, \mathcal{T}^{(l)}) \longrightarrow \text{H}^1(F_2, \mathcal{T}^{(l)}) \longrightarrow \bigoplus_{\vec{v} \in \mathbb{C}} \mathbb{Z}_l \longrightarrow \text{H}^2(\mathcal{C}, \mathcal{T}^{(l)})$$
(which has to be interpreted accurately!?) shows that
\[ H^1(C, \mathbb{T}^0) = (C')^2 \] which should coincide with \( K_1 C_2 \)
as \( l \) is out of \( C \). (Calvin-Moore shows that
\[ K_2 F \rightarrow \bigoplus_{v \in \mathcal{L}} \mu_v \], hence \( K_1 C = C' \)).

According to Tate

\[ H^1(C, \mathbb{T}^0) \sim \rightarrow H^1(F, \mathbb{T}^0) \]

has rank \( n_2 \) over \( \mathbb{Z}_l \) and this fits nicely with
Soulé's determinant of \( K_2(C) \otimes \mathbb{Q} \). Finally Tate
knows that

\[ K_2(C) = H^2(C, \mathbb{T}^0)_{tor} \]

so it seems (conjecturally) that \( H^2(C, \mathbb{T}^0) \) should
be finite and that

\[ \Omega_2(K_2 F) = H^2(F, \mathbb{T}^0) \].

---

Let \( \overline{C} \) be the
integral closure of \( C \) in \( F(\mu_{l^n}) \). We assume \( \mu_{l^n} \subset F \)
or \( \mu_{l^n} \subset F \) if \( l = 2 \), so that the Galois group
\( \Gamma \) of \( F(\mu_{l^n}) / F \) is \( \cong \mathbb{Z}_l \). Let \( X = H_1(\overline{C}, \mathbb{Z}_l) \),
the Galois group of the maximal pro-\( l \)-abelian extension of
\( F(\mu_{l^n}) \) unramified outside of \( l \). Then
for any \( \mathbb{Z}_2 \)-module \( A \).
Hence we have short exact sequences:

\[
0 \to H^1(\Gamma; H^{d-1}(\mathcal{C}, T^{(0)})) \to H^0(\mathcal{C}, T^{(0)}) \to H^0(\mathcal{C}, T^{(2)}) \to 0.
\]

Now it should be the case that \( \mathcal{C} \) has cohomological dimension 1 (?), hence we hope for

\[
0 \to T^{(0)}/\Gamma \to H^1(\mathcal{C}, T^{(0)}) \to \text{Hom}_F(\mathcal{C}, T^{(0)}) \to 0.
\]

\[
\text{like the finite field contribution}
\]

\[
\text{(c) } K_{2r-1}(\mathcal{C})
\]

\[
\text{(d) } \mathbb{Z}_2^{r_2} \text{ if } r \text{ even}
\]

\[
\mathbb{Z}_2^{n+r_2} \text{ if } r \text{ odd (by Borel)}
\]

\[
\text{Hom}(\mathcal{C}, T^{(0)})/\Gamma \cong H^2(\mathcal{C}, T^{(0)})
\]

\[
\text{(e) } K_{2r-2}(\mathcal{C})
\]

Remarks: Somehow the odd \( K \) groups are trivial and don't depend on the number of points left out of \( \mathcal{C} \).

Conjecture: \( K_{\text{odd}}(\mathcal{C}) \cong K_{\text{odd}}(F) \) (in dim > 1,\( F \) not complete).

This agrees with \( H^1(\mathcal{C}, T^{(0)}) \cong H^1(F, T^{(0)}) \).
March 6, 1971

More Math.

How Mather deloops $BG$, $G = \text{orientation-preserving diffeomorphisms of } \mathbb{R}$.

**Bisimplicial version.** Let $\overline{W}(G)$ be the standard simplicial "classifying space" for $G$:

$$\overline{W}(G)_\nu = G^\nu$$

$$d_0(\tau_{\nu}^1, \ldots, \tau_{\nu}^\nu) = (\tau_{\nu+1}^1, \ldots, \tau_{\nu}^\nu)$$

$$d_\nu(\tau_{\nu}^1, \ldots, \tau_{\nu}^\nu) = (\tau_{\nu}^1, \ldots, \tau_{\nu-1}^\nu)$$

Let

$$\overline{W}(G)^{(p)}_\nu \subset \overline{W}(G)^{(p)}_\nu = (G^\nu)_p$$

be the subset consisting of $(\tau_{\nu}^1, \ldots, \tau_{\nu}^\nu)$ such that

$$I(\tau_{\nu}^1) < I(\tau_{\nu}^2) < \cdots < I(\tau_{\nu}^p),$$

where $I(\tau_{\nu}^1, \ldots, \tau_{\nu}^\nu)$ is the smallest closed interval of $\mathbb{R}$ outside of which the $\tau_{\nu}^i$ are identity.

Clearly $\overline{W}(G)^{(p)}$ is a simplicial subset of $\overline{W}(G)^{(p)}$.

If we let $G(I)$ be the subgroup of $G$ consisting of the diffeos with support in $I$,

$$\overline{W}(G)^{(p)} = \bigcup_{I_1 < I_2 < \cdots < I_p} \overline{W}(G(I_1) \times \cdots \times G(I_p))$$
On the other hand there are homomorphisms
\[ G(I) \times G(I') \rightarrow G(I') \]
if \( I' \) is the smallest interval containing \( I \cup I' \). This gives us maps
\[ \overline{W}(G(I_1) \times \cdots \times G(I_p)) \rightarrow \overline{W}(G(I_1) \times \cdots \times G(I_j + I_{j+1} \cup \cdots \times G(I_p)) \]
which allow us to form a bi-simplicial set
\[ (p, q) \rightarrow \overline{W}(G)^{(p)} \]
with face operators
\[ d^h_j (g_1, g_2, \cdots, g_p) = (g_1, \cdots, g_i^{g_j}, g_{i+1}^{g_j}, \cdots, g_p^{g_j}) \]
\[ d^v_i (g_1, \cdots, g_i^1), \cdots, (g_1, \cdots, g_i^p)) = (g_1, \cdots, g_i^{g_j}, \cdots, g_i^{g_j}) \]

Notation: \( \overline{W}(G)^{(k)} \) for this bi-simplicial set.

Now let \( G_p \) be the group of diffeos. of \( \Delta(p) \) which preserve the faces and which are the identity in a neighborhood of each vertex. Clearly \( \{ G_p \} \) is a simplicial group, and
$G_1 = G$ once we choose a diffeom. of $\mathbb{R}$ with $(0,1)$. Unfortunately there is no foliation of the $\Delta(p)$-bundle over $BG_p$ of codimension 1, so this doesn’t seem to be very promising.
Let $G_n$ be the group of diffeomorphisms of $\mathbb{R}$ which are the identity in a neighborhood of each of the points $0, 1, \ldots, n$. Let $C_n$ be the category whose objects are sets $I$ endowed with a $G_n$-equivalence class of bijections $u : I \to \mathbb{R}$ and whose morphisms are isomorphisms in the obvious sense. Thus $C_n$ is equivalent to the category with the single object $\mathbb{R}$ and the group $G_n$ for morphisms.

The face operator $d_1 : C_n \to C_{n-1}$ is defined as follows: First consider $d_1 : C_2 \to C_1$. Choose a diffeomorphism of $\mathbb{R}$ with $\mathbb{R}$ coinciding with $y = x$ near 0 and with $y = x - 1$ near 2. Then given a bijection $u : I \to \mathbb{R}$, we compose it with this chosen diffeomorphism (call it $D_1$) to get a bijection $D_1u : I \to \mathbb{R}$. The important thing is that $D_1$ is unique up to a canonical element of $G_1$. One defines $d_0$ and $d_2$ by deleting $D_0$ and $(1, 2, \ldots, n-1)$, parametrizing the rest in the obvious way.

In general given $i$, 1 $\leq i \leq n-1$, one chooses a diffeomorphism $D_1$ of $\mathbb{R}$ coinciding with $y = x$ for $x < i$ and near $i$ and coinciding with $y = x - 1$ for $x > i + 1$ and near $i + 1$. The choice of $D_1$ is unique up to composition with an element of $G_{n-1}$.

The verification that the face identities hold should be straightforward, due to essential uniqueness of the $D_1$. Degeneracies have to be added in the stupid way, as they do not seem to arise naturally geometrically.
Let $X$ be a manifold endowed with a normally-oriented foliation of codimension 1. Assume that $X$ can be triangulated in such a way that over any simplex $\sigma$ the foliation is quasi-linear, i.e. after a diffeomorphism of $\sigma$ preserving faces it is defined by a linear function in the simplex. Choose a function defining the foliation in a nbhd. of each vertex and taking the value zero at the vertex. I assume that each one simplex is transversal to the foliation, whence it obtains an orientation from that of the foliation. Now each one simplex has a parameterization $\varphi: \sigma = [0,1]$, unique up to an element of $G$, $\varphi(a) = f_0(a)$ for $a$ near $v_0$, and $\varphi(b) = f_1(b) + 1$ for $b$ near $v_1$, where $\sigma = [v_0, v_1]$. To choose such a parameterization $\varphi_0$. Now given a 2-simplex, one has identifications

\[
\begin{array}{c}
[0,1] \\
| \\
[0,1] \\
[0,1] \\
| \\
[0,1] \\
[0,1]
\end{array}
\]

hince if one chooses a fixed diffeomorphism of $[0,2]$ with $[0,1]$ one obtains a diffeo of $[0,1]$ with compact support as follows: One compares the function in the above simplex furnished by the foliation with the one given by the diffeom. Not like twisting situation because neither first nor last face appears
Basic geometric object consists of an \( n \)-simplex \( \Delta(n) \) with ordered vertices \( 0, 1, \ldots, n \) and a codimension 1 foliation on \( \Delta(n) \) defined by a function \( \phi : \Delta(n) \rightarrow \mathbb{R} \) conjugate by a \( \Gamma \)-homeo to a linear function on \( \Delta(n) \) such that \( \phi(0) < \phi(1) < \ldots < \phi(n) \).
March 10, 1971:

Let $G$ be a topological group and let $X$ be a space. Suppose $P$ is a $G$-torsor over $X$. Then we can form the simplicial space $Nerve(P_G)$:

$$P \times G^2 \Longrightarrow P \times G \Longrightarrow P$$

over $X$; it is also the Cech complex of the map $P \to X$. We have maps of simplicial spaces:

$$\xymatrix{ Nerve(P G) \ar[d] & \ar[l] \ar[r] \ar[d] \ar[dr] & \ar[l] \ar[r] \text{Nerve}(P \times G) \ar[rr] & & \text{Nerve}(P \rtimes G) \ar[d] }$$

the first of which is some sort of homotopy equivalence, and the latter has fibre $P$. Perhaps, as a topologist, I should realize these simplicial spaces, and identify them:

$$\xymatrix{ \ar@/_/[d] \ar[r] & X \ar[l] \ar[d] \ar[dl] & \ar[l] \ar[r] \ar[d] \ar[dr] \ar@/_/[l] & P \times G & \ar[r] \ar[l] \ar[d] \ar[dr] \ar[u] & \ar[l] \ar[r] & P \rtimes G \ar[l] \ar[d] \ar[u] }$$

Because the former map is a homotopy equivalence, we obtain a map $X \to BG$ in the homotopy category, which one calls the classifying map for $P$. It is associated to an equivariant map $P \to PG$. 
Better diagram maybe:

\[
P \leftarrow P \times PG \rightarrow PG
\]
\[
| \quad | \quad |
\]
\[
X \leftarrow P \times^{\mathcal{G}} PG \rightarrow BG
\]

Important:

\[
\text{Hom}_G(P, PG) \cong \Gamma(X, P \times^{\mathcal{G}} PG)
\]

is contractible. \textit{(X "cofibrant")}.

Eventual problem is to make sense of this classification theorem in more general circumstances, e.g. \(X\) should be a topos.

Stasheff problem: Fix a space \(F\) and consider maps \(Z \rightarrow X\) which are locally fiber homotopically trivial with fibre \(F\). According to Segal, this means that there is a covering \(\mathcal{U} = \{U\}_i\) of \(X\) and fiber homotopy equivalences for each \(U\):

\[
Z|U \leftarrow U \times F
\]

I think this means that \(Z \rightarrow X\) is a quasi-fibration with fibre \(F\).
Given such a map \( \pi: Z \to X \), let

\[
P = \text{Hoe}_X (X \times F, P)
\]

\[
M = \text{Hoe}_F (F, F)
\]

where the former is the space over \( X \) whose fibre at \( x \) is the space of homotopy equivalences of \( F \) with \( Z [x] \), and the latter is the space of self homotopy equivalences of \( F \). It is clear that \( M \) acts on \( P \) on the right and that we again have a diagram of simplicial spaces

\[
\begin{array}{ccc}
P & \xleftarrow{\text{Nerve}(P \times M, M)} & \text{Nerve}(M, M) \\
\downarrow & & \downarrow \mu \\
X & \xleftarrow{\text{Nerve}(P, M)} & \text{Nerve}(X, M)
\end{array}
\]

It should be true that the bottom left arrow is a homotopy equivalence. The proof might consist in showing that the canonical map

\[
\text{Nerve}(P, M) \to \text{Gek}(P \to X)
\]

is a homotopy equivalence dimension-wise. Indeed, in dim \( v \), it is

\[
P \times M^v \to (P/X)^{v+1}
\]

\[
(p, m_1, \ldots, m_v) \mapsto (p \circ m_1, p \circ m_2, \ldots, p \circ m_v, m_v)
\]
and locally $P$ is fiber-homotopically equivalent to
\[ \text{Heq}_{/X}(X \times F, X \times F) = \text{Heq}_F(F, F) = M, \]
thus this last map is locally a fiber homotopy equivalence.

Thus if what precedes is correct, we have a definite map in the homotopy category from $X$ to $\text{Nerve}(\text{pt}, M)$ which "classifies" $Z$ because of the quasi-fibrations:

\[
\begin{array}{cccccc}
Z & \leftarrow & \text{Nerve}(P \times M, M) \times M F & \rightarrow & \text{Nerve}(M, M) \times M F \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & \text{Nerve}(P, M) & \rightarrow & \text{Nerve}(\text{pt}, M)
\end{array}
\]

dimension-wise:

\[
\begin{array}{cccccc}
Z & \leftarrow & P \times M^\nu \times F & \rightarrow & M^\nu \times F \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & P \times M^\nu & \rightarrow & M^\nu
\end{array}
\]
Let $\Gamma$ denote a topological groupoid:

$$\Gamma \times_{\mathbb{R}} \Gamma \cong \Gamma \Rightarrow \mathbb{R}$$

(basic example: the pseudogroup of diffeomorphism of $\mathbb{R}^n$)

Given a space $X$, we consider the fibred category over $\text{Open}(X)$ assigning to $U$ the category $\text{Hom}(U, \Gamma)$:

$$\text{Hom}(U, \Gamma) \times_{\mathbb{R}} \text{Hom}(U, \Gamma) \Rightarrow \text{Hom}(U, \mathbb{R}).$$

This is a presheaf of categories in the strict sense over $X$, since the morphisms glue it is a pre-stack. We form the associated stack $\text{St}(\Gamma)$. Then the category $\text{St}(\Gamma)(X)$ is the category of $\Gamma$ structures on $X$. The set of isomorphism classes we denote $\pi^1(X, \Gamma)$.

**Example:** Let $\Gamma$ be the top. category defined by a topological group $G$. Then $\text{St}(\Gamma)(X)$ is the category of principal $G$-bundles over $X$. 
Note: The stack $\text{St}(\Gamma)$ in the general case does not have the property that any two objects are locally isomorphic.

In the example where $\Gamma$ comes from $G$ one constructs a classifying map in the following way. Given a principal $G$-bundle $P$ over $X$ one has maps of simplicial spaces

$$
\begin{array}{ccc}
X & \leftarrow & P \times_x G \xrightarrow{\cong} G \times G \\
\cong & & \cong \\
X & \leftarrow & P \times G \xrightarrow{\cong} G \\
\cong & & \\
X & \leftarrow & P \xrightarrow{\cong} \{pt\}
\end{array}
$$

In other words, over $X$ we have the nerve of the category defined by $(P, G)$ which is homotopy equivalent to $X$ and this category maps to $(\{pt\}, G)$. Thus we get a situation

$$
X \xleftarrow{\text{hteq.}} \left| N(P, G) \right| \xrightarrow{\cong} \left| N(\{pt\}, G) \right| = BG
$$

Now I want to imitate this example in the case of say the pseudo group $\Gamma$. 
First attempt: Let $\mathcal{G}$ be a $\Gamma$-structure on $X$. Then there is an open covering $\mathcal{U}$ of $X$ and isomorphisms in $\text{St}(\Gamma)(\mathcal{U})$ between $\mathcal{G}|_U$ and a function $f_u: U \to R$

$$\varphi_u: \mathcal{G}|_U \to f_u \quad \text{for} \quad U \in \mathcal{U}$$

Also given $U, V \in \mathcal{U}$ we obtain a map

$$\varphi_{u,v}: U \cap V \to \Gamma$$

realizing the isomorphism $\varphi_u \varphi_{v}^{-1}$:

$$f_v|_{U \cap V} \xleftarrow{\varphi_v} \mathcal{G}|_{U \cap V} \xrightarrow{\varphi_u} f_u|_{U \cap V}$$

Since the pre-stack on $X$ defined by $\Gamma$ is a full-subcategory of $\text{St}(\Gamma)$, the isom. $\varphi_u \varphi_{v}^{-1}$ is defined by a map $U \cap V \to \Gamma$.

For each open set $W$ of $X$ I consider the maps $f: W \to R$ which are isomorphisms in $\text{St}(\Gamma)(W)$ to $\mathcal{G}|_W$. In other words we consider those functions which define the structure over $W$.

Now I want to know if this is a sheaf, i.e. given $W = U \cup W_i$ and $f: W \to R$ such that $f|_{W_i} \approx \mathcal{G}|_{W_i}$ in some way for each $i$, does it follow that $f \approx \mathcal{G}|_W$?

Question: To what extent is a $\Gamma$-structure on $X$ the same as a subsheaf of functions from $X$ to $R$? If $\Gamma$ comes from $G$, then $R$ is a
point so that all the $\Gamma$-structures give the same sheaf of functions. On the other hand a $\mathfrak{g}$-foliation on a manifold $X$ is clearly determined by the sheaf of functions to $R^8$ flat along the leaves.)

In any case denote by $F_\mathfrak{g}$ the sheaf of functions which locally define the $\Gamma$ structure. Then we construct a simplicial object and a map:

$$
\begin{align*}
& F \times_{F_R} \Gamma \\
\downarrow & \downarrow \\
& F \times_R \Gamma \\
\downarrow & \\
& \Gamma \\
\downarrow & \\
& R \\
\downarrow & \\
& X
\end{align*}
$$

The problem with this is that the simplicial object is not usually acyclic over $X$; for example if $X=pt$ and we take constant functions. On the other hand with the differentiable pseudo-group, if the functions come from a foliation, then

$$
F_z \times_R \Gamma \sim F_x \times F_x
$$

$$(f, x) \mapsto (f, xf)
$$

and the simplicial object is acyclic.
The principal bundle associated to a $\Gamma$-structure:

Suppose given a $\Gamma$-cocycle on $X$. This means I have a covering $U \in \mathcal{U}$ of $X$ and for each $U \in \mathcal{U}$ a map $f_U: U \rightarrow \Gamma$ and for each pair $U \cap V$ a map $g_{UV}: U \cap V \rightarrow \Gamma$

such that

\[ g_{UV} \circ f_U = f_V \quad \text{(i.e. source $g_{UV} = f_U$)} \]

and

\[ g_{VW} \circ g_{UV} = g_{UV} \quad \text{on $U \cap V \cap W$.} \]

To obtain the principal bundle we glue together

\[ P = \bigsqcup_{U \in \mathcal{U}} (U \times \Gamma, (f_U, s)) \quad \text{equivalence relation} \]

the equivalence relation is as follows. Given $x \in U$ $\gamma \in \Gamma$ such that $f_U(x) = \gamma(x)$ let $(x, \gamma)_U$ denote the associated elt in the disjoint union. Then

\[ (x, \gamma)_U \sim (x', \gamma')_U \]

if

\[ f_U(x) g_{U}(x) = f_{U'}(x) \]

\[ g_{U'}(x) g_{U}(x) = g_{U}(x) \]

\[ \gamma(x) \gamma_U(x) = \gamma(x) \]
The transitivity condition guarantees this is an equivalence relation. The equivalence relation implies that \( Y(x) \) and \( V(x) \) have the same target, hence we have \( \Gamma \) maps

\[
P \xrightarrow{\Gamma} R.
\]

(Note that for a pseudo-group \( s: \Gamma \rightarrow R \) is etale, hence \( P \) is etale over \( X \) in this case.)

Now the desired resolution of \( X \) is

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
P \times_R \Gamma & \rightarrow & \Gamma \\
\downarrow & \downarrow & \downarrow \\
P & \rightarrow & R \\
\downarrow & & \\
X & &
\end{array}
\]

**Definition:** A \( \Gamma \)-torsor over \( X \) is a space \( P \) endowed with maps

\[
p: P \rightarrow X \\
q: P \rightarrow R \\
\Delta: P \times_R \Gamma \rightarrow P
\]
which locally over $X$ is isomorphic to

$$P = X \times_R \Gamma$$

for some $f: X \to R$ with maps

$$\text{pr}_1: X \times_R \Gamma \to X$$

$$t \cdot \text{pr}_2: X \times_R \Gamma \to R$$

$$\text{id} \times \Delta: X \times_R \Gamma \times_R \Gamma \to X \times_R \Gamma$$

---

**Remarks:**

1. A section $s: X \to P$ of $P$ gives a "trivialization"

\[ \begin{array}{ccc}
X \times_R \Gamma & \xrightarrow{\alpha} & P \\
\downarrow{\text{sg}} & & \downarrow{\Delta} \\
X \times R & \xrightarrow{s \times \text{id}} & P \times_R \Gamma \\
\end{array} \]

This is clear because $\Gamma$ is a groupoid (these things have only to be checked in sets).

2. The difference of two sections is a morphism $X \to \Gamma$. Indeed

$$p \times P \xleftarrow{\sim} p \times R$$
March 13, 1971

Review proof of classification theorem for principal $G$-bundles.

Given $P$ over $X$, we form the mixing diagram

$$
\begin{array}{ccc}
P & \xleftarrow{p_1} & \mathbf{P \times PG} & \xrightarrow{p_2} & \mathbf{PG} \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{p_1} & \mathbf{P \times^{G} PG} & \xrightarrow{p_2} & \mathbf{BG}
\end{array}
$$

As $p_2$ is a homotopy equivalence we obtain an element of $[X, \mathbf{BG}]$. This defines a map

$$H^1(X, G) \longrightarrow [X, \mathbf{BG}].$$

On the other hand we have a map

$$[X, \mathbf{BG}] \longrightarrow H^1(X, G)$$

by associating to $f$ the bundle $f^*(PG)$. (Use here that $f^*(P)$ depends only on the homotopy class of $f$ which results from the fact that $P \to X$ is a fibration.)

The composition

$$H^1(X, G) \longrightarrow [X, \mathbf{BG}] \longrightarrow H^1(X, G)$$
is the identity of the mixing diagram. Indeed to $cl(P) \in H^1(X,G)$ we associate the homotopy class of the map $p_2 s$, where $s$ is a section of $p_2$, so we want to show that

$$(p_2 s)^* cl(P) = cl(P)$$

\[ s^* cl(P \times PG) = s^* p_2^* cl(P) = (p_4 s)^* cl(P). \]

so done as $p_4^* = id$.

To show that the composition

\[ [X, BG] \rightarrow H^1(X,G) \rightarrow [X, BG] \]

is the identity, we form the mixing diagram for the bundle $PG$ over $BG$.

\[ PG \xrightarrow{\Delta} PG \times PG \rightarrow PG \]

\[ BG \xrightarrow{p_4} PG \times PG \rightarrow BG \]

Take $s$ to be induced by the diagonal, whence $p_2 s = id$ and done.
Logic of above argument:

1) Over $BG$ there is a canonical bundle $PG$, hence there is a map

$$[X, BG] \longrightarrow H^1(X, G)$$

as $H^1(?, G)$ is a homotopy functor.

2) From the mixing diagram and contractibility of $PG$, there is a map

$$H^1(X, G) \longrightarrow [X, BG]$$

3) The composite

$$[X, BG] \longrightarrow H^1(X, G) \longrightarrow [X, BG]$$

is the identity because given $f: X \longrightarrow BG$ it induces an equivariant $P=f^*(PG)$ which comes provided with an map $P \longrightarrow PG$; hence there is a section $s: X \longrightarrow P \times^G PG$ with $p_2s=f$; as the homotopy class of $p_2s$ is the map 2) applied to $f$, we are done.

4) The composite

$$H^1(X, BG) \longrightarrow [X, BG] \longrightarrow H^1(X, BG)$$

is the identity because the map $p_2s$ constructed in the mixing diagram pulls back $PG$ to $P$. 
Mixing diagram for a principal $\Gamma$-bundle:

\[
\begin{array}{c}
P \times P \Gamma : \quad P \times \Gamma \times \Gamma \times \Gamma \xrightarrow{\Delta \times \Delta} P \times \Gamma \times \Gamma \xrightarrow{\Delta \times \Delta} P \times \Gamma \\
\downarrow P_{123} \quad \quad \downarrow P_{12} \quad \quad \downarrow P_1 \\
(P \times P \Gamma) \times \Gamma : \quad P \times \Gamma \quad \xrightarrow{\Delta \times \Delta} \quad P \times \Gamma \quad \xrightarrow{\Delta} \quad P
\end{array}
\]

This leads to a diagram of simplicial spaces

\[
\begin{array}{ccc}
P & \leftarrow & P \times P \Gamma \\
\downarrow & & \downarrow \\
\Gamma & \leftarrow & (P \times P \Gamma) \times \Gamma
\end{array}
\]

where the right horizontal maps are the projections forgetting the $P$-factors.

Consider the classification theorem more formally, assuming that there
is a reasonable realization functor. Then $\bar{\eta}_1$ is a homotopy equivalence, so choosing a homotopy inverse $s$ one obtains a map $\bar{p}_s : X \to B\Gamma$. This gives a map

$$H^1(X, \Gamma) \to [X, B\Gamma]$$

and homotopic $\Gamma$-structures give rise to homotopic maps. (The proof of this will require some argument. The point is that if we have a $\Gamma$-bundle $P$ over $X \times I$, then we have

$$\begin{array}{ccc}
(P \times_{\mathcal{R}} P\mathcal{T})_{\xi} \mathcal{R} & \longrightarrow & (P \times_{\mathcal{R}} P\mathcal{T})_{\xi} \mathcal{R} \\
\downarrow \bar{p}_1 & & \downarrow \bar{p}_1 \\
X \times I & \longrightarrow & X \times I
\end{array}$$

where the vertical maps are homotopy equivalences. It will be necessary to construct compatible homotopy inverses. (Details involving sections of $\bar{p}_1$ will be constructed using partitions of unity, so what is needed should be found in the Eilenberg–Steinberg book.)

Thus we have a map

$$\Phi : H^1(X, \Gamma)/\text{hom.} \longrightarrow [X, B\Gamma].$$

On the other hand $P\mathcal{T}$ defines an
element of $H^1(\mathcal{B}^\Gamma, \Gamma)$, hence by functoriality we have a map

$$\overline{\pi} : [X, \mathcal{B}^\Gamma] \longrightarrow H^1(X, \Gamma)/\text{ker}.$$ 

Proof that $\overline{\pi} \pi = \text{id}$: Start with $f : X \to \mathcal{B}^\Gamma$. Then $\overline{\pi}(\text{cl}(f)) = \text{cl}(P)$ where $P = f^* \mathcal{P}^\Gamma$. Moreover there is a map $\overline{\pi} : P \to \mathcal{P}^\Gamma$ covering $f$. I claim this induces a section $s$ of $\overline{\mathcal{P}}^\Gamma$ in the mixing diagram such that $\overline{\pi} \overline{\pi} s = f$. As $\overline{\pi}(\text{cl}(P)) = \text{cl}(\overline{\mathcal{P}}^\Gamma s)$ by definition one wins. (This claim requires work due to the fact that the model $P \times_R \mathcal{P}^\Gamma$ is not clean somehow)

It is necessary to digress and work out the mixing diagram more carefully.

Given principal $\Gamma$-bundles $P = (X, P, p : P \to X, \varrho : P \to R, \mu : P \times_R \mathcal{P} \to P)$ and $P' = (X', P', p' : P' \to X', \varrho' : P' \to R, \mu' : P' \times_R \mathcal{P} \to P')$ we wish to define their product which perhaps should be denoted $P \times_R P'$. For the total space take

$$P \times_R P' = \left\{ \frac{\varrho(x, \lambda')}{\varrho'(x', \lambda')} \bigg| \varrho(x) = \varrho(x') \in R \right\} \quad (\text{denote this} \quad P \times_R P')$$

and make $\Gamma$ act by

$$((\lambda, \lambda'), \gamma) = (\lambda \gamma, \lambda' \gamma) \quad \text{if} \quad \varrho(x) = \varrho(x') = \varrho(\lambda)$$
Denote the orbit space of the action by $P \times \Gamma P'$. I want to show that the projection

$$p^* : P \times \Gamma P' \to P \times \Gamma P'$$

together with the canonical maps

$$q^* : P \times \Gamma P' \to \mathbb{R}$$

$$\mu^* : (P \times \Gamma P') \times \Gamma \to P \times \Gamma P'$$

constitute a principal $\mathbb{R}$-bundle. The question is local over $X$ (also over $X'$), so we can assume

$$P = X \times_{\mathbb{R}} \Gamma$$

for some $f : X \to \mathbb{R}$. Then

$$P \times_{\mathbb{R}} \Gamma P' = \{ (x, y, \lambda') \mid t(x) = q'(\lambda') \}$$

and the action is:

$$f(x, y, \lambda') = f(x, y, \lambda')$$

where $f(x, y, \lambda') = \Delta(x, y)$. In each orbit there is a unique representative of the form $(x, \text{id}, \lambda')$ where $f(x) = s(\text{id}) = t(\text{id}) = q'(\lambda')$. Thus

$$(X \times_{\mathbb{R}} \Gamma) \times \Gamma P' = X \times_{\mathbb{R}^2} P'$$

and
\[ \rho^* : \mathbb{P} \times \xi' \longrightarrow X \times \xi' \]

\[ (x, \gamma, \lambda') \longrightarrow (x, \lambda' \gamma^{-1}) \]

Next one has

\[ (X \times_R \xi') \times_R \Gamma \overset{\sim}{\longrightarrow} \mathbb{P} \times \xi' \]

\[ ((x, \lambda'), \gamma_1) \longrightarrow (x, \gamma_1, \lambda' \gamma_1) \]

(here \( X \times_R \xi' \longrightarrow \mathbb{P} \) sends \( (x, \lambda') \mapsto f(x) = g'(\lambda') \) and \( s(\gamma_1) = f(x) \)) because \( \Gamma \) is a groupoid. This proves that we have defined a principal \( \Gamma \)-bundle. Thus we have the mixing diagram

\[ P \xleftarrow{p_{1R}} P \times_R \xi' \xrightarrow{p_{2R}} \mathbb{P} \]

\[ \text{curv.} \quad \text{curv.} \quad \text{curv.} \]

\[ X \xleftarrow{p_1} P \times_R \Gamma \xrightarrow{(p_2)} X' \]

(These are the two 'projections of \( P \times \xi' \) onto the factors).

Moreover if \( P' = U \times_R \Gamma \), then

\[ (p_1)^{-1} U = U \times_R \xi' \]

For the classification theorem we assume that the map \( g' : \mathbb{P}' \longrightarrow \mathbb{P} \) is a fibre homotopy equivalence, i.e. \( \exists \) a section and fibre-wise retraction to the section...
Then it follows that $\tilde{P}_1$ is a quasi-fibration with contractible fibres (I hope), hence $\tilde{P}_1$ is a homotopy equivalence. In any case, $\tilde{P}_1$ induces an isomorphism on cohomology by Leray.

\[ (\tilde{P}_2 s)^* \text{cl } P' = s^* \text{cl } (\tilde{P}_1 s) = s^* \tilde{P}_1^* \text{cl } P = \text{cl } (\tilde{P}_1 s) = \text{cl } P \]

the last step because $\tilde{P}_1 s \sim \text{id}_X$. Thus we have defined a map

\[ H^1(X, \Gamma) \xrightarrow{\varphi} [X, X'] \]

by sending $\text{cl } P$ to the homotopy class of $\tilde{P}_2 s$. Again modulo construction of $s$, this induces

\[ H^1(X, \Gamma)/\text{hom.} \xrightarrow{\bar{\varphi}} [X, X'] \]

and we have shown that if $\bar{\varphi}$ is the map,

\[ [X, X'] \xrightarrow{\varphi} H^1(X, \Gamma)/\text{hom.} \]

defined by $\bar{\varphi}(\text{cl } f) = \text{cl } (f^* P')$, then $\bar{\varphi} \varphi = \text{id}$. On the other hand given $f : X \to X'$, $\bar{\varphi}(\text{cl } f) = \text{cl } (f^* P')$. 
setting $P = f^* P'$, we have a map $P \to P'$ over $f$,
hence a section $s$ of $\tilde{P}_1$ defined by the graph,
such that $\tilde{P}_2 s = f_j$ as $\Phi \Phi (\text{cl} f) = \text{cl} (\text{cl} f^* P') = \text{cl} (\tilde{P}_2 s)$, we see $\Phi \Phi = \text{id}$ also.

Things to be checked carefully using appropriate
para-compactness assumptions on $X$.

i) If $f' : P' \to R$ is a fiber-homotopy-equivalence
then in the mixing diagram $\tilde{P}_1$ is a homotopy equivalence.

ii) Given $P$ over $X \times I$, one can find
compatible homotopy inverses for the vertical arrows

$(P |_{X \times I}) \times f P' \quad \longrightarrow \quad P \times f P'$

\[ X \times I \quad \longrightarrow \quad X \times I \]

iii) The Milnor construction fulfills these
requirements.
March 17, 1971

I want to understand Mather's theorem that the group of homeomorphisms of \( R^n \) with compact support has no cohomology. Call this group \( G \).

First I show that \( G \) is trivial. Let \( g \in G \) have support in a ball \( B_0 \), and choose balls \( B_1, B_2, \ldots \) which are disjoint and converge to a limit \( \Delta \).

\[
\begin{array}{c}
B_0 \\
\quad \\
B_1 \\
\quad \\
B_2 \\
\quad \\
n \quad \\
\end{array}
\]

all in some bounded region \( \Delta \). Let \( x \in G \) have support in \( \Delta \) and carry \( B_0 \) onto \( B_1 \), \( B_1 \) onto \( B_2 \), etc. Then if \( \tau(x) = z x z^{-1} \) we have that

\[
\tau(g) = z g z^{-1}
\]

has support in \( B_1 \),

\[
\tau^2(g)
\]

and the infinite product

\[
\tau(g) = \tau(g) \tau^2(g) \ldots
\]

converges defining an element of \( G \). Then

\[
\tau^{-1}(g) = g \tau(g) \iff g = \tau(g)^{-1} z^{-1} \tau(g) z
\]
showing that $g$ is a commutator.

(Observation: One knows that the inner auto.
$g \mapsto zg\bar{z}^{-1}$ acts trivially on the homology with constant coefficients of a group $G$. The homotopy operator is

$$h(g_1, \ldots, g_k) = (z_1^{-1}, z_1g_1^{-1}z_1, z_2g_2^{-1}z_1, \ldots, z_kg_k^{-1}z_1)$$

$$- (g_1^{-1}, z_2g_2^{-1}z_1, \ldots, z_kg_k^{-1}z_1)$$

$$\quad \quad \quad \quad \quad \quad \quad +(-1)^k(g_1, \ldots, g_k, z_1^{-1})$$

I think. It checks for $k=1$.)

Observe that the same argument shows that

$\hat{\mathcal{G}}_{*g} = 0$, where $\hat{\mathcal{G}} = \text{differs}$ of $IR$ with support contained in $(-\infty, a]$ for some $a$. Indeed give $g \in \hat{\mathcal{G}}$
choose $z \in G$ such that $z(x) = x-1$ for

$x < a$ where $(-\infty, a]$ contains the support of $g$. Then $\hat{\mathcal{G}}_{*g} = \hat{\mathcal{G}}_{*g}z^{-k}$ sends $x \mapsto g(x+k)-k$ and
has support in $(-\infty, a-k]$ so

$T(g) = \hat{\mathcal{G}}_{*g} = \hat{\mathcal{G}}_{*g}z^{-k}$... converges in $\hat{\mathcal{G}}$, and $g = T(g)z^{-1}T(g)z$. (Cleaner
to note that the translation $x \mapsto x-1$ is the product of
$z$ and something centralizing the subgroup with support in $(-\infty, a]$.)
Question: Let \( G \) be a group endowed with an endomorphism \( \sigma \) such that \( x \mapsto x \sigma(x)^{-1} \) is bijective. If \( \sigma \) acts trivially on \( H^k(G) \) does it follow that \( H^+(G) = 0 \)? (true for \( H^1(G) \))

Mathies proof that \( H^+_+(G) = 0 \) uses the algebra structure and the fact that the monoid of isom. classes of \( G \)-bundles over \( X \) finite, then \( \exists x \in X \) s.t. \( x + y \equiv x \), hence any primitive \( \alpha \)-class \( c \) will satisfy
\[
(c(x) + c(y)) = c(x + y) = c(x) \Rightarrow c(y) = 0
\]
Given two real numbers \( a \leq b \), let \( G_{ab} \) be the diffeomorphisms of \( \mathbb{R} \) with support in \([a, b]\). Then we have a 2-category:

Objects: real nos. \( a \)

1-maps: for each pair \( a, b \), there is a unique 1-arrow \( a \to b \) if \( a \leq b \), and none otherwise

2-maps: the two arrows from \( a \to b \) to \( a \to b \) are the elements of \( G_{ab} \).

In other words, \( Hom(a, b) \) is the category associated to \( G_{ab} \) if \( a \leq b \) and is empty otherwise.

Using Segal's classifying space we obtain a simplicial space

\[
\cdots \overset{\ast}{\longrightarrow} G_{a} \times G_{b} \overset{\ast}{\longrightarrow} G_{ab} \overset{\ast}{\longrightarrow} pt
\]

which is a topological category with objects \( a \in \mathbb{R} \) and morphisms \( G_{ab} \) from \( a \) to \( b \). Hopefully the realization of this simplicial space is Mathew's \( B(BG) \).

Assuming this we want a map of \( B(BG) \) into \( \Gamma \), that is, a \( \Gamma \)-structure on the realization of \((x)\). Recall that the basic map \( 
\Sigma B_{ab} \to \Gamma
\) is obtained by taking the canonical foliation on the bundle \( P_{ab} \times [a, b] \).
and identifying $\mathbb{C}^n \times \mathbb{C}^n [a, b] \cong \mathbb{C}^n \times [0, 1]$ using the triviality of $[0, 1]$-bundles. This suggests that the map from the realization of $(\ast)$ to $B\Gamma$ should be obtained as follows.

Given real numbers $a_0 \leq a_1 \leq \ldots \leq a_n$, we make

$$G_{a_0, a_1, \ldots, a_n} = G_{a_0, a_1} \times G_{a_1, a_2} \times \ldots \times G_{a_{n-1}, a_n}$$

act on $\Delta(n)$ as follows. Let $f: \Delta(n) \to R$ be the function with

$$f(t_0, \ldots, t_n) = \sum_{i=0}^{n} t_i a_i.$$

Consider the case where $a_0 < a_1 < \ldots < a_n$. Then each of the slices

$$f^{-1}(x) \hspace{1cm} a_0 < x < a_n$$

is an $(n-1)$-simplex and each region

$$f^{-1}(a_{i-1}, a_i)$$

is homeomorphic to

$$f^{-1}(x) \times (a_{i-1}, a_i) \hspace{1cm} a_{i-1} < x < a_i$$

in an "obvious" way.
such that things are compatibly chosen. Let \( U_0 \) be given such a gadget \( \Sigma_0 \), \( \Phi_0 \), \( x_0 \).

For \( x \), suppose given such a gadget \( \Sigma(x), \Phi(x), x \).

Then instead of the realization of \( \Phi(x) \), let \( \Phi' \) be the homomorphism (canonical) that transports \( \Phi(x) \) into \( \Phi' \).

That means that we have a covering \( \Sigma = [U_0, \Phi_0] \) of \( BG \) by \( PG \). and for a \( g \in \text{Princ} \) and for an \( x \in \text{Princ} \), make: \( PG \), \( U_0 \), \( \Phi_0 \), \( x_0 \).
which lie $X$ and has partitions of 1 for its sections we can form the space

$$(++) \bigcup \bigcup_{g_0 \cdots g_n} P_{a_0 \cdots a_n} \times G_{a_0 \cdots a_n} \Delta(g)$$

which is homeomorphic to the realization, once we choose compatible views:

$$P_{a_0 \cdots a_n} \times G_{a_0 \cdots a_n} \Delta(g) \simeq \bigcup_{a_0 \cdots a_n} \times \Delta(g).$$

Picture:

The point is that the realization of $(++)$ carries a canonical $\Gamma$-structure.

Now in the universal cone we take the space.
which comes provided with a canonical gadget 
\\{u_0, p_{ab}, m_{ab}\} \text{ and moreover such that the corresponding }
\text{(++) realization has a canonical section.}

\textbf{Conclusion:} To prove Mather's theorem we must show how a $\Gamma$-structure over $X$ is homotopic to one produced by a gadget \(\{u_0, p_{ab}, m_{ab}\}\) together with a section of the associated (**) space.
March 18, 1971

Marv Mather

Let $G_{ab}$ denote the cliques of $R$ with support in $[a,b]$ and

$$G_{a_0 \cdots a_8} = G_{a_0 a_1} \times \cdots \times G_{a_7 a_8}$$

Letting $BG = \{W(G)\}$ we have a topological category

$$\prod_{a_0 < a_1} BG_{a_0 a_1} \implies \prod_{a_0 < a_1} BG_{a_0 a_1} \implies \prod_{a_0} pt$$

whose realization we wish to show is $BG$. Therefore we want to produce a $\Gamma$-structure over the realization.

The realization, which we denote $\mathcal{B}(BG)$, consists of a union

$$\bigcup_{g \geq 0} \bigcup_{a_0 < \cdots < a_8} BG_{a_0 \cdots a_8} \times \Delta(g)$$

where the identifications are made by face operators. Thus every point $i$ of $\mathcal{B}(BG)$ determines a sequence

$$\alpha(i) = (a_0 < \cdots < a_8)$$

of real numbers, a point

$$b(i) \in BG_{a_0 \cdots a_8}$$

and a point

$$t(i): \begin{array}{c} t_0 + \cdots + t_g \end{array} = 1 \quad 0 < t_g$$
of the interior of $\Delta(q)$. As with the Milnor model for BG, one agrees to coalesce $\hat{t}(i)$ and $t(i)$ into a finite sum

$$\sum_{a \in \mathbb{R}} t_a = 1 \quad 0 \leq t_a.$$ 

The sequence $a_0 < \ldots < a_q$ being precisely the $a$ for which $t_a > 0$. Thus every point $i$ of $B(BG)$ determines a point $t(i) = \{ t_a(i), a \in \mathbb{R} \} > 0$. Thus it is the coarsest such that the functions $t_a$ and $b$ are continuous.

To give a map from $X$ to $B(BG)$ means we give a partition of unity on $X$

$$\sum_{a \in \mathbb{R}} f_a = 1$$

and for each sequence $a_0 < \ldots < a_q$ a map $f(i) \in B_{a_0 \ldots a_q}$.