Deformation of torsors

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1. Extensions of Picard stacks.

One works in some fixed topos $\mathcal{T}$, $A$ is a ring of $\mathcal{T}$, the Picard stacks considered in the sequel will be $A$-linear.

1.1. Let $u_i : X_i \rightarrow Y$ be morphisms of Pic. stacks, $i = 1, 2$, one defines the fibered product

$$X = X_1 \times_Y X_2$$

as the (Pic st) of triples $(x_1, x_2, s)$ where $x_i$ is an object of $X_i$ and $s$ an isomorphism $\mathbb{X}(X_1, X_2, s)$ in $\mathcal{Y}$: maps are the obvious ones, addition is defined by addition of coordinates with the usual grain of salt.

One has a canonical square

$$\begin{array}{ccc}
X & \xrightarrow{pr_2} & X_2 \\
\downarrow{pr_1} & & \downarrow{u_2} \\
X_1 & \xrightarrow{u_2} & Y
\end{array}$$

commutative up to can isom $(pr_1(x_1, x_2, s) = x_1)$, and for any Pic $L$, the induced map

(1.1.1) $\text{Hom}(L, X) \rightarrow \text{Hom}(L, X_1) \times_{\text{Hom}(L, Y)} \text{Hom}(L, X_2)$

is an equivalence ($\text{Hom}$ denotes the Pic st of morphisms from to).

1.2. Let $u : M \rightarrow N$ be a map of Pic st, one defines

(1.2.1) $\text{Ker}(u) = M \times_N 0$.

Put $L = \text{Ker}(u)$. One has an (A-linear) map

$$d : \text{Aut}(O_N) \rightarrow \text{Isob}(L)$$

$$d(s) = c_1(O_M, s)$$

(one may assume $u(O_M) = O_N$),

where $\text{Aut} = \text{sheaf of autom}$, $\text{Isob} = \text{sheaf of isom. cl. of obj.}$

The sequence

(1.2.2) $0 \rightarrow \text{Aut}(O_L) \rightarrow \text{Aut}(O_M) \rightarrow \text{Aut}(O_N) \rightarrow \text{Isob}(L) \rightarrow$

$$\rightarrow \text{Isob}(M) \rightarrow \text{Isob}(N)$$.
is exact. (Proof left to the reader).

1.3. A $O$-sequence of $Pic$ consists of maps of $Pic$

$$(1.3.1) \quad L \xrightarrow{u} M \xrightarrow{v} N$$

together with an isom $s : rv \to O$. By the universal property (1.1.1)

one then has a well defined map

$$(\nu) \quad L \to \text{Ker}(v).$$

(completed by $O$ at both ends.)

One says that (1.3.1) is an exact sequence if (\nu) is an equivalence

and $\text{Isoh}(v)$ is an epimorphism. One then has an exact sequence

$$(1.3.2) \quad O \to \text{Aut}(O_L) \to \text{Aut}(O_M) \to \text{Aut}(O_N) \to \text{Isoh}(L) \to \text{Isoh}(M)$$

$\to \text{Isoh}(N) \to O$.

Example. Let

$$E = (0 \to X \to Y \to Z \to O)$$

be an exact sequence of $\mathcal{O}(A)$. Then $st(E)$ is an exact sequence

of $Pic$. Conversely any exact sequence of $Pic$ is equivalent to $st(E)$ for a suitable $E$. (The first assertion is straightforward, for the second use the fact that a $O$-quasi-iso morphism of $\mathcal{O}(A)$ can be represented by a map surjective in each degree). In this dictionary, (1.3.2) corresponds to the usual "snake sequence".

1.4. If

$$E = (0 \to L \to M \to N \to O)$$

is an exact sequence of $Pic$ and $p : N' \to N$ a map of $Pic$, one defines the pull-back

$$E' = E_{p} = (0 \to L' \to M' \to N' \to O)$$

by $M' = M 	imes_{N} N'$, $L' = \text{Ker}(M' \to N')$. The map $L' \to L$ is an equivalence.

1.5. An extension of $A$ by a $Pic$ $M$ is also called a torsor

under $M$. In down-to-earth terms, this amounts to the data:

- a stack $X$,

- a sum $+ : M \times X \to X$ associative up to given isom satisfying...
pentagon,
together with the axioms:
- X is locally non void
- the can map \( M \times X \to X \times X \) is an equivalence.

In fact, given an extension
\[
0 \to M \to E \to A \to 0
\]
one defines \( X \) as the stack of \( x \in \text{Ob } E \) s.t. \( p(x) = 1 \). Conversely, given a torsor \( X \) under \( M \), one defines \( E = \sum_{x \in \text{Ob } A} \mathcal{A} x \).

More generally, one has an interpretation of an extension of a \( \mathbb{Z} \)-Module by a Pic st as a family of torsors \( X_n, X_p, X_q \to X_{p+q} \), in the Grothendieck style (SGA 7 ...).

1.6. By the example above, to an extension
\[
E = (0 \to Y \to E \to X \to 0)
\]
of Pic st is associated an element
\[(1.6.1) \quad \text{cl}(E) \in \text{Ext}^1_A(X,Y) \quad , \quad \text{(hyperext)}\]
whose vanishing is necessary and sufficient for the splitting of \( E \) i.e. the existence of \( \exists x \in X \) an \( s_\omega : X \to E \) s.t. \( ps \simeq \text{id}_X \).

When \( X = A \), i.e. \( E \) is a torsor under \( Y \), giving such an \( s \) is simply giving a global object in \( E \); if \( e \) is a global object then "\( t \mapsto t-e \)"
identifies \( E \) to \( Y \).

2. First geometrical applications.

2.1. If \( X \to Y \) is a morphism of ringed topoi and I an \( \mathcal{O}_X \)-Module,
\[
\text{Ext}(X/Y, I)\]
denotes the Pic st of \( Y \)-extensions of \( X \) by \( I \). Recall that
\[
\text{Ext}(X/Y, I) = \text{Hom}(\text{st}(I_{\mathcal{L}/X/Y}), I[I])
\]
\[(*) \quad \text{"}\text{st } \Rightarrow \text{st}\text{" I}\]
2.2. Let
\[ \xymatrix{ X \\ Y \\ Y' } \]
be a diagram of ringed topoi, with \( j \) is an extension of \( Y \) by \( J \), and let \( I \) be an \( O_X \)-module. One has a 0-sequence
\[(2.2.1)\quad 0 \to \text{Ext}(X/Y,I) \to \text{Ext}(X/Y',I) \to \text{Hom}(f^*J,I) \to 0.\]
Claim: (2.2.1) is exact, i.e. \( \text{Ext}(X/Y,I) \cong \text{Ker}(c) \).
Proof. Observe that a \( Y \)-extension of \( X \) by \( I \) is the same as a \( Y' \)-extension of \( X \) by \( I \) whose characteristic homomorphism is zero. (Reduce to a question of rings in the same topos.)

2.3. Put \( I = f^*J \). By a deformation of \( f \) over \( Y' \) one means an object \( X' \) of \( \text{Ext}(X/Y',I) \) such that \( c(X') = 1 \in \text{End}(f^*J) \). The deformations of \( f \) over \( Y' \) make a stack
\[ \text{Def}(X/Y',I)_1 \]
where \((-)_1\) means "fibre over 1" and \( \text{Def}(X/Y,j) \) is defined as the fibre-product
\[(2.3.1)\quad \text{Def}(X/Y,j) \to O_X \\
\phantom{(2.3.1)} \downarrow \qquad \downarrow 1 \\
\phantom{(2.3.1)} \text{Ext}(X/Y',I) \to \text{Hom}(f^*J,I).\]
So, one has an exact sequence
\[(2.3.2)\quad 0 \to \text{Ext}(X/Y,I) \to \text{Def}(X/Y,j) \to O_X \to 0.\]
The obstruction to the existence of a local deformation of \( f \) is, as one knows, the section of \( \text{Ext}^2(L_{X/Y},I) \) image of \( 1 \in \text{Hom}(f^*J,I) \) by the canonical hom. Suppose this obstruction vanishes. Then one can add a zero to the right of (2.3.2), in other words \( \text{Def}(X/Y,j) \) is a torsor under \( \text{Ext}(X/Y,I) \). Hence, if moreover this torsor is trivial, i.e. a global deformation exists, one
3. Equivariant deformations.

3.0. Let $S$ be a scheme (more generally a ringed topos), and $G$ a group scheme over $S$. For an equivariant morphism $f : X \to Y$ of $G$-schemes/$S$, one would like to define $L_{X/Y}$ as a complex of $G$-$O_X$-modules. This will be done after some preliminaries. 

(Unless otherwise stated, $X$ is a diagram in $\text{Sch}/S$, $\text{fpqc}$ say. If $X$ is a diagram in $\text{Sch}/S$, one will still denote by $X$ the corresponding 'contravariant' topos. As usual, $BG$ denotes the classifying topos of $G$, i.e., the topos of local systems on the nerve $G(1)$ of $G$. The inclusion of the cat of local systems into the cat of all sheaves on $G(1)$ defines a morphism of topoi 

\[(3.1.1) \quad \phi : G(1) \to BG \]

This is in fact a morphism of ringed topoi, and $\phi^* \phi^!$ is exact. Moreover, using the general nonsense of cohomological descent, one can prove that the adjunction map

\[(3.1.2) \quad \text{Id} \to \text{R}$\end{equation}$\,$\text{R}^!$\]](\end{equation}) is an isomorphism and the image of $\phi^* : D^+(BG) \to D^+(G(1))$ (which is fully faithful by the preceding) consists of complexes whose cohomology sheaves are local systems.

Generalization. Let $X$ be a $G$-scheme. One defines

\[ G(1)/X = \text{nerve of } G \text{ acting on } X \]

One has a natural map

\[(3.1.3) \quad \phi : G(1)/X \to BG/X \]

with analogous properties to the $\phi$ (3.1.1).

3.2. Let now $f : X \to Y$ be an equivariant morphism as in (3.0). So one has a morphism $G(1)/f$ of the corresponding nerves, let's denote it simply by $f' : X' \to Y'$. Let's denote by $L_{X'/Y'}$, the

\[(3.1) \quad \text{whose objects are families of sheaves } E_i \text{ over } X_i, f^* E_j \to E_i \quad \text{for } f : X_i \to X_j \]
cotangent complex of the morphism of small ringed zariski topoi defined by \( f' \). Assume

(i) either \( G \) or \( f \) flat,

(ii) \( L^X_Y \) of finite tor-amplitude.

Then, by the base-change theorem, \( L^X'_{/Y} \) has finite tor-amplitude and its cohomology sheaves are quasi-coherent \( G_\mathcal{O}_X \)-modules. Therefore, by (3.1) and the remark below, \( L^X'_{/Y} \), can be identified via \( \phi \) to a complex denoted

\[
(3.2.1) \quad L^X_{/Y} \in \text{ob} \ D^b(BG_{/X})
\]

Remark 3.2.2. One has natural morphisms of ringed topoi

\[
\begin{array}{ccc}
\text{Zar} & \leftarrow & \text{Et} \leftarrow \text{Flat} \\
\uparrow & & \uparrow \\
\text{zar} & \leftarrow & \text{et} \leftarrow \text{flat}
\end{array}
\]

where a capital means "large". Applying \( D^b(-)_{\text{qcoh}} \) (where qcoh means "quasi-coherent cohomology"), one finds a diagram of equivalences (it's trivial on the columns and well known by descent on the rows). The reason why one worked with the small zariski topos is that a localization map has zero cotangent complex. (Note: "small étale" would work well too).

3.2.3. If (ii) is not satisfied, define \( L^X_{/Y} \) in \( \text{pro-} D^b(BG_{/X}) \).

3.2.4. The image of \( L^X_{/Y} \) by the forgetful functor \( D(BG_{/X}) \to D(X) \) is \( L^X_{/Y} \).

3.3. Yoga of diagram deforming: let \( Z \rightarrow \overline{Z} \) defined by a nilpotent ideal, if a map \( \overline{f} : \overline{X} \rightarrow \overline{Y} \) of flat \( \overline{Z} \)-schemes reduces to an isom mod \( I \), it was already one.
3.4. Let

\[ \xymatrix{ X \\
\ar@{->}[r]_f \ar@{-->}[r]_j & Y } \]

be a diagram of G-S-schemes, where j is defined by \( j^2 = 0 \), and \( f \) is flat. By the yoga of diagram deforming, the obstruction to deforming \( f \) over \( Y \) (as a map of flat G-schemes) is a class

\[ \omega(X,j) \in \text{Ext}^2(BG/X,1) \]

expressible as a cup-product \( \cdots \), and when \( \omega(X,j) = 0 \), cl. of def make a torsor under \( \text{Ext}^1(BG/X,1) \) and the autom of a def = \( \text{Ext}^0(\cdots) \).

3.5. Pic st interpretation. To avoid all base change troubles, assume \( G \) flat. Then, if \( X \to Y \) is a morphism of G-S-schemes, \( G \) acts on the Pic st \( \text{Ext}(X/Y,1) \), hence on \( \text{Ext}(X/Y,1) = \text{Hom}(t^{[i]}[2X/Y],BG) \) for any \( G \)-Module \( I \). As a result, in the situation of (3.4), the exact sequence (2.3.2) is in fact an exact sequence of \( G \)-Pic st. Hence, if \( X \) can locally (upstairs) be deformed as a scheme/\( Y \), then \( \text{Def}(X/Y,j) \) is a torsor under \( \text{Ext}(X/Y,1) \) on \( BG/X \). A global object of this torsor, i.e. a global equivariant deformation of \( f \), identifies \( \text{Def}(X/Y,j) \) to \( \text{Ext}(X/Y,1) \) (as G-stacks), hence the isom. cl. of equiv. def. ism (resp. the autom. group of a given equiv. def.) to \( \text{Ext}^1(BG/X,1) \) (resp. \( \text{Ext}^0(\cdots) \)). (Use that

\[ H^i(BG/X,\text{Ext}(X/Y,1)) = \text{Ext}^{i+1}(BG/X,1) \]

for \( i = 0, -1 \).)

\(^{(1)}\) Let L be a Pic st over \( X \), an action of \( G \) on \( L \) consists of an action of \( G(T) \) on \( L_T \) for each \( T/S \) ("functorial" in \( T \)); equivalently a \( G \)-action of \( L \) is a descent data on \( L \) with respect to the nerve of \( G/X \) i.e. an equivalence \( d^m_{01} \to d^m_{01}(X) \), a 2-map on \( G \times X \), with cocycle condition on \( G \times G \times G \times X \). One has a dictionary:

\[ (G\text{-Pic st}/X) \leftrightarrow \text{Ext}^1(G,0)(BG/X) \]
3.6. Case of a $G$-torsor. From now on, $G$ will be assumed to be flat.
Suppose in (3.4) $Y$, $Y$ are trivial $G$-schemes, and $X$ is a torsor on $Y$
under $G$ (for the flat topology). One seeks the obstruction to
deforming $X$ on $Y$ as a torsor under $G$. I claim that deforming $X$
as a torsor is the same as deforming $X$ as a $G$-scheme over $Y$. In fact,
let/\( \exists X/Y \) be an equivariant def. of $X/Y$, then first of all $X$ is a
pseudo-torsor, i.e. $G_X \times X \to X \times X$, (because of (3.3)), and it
has a local fppf section because $X \to Y$ is flat and surjective.
Now, since $X$ is a torsor, one has a natural equivalence

\[(3.6.1) \quad \mathcal{E} / X \to Y,\]

so $L_{X/Y}$ is induced via (3.6.1) by a well defined

\[(3.6.2) \quad \mathcal{I}_{X/Y} \notin \text{ob D}(Y),\]

called the invariant cotangent complex. Here, to avoid an irrelevant

\[(3.6.3) \quad \text{Ext}^4(BG/x L_{X/Y}, J) = \text{Ext}^4(Y, \mathcal{I}_{X/Y}, J),\]

the obstruction

\[(3.6.4) \quad \omega(X, j) \notin \text{Ext}^2(\mathcal{I}_{X/Y}, J),\]

etc.

Pic at interpretation. Denote by $\text{Def}_G(X/Y, J)$ the Pic at on $\overline{Y}$ of
local downstairs, global upstairs, equivariant deformations of $X$ on
$Y$. One has (for arbitrary $G$-maps $f$, $j$)

\[(3.6.5) \quad \text{Def}_G(X/Y, J) = \bigwedge \text{Def}(X/Y, J)_L,\]

where $\bigwedge$ is the invariant direct image functor, i.e. the composition

\[G-\text{Picst}(X) \xrightarrow{\bigwedge} G-\text{Picst}(Y) \xrightarrow{f^*} \text{Picst}(Y),\]

for a Picst L on $Y$, $G^G(X)$ is the Picet of pairs $(x, s)$, $x \notin \text{ob L}$,
$s : d^Y_0 x \sim a d^X_1 x$ (a : $d^M_0 \to d^M_1$ the structural equivalence), s. t.
the suitable cocycle condition holds on $G \times G \times Y$; in the dictionary,
$\Gamma (L) \xrightarrow{\bigwedge t} G^{G}(L)$.
Returning now to the situation of (3.6), one deduces from (3.5):

Prop. 3.6.6. Suppose there exists a global equivariant deformation $\tilde{X}$ of $X$ (which therefore will be automatically a torsor under $G$ as seen above), then $\tilde{X}$ defines a canonical equivalence

$$\text{Def}_G(X/Y,j) \xrightarrow{\sim} \text{RHom}(\chi_{X/Y},j) \left[1\right].$$

In fact, the right hand side is just $\chi_Y \text{RHom}(\chi_X/Y, i^*_Y j) \left[1\right]$ by the projection formula.

Cor. 3.6.7. (Mazur-Roberts). Assume $G$ commutative. Then, in the derived cat. of $\sE$-Mod. (for the flat top) on $Y$, there exists a canonical isomorphism

$$(0 \to G \to j^*_Y G \to 0) \xrightarrow{\sim} j^*_Y \text{RHom}(\chi_G, j),$$

where $\chi_G = \chi_{G_Y} \times Y$, and $G_Y$ is placed in degree 0.

Proof. In (3.6.6), take $\tilde{X} = G_Y$, $X = G_Y$. The stack (on $Y$) of equivariant deformations of $X$ is nothing else but the stack of torsors under $G_Y$ trivialized along $Y$, so, by the dictionary, it corresponds to the complex $(G_Y \to j^*_Y G)$ where $G_Y$ is placed in degree $-1$. Therefore the equivalence of (3.6.6) yields the desired isomorphism.

Lemma 3.6.8. Let $(0 \to L_1 \to L_0 \to 0)$ be a complex of abelian sheaves in some topos. One has

$$\text{st}(L) = \text{Ker}(\text{st}(L_1 \left[1\right]) \to \text{st}(L_0 \left[1\right]))$$

$$= \text{st}(x, s), x \text{ a torsor under } L_1, s : d_x \xrightarrow{\sim} 0 \text{ (a trivialization of the torsor under } L_0 \text{ image of } x \text{ by } d).$$

Proof. Left to the reader.

3.7. The Atiyah extension.

The obstruction $\omega(X,j)$ of (3.4) is the cup-product of the class of $j$, $e(j) \in \text{Ext}^1(L_Y/S, j)$, by the Kodaira-Spencer class

$$c(X/Y/S) \in \text{Ext}^1(L_X/Y, \chi_{X/Y} \left[1\right]/S).$$

Assume now, as in (3.6), that $Y$, $\tilde{Y}$ are trivial $G$-schemes and $X$ is
a torsor under $G$. Then, by descent, one has
\[ \text{Ext}^i\left( L_{X/Y}, \Gamma_{L_Y/S} \right) = \text{Ext}^i\left( \mathcal{X}_{X/Y}, \Gamma_{L_Y/S} \right) \]
(at least if $L_Y/S$ is in $D^b$), so $c(X/Y/S)$ is a class
\[ (3.7.1) \quad c(X/Y/S) \in \text{Ext}^1(X_{X/Y}, L_Y/S) \]
When $G$ and $Y$ are smooth, this class is easily seen to coincide with the class of the Atiyah extension
\[ (\omega) \quad 0 \to \mathcal{S}_{X/Y} \to \Lambda(X) \to \mathcal{O}_{X/Y} \to 0 \]
defined by descent to $Y$ of the exact sequence of differentials on $X$; when $G$ is smooth but not necessarily $Y$, $(\omega)$ is not still defined, and is just the image of $c(X/Y/S)$ by $L_Y/S \to \mathcal{S}_{Y/S}$. Moreover, $\mathcal{O}_{X/Y}$ is known to be isomorphic to the sheaf $X^\text{Lie}(G)_Y$ obtained from the invariant differential forms on $G$ by taking the inverse image on $Y$ and twisting by $X$ via the adjoint operation. This can be generalized as follows.

By the classifying property of $BG$, the $G$-torsor $X$ over $Y$ defines a map $Y \to BG$ s.t. $X$ is the inverse image by $u$ of the universal torsor $PG$ over $BG$ (recall $u^\natural$ consists in inducing on $Y$ and twisting by $X$), in other words one has a "commutative" diagram with a cartesian square
\[ (3.7.2) \]
\[ \begin{array}{ccc}
\text{PG} & \xleftarrow{u^\natural} & X \\
\downarrow & & \downarrow \\
BG & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
S & & S
\end{array} \]
Now one disposes of
\[ (3.7.3) \quad \mathcal{X}_{BG} = \mathcal{X}_{PG/BG} \in \text{ob} D(BG) \]
defined by means of nerves like in (3.2), (3.6), and it is clear that one has
\[ (3.7.4) \quad \mathcal{X}_{X/Y} \cong u^\natural\mathcal{X}_{BG} \]
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Status of stability problem.

Given ring $\Lambda$ we associate a simplicial complex $X(\Lambda,n)$ for unimodular vectors in $\Lambda^n$.

**Problem 1**: Show that $GL_n\Lambda$ acts transitively on the $(j-1)$-simplices for $1 \leq j \leq n-d$, $d = \dim \text{Max}(\Lambda)$, and that the homology of $X(\Lambda,n)$ begins in dimension $n-1-d$.

This is OKAY when $\text{spec} \Lambda$ is linear, projection arguments can be used, e.g. $\Lambda = k[x_1, \ldots, x_m]$ where $k$ is finite field. Need somehow to do non-linear projections à la Nagata.

**Problem 2**: Deduce stability from Problem 1.

OKAY for coefficients in $\mathbb{Q}$ if $\Lambda$ over $\mathbb{Z}[\xi^{-1}]$ some $l$. For $\mathbb{Z}/l\mathbb{Z}$ coeffs. and $l'|\text{rad}\Lambda$ we need a formula for differential of the spectral sequence. There is some indication that I can push standard range

$$H_i(GL_n) \rightarrow H_i(GL_n)$$

iso: $i \leq n-d-1$

surj: $i = n-d-1$

through for $l$ odd, but not for $l = 2$. (Orthogonal groups $O_n(\mathbb{F}_2)$, $4|8-1$, not surjective for $i = n-1$.)

Symmetric groups $S_n$, $GL_2(\mathbb{F}_2)$ not surj. for $i = n-1$. 
Theorem: Let $\Lambda$ be a perfect ring of characteristic $p$. Then $K_i \Lambda \otimes \Lambda$ is uniquely $p$-divisible for $i > 0$.

Proof: The Frobenius auto. $F: \Lambda \to \Lambda$, $Fx = x^p$ induces an auto of $K_i \Lambda$ which coincides with $\Phi^i$. (Known for representations, hence in general.) One knows for any element $x \in K_i \Lambda$ that for some $n$

$$(\Phi^p - x) \cdots (\Phi^p - x^n) x = 0$$

Let $F_n$ be the set of $x$ in $K_i \Lambda$ satisfying this equation. Then $F_n$ is stable under the Adams operation, $F_{n-1} \subseteq F_n$, and

$$\Phi^p = x^n \text{ on } F_n/F_{n-1}.$$ 

Since $\Phi^p$ is an automorphism, we see that $F_n/F_{n-1}$ is uniquely $p$-divisible for each $n$, hence $K_i \Lambda = 0$ $F_n$ is also uniquely $p$-divisibles.

Complement: $K_0 \Lambda$ also uniquely $p$-divisible by the same argument (namely $\Phi^p$ has eigenvalue $1$ on $i$-th quotient of the $K$-filtration.)

stability problem:

Recall that our chain complex is

\[ C_{i-1}(X) = \mathbb{Z}_l[GL_n] \times \bigoplus_{0 \leq i < n} \mathbb{Z}_l[\Sigma_i] \text{ sgn} \otimes 1 \]

and that under the assumptions made,

\[ H_*(GL_n, C_{i-1}(X)) = H_*(\Sigma_i, \text{ sgn}) \otimes H_*(GL_{n-i}, \mathbb{Z}/l) \]

It therefore becomes important to know something about \( H_*(\Sigma_i, \text{ sgn}) \).

Proof: If \( l \) odd, then \( H_*(\Sigma_i, \text{ sgn}) = 0 \) for \( n \neq 0, 1 \) (l).

Proof: The index of \( \Sigma_{n-2} \times \Sigma_2 \) in \( \Sigma_n \) is \( \frac{n(n-1)}{2} \), prime to \( l \). Hence \( \exists \) surjective map

\[ H_*(\Sigma_{n-2} \times \Sigma_2, \text{ sgn}) \Rightarrow H_*(\Sigma_{n-2}, \text{ sgn}) \]

\[ H_*(\Sigma_{n-2}, \text{ sgn}) \otimes H_*(\Sigma_2, \text{ sgn}) \Rightarrow 0 \]
and \( H_x(\Sigma_2, \text{sgn}/l) = 0 \). q.e.d.

Prop. 2: transfer: \( H_x(\Sigma_{ml+1}, \text{sgn}/l) \sim H_x(\Sigma_{ml}, \text{sgn}/l) \)
(canonical map other way is also isomorphism).

Proof. Injectivity of transfer + surjectivity of restriction clear as index \( ml+1 \) is prime to \( l \).
Now quite generally when \( H < G \) contains the Sylow subgp, one has Brauer type \( \mathcal{E} \) then:

\[
H^*(G, M) \to H^*(H, M) \Rightarrow \prod H^*(H \times H^{-1}, M)
\]

but in this case all the intersections \( H \times H^{-1} \) are of form \( \Sigma_{ml-1} \) and by the proposition these have trivial cohomology except when \( x \) normalizes \( H \), which doesn't happen here as \( \Sigma_{ml} \) is its own normalizer in \( \Sigma_n \).

Remark: Prop. 1+2 also holds for \( l = 2 \), prop. 1 holds trivially (every \( n \equiv 0, 1 \mod 2 \)), and proof of prop. 2 same.
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Computation of the differential $d^1$. Thus we have the map

$$d: C_i(X) \rightarrow C_{i-1}(X)$$

$$d [e_0, \ldots, e_i] = \sum_{\nu=0}^{i} (-1)^{\nu} [e_0, \ldots, \hat{e}_{\nu}, \ldots, e_i]$$

and in mod $l$ cohomology it induces a map

$$H_*(\Sigma_{i+1}, ogn) \otimes H_*(GL_{n-i-l}) \rightarrow H_*(\Sigma_i, ogn) \otimes H_*(GL_i).$$

I claim that this map is the tensor product of the induction

$$H_*(GL_{n-i-l}) \rightarrow H_*(GL_i)$$

and the transfer

$$H_*(\Sigma_{i+1}, ogn) \rightarrow H_*(\Sigma_i, ogn).$$

In virtue of Kunneth it means that we have a comm. diag

$$\begin{array}{ccc}
H_*(GL_n, C_i(X)) & \xrightarrow{\text{tr}} & H_*\left(\frac{\Sigma_i}{0}, \frac{0}{GL_{n-i-1}}\right) \\
\uparrow & & \uparrow \\
H_*\left(\frac{\Sigma_i}{0}, \frac{0}{GL_{n-i-1}}\right) ogn \otimes 1 & \xrightarrow{\text{in}} & H_*\left(\frac{\Sigma_i}{0}, \frac{0}{GL_{n-i-1}}\right) ogn \otimes 1
\end{array}$$
To prove this is the case we generalize slightly and consider the morphism of homological functors

\[ H^*_\ast(GL_n, C_i(X) \otimes M) \rightarrow H^*_\ast(GL_n, C_{i-1}(X) \otimes M). \]

of the \( GL_n \)-module \( M \). Actually we shall work with the dual coh. functors

\[ \text{Ext}^*_\ast(GL_n, C_{i-1}(X), M) \rightarrow \text{Ext}^*_\ast(GL_n, C_i(X), M). \]

Such a transf. is determined by what it does in dimension 0.

\[ \text{Hom}_{GL_n}(C_{i-1}(X), M) = \text{set of functions } f(\sigma_{1, \ldots, n}) \]

defined on independent vectors with values in \( M \) which are \( GL_n \)-equivariant and alternating.

\[ \{ m \in M \mid (\sigma_{1, \ldots, n})^t m = (-1)^{\sigma} m \quad \sigma \in \Sigma_i \} \]

The rule giving this iso assigns to \( f \) the element \( m = f(e_1, \ldots, e_i) \) where \( e_1, \ldots, e_n \) is standard base for \( V \).

Now the differential \( \delta: \text{Hom}_{GL_n}(C_{i-1}(X), M) \rightarrow \text{Hom}_{GL_n}(C_i(X), M) \)

is:
\[(\delta f)(\sigma_1, \ldots, \sigma_{i+1}) = \sum_{\nu=1}^{i+1} (-1)^{\nu-1} f(\sigma_1, \ldots, \sigma_\nu, \ldots, \sigma_{i+1})\]

hence

\[(\delta m) = (\delta f)(e_1, \ldots, e_{i+1}) = \sum_{\nu=1}^{i+1} (-1)^{\nu-1} f(e_1, \ldots, \hat{e}_\nu, \ldots, e_{i+1})\]

Let \(\sigma_\nu \in \Sigma_{i+1}\) be the permutation

\[
\begin{array}{c}
\sigma_\nu = \begin{pmatrix}
1 & & & i+1 \\
\vdots & & \times & \times \\
1 & \cdots & \nu & i+1
\end{pmatrix}
\end{array}
\]

\[\text{sgn}(\sigma_\nu) = (-1)^{i-\nu+1}\]

so that \([e_1, \ldots, \hat{e}_\nu, \ldots, e_{i+1}] = \sigma_\nu [e_1, \ldots, e_i]\]. Then

\[f(e_1, \ldots, \hat{e}_\nu, \ldots, e_{i+1}) = \sigma_\nu m.\]

so

\[\delta m = \sum_{\nu=1}^{i+1} (-1)^{\nu-1} \sigma_\nu m\]

Now reinterpret this as follows.

\[
\begin{array}{c}
\Ext^0_{\text{GL}_n}(C_{i-1}(X), M) = H^0\left(\frac{\Sigma_i}{\times \text{GL}_{n-i}}, \text{sgn} \otimes M\right)
\end{array}
\]

\[
\begin{array}{c}
\Ext^0_{\text{GL}_n}(C_i(X), M) = H^0\left(\frac{\Sigma_{i+1}}{\times \text{GL}_{n-i-1}}, \text{sgn} \otimes M\right)
\end{array}
\]
and the map on the left corresponding to \( \sigma \) sends
\[
1 \otimes m \rightarrow (-1)^{i-1} \left( \sum_{\nu=1}^{i+1} \sigma_{\nu} \right) 1 \otimes m.
\]

since the \( \sigma_{\nu} \) are coset representatives for the subgroup
\[
\begin{pmatrix}
\Sigma_i & 0 \\
0 & 1 \\
\times & \text{GL}_{n-i-1}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\Sigma_{i+1} & 0 \\
0 & 1 \\
\times & \text{GL}_{n-i-1}
\end{pmatrix}
\]

where \( \times \) is the composition of
\[
H^0 \left( \Sigma_i \bigg/ \text{GL}_{n-i}, \operatorname{sgn}_{\Sigma_i} \otimes M \right) \rightarrow H^0 \left( \begin{pmatrix}
\Sigma_i & 0 \\
0 & 1 \\
\times & \text{GL}_{n-i-1}
\end{pmatrix}, \operatorname{sgn}_{\Sigma_{i+1}} \otimes M \right)
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\times & \times
\end{pmatrix} m = (-1)^{\nu} m
\]

followed by
\[
H^0 \left( \begin{pmatrix}
\Sigma_{i+1} & 0 \\
0 & 1 \\
\times & \text{GL}_{n-i-1}
\end{pmatrix}, \operatorname{sgn}_{\Sigma_{i+1}} \otimes M \right) \rightarrow H^0 \left( \begin{pmatrix}
\Sigma_{i+1} & 0 \\
0 & 1 \\
\times & \text{GL}_{n-i-1}
\end{pmatrix}, \text{sgn}_{\Sigma_{i+1}} \otimes M \right)
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\times & \times
\end{pmatrix} m = (-1)^{\nu} m
\]

\[
\sum_{\nu=1}^{i+1} \sigma_{\nu} (m).
\]
Then since $\sigma_\nu$ has sign $(-1)^{i+1-\nu}$ we have

$$1 \otimes (-1)^\nu \sigma_\nu m = (-1)^{i+1} \sigma_\nu (1 \otimes m)$$

so that $x$ is $(-1)^{i+1}$ times $s$. As $x$ and $s$ are coh. functors of $M$ we see that up to sign

$$\operatorname{Ext}^*_X(C_{i-1}(X), M) \to \operatorname{Ext}^*_X(C_i(X), M)$$

is the composite of

$$H^* \left( \Sigma_i \otimes M \right) \to H^* \left( \frac{\Sigma_i}{\bigoplus \Sigma_i} \otimes M \right)$$

transfer

$$H^* \left( \frac{\Sigma_{i+1}}{\bigoplus \Sigma_i} \otimes M \right).$$
On the stability range:

\[ l=2 \quad \text{not surj for} \quad n=2^a \quad \text{because} \]

\[ W_{2^n-1} \in H^{2^n-1}(\Sigma_{2^n}) \]

is non-zero (restrict to \( e_A \), \( A = (\mathbb{Z}/2\mathbb{Z})^a \)) yet it dies on \( \Sigma_{2^n-1} \).

Second instance of this is for \( H_3(\Sigma_3) \rightarrow H_3(\Sigma_4) \), I recall the spectral sequence that I constructed

\[ E^1_{pq} = H_q(\Sigma_p \times \Sigma_{n-p}) \Rightarrow H_*(pt) \]

Consider \( H_1(\Sigma_1) \rightarrow H_1(\Sigma_2) \) situation: \( n=2 \).

\[
\begin{array}{c|c}
0 & 0 \quad H_1(\Sigma_2 \times \Sigma_0) \\
0 & 0 \quad H_0(\Sigma_2 \times \Sigma_0) \\
\end{array}
\]

\[
\begin{array}{c|c}
H_1(\Sigma_1 \times \Sigma_1) & \rightarrow H_1(\Sigma_0 \times \Sigma_2) \\
H_0(\Sigma_1 \times \Sigma_1) & \rightarrow H_0(\Sigma_0 \times \Sigma_2) \\
\end{array}
\]

Periodic
Next case: \[ H_3(\Sigma_3) \rightarrow H_3(\Sigma_4) \]

\[
\begin{align*}
H_2(\Sigma_3 \times \Sigma_2) &\rightarrow H_2(\Sigma_2 \times \Sigma_2) \\
H_1(\Sigma_3 \times \Sigma_2) &\rightarrow H_1(\Sigma_2 \times \Sigma_2) \\
H_0(\Sigma_3 \times \Sigma_2, \Sigma_2) &\rightarrow H_0(\Sigma_3 \times \Sigma_2, \Sigma_2)
\end{align*}
\]

Conclusion: The differentials in this spectral sequence are terrible.
February 17, 1971

A curve \( X \) over a finite field \( k \), \( \overline{k} = \text{alg. closure of } k \), \( \overline{X} = \overline{k} \times_k X \), \( q = \text{card } (k) \). Then (at least for \( G \) finite of order prime to \( p = \text{char } (k) \)) we have

\[
R_X(G) \sim R_{\overline{X}}(G)^{\text{Gal}(\overline{k}/k)} = \left( R_{\overline{X}}(G) \otimes K(\overline{X}) \right)^{\text{Gal}(\overline{k}/k)}
\]

Since

\[
K(\overline{X}) = \mathbb{Z} \oplus \mathbb{Z} \oplus A(\overline{k})
\]

where \( A(\overline{k}) \) is the Jacobian of \( \overline{X} \), and this isomorphism with Galois as \( I \) a divisor of degree 1 on \( X \), one has an isomorphism

\[
R_X(G) \cong R_k(G) \oplus R_k(G) \oplus \left( R_{\overline{X}}(G) \otimes A(\overline{k}) \right)^{\text{Gal}(\overline{k}/k)}
\]

Now I want to compute the char. classes of \( \tilde{R}_X \) with values in mod \( l \) coh., \( l \neq p \). Recall

\[
\tilde{R}_X(G) \oplus K(X) = R_X(G)
\]

Now an exponential class
\[ \Theta : \tilde{R}_X(\bar{\Omega}) \longrightarrow H^0(\bar{\Omega}, \mathcal{S}) \quad \quad \quad s_0 = \mathbb{Z}/l\mathbb{Z} \]

factors through the \( l \)-adic completion of \( \tilde{R}_X \):
\[
\tilde{R}_X(G) \cong \tilde{R}_K(G) \oplus \tilde{R}_K(G) \oplus (\tilde{R}_K^{-}(G) \otimes A(\bar{K}))^{\text{Gal}(\bar{K}/K)}.
\]

\[0 \longrightarrow \mathfrak{A}(\bar{K}) \longrightarrow A(\bar{K}) \xrightarrow{l^\mathfrak{A}} A(\bar{K}) \longrightarrow 0\]

\[0 \longrightarrow \tilde{R}_K^{-}(G) \otimes \mathfrak{A}(\bar{K}) \longrightarrow \tilde{R}_K^{-}(G) \otimes A(\bar{K}) \xrightarrow{l^\mathfrak{A}} \tilde{R}_K^{-}(G) \otimes A(\bar{K}) \longrightarrow 0\]

Now take cohomology with \( \text{Gal}(\bar{K}/k) \). Claim that
\[
H^4(\text{Gal}(\bar{K}/k), \tilde{R}_K^{-}(G) \otimes A(\bar{K})) \]

is a \( p \)-torsion group. Indeed \( \text{Gal}(\bar{K}/k) = \hat{\mathbb{Z}} \) with Frobenius as generator, so the \( H^4 \) is the coinvariants, hence is (Pontryagin) dual to the invariants in
\[
\text{Hom}(\tilde{R}_K^{-}(G), A(\bar{K})^\vee)\]

Now the \( l \)-primary part of \( A(\bar{K})^\vee \) is \( \cong (\mathbb{Z}/l\mathbb{Z})^{2g} \) and Frobenius acts with \( l \)-primary eigenvalues, so the eigenvalues by Weil are alg. integers of absolute value \( l^{\frac{g}{2}} \). But Frobenius permutes the basis of irreducible reps. around so the eigenvalues are roots of 1. Conclude there are no invariants.
to the long exact sequence in col. gives

\[ (\tilde{\mathbb{R}}_k(G) \otimes \overline{A}(k))^\text{Gal} \otimes \mathbb{Z}/\ell^n \mathbb{Z} \to (\tilde{\mathbb{R}}_k(G) \otimes \mathbb{Z}_\ell^{\nu} A(K))^\text{Gal} \]

Taking the inverse limit over \( n \), and using the fact that the above groups are finite, we get

\[ (\tilde{\mathbb{R}}_k(G) \otimes \overline{A}(k))^\text{Gal} \otimes \mathbb{Z}_\ell \to (\tilde{\mathbb{R}}_k(G) \otimes T_\ell(A))^\text{Gal} \]

(a finite step)

where cheap Tate

\[ T_\ell(A) = \lim_{\leftarrow \ell} \ell^\nu A(K). \]

Now suppose that we have an exponential class

\[ \Theta : (\tilde{\mathbb{R}}_k(G) \otimes T_\ell(A))^\text{Gal} \to H^0(G, S) \times \]

In particular for each \( t \in T_\ell(A) \) we have an exponential class

\[ \tilde{\mathbb{R}}_k(G) \to H^0(G, S) \]

\[ u \mapsto \Theta(u \otimes t). \]

Such a class by our previous work is the same thing as a \[ \mathbb{R} \text{-} \text{power series} \]
\[ \sum_{i=0}^{\infty} s_i x^i \quad \text{where} \quad s_i \in S_{2i} \quad \text{and} \quad s_0 = 1. \]

Recall how this series is obtained: let \( \bar{k}^x \) act on \( \bar{k} \); it gives a canonical element \( u \) in \( R_{\bar{k}}^{+}(\bar{k}^x) \); applying \( \Theta(u \otimes t) \) gives an element of

\[ H^0(\bar{k}^x, S_i) = \prod_{i=0}^{\infty} H^i(\bar{k}^x, S_i) = \prod_{i=0}^{\infty} x^i S_{2i} \]

where \( x \in H^2(\bar{k}^x) \) is the element represented by the extension

\[ 0 \rightarrow \mu_\ell \rightarrow \bar{k}^x \rightarrow \bar{k}^x \rightarrow 0 \]

plus an iso. \( \mathbb{Z}/\ell \mathbb{Z} \subseteq \bar{k}^x \). \( x = c_1(u) \)

Therefore for each \( t \in T_\ell(A) \) we get a power series

\[ \Theta(u \otimes t) = \sum_i s_i(t) x^i; \]

denote by \( \varphi(t) \) this series. Then

\[ T_\ell(A) \xrightarrow{\Phi} 1 + \prod_{i=0}^{\infty} x^i S_{2i} \]

\[ t \mapsto \varphi(t) \]

is a homomorphism.

Next we must figure out action of Galois.
First put earlier results in better form. Given

\[ \Theta : \tilde{R}_k^-(G) \otimes T_\ell(A) \longrightarrow H^0(G, S_\nu) \]

one obtains a homomorphism

\[ \psi^\Theta : T_\ell(A) \longrightarrow H^0(\tilde{R}_k^x, S_\nu)^x = \left(1 + \prod_{i > 0} x_i S_{2i} \right)^x \]

by applying \( \Theta \) to \( u - 1 \) where \( u \) is standard form of \( \tilde{R}_k^x \) on \( \tilde{R}_k^x \). Now we have that generator of \( \text{Gal}(\tilde{k}/k) \) acts on \( \tilde{R}_k^-(G) \otimes T_\ell(A) \) as \( \Pi^b \otimes \sigma \) where \( \sigma \) is the auto of \( T_\ell(A) \) produced by Galois action in \( \text{Pic}(\tilde{k}) \). (I think this means that \( \sigma \) is the inverse of the geometric Frobenius.) Now \( \Theta \) is invariant mean commutativity in

\[
\begin{array}{ccc}
\tilde{R}_k^-(G) \otimes T_\ell(A) & \xrightarrow{\Pi^b \otimes \sigma} & H^0(G, S_\nu) \\
\Theta & \searrow & \swarrow \\
\tilde{R}_k^-(G) \otimes T_\ell(A) & \xrightarrow{\Theta} & \end{array}
\]

But for \( G = \tilde{R}_k^x \), \( \Pi^\theta = \lambda^x \) where \( \lambda : k^* \rightarrow k^* \) is raising to \( \theta \)-th power. \( \lambda \) acts trivially on cohomology provided we assume \( \mu_\bar{\ell} < k \). Therefore we see that
\( T_e(A) \longrightarrow H^0(\mathbb{R}^x, S_x)^x \)

\[ \downarrow \sigma \quad \quad \quad \quad \quad \quad \quad \longrightarrow \]

\( T_e(A) \longrightarrow H^0(\mathbb{R}^x, S_x)^x \)

commutes. So

\[ \exp \text{ classes} (\mathbb{R}_x, H^0(S_x)) = \text{Hom} \left( T_e(A)_{\text{Gal}}, \left( 1 + \prod_{i > 0} x_i S_{2i} \right)^x \right) \]

showing the homology is quite far from being a polynomial ring.

Assume \( \exists X \) such that \( T_e(A)_{\text{Gal}} = \mathbb{Z}/l\mathbb{Z} \).

Then an exponential class is a series

\[ \sum x^i s_i \quad s_0 = 1 \quad s_i \in S_{2i} \]

such that

\[ 1 = \left( \sum x^i s_i \right)^l = \sum x^{li} s_i^l \]

\[ \Rightarrow s_i^l = 0 \quad \text{for all } i \geq 1. \]

Thus the Hopf algebra of homology is

\[ \mathbb{Z}/l\mathbb{Z} \left[ z_1, z_2, \ldots \right] / (z_1^l, z_2^l, \ldots) \]

with

\[ \Delta z_n = \sum_{i+j = n} z_i \otimes z_j \]

Does this belong to any space?
February 20, 197[1], Conjectures about $K_a(X)$, $X$ curve over $\mathbb{F}_q$:

1) The $K$-groups should split into three parts

$$K_a(X) = K_a(k) \oplus K_a^{pr}(X) \oplus K_a(k)$$

$k = \mathbb{F}_q$

where the outer two summands come from

$$i^*: K_*(X) \to K_*(k)$$

inclusion of a point and

$$f^*: K_*(X) \to K_*(k)$$

$f: X \to \text{Spec}(k)$ being the canonical map. The primitive part comes from the Jacobian of $\bar{X}$.

2) Formula for $K_*^{pr}(X)$: This should be a finite group of order prime to the characteristic $p$, and we consider only the $l$-primary part. Let $T_l$ be the Tate module of rank $2g$ over $\mathbb{Z}_l$ associated to the Jacobian of $\bar{X}$, and denote by $F$ the Frobenius automorphism of $T_l$ so that

$$\zeta^{pr}(s) = \det (1-q^{-s}F)$$

($\zeta^{pr}$ equals part of $\zeta$ not involving $H^0$ and $H^2$.)
Now form the "space" $U \otimes T_\ell$; it is a product of $2g$ copies of the $l$-adic completion of $U$.

On $U \otimes T_\ell$ we put the endomorphism $\sigma = \Phi^g \otimes F$

and we form the fibre

$$E(\sigma) \longrightarrow U \otimes T_\ell \overset{\sigma-1}{\longrightarrow} U \otimes T_\ell$$

I conjecture that

$$K_{a}^{pr}(X)_{(\ell)} = \prod_{a} E(\sigma) \quad a \geq 0.$$ 

Can check this:

$$K_{0}^{pr}(X)_{(\ell)} = T_\ell \langle \sigma-1 \rangle = \text{Jac}(X)^{\text{Gal}} = \text{Pic}^{0}(X)$$

Also

$$K_{1}^{pr}(X)_{(\ell)} = 0$$

and

$$\text{card } K_{2i}^{pr}(X)_{(\ell)} = |\text{det } (1 - g^{i}F)| \quad 1 \leq i \leq \ell$$

Hence

$$K_{2i+1}^{pr}(X) = 0$$

and

$$\text{card } K_{2i}^{pr}(X) = \text{det } (1 - g^{i}F)$$

$$= \zeta^{pr}(-i)$$

For $i = 1$, this is compatible with Tate's computation of $K_{2}(F)$ and the exact sequence

$$0 \longrightarrow K_{2}(X) \longrightarrow K_{2}(F) \overset{\text{N}}{\longrightarrow} \bigoplus_{v \in X} K_{v}^{0} \longrightarrow K_{1}(X).$$
3) Behavior under base extension. If \([k_1:k] = d\), then the endo. \(\sigma\) for \(k_1\) is \(\sigma^d\):

\[
\begin{align*}
E(\sigma) & \longrightarrow U \otimes T_e \\
& \downarrow \quad 1 \\
E(\sigma^d) & \longrightarrow U \otimes T_e
\end{align*}
\]

\[
\begin{align*}
\pi_{2i+1} U \otimes T_e & \longrightarrow \pi_{2i+1} U \otimes T_e \\
& \downarrow \quad 1 + \sigma + \cdots + \sigma^{d-1} \\
\pi_{2i+1} U \otimes T_e & \longrightarrow \pi_{2i+1} U \otimes T_e
\end{align*}
\]

Serre's lemma shows that

\[
K_{2i}^{pr}(X) \longrightarrow \longrightarrow K_{2i}^{pr}(X \times k, k_1)
\]

provided \(1 + \sigma + \cdots + \sigma^{d-1}\) has none of its eigenvalues equal to zero, which is clear because after tensoring with \(A\), it becomes an iso.

Therefore as in the case of a finite field we get
\[ K_{2i}^{pr}(\overline{X}) \cong \Theta^i(\mathbb{Q}/\mathbb{Z})^{2g} \]

that the Frobenius acts as \( \phi^i F \), and that

\[ K_{2i}^{pr}(\overline{X}) \cong K_{2i}^{pr}(\overline{X}) \text{Gal}(k/k) \]

Furthermore it is clear that the restriction of scalars homomorphism from \( k_1 \) down to \( k \) is given by the norm.

Conjecture: Let \( \overline{X} \) be of finite type over the algebraic closure \( \overline{k} \) of a finite field. Then the \( K \)-groups satisfy periodicity:

\[ K_i(\overline{X}) \cong K_{i+2}(\overline{X}) \quad i \geq 0. \]

Moreover if \( \overline{X} \) is smooth, then \( K_+(\overline{X}) \) should have no \( p \)-torsion and be \( l \)-divisible for all \( l \).

Maybe one should look at things this way: Form the "topological" \( K \)-groups: \( [\Sigma^* X_{et}, BU_{et}(\overline{\mathbb{Q}})] \) as suggested by Friedlander. Denote them by \( K_{\text{top}}(\overline{X}) \); they are free \( \mathbb{Z}_l \)-modules if \( H^*(\overline{X}, \mathbb{Z}_l) \) is torsion-free which will assume. Then there should be an \( l \)-divisible type procedure for converting free \( \mathbb{Z}_l \)-modules into \( \mathbb{Q}_l/\mathbb{Z}_l \)'s in one lower dimension, and this should give the \( K \)-groups of \( \overline{X} \) over \( k \). Now one can take invariants under Frobenius to get the \( K \)-groups over \( k \).
Basic internal consistency of this scheme with $I$-function: Again $X$ is a curve over $k$. Then we have a map

$$c_2^\#: K_2(X) \longrightarrow \varprojlim H^2(X, \mu_{2^n}^\otimes)$$

Exact sequence:

$$0 \rightarrow H^1(\text{Gal}, H^1(\overline{X}, \mu_{2^n}^\otimes)) \rightarrow H^2(X, \mu_{2^n}^\otimes) \rightarrow H^0(\text{Gal}, H^2(\overline{X}, \mu_{2^n}^\otimes)) \rightarrow 0$$

Use.

$$H^1(\overline{X}, \mu_{2^n}) \rightarrow \ell^n \text{Pic}^0(X)$$

$$\mathbb{Z}/\ell^n = \text{Pic}(X)/\ell^n \text{Pic}(X) \rightarrow H^2(\overline{X}, \mu_{2^n})$$

hence one gets

$$H^1(\text{Gal}, T(e(\text{Jac}(\overline{X}))(1))) \rightarrow \varprojlim H^2(X, \mu_{2^n}^\otimes)$$

$$\rightarrow \left[ \text{Pic}^0(\overline{X})(1) \right]_{\text{Gal}(\mathbb{F}_l/k)}$$

and this by Tate should be an isom: $K_2^{\text{pr}}(X) = \left[ \text{Pic}^0(\overline{X})(1) \right]_{\text{Gal}(\mathbb{F}_l/k)}$

Generalizing this conjecturally, we expect that the groups $K_2^{\text{pr}}(X)$ and $\varprojlim H^2(X, \mu_{2^{n+1}}^\otimes)$
are isomorphic and that

\[ c^*_i : K^{p_i}_{2i} (X) \rightarrow \frac{\mathbb{L}}{\mathbb{L}} H^{2i} (X, \mathbb{Z}_9^{\otimes (i+1)}) \]

is multiplication by \( \pm (i)! \). This conjecture agrees with our earlier conjecture and with the \( S \)-function nonsense.

What is incredibly mystifying is the way the \( S \)-function enters into the theory. At the moment we relate \( K \) to \( S \) by these steps:

A.) \( K \rightarrow H^* (\_ , T^\otimes i) \) via Chern classes

B.) \( H^* (\_ , T^\otimes i) \) to values of \( S \) at \(-i \pm \varepsilon\) via Lefschetz formula in etale cohomology.

In higher dimensions the relations aren't so easy to decipher. Ideally one might expect the \( \otimes \) motive \( X^i \) of \( i \)-th coh. of a non-sing. proj. var. \( X \) over \( k \) to have the following \( K \)-groups:

\[ K_\ast (X^i) = K_\ast (\overline{X}^i)^{Gal} \]

\[ K_a (\overline{X}^i) = \begin{cases} H^a (\overline{X}) \otimes \mathbb{Q}/\mathbb{Z} (j) & a = i \otimes 1 + 2j \\ 0 & a \neq i-1 \mod 2 \end{cases} \]

Hence

\[ \text{card } K_{2j-1+i} (X^i) = \det \left( 1 - q^i F \right) = \frac{1}{\chi^i} \sum_{j} (-j)^{C_{j+i}} \]
Intriguing possibility: Over \( S \)-rings of \( S \)-integers in number fields one has cohomology and hopefully a relation between K-theory and cohomology. Empirically one has by Lichtenbaum a relation between the K-groups and the \( S \) function. So maybe one will eventually establish this relation fulfilling the dream of a cohomological interpretation of \( S \)-functions.
February 21, 1974

today I removed stuff on equivariant coh. from
desk to bookcase and brought back finite groups
of rational points paper for writing. Projects:

**Part III to Spectrum Paper:**
1. localization thm. + (fixpoint formula, maybe)
2. maximal strata given by Centralizers
3. Central elementary $A$ + depth - primary spaces
4. structure of an $A$-space + recovery of Euler
   classes for $p$ odd.

Remaining topics not much developed
1. Comm subring of $H^*$ has same spectrum?
2. Tate cohomology (duality if I any)
3. Multiplicative transfer + Riemann Roch
4. Euler characteristic for $H^*_G(X)$
5. Characteristic classes in $H^*(X/G)$ for actions.
   (Sullivan mod 2 Whitney classes).

should write a paper on
symmetric groups, $h$-symmetry operations
cohomology ops.
February 22, 1979

Problem: Let \( G = \{ G_v \} \) be a simplicial gp, and let \( E = \{ E_v \} \) be a simplicial \( A \)-module on which \( G \) acts. Assume that the normalization \( N.E \) is a bdd complex which has fin. gen. proj. \( A \)-modules in each degree. Does then \( E \) give rise to an element of \( [BG, BGL(A)^+] \)?

We want this element to agree with this special case: if \( G \) constant, then it should be the alternating sum of the representations \( N \cdot E \) of \( G \).

Question: Given a simplicial set \( X \) consider animals \( E = P \times G E \) where \( P \) is a principal \( G \)-bundle over \( X \), and \( E \) is a simplicial \( A \)-module which is "perfect" and which has an action of \( G \). Do such things define elements of \( K(X; A) = [X, K_0 A \times BGL(A)^+] \) and is every such \( K \)-element so realized?

**Definition:** A **topological category** $\mathcal{C} = \text{category}$ consists of two spaces $\text{Ar}\mathcal{C}$ and $\text{Ob}\mathcal{C}$ and four maps

$$
\begin{align*}
\text{Ar}\mathcal{C} & \rightarrow \text{Ob}\mathcal{C} \\
\text{Ob}\mathcal{C} & \rightarrow \text{Ar}\mathcal{C} \\
\text{Ar}\mathcal{C} \times \text{Ar}\mathcal{C} & \rightarrow \text{Ar}\mathcal{C} \\
\text{Ar}\mathcal{C} \times \text{Ob}\mathcal{C} & \rightarrow \text{Ob}\mathcal{C}
\end{align*}
$$

satisfying habitual identities.

**Definition:** A **topological groupoid** $\mathcal{C} = \text{topological category}$ such that $\exists$ continuous inverse $\iota : \text{Ar}\mathcal{C} \rightarrow \text{Ar}\mathcal{C}$ (represents a functor from (spaces) to (groupoids)).

Recall convention that arrows are drawn $\rightarrow$. Thus a left $\mathcal{C}$-space is a space $X \rightarrow \text{Ob}\mathcal{C}$ with an associative unitary action

$$
\text{Ar}\mathcal{C} \times \text{Ob}\mathcal{C} \times X \rightarrow X
$$

is the analogue of a covariant functor to sets. A right $\mathcal{C}$-space is analogous to a contravariant functor.

**Definition:** $\mathcal{C}$-torsor over $\mathcal{C}$ (or with base) a space $X$ which locally on $X$ is of the form

$$
\text{Ar}\mathcal{C} \times \text{Ob}\mathcal{C}
$$

for some map $X \rightarrow \text{Ob}\mathcal{C}$. 
Remarks: 1. C-torsors form a stack over (spaces). It is the stack generated by the pre-stack assigning to each space \( X \) the \( C(X) \). Observe two C-torsors are not necessarily locally isomorphic in general (they are, if \( C \) is a top. group).

2. To give an isomorphism \( P \cong \ar C \times \text{Ob}\ e \ X \) of C-torsors over \( X \) is the same as giving a section \( X \rightarrow P \). The difference of two such sections is a well-defined map \( X \rightarrow \ar C \), i.e. an arrow in \( C(X) \), because

\[
\ar C \times \begin{array}{c}
P \\
\text{Ob}
\end{array} \rightarrow P \times \text{Ob} \ circ \ P
\]

Consequently isomorphism classes of C-torsors are the same as Čech cohomology \( \check{H}^1(X; C) \).

3. Given two C-torsors: \( P \) over \( X \), \( P' \) over \( X' \), we can form their direct product leading to mixing diagram

\[
P \quad \rightarrow \quad P \times P' \quad \rightarrow \quad P'
\]

\[
X \quad \rightarrow \quad (P \times P') \quad \rightarrow \quad X'
\]

This in turn leads to a theory of universal bundles: \( P \) over \( X \) is universal if the map \( P \rightarrow \ar C \) is a fiber homotopy equivalence over \( \text{Ob} \ C \).

Problem: C-torsors for a topological category.
Example: Haefliger structures of codimension $\mathcal{G}$. Let $\Gamma_0$ denote the top. groupoid with $\text{Ob} \Gamma_0 = \mathbb{R}^0$, $\text{Ar} \Gamma_0$ the etale space over $\mathbb{R}^0$, $\Gamma_0$ whose sheaf of sections is the sheaf of $C^\infty$ maps $\mathbb{R}^0 \to \mathbb{R}^0$ which are local diffeomorphisms. $\Gamma_0 = \text{pseudo-group}$ of local diffeos of $\mathbb{R}^0$.

Let $X$ be a manifold endowed with a (smooth) foliation of codimension $\mathcal{G}$. Locally $\mathcal{E}$ submersions $f: X \to \mathbb{R}^0$ whose fibres are the leaves of the foliation; let $P$ be the sheaf of such submersions. Then $P$ is a $\Gamma_0$-torus, with action

$$\left(\text{Ar} \Gamma_0 \times \text{Ob} \Gamma_0\right) \times P \to P$$

given by composition of $f: X \to \mathbb{R}^0$ with a diffeomorphism of $\mathbb{R}^0$.

Now in general consider a vector bundle (smooth) $\mathcal{E}$ over a smooth manifold $X$ endowed with a foliation transversal to the fibres and a continuous section $s$.

Then $\mathcal{E}$ has a $\Gamma_0$-torus which can be pulled back via the section $s$. Thus $\Gamma_0$-torus same as Haefliger structure for the pseudo-group $\Gamma_0$. 
(Now I leave topological categories with "thick" arrow spaces such as topological groups which are not discrete, I want to consider C-sheaves without having to go to gross topos.)

So now consider a topological category $C$ such that source: $\text{Ar}C \rightarrow \text{Ob}C$ is etale. Then by $C^\wedge$ I mean the category of etale spaces $F \rightarrow \text{Ob}C$ with right $C$ action. Thus if $\text{Ob}C$ is discrete, $C$ is an ordinary category, and $C^\wedge$ is the category of contravariant functors from $C$ to (sets).

**Example:** $C = \Gamma$; any sheaf on $R^3$ intrinsically associated to the differential structure is a $\Gamma$-sheaf, such as $\Omega^q$, $\Theta$, jets, etc.

$C^\wedge$ is a topos. This is clear (more or less) because of the functor $C^\wedge \rightarrow \text{Top}(\text{Ob}C)$ forgetting the action, which commutes with everything. Generators of the form $\text{Ar}C \times \text{Ob}C \times U$, $U \subset \text{Ob}C$.

**Definition:** $C$ as above, a $C$-torsor over a space $X$ is a morphism of topos

$$f: \text{Top}(X) \longrightarrow C^\wedge.$$ 

(This definition is too virtuous to be understood.)

$f$ as above consider $\text{Ar}C$ as an object of $C^\wedge$.
via the source map. Then \( P = f^*(\mathcal{A}r\mathcal{C}) \) is a sheaf over \( X \). We have a morphism of induced toposes

\[
\begin{align*}
\text{Top}(P) & \longrightarrow \text{Top}(\text{Ob}\mathcal{C}) \\
\text{Top}(X)_{/P} & \longrightarrow \mathcal{C}^\mathcal{C}/\mathcal{A}r\mathcal{C}.
\end{align*}
\]

Assuming \( \text{Ob}\mathcal{C} \) is a sober space we get a map

\[ g: P \longrightarrow \text{Ob}\mathcal{C}. \]

It has the property

\[ f^*(F \times_{\text{Ob}\mathcal{C}} \mathcal{A}r\mathcal{C}) = g^*F = F \times_{\text{Ob}\mathcal{C}} P, \]

for any \( F \) in \( \text{Top}(\text{Ob}\mathcal{C}) \). In particular, taking \( F = \mathcal{A}r\mathcal{C} \) we have

\[
\begin{align*}
f^*(\mathcal{A}r\mathcal{C} \times_{\text{Ob}\mathcal{C}} \mathcal{A}r\mathcal{C}) &= \mathcal{A}r\mathcal{C} \times_{\text{Ob}\mathcal{C}} P \\
f^*(\mathcal{A}r\mathcal{C}) &= P
\end{align*}
\]

and we get a left action of \( \mathcal{A}r\mathcal{C} \) on \( P \). Moreover if \( F \) is in \( \mathcal{C}^\mathcal{C} \) we have an exact diagram

\[
\begin{align*}
\mathcal{C}^\mathcal{C} \times_{\text{Ob}\mathcal{C}} \mathcal{A}r\mathcal{C} & \longrightarrow \mathcal{C}^\mathcal{C} \times_{\text{Ob}\mathcal{C}} \mathcal{A}r\mathcal{C} \\
\mathcal{C}^\mathcal{C} \times_{\text{Ob}\mathcal{C}} \mathcal{A}r\mathcal{C} & \longrightarrow F
\end{align*}
\]

so as \( f^* \) is a left adjoint we have exact diagram
\[ f^* \left( F \times \text{Ob} \mathcal{E} \times \text{Ob} \mathcal{E} \right) \Rightarrow f^* \left( F \times \text{Ob} \mathcal{E} \right) \Rightarrow f^*(F) \]

\[ \text{whence (modulo checking the maps) we have} \]

\[ f^*(F) = F \times \mathcal{E} P \]

in other words, \( f^* \) is given by twisting with respect to \( \mathcal{E} P \).

Conversely, given a \( \mathcal{E} \) etale over \( X \) and a left \( \mathcal{E} \)-action on \( \mathcal{E} \) \( P \), we can define \( f^* \) by this formulas. For \( f^* \) to constitute a morphism of topoi it must commute with finite \( \lim \)'s. Can check this over each \( x \in X \).

\[
(F \times \mathcal{E} P)_x = \text{Coker} \left\{ F \times \text{Ob} \mathcal{E}, P_x \rightsquigarrow F \times \text{Ob} \mathcal{E}, P_x \right\}
\]

\[
= \lim_{(p, y)} F_y
\]

where \((p, y)\) runs over the category whose objects are pairs with \( y \in \text{Ob} \mathcal{C} \) and \( p \in P_x \) over \( y \), evident morphisms (i.e. the cofibered category over \( \mathcal{C} \) determined by the functor \( y \mapsto (p, y) \)). In order that this be exact it is necessary and sufficient that the category \( \mathcal{E} (p, y) \)
be cofiltering, i.e. that the functor $P_x$ be a pro-object in $C$. Thus have checked.

**Proposition:** Let $C$ be a topological category such that source: $\text{Ar} C \rightarrow \text{Ob} C$ is etale, and let $C^\wedge$ be the topos of sheaves over $\text{Ob} C$ with right $C$-action. Assume $\text{Ob} C$ sober.

(i) A point in $C$ is the same as a pro-object in the underlying discrete category.

(ii) A morphism of topoi $f : \text{Top}(X) \rightarrow C^\wedge$ is the same as a sheaf $P$ over $X$ with left $C$-action whose stalks give rise to pro-representable functors on $C$. The morphism $f$ is given by

$$f^*(F) = F \times^C P.$$ 

The direct image: Let $f : P \rightarrow \text{Ob} C$ be the map induced by $f$. Then

$$f_*(F') = f'_*(F' \times_X P)$$

$$R^nf_*(F') = R^nf'_*(F' \times_X P).$$

(Last assertion results from the fact that $f_*$ is compatible with localizations hence in the cartesian square

$$\begin{array}{ccc}
\text{Top}(P) & \xrightarrow{f'} & \text{Top}(\text{Ob} C) \\
\downarrow & & \downarrow \\
\text{Top}(X) & \xrightarrow{f} & C^\wedge
\end{array}$$

we have base changes.)
Thus given a $C$-torsor $P$ over $X$ we have an induced map of cohomology

$$H^i(C; F) \to H^i(X, F \times \mathbb{C}P)$$

(it is the morphism belonging to the morphism of topoi

$$f: \text{Top}(X) \to \mathbb{C}^\times.$$ )

**Corollary:** Assume $F \in \mathbb{C}^\times$ is such that as a sheaf on $\text{Ob} \mathcal{C}$ it is acyclic w.r.t. the map $f': P \to \text{Ob} \mathcal{C}$, i.e.

$$R^qf^*(f^*F) = \begin{cases} 0 & q > 0 \\ F & q = 0. \end{cases}$$

Then

$$H^i(C; F) \to H^i(X, F \times \mathbb{C}P).$$

**Proof:** Immediate consequence of the Leray spectral sequence for $f$. The point is that $R^qf_*(f^*F)$ when "lifted" to $\text{Ob} \mathcal{C}$ (i.e., you forget the $\mathbb{C}^\times$-action) is the sheaf $R^qf_*(f^*F)$.

($\eta$-acyclic variation on the preceding).
Remark: If $C$ is an etale groupoid, then pro-representable functors on $C$ are representable, hence the above two notions of $C$-torsors are equivalent.

**Problem:** Take $C = \Gamma$ and construct a $\Gamma$-torsor $P$ over a CW complex $X$ such that the map $f : P \to \text{Ob} \Gamma = \mathbb{R}^3$ is acyclic for constant sheaves. One wants the map $f'$ to admit a fibrewise deformation to a section (i.e. quasi-fibration with contractible fibres), in which case it would be acyclic for all sheaves on $\mathbb{R}^3$.

**Remarks:** If $C$ is an etale groupoid, then Obj of $C$ may be identified with characteristic sheaves for $C$-torsors, i.e. functors $F$ which assign to a $C$-torsor $P \to X$ a sheaf $F(P,X)$ on $X$ in a functorial way. (Cartesian functors from stack of $C$-torsors to stack of sheaves)

**Example:** If $P$ comes from a codimension $k$ foliation on $X$, then

$$\Omega^\cdot_{\mathbb{R}^3} \times \Gamma^\cdot(P)$$

is the de Rham complex of forms locally constant along the leaves of the foliation.
Let $\mathcal{C}$ and $\mathcal{C}'$ be topological categories with stalk source maps and $u: \mathcal{C} \to \mathcal{C}'$ a functor. Then have $u^*: \mathcal{C}'^\wedge \to \mathcal{C}^\wedge$ given by

$$u^*F' = F' \times \text{Ob} \mathcal{C}.'$$

This being compatible with finite proj. limits and arb. ind. lims., it constitutes a morphism of topos $u: \mathcal{C}^\wedge \to \mathcal{C}'^\wedge$.

Suppose $u$ now such that $\text{Ob}(u): \text{Ob} \mathcal{C} \to \text{Ob} \mathcal{C}'$ is etale whence we have $\text{Ob}(u)_! : \text{Top}(\text{Ob} \mathcal{C}) \to \text{Top}(\text{Ob} \mathcal{C}')$. Then we have adjoint functors

$$\mathcal{C}^\wedge \xrightarrow{u_!} \mathcal{C}'^\wedge \xleftarrow{u_*}$$

where

$$(u_!F) = \text{Coker}\left\{ F \times \text{Ar} \mathcal{C} \times \text{Ar} \mathcal{C}' \to F \times \text{Ar} \mathcal{C}' \right\}$$

and

$$(u_!F)_y = \text{Coker}\left\{ \prod_{y \to u(x)} F \to \prod_{y \to u(x)} F \right\} \xrightarrow{\text{lim}} F_x \in \mathcal{H}_x^\imath(C', F).$$

* denotes the category of arrows of $u(x)$. 
Proof of $(\ast)$:

$$
\begin{array}{c}
\text{Top}(\mathcal{Ob} C) \xrightarrow{\mathcal{Ob}(u)} \text{Top}(\mathcal{Ob} C') \\
\downarrow f \downarrow \mathcal{u} \downarrow f' \downarrow \mathcal{u}' \\
\mathcal{C} \xrightarrow{u} \mathcal{C}'
\end{array}
$$

\[ f'_! F = F \times_{\mathcal{Ob} C} \mathcal{Ar} C \quad \text{(immediate)} \]

\[ u'_! f'_! F = f'_! \left( \mathcal{Ob}(u)'_! F \right) \]

(recall $\mathcal{Ob}(u)'_! F$ is the composite etale map $F \to \mathcal{Ob} C \to \mathcal{Ob} C'$)

\[ u'_! f'_! F = F \times_{\mathcal{Ob} C} \mathcal{Ar} C'. \]

Now in general we have exact situation

\[ (f'_! f'^{*})^2 F \Rightarrow f'_! f'^{*} F \Rightarrow F \]

because $f'^{*}$ is faithfully exact, and when applied to this gadget it becomes homotopically trivial. Thus since $u'_!$ is left exact:

\[ u'_! (f'_! f'^{*})^2 F \Rightarrow u'_! f'_! f'^{*} F \Rightarrow u'_! F \]

\[ (F \times \mathcal{Ar} C) \times \mathcal{Ar} C' \Rightarrow F \times \mathcal{Ar} C' \]

\[ \text{g.e.d.} \]
From now on we work with abelian sheaves and write \( u^! \) instead of \( u_{ab} \). Thus
\[
(f_1^*F)_x = \bigoplus_{x \to x'} F_{x'} \quad \text{exact.}
\]

\[
(u^! F)_y = \operatorname{lim}_{y \to u(x)} F_x \quad \text{limit taken as abelian functor}
\]

\[
= H_0(y\downarrow \mathcal{C}, F)
\]

where \( y\downarrow \mathcal{C} \) is the category of arrows \( y \to u(x) \), \( x \) varying in \( \mathcal{C} \).

Existence of derived functor \( Lu^! \): Standard resolution
\[
(\ast) \quad (f_1^* f^*)^n F \Rightarrow (f_1^* f^*) F \rightarrow F
\]

exact functors of \( F \), (compatible with filtered limits). Let \( \underline{Lu}^! (F) \): the complex \( v \mapsto u^! (f_1^* f^*)^{v+1} F \)

\[
\underline{Lu}^! (F) = q\text{-th homology group.}
\]

This is an exact functor, effaceable since of \( f_1^* F \) and the complex (\( \ast \)) splits for \( F = f_1^* M \).
Stalk formulas:

\[ L_{qE}(F) = \lim_{y \to u(x)} F_x \]

\[ = H_q(y \downarrow C, F) \]

Both sides are homological functors, hence need only establish effacement on the right.

\[
\begin{array}{ccc}
\{(y \to u(x), x \to x')\} & \xrightarrow{p_2} & \text{Ob } C = \{ \ast' \text{, no arrows} \} \\
\downarrow p_1 \downarrow f & & \downarrow f \\
\{y \to u(x)\} & \xrightarrow{j} & C
\end{array}
\]

\[
(j^* f_! M)_{y \to u(x)} = (f_! M)_{x' \to x} = \bigoplus_{x' \to x} M_{x'}
\]

\[
(p_1^! p_2^* M)_{y \to u(x)} = \lim_{x \to x'} M_{x'} = \bigoplus_{x \to x'} M_{x'}
\]

(P1 is filtered, hence \( p_1^! \) can be computed as the limit over the fibre.) \( p_1^! \) is exact.

\[ H_q(y \downarrow C, p_1^! p_2^* M) = H_q(\{(y \to u(x), x \to x')\}; p_2^* M) \]

But the category \( \{(y \to u(x), x \to x')\} \) is a disjoint sum over the different maps \( y \to u(x') \) and different \( x' \) of categories with a final object, so the homology is trivial as \( p_2^* M \) is constant for \( x' \) fixed. (Observe - easy to make a mistake here as the \( \lim \) functor
will be exact for a category with an initial object.)

Another proof of stalk formula: From the explicit construction of $\mathcal{L}u_1$ and have $\mathcal{L}u_1(F)_y$ will be the complex

\[
\begin{array}{c}
\xrightarrow{d}
\end{array}
\]

\[y \rightarrow (x_y) \rightarrow F_x \rightarrow 0\]

which is the complex of chains for the category $y \downarrow C$ with coefficients in the functor $(y \rightarrow (x_y)) \mapsto F_x$. Thus knowing this chain complex calculates $H_*(y \downarrow C, F)$ even (already a special case of formula for $\mathcal{L}u_1$ where $u: y \downarrow C \to C$).

(both of the above proofs look hard to write down)

Remaining points:

1) Adjunction

\[\text{Hom}(\mathcal{L}u_1(F), F') = \text{Hom}(F_1^\text{??}, F')\]

\[\text{under suitable conditions on } F, F'.\]

2) Stalk formula when $u$ is pre-fibred:

\[\mathcal{L}u_1(F)_y = \mathcal{L} \lim_{x \rightarrow y} F_x = H_\ast(C_y, F).\]
Consequence of fact that \( \mathcal{C}_y \mapsto g/C \) has the appropriate adjoint.

3) The \((f_1, f^*)\) resolution furnishes a spectral sequence

\[
E_1^{pq} = H^0(A_p C; \text{(last })^*F) \Rightarrow H^{p+q}(C; F)
\]

\[
a_p C = \underbrace{\text{Am } C \times \cdots \times \text{Am } C}_{p \text{ times}}
\]

When \( C \) is a groupoid this is the Čech spectral sequence for the covering \( \text{Am } C \rightarrow \text{Ob } C \).