Deformation of torsors Jan 27, 1971

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1. Extensions of Picard stacks.

One works in some fixed topos T, A is a ring of T, the Picard stacks considered in the sequel will be A-linear.

1.1. Let $u_i : X_i \to Y$ be morphisms of Pic. stacks, i = 1, 2, one defines the fibered product

$$X = X_1 \times_Y X_2$$

as the (Pic st) of triples (x_1, x_2, s) where x_i is an object of X_i and s an isomorphism $s : u_1(x_1) \to u_2(x_2)$ in Y : maps are the obvious ones, addition is defined by addition of coordinates with the usual grain of salt.

One has a canonical square

$$\begin{array}{ccc} X & \xrightarrow{pr_2} & X_2 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle u_2} \\ X_1 & \xrightarrow{u_1} & Y \end{array}$$

commutative up to can isom $(pr_i(x_1, x_2, s) = x_i)$, and for any Pic st L, the induced map

(1.1.1) $\underline{Hom}(L,X) \to \underline{Hom}(L,X_1) \times_{\underline{Hom}(L,Y)} \underline{Hom}(L,X_2)$

is an equivalence (Hom denotes the Pic st of morphisms from to).

1.2. Let $u : M \to N$ be a map of Pic st, one defines

(1.2.1) $\mathrm{Ker}(u) = M \times_N 0$.

Put L = Ker(u). One has an (A-linear) map

$$d : \underline{Aut}(0_N) \longrightarrow \underline{Isob}(L)$$

$$d(s) = cl(0_M, s) \qquad \text{(one may assume } u(0_M) = 0_N) ,$$

where Aut = sheaf of autom, Isob = sheaf of isom. cl. of obj.

The sequence

(1.2.2) $0 \to \underline{Aut}(0_L) \to \underline{Aut}(0_M) \longrightarrow \underline{Aut}(0_N) \xrightarrow{d} \underline{Isob}(L) \to$

$\to \underline{Isob}(M) \longrightarrow \underline{Isob}(N)$.

is exact. (Proof left to the reader).

1.3. A 0-<u>sequence</u> of Pic st consists of maps of Pic st

(1.3.1) $L \xrightarrow{u} M \xrightarrow{v} N$

together with an isom $s : vu \to 0$. By the universal property (1.1.1) one then has a well defined map

(∗) $L \to Ker(v)$.

One says that (1.3.1) (completed by 0 at both ends) is an <u>exact sequence</u> if (∗) is an equivalence and <u>Isob</u>(v) is an epimorphism. One then has an exact sequence

(1.3.2) $0 \to \underline{Aut}(0_L) \to \underline{Aut}(0_M) \to \underline{Aut}(0_N) \to \underline{Isob}(L) \to \underline{Isob}(M) \to \underline{Isob}(N) \to 0$.

<u>Example.</u> Let

$$E = (0 \to X \to Y \to Z \to 0)$$

be an exact sequence of $C^{[-1,0]}(A)$. Then st(E) is an exact sequence of Pic st. Conversely any exact sequence of Pic st is equivalent to st(E) for a suitable E. (The first assertion is straightforward, for the second use the fact that a 0-quasi-isomorphism of $C^{[-1,0]}(A)$ can be represented by a map surjective in each degree). In this dictionary, (1.3.2) corresponds to the usual "snake sequence".

1.4. If

$$E = (0 \to L \to M \to N \to 0)$$

is an exact sequence of Pic st and $p : N' \to N$ a map of Pic st, one defines the <u>pull-back</u>

$$E' = E*p = (0 \to L' \to M' \to N' \to 0)$$

by $M' = M \times_N N'$, $L' = Ker(M' \to N')$. The can map $L' \to L$ is an equivalence.

1.5. An extension of A by a Pic st M is also called a <u>torsor</u> under M. In down-to-earth terms, "this amounts to the data":

- a stack X ,
- a sum $+ : M \times X \to X$ associative up to given isom satisfying

pentagon,

together with the axioms:

- X is locally non void
- the can map $M \times X \longrightarrow X \times X$ is an equivalence.

In fact, given an extension

$$0 \longrightarrow M \longrightarrow E \xrightarrow{p} A \longrightarrow 0 \qquad ,$$

one defines X as the stack of $x \in$ ob E s. t. $p(x) = 1$. Conversely, given a torsor X under M, one defines $E = \coprod_{a \in A} aX$.

More generally, one has an interpretation of an extension of a Z-Module by a Pic st as a family of torsors X_n, $X_p + X_q \longrightarrow X_{p+q}$, in the Grothendieck style (SGA 7 ...).

1.6. By the example above, to an extension

$$E = (0 \longrightarrow Y \longrightarrow E \xrightarrow{p} X \longrightarrow 0)$$

of Pic st is associated an element

(1.6.1) $\quad cl(E) \in Ext^1_A(X,Y) \quad , \quad$ (hyperext)

whose vanishing is necessary and sufficient for the <u>splitting</u> of E i.e. the existence of ~~xxxxxxxxx~~ an $s : X \longrightarrow E$ s.t. $ps \xrightarrow{\sim} id_X$.

When X = A, i.e. E "is" a torsor under Y, giving such an s is simply giving a global object in E ; if e is a global objett then "$t \mapsto t-e$" identifies E_1 (*) to Y.

2. <u>First geometrical applications</u>.

2.1. If $X \longrightarrow Y$ is a morphism of ringed topoi and I an $\underline{O}_X$-Module,

$$\underline{Ext}(X/Y,I)$$

denotes the Pic st of Y-extensions of X by I. Recall that

$$\underline{Ext}(X/Y,I) = \underline{Hom}(st(t_{[-1}L_{X/Y}),I[1]) \qquad .$$

(*) "fibre over 1"

2.2. Let

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \quad \\ Y \overset{j}{\hookrightarrow} Y' \end{array}$$

be a diagram of ringed topoi, where j is an extension of Y by J, and let I be an $\underline{O}_X$-Module. One has a 0-sequence

(2.2.1) $\quad 0 \to \underline{\mathrm{Ext}}(X/Y,I) \to \underline{\mathrm{Ext}}(X/Y',I) \xrightarrow{c} \underline{\mathrm{Hom}}(f^*J,I)$.

Claim : (2.2.1) is exact, i.e. $\underline{\mathrm{Ext}}(X/Y,I) \xrightarrow{\approx} \mathrm{Ker}(c)$.

Proof. Observe that a Y-extension of X by I is the same as a Y'-extension of X by I whose characteristic homomorphism is zero. (reduce to a questions of rings in the same topos).

2.3. Put $I = f^*J$. By a *deformation of* f *over* Y' one means an object X' of $\underline{\mathrm{Ext}}(X/Y',I)$ such that $c(X') = 1 \in \underline{\mathrm{End}}(f^*J)$. The deformations of f over Y' make a stack

$$\underline{\mathrm{Def}}(X/Y,j)_1 \quad ,$$

where $(-)_1$ means "fibre over 1" and $\underline{\mathrm{Def}}(X/Y,j)$ is defined as the fibre-product

(2.3.1)
$$\begin{array}{ccc} \underline{\mathrm{Def}}(X/Y,j) & \longrightarrow & \underline{O}_X \\ \downarrow & & \downarrow {\scriptstyle 1} \\ \underline{\mathrm{Ext}}(X/Y',I) & \xrightarrow{c} & \underline{\mathrm{Hom}}(f^*J,I) \end{array} \quad .$$

So, one has an exact sequence

(2.3.2) $\quad 0 \to \underline{\mathrm{Ext}}(X/Y,I) \to \underline{\mathrm{Def}}(X/Y,j) \to \underline{O}_X$.

The obstruction to the existence of a local deformation of f is, as one knows, the section of $\underline{\mathrm{Ext}}^2(L_{X/Y},I)$ image of $1 \in \underline{\mathrm{Hom}}(f^*J,I)$ by the canonical hom; *Suppose this obstruction vanishes*. *Then* one can add a zero to the right of (2.3.2), in other words $\underline{\mathrm{Def}}(X/Y,j)_1$ *is a torsor under* $\underline{\mathrm{Ext}}(X/Y,I)$. Hence, if moreover this torsor is trivial,

… object (i.e. a global deformation exists), one

3. Equivariant deformations.

3.0. Let S be a scheme (more generally a ringed topos), and G a group scheme over S. For an equivariant morphism $f : X \to Y$ of G-schemes/S, one would like to define $L_{X/Y}$ as a complex of G-$\underline{O}_X$-Modules. This will be done after some preliminaries.

3.1. Let us work with some fixed topology on Sch/S, (unless otherwise stated), fppf say. If X is a diagram in Sch/S, one will still denote by X the corresponding "contravariant" topos.(1) As usual, BG denotes the classifying topos of G, i.e. the topos of local systems on the nerve G(1) of G. The inclusion of the cat of local systems into the cat of all sheaves on G(1) defines a morphism of topoi

(3.1.1) $\phi : G(1) \longrightarrow BG$.

This is in fact a morphism of ringed topoi, and ϕ^* is exact. Moreover, using the general nonsense of cohomological descent, one can prove that the adjunction map

(3.1.2) $\mathrm{Id} \to R\phi_*\phi^*$

is an isomorphism and the image of $\phi^* : D^+(BG) \to D^+(G(1))$ (which is fully faithful by the preceding) consists of complexes whose cohomology sheaves are local systems.

Generalization. Let X be a G-scheme. One defines

$G(1)_{/X}$ = nerve of G acting on X .

One has a natural map

(3.1.3) $\phi : G(1)_{/X} \longrightarrow BG_{/X}$,

with analogous properties to the ϕ (3.1.1).

3.2. Let now $f : X \to Y$ be an equivariant morphism as in (3.0). So one has a morphism $G(1)_{/f}$ of the corresponding nerves, let's denote it simply by $f' : X' \to Y'$. ~~Assume~~ Let's denote by $L_{X'/Y'}$ the

(1) whose objects are families of sheaves E_i over X_i, $f^*E_j \to E_i$ for $f : X_i \longrightarrow X_j$)

cotangent complex of the morphism of <u>small</u> ringed <u>zariski</u> topoi defined by f'. Assume

(i) either G or f flat,

(ii) $L_{X/Y}$ of finite tor-amplitude.

Then, by the base-change theorem, $L_{X'/Y'}$ has finite tor-amplitude and its cohomology sheaves are quasi-coherent G-$\underline{O}_X$-Modules. Therefore, by (3.1) and the remark below, $L_{X'/Y'}$ can be identified via ϕ to a complex denoted

(3.2.1) $\quad \underline{\underline{L}}_{X/Y} \in \mathrm{ob}\, D^b(BG_{/X})$.

<u>Remk</u>. 3.2.2. One has natural morphisms of ringed topoi

$$\begin{array}{ccccc} -_{Zar} & \longleftarrow & -_{Et} & \longleftarrow & -_{Flat} \\ \uparrow & & \uparrow & & \uparrow \\ -_{zar} & \longleftarrow & -_{et} & \longleftarrow & -_{flat} \end{array},$$

where a capital means "large". Applying $D^b(-)_{qcoh}$ (where qcoh means "quasi-coherent cohomology"), one finds a diagram of equivalences (it's trivial on the columns and well known by descent on the rows). The reason why one worked with the small zariski topoi is that a localization map has zero cotangent complex. (Note : "small étale" would work well too).

3.2.3. If (ii) is not satisfied, define $\underline{\underline{L}}_{X/Y}$ in pro-$D^b(BG_{/X})$.

3.2.4. The image of $\underline{\underline{L}}_{X/Y}$ by the forgetful functor $D(BG_{/X}) \longrightarrow D(X)$ is $L_{X/Y}$.

3.3. Yoga of <u>diagram deforming</u> : let $Z \hookrightarrow \overline{Z}$ defined by a nilpotent Ideal (I), if a map $\overline{f} : \overline{X} \longrightarrow \overline{Y}$ of flat $\overline{Z}$-schemes reduces to an isom mod I, it was already one.

3.4. Let

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & & \\ Y & \overset{j}{\hookrightarrow} & \overline{Y} \end{array}$$

be a diagram of G-S-schemes, where j is defined by J s.t. $J^2 = 0$, and f is flat. By the yoga of diagram deforming, the obstruction to deforming f over $\overline{Y}$ (as a map of flat G-schemes) is a class

$$\omega(X,j) \in \mathrm{Ext}^2(BG_{/X}; \underline{\underline{L}}_{X/Y}, J) \quad ,$$

expressable as a cup-product ..., and when $\omega(X,j) = 0$, cl. of def make a torsor under $\mathrm{Ext}^1(\ )$ and the autom of a def = $\mathrm{Ext}^0(\ \ )$.

3.5. Pic st interpretation. To avoid all base change troubles, assume G flat. Then, if $X \to Y$ is a morphism of G-S-schemes, G acts[(1)] on the Pic st $t_{[-1}\underline{\underline{L}}_{X/Y}$, hence on $\underline{\mathrm{Ext}}(X/Y,I) = \underline{\mathrm{Hom}}(t_{[-1}\underline{\underline{L}}_{X/Y}, I[1])$ for any G-$\underline{O}_X$-Module I. As a result, in the situation of (3.4), the exact sequence (2.3.2) is in fact an exact sequence of G-$\underline{O}_X$-Pic st. Hence, if X can locally (upstairs) be deformed as a scheme/Y, then $\underline{\mathrm{Def}}(X/Y,j)$ is a torsor under $\underline{\mathrm{Ext}}(X/Y,J)$ on $BG_{/X}$. A global object of this torsor, i.e. a global equivariant deformation of f, identifies $\underline{\mathrm{Def}}(X/Y,j)$ to $\underline{\mathrm{Ext}}(X/Y,J)$ (as G-stacks), hence the isom. cl. of equiv. def. (resp. the autom. group of a given equiv. def.) to $\mathrm{Ext}^1(BG_{/X}; \underline{\underline{L}}_{X/Y}, J)$ (resp. $\mathrm{Ext}^0(\ \ )$). (Use that

$$H^i(BG_{/X}, \underline{\mathrm{Ext}}(X/Y,J)) = \mathrm{Ext}^{i+1}(BG_{/X}; \underline{\underline{L}}_{X/Y}, J)$$

for $i = 0, -1$.).

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(1) Let L be a Pic st over X, an action of G on L consists of an action of G(T) on L_T for each T/S ("functorial" in T) ; equivalently a G-action on L is a descent data on L with respect to the nerve of G/X i.e. an equivalence $d_0^*L \to d_1^*L$ (on $G \times X$), plus a 2-map on $G \times G \times X$, with cocycle condition on $G \times G \times G \times X$. One has a dictionary :

$$(\text{G-Pic st}/X) \longleftrightarrow C^{[-1,0]}(BG_{/X}) \quad .$$

3.6. Case of a G-torsor. From now on, G will be assumed to be flat. Suppose in (3.4) Y, $\overline{Y}$ are trivial G-schemes, and X is a torsor on Y under G (for the flat topology). One seeks the obstruction to deforming X on $\overline{Y}$ as a torsor under G. I claim that deforming X as a torsor is the same as deforming X as a G-scheme over $\overline{Y}$. In fact, let $\overline{X}/\overline{Y}$ be an equivariant def. of X/Y, then first of all $\overline{X}$ is a pseudo-torsor, i.e. $G_{\overline{X}} \times \overline{X} \xrightarrow{\sim} \overline{X} \times \overline{X}$, (because of (3.3)), and it has a local fppf section because $\overline{X} \longrightarrow \overline{Y}$ is flat and surjective. Now, since X is a torsor, one has a natural equivalence

(3.6.1) $\quad BG_{/X} \longrightarrow Y$,

so $\underline{\underline{L}}_{X/Y}$ is induced via (3.6.1) by a well defined

(3.6.2) $\quad \mathcal{X}_{X/Y} \in \text{ob } D(Y)$,

called the invariant cotangent complex. Here, to avoid an irrelevant grain of salt, it's nice to assume G is loc. of f. p., because then $\underline{\underline{L}}_{X/Y}$ (hence $\mathcal{X}_{X/Y}$) is of perfect amplitude $\subset [-1,0]$. One has (descent)

(3.6.3) $\quad \text{Ext}^i(BG_{/X};\underline{\underline{L}}_{X/Y},J) = \text{Ext}^i(Y;\mathcal{X}_{X/Y},J)$,

hence the obstruction

(3.6.4) $\quad \omega(X,j) \in \text{Ext}^2(\mathcal{X}_{X/Y},J)$,

etc.

Pic st interpretation. Denote by $\underline{\text{Def}}_G(X/Y,j)_1$ the Pic st on $\overline{Y}$ of local downstairs, global upstairs, equivariant deformations of X on $\overline{Y}$. One has (for arbitrary G-maps f, j)

(3.6.5) $\quad \underline{\text{Def}}_G(X/Y,j)_1 = j_* f^G_* \underline{\text{Def}}(X/Y,j)_1$,

where f^G_* is the invariant direct image functor, i.e. the composition

$$\text{G-Picst}(X) \xrightarrow{f_*} \text{G-Picst}(Y) \xrightarrow{\underline{\Gamma}^G} \text{Picst}(Y) \quad ;$$

for a Picst L on Y, $\underline{\Gamma}^G(L)$ is the Picst of pairs (x,s), $x \in \text{ob } L$, $s : d_0^* x \xrightarrow{\sim} a\, d_1^* x$ $\quad (a : d_0^* L \rightarrow d_1^* L$ the structural equivalence), s. t. the suitable cocycle condition holds on G x G x Y ; in the dictionary, $\underline{\Gamma}^G(L)^\flat \simeq t_{\mathfrak{g}} R\underline{\Gamma}^G(L^\flat)$.

Returning now to the situation of (3.6), one deduces from (3.5) :

Prop. 3.6.6. Suppose there exists a global equivariant deformation $\overline{X}$ of X (which therefore will be automatically a torsor under G as seen above), then $\overline{X}$ defines a canonical equivalence

$$\underline{\mathrm{Def}}_G(X/Y,j)_1 \xrightarrow{\sim} j_* \underline{\mathrm{RHom}}(\chi_{X/Y}, J)[1] \quad .$$

In fact, the right hand side is just $(j_*) Rf_*^G \underline{\mathrm{RHom}}(\underline{\underline{L}}_{X/Y}, f^*J)[1]$ by the projection formula.

Cor. 3.6.7. (Mazur-Roberts). Assume G commutative. Then, in the derived cat. of $\mathbb{Z}$-Mod. (for the flat top) on $\overline{Y}$, there exists a canonical isomorphism

$$(0 \to G_{\overline{Y}} \to j_* G_Y \to 0) \xrightarrow{\sim} j_* \underline{\mathrm{RHom}}(\chi_G, J) \quad ,$$

where $\chi_G = \chi_{G_Y/Y}$, and $G_{\overline{Y}}$ is placed in degree 0.

Proof. In (3.6.6), take $\overline{X} = G_{\overline{Y}}$, $X = G_Y$. The stack (on $\overline{Y}$) of equivariant deformations of X is nothing else but the stack of torsors under $G_{\overline{Y}}$ trivialized along Y, so, by the dictionary (see lemma below), it corresponds to the complex $(G_{\overline{Y}} \to j_* G_Y)$ where $G_{\overline{Y}}$ is placed in degree -1. Therefore the equivalence of (3.6.6) yields the desired isomorphism.

Lemma 3.6.8. Let $L = (0 \to L_1 \xrightarrow{d} L_0 \to 0)$ be a complex of abelian sheaves in some topos. One has

$$\mathrm{st}(L) = \mathrm{Ker}(\mathrm{st}(L_1[1]) \to \mathrm{st}(L_0[1]))$$

$= \mathrm{st}$ of (x,s), x a torsor under L_1, $s : d_* x \xrightarrow{\sim} 0$ (a trivialization of the torsor under L_0 image of x by d).

Proof. Left to the reader.

3.7. The Atiyah extension.

The obstruction $\omega(X,j)$ of (3.4) is the cup-product of the class of j, $e(j) \in \mathrm{Ext}^1(L_{Y/S}, J)$, by the Kodaira-Spencer class

$$c(X/Y/S) \in \mathrm{Ext}^1(\underline{\underline{L}}_{X/Y}, f^*L_{Y/S}) \quad .$$

Assume now, as in (3.6), that Y, $\overline{Y}$ are trivial G-schemes and X is a torsor under G. Then, by descent, one has

$$\mathrm{Ext}^i(\underline{\underline{L}}_{X/Y}, f^* L_{Y/S}) = \mathrm{Ext}^i(\chi_{X/Y}, L_{Y/S})$$

(at least if $L_{Y/S}$ is in D^b), so c(X/Y/S) is a class

(3.7.1) $\quad c(X/Y/S) \in \mathrm{Ext}^1(\chi_{X/Y}, L_{Y/S})$.

When G and Y are smooth, this class is easily seen to coincide with the class of the Atiyah extension

(*) $\quad 0 \to \Omega^1_{Y/S} \to \mathrm{At}(X) \to \omega_{X/Y} \to 0$

defined by descent to Y of the exact sequence of differentials on X; when G is smooth but not necessarily Y, (*) is still defined, and is just the image of c(X/Y/S) by $L_{Y/S} \to \Omega^1_{Y/S}$. Moreover, $\omega_{X/Y}$ is known to be isomorphic to the sheaf $X \times^G \mathrm{Lie}(G)_Y$ obtained from the invariant differential forms on G by taking the inverse image on Y and twisting by X via the adjoint operation. This can be generalized as follows.

By the classifying property of BG, the G-torsor X over Y defines a map $Y \xrightarrow{u} BG$ s.t. X is the inverse image by u of the universal torsor PG over BG (recall u^* consists in inducing on Y and twisting by X), in other words one has a "commutative" diagram with a cartesian square

(3.7.2)

$$\begin{array}{ccc} PG & \longleftarrow & X \\ \downarrow & & \downarrow \\ BG & \xleftarrow{u} & Y \\ & \searrow \;\; S \;\; \swarrow & \end{array}$$

Now one disposes of

(3.7.3) $\quad \underline{\underline{\chi}}_G = \chi_{PG/BG} \in \mathrm{ob}\, D(BG)$,

defined by means of nerves like in (3.2), (3.6), and it is clear that one has

(3.7.4) $\quad \chi_{X/Y} \simeq u^* \underline{\underline{\chi}}_G$.

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Status of stability problem.

Given ring Λ we associate a simplicial complex $X(\Lambda, n)$ for unimodular vectors in Λ^n.

Problem 1: Show that $GL_n\Lambda$ acts transitively on the $(j-1)$-simplices for $1 \leq j \leq n-d$, $d = \dim \mathrm{Max}(\Lambda)$, and that the homology of $X(\Lambda, n)$ begins in dimension $n-1-d$.

This is OKAY when ~~[illegible]~~ generic linear projection arguments can be used, e.g. $\Lambda = k[x_1, \ldots, x_m]$ where k is an infinite field. Need somehow to do non-linear projections à la Nagata.

Problem 2: Deduce stability from problem 1.

OKAY for coefficients in $\mathbb{Q}$ if Λ over $\mathbb{Z}[\ell^{-1}]$ some ℓ. For $\mathbb{Z}/\ell\mathbb{Z}$ coeffs. and $\ell^{-1} \in \Lambda$ we need a formula for differential of the spectral sequence. There is some indication that I can push standard range

$$H_i(GL_{n-1}) \longrightarrow H_i(GL_n) \quad \begin{matrix} \text{iso.} & i < n-d-1 \\ \text{surj} & i = n-d-1 \end{matrix}$$

through for ℓ odd but not for $\ell = 2$. (Orthogonal groups $O_n(\mathbb{F}_q)$, $4 | q-1$, not surjective for $i = n-1$.) symmetric groups Σ_n, $GL_2(\mathbb{F}_2)$ not surj. for $i = n-1$.

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Theorem: Let Λ be a perfect ring of characteristic p. Then $K_i\Lambda$ is uniquely p-divisible for $i>0$.

Proof: The Frobenius auto. $F: \Lambda \longrightarrow \Lambda$, $F\lambda = \lambda^p$ induces an auto. of $K_i\Lambda$ which coincides with Ψ^p. (Known for representations, hence in general.) One knows for any element $x \in K_i\Lambda$ $i>0$ that for some n

$$(\Psi^p - p)\cdots(\Psi^p - p^n)x = 0$$

Let F_n be the set of x in $K_i\Lambda$ satisfying this equation. Then F_n is stable under the Adams operation, $F_{n-1} \subset F_n$, and

$$\Psi^p = p^n \quad \text{on} \quad F_n/F_{n-1}.$$

Since Ψ^p is an automorphism, we see that F_n/F_{n-1} is uniquely p-divisible for each n, hence $K_i\Lambda = \bigcup F_n$ is also uniquely p-divisible.

Complement: $\tilde{K}_0\Lambda$ also uniquely p-divisible by the same argument (namely Ψ^k has eigenvalues k^i on i-th quotient of the γ-filtration.)

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stability problem:

Recall that our chain complex is

$$C_{i-1}(X) = \mathbb{Z}/\ell[GL_n] \times_{\mathbb{Z}/\ell\left[\begin{smallmatrix}\Sigma_i & 0 \\ * & GL_{n-i}\end{smallmatrix}\right]} (\mathrm{sgn} \otimes 1)$$

and that under the assumptions made

$$H_*(GL_n, C_{i-1}(X)/\ell) = H_*(\Sigma_i, \mathrm{sgn}/\ell) \otimes H_*(GL_{n-i}, \mathbb{Z}/\ell)$$

It therefore becomes important to know something about $H_*(\Sigma_n, \mathrm{sgn}/\ell)$. $\mathrm{sgn}/\ell = \mathbb{Z}/\ell\mathbb{Z}$ with σ acting as $(-1)^{\sigma}$.

Prop 1: If ℓ odd, then $H_*(\Sigma_n, \mathrm{sgn}/\ell) = 0$ for $n \not\equiv 0,1\ (\ell)$.

Proof: The index of $\Sigma_{n-2} \times \Sigma_2$ in Σ_n is $\frac{n(n-1)}{2}$, prime to ℓ. Hence $\exists$ surjective map

$$H_*(\Sigma_{n-2} \times \Sigma_2, \mathrm{sgn}/\ell) \twoheadrightarrow H_*(\Sigma_{n-2}, \mathrm{sgn}/\ell)$$

$$\parallel$$

$$H_*(\Sigma_{n-2}, \mathrm{sgn}/\ell) \otimes \underbrace{H_*(\Sigma_2, \mathrm{sgn}/\ell)}_{=0}$$

and $H_*(\Sigma_2, sgn/\ell) = 0$. q.e.d.

Prop 2: transfer: $H_*(\Sigma_{m\ell+1}, sgn/\ell) \xrightarrow{\sim} H_*(\Sigma_{m\ell}, sgn/\ell)$ (canonical map other-way is also isomorphism).

Proof. Injectivity of transfer + surjectivity of restriction clear as index $m\ell+1$ is prime to ℓ. Now quite generally when $H \subset G$ contains the Sylow subgp, one has Brauer type thm:

$$H^*(G, M) \longrightarrow H^*(H, M) \rightrightarrows \prod H^*(H \cap xHx^{-1}, M)$$

but in this case all the intersections $H \cap xHx^{-1}$ are of form $\Sigma_{m\ell-1}$ and by the proposition these have trivial cohomology except when x normalizes H, which doesn't happen here as Σ_{n-1} is its own normalizer in Σ_n.

Remark: Prop. 1+2 also holds for $\ell = 2$, prop. 1 holds trivially (every $n \equiv 0, 1 \bmod 2$), and proof of prop. 2 same.

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Computation of the differential d^1. Thus we have the map

$$d: C_i(X) \longrightarrow C_{i-1}(X)$$

$$d\,[e_0, \cdots, e_i] = \sum_{\nu=0}^{i} (-1)^{\nu} [e_0, \cdots, \hat{e}_\nu, \cdots, e_i]$$

and in mod ℓ cohomology it induces a map

$$H_*(\Sigma_{i+1}, sgn) \otimes H_*(GL_{n-i-1}) \longrightarrow H_*(\Sigma_i, sgn) \otimes H_*(GL_i).$$

I claim that this map is the tensor product of the instriction

$$H_*(GL_{n-i-1}) \longrightarrow H_*(GL_i)$$

and the transfer

$$H_*(\Sigma_{i+1}, sgn) \longrightarrow H_*(\Sigma_i, sgn).$$

In virtue of Kunneth it means that we have a comm. diag

$$H_*(GL_n, C_i(X)) \longrightarrow H_*(GL_n, C_{i-1}(X))$$

$$H_*(GL_n, C_i(X)) = H_*\left(\begin{array}{c|c} \Sigma_{i+1} & 0 \\ \hline * & GL_{n-i-1} \end{array}, sgn \otimes 1\right)$$

$$H_*(GL_n, C_{i-1}(X)) = H_*\left(\begin{array}{c|c} \Sigma_i & 0 \\ \hline * & GL_{n-i} \end{array}, sgn \otimes 1\right)$$

$$H_*\left(\begin{array}{c|c} \Sigma_{i+1} & 0 \\ \hline * & GL_{n-i-1} \end{array}, sgn \otimes 1\right) \xrightarrow{tr} H_*\left(\begin{array}{c|c} \begin{array}{c|c} \Sigma_i & 0 \\ \hline 0 & 1 \end{array} & 0 \\ \hline * & GL_{n-i-1} \end{array}, sgn \otimes 1\right) \xrightarrow{in} H_*\left(\begin{array}{c|c} \Sigma_i & 0 \\ \hline * & GL_{n-i} \end{array}, sgn \otimes 1\right)$$

To prove this is the case we generalize slightly and consider the morphism of homological functors

$$H_*(GL_n, C_i(X) \otimes M) \longrightarrow H_*(GL_n, C_{i-1}(X) \otimes M).$$

of the GL_n-module M. Actually we shall work with the dual coh. functors

$$\mathrm{Ext}^*_{GL_n}(C_{i-1}(X), M) \longrightarrow \mathrm{Ext}^*_{GL_n}(C_i(X), M).$$

Such a transf. is determined by what it does in dimension 0.

$\mathrm{Hom}_{GL_n}(C_{i-1}(X), M) =$ set of functions $f(v_1, \ldots, v_n)$ defined on independent vectors with values in M which are GL_n equivariant and alternating.

$$\cong \left\{ m \in M \;\middle|\; \begin{pmatrix} \sigma & 0 \\ * & * \end{pmatrix} m = (-1)^\sigma m \quad \sigma \in \Sigma_i \right\}.$$

The rule giving this isom assigns to f the element $m = f(e_1, \ldots, e_i)$ where $e_1, \ldots, e_n$ is standard base for Λ^n.

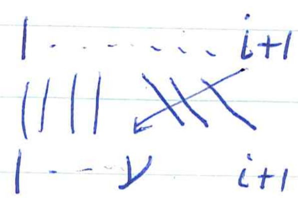
Now the differential $\delta : \mathrm{Hom}_{GL_n}(C_{i-1}(X), M) \to \mathrm{Hom}_{GL_n}(C_i(X), M)$ is:

$$(\delta f)(\sigma_1,\dots,\sigma_{i+1}) = \sum_{\nu=1}^{i+1}(-1)^{\nu-1} f(\sigma_1,\dots,\hat{\sigma}_\nu,\dots,\sigma_{i+1})$$

hence

$$\begin{aligned}(\delta m) &= (\delta f)(e_1,\dots,e_{i+1})\\ &= \sum_{\nu=1}^{i+1}(-1)^{\nu-1} f(e_1,\dots,\hat{e}_\nu,\dots,e_{i+1})\end{aligned}$$

Let $\sigma_\nu \in \Sigma_{i+1}$ be the permutation



$$\operatorname{sign}(\sigma_\nu) = (-1)^{i-\nu+1}$$

so that $[e_1,\dots,\hat{e}_\nu,\dots,e_{i+1}] = \sigma_\nu[e_1,\dots,e_i]$. Then

$$f(e_1,\dots,\hat{e}_\nu,\dots,e_{i+1}) = \sigma_\nu m.$$

so

$$\delta m = \sum_{\nu=1}^{i+1}(-1)^{\nu-1}\sigma_\nu m$$

Now reinterpret this as follows.

$$\begin{array}{ccc}\operatorname{Ext}^0_{GL_n}(C_{i-1}(X), M) & = & H^0\left(\begin{array}{c|c}\Sigma_i & 0\\ \hline * & GL_{n-i}\end{array}, \operatorname{sgn}\otimes M\right)\\ \downarrow \delta & & \downarrow \alpha\\ \operatorname{Ext}^0_{GL_n}(C_i(X), M) & = & H^0\left(\begin{array}{c|c}\Sigma_{i+1} & 0\\ \hline * & GL_{n-i-1}\end{array}, \operatorname{sgn}\otimes M\right)\end{array}$$

~~and [redacted] the map on the left corresponding to δ sends~~

~~$$1\otimes m \longmapsto (-1)^{i-1}\left(\sum_{\nu=1}^{i+1}\sigma_\nu\right)1\otimes m.$$~~

~~[redacted] since the σ_i are coset representatives for the subgroup~~

~~$$\left(\begin{array}{cc|c}\Sigma_i & 0 & \\ 0 & 1 & \\ \hline * & & GL_{n-i-1}\end{array}\right) \hookrightarrow \left(\begin{array}{c|c}\Sigma_{i+1} & 0\\ \hline * & GL_{n-i-1}\end{array}\right)$$~~

where α is the composition of

$$H^0\left(\begin{array}{c|c}\Sigma_i & 0\\ \hline * & GL_{n-i}\end{array}, p_1^*\mathrm{sgn}_{\Sigma_i}\otimes M\right) \xrightarrow{\text{res}} H^0\left(\begin{array}{cc|c}\Sigma_i & 0 & 0\\ 0 & 1 & 0\\ \hline * & * & GL_{n-i-1}\end{array}, \mathrm{sgn}_{\Sigma_{i+1}}\otimes M\right)$$

$$m \qquad\qquad\qquad\qquad m$$

$$\ni \begin{pmatrix}\sigma & 0\\ * & *\end{pmatrix} m = (-1)^\sigma m \qquad\qquad \left(\begin{array}{cc|c}\sigma & 0 & 0\\ 0 & 1 & 0\\ \hline * & * & *\end{array}\right) m = (-1)^\sigma m$$

followed by

$$H^0\left(\begin{array}{cc|c}\Sigma_i & & 0\\ 0 & 1 & 0\\ \hline * & * & GL_{n-i-1}\end{array}, \mathrm{sgn}_{\Sigma_{i+1}}\otimes M\right) \xrightarrow{tr.} H^0\left(\begin{array}{c|c}\Sigma_{i+1} & 0\\ \hline * & GL_{n-i-1}\end{array}, \mathrm{sgn}_{\Sigma_{i+1}}\otimes M\right)$$

$$\left(\begin{array}{c|c}\sigma\oplus 1 & 0\\ \hline * & *\end{array}\right) m = (-1)^\sigma m \longmapsto \sum_{\nu=1}^{i+1}\sigma_\nu(m).$$

Then since σ_ν has sign $(-1)^{i+1-\nu}$ we have

$$1 \otimes (-1)^{\nu} \sigma_\nu m = (-1)^{i+1} \sigma_\nu (1 \otimes m)$$

so that α is $(-1)^{i+1}$ times δ. As α and δ are coh. functors of M we see that up to sign

$$\operatorname{Ext}^*_{GL_n}(C_{i-1}(X), M) \longrightarrow \operatorname{Ext}^*_{GL_n}(C_i(X), M)$$

is the composite of

$$H^*\left(\begin{array}{c|c} \Sigma_i & \\ \hline * & GL_{n-i} \end{array}, \operatorname{sgn}_{\Sigma_i} \otimes M\right) \xrightarrow{\text{res}} H^*\left(\begin{array}{c|c|c} \Sigma_i & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ \hline * & * & GL_{n-i-1} \end{array}, \operatorname{sgn}_{\Sigma_{i+1}} \otimes M\right)$$

$$\xrightarrow{\text{transfer}} H^*\left(\begin{array}{c|c} \Sigma_{i+1} & 0 \\ \hline * & GL_{n-i-1} \end{array}, \operatorname{sgn}_{\Sigma_i} \otimes M\right).$$

On the stability range:

$l=2$. $H_{n-1}(\Sigma_{n-1}) \longrightarrow H_{n-1}(\Sigma_n)$ not surj for $n=2^a$ because

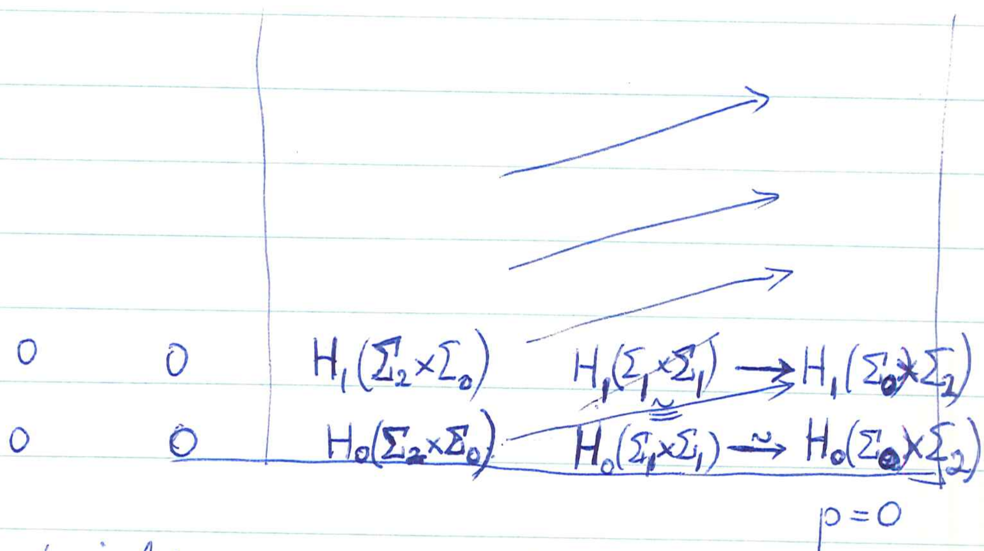
$$w_{2^a-1}(\text{st rep on } \mathbb{R}^{2^a}) \in H^{2^a-1}(\Sigma_{2^a})$$

is non-zero (restricts to e_A $A=(\mathbb{Z}/2\mathbb{Z})^a$) yet it dies on Σ_{2^a-1}.

second instance of this is for $H_3(\Sigma_3) \longrightarrow H_3(\Sigma_4)$. I recall the spectral sequence that I constructed

$$E^1_{pq} = H_q(\Sigma_p \times \Sigma_{n-p}) \Longrightarrow H_*(pt)$$

Consider $H_1(\Sigma_1) \longrightarrow H_1(\Sigma_2)$ situation: $n=2$.



periodic

Next case $H_3(\Sigma_3) \longrightarrow H_3(\Sigma_4)$

$$H_3(\Sigma_1 \times \Sigma_3) \hookrightarrow H_3(\Sigma_0 \times \Sigma_4)$$

$$H_2(\Sigma_3 \times \Sigma_1) \hookrightarrow H_2(\Sigma_2 \times \Sigma_2) \qquad H_2(\Sigma_1 \times \Sigma_3) \xrightarrow{\sim} H_2(\Sigma_0 \times \Sigma_4)$$

$$H_1(\Sigma_4) \quad H_1(\Sigma_3 \times \Sigma_1) \hookrightarrow H_1(\Sigma_2 \times \Sigma_2) \qquad H_1(\Sigma_1 \times \Sigma_3) \xrightarrow{\sim} H_1(\Sigma_0 \times \Sigma_4)$$

$$H_0(\Sigma_4 \times \Sigma_0) \to H_0(\Sigma_3 \times \Sigma_1) \xrightarrow{\sim} H_0 \qquad \cong$$

Conclusion: The differentials in this spectral sequence are terrible!

February 17, 1971.

X curve over a finite field k, $\bar{k}$ = alg. closure of k, $\bar{X} = \bar{k} \times_k X$, q = card(k). Then (at least for G finite of order prime to $p = \text{char}(k)$) we have

$$R_X(G) \xrightarrow{\sim} R_{\bar{X}}(G)^{Gal(\bar{k}/k)}$$

$$= \left(R_{\bar{k}}(G) \otimes K(\bar{X})\right)^{Gal(\bar{k}/k)}$$

since

$$K(\bar{X}) \cong \underset{\text{gen rank}}{\mathbb{Z}} \oplus \underset{x}{\mathbb{Z}} \oplus A(\bar{k})$$

where A Jacobian of $\bar{X}$, and this isomorphism with Galois as $\exists$ a divisor of degree 1 on X, one has an isomorphism

$$R_X(G) \cong \underset{\text{gen. pt.}}{R_k(G)} \oplus \underset{x}{R_k(G)} \oplus \left(R_{\bar{k}}(G) \otimes A(\bar{k})\right)^{Gal(\bar{k}/k)}.$$

Now I want to compute the char. classes of $\tilde{R}_X$ with values in mod l coh., $l \neq p$. Recall

$$\tilde{R}_X(G) \oplus K(X) = R_X(G)$$

Now an exponential class

$$\theta : \tilde{R}_X(?) \longrightarrow H^0(?, S_\circ) \qquad S_\circ = \mathbb{Z}/\ell\mathbb{Z}$$

factors through the ℓ-adic completion of $\tilde{R}_X$:

$$\tilde{R}_X(G) \cong \tilde{R}_k(G) \oplus \tilde{R}_k(G) \oplus \left(\tilde{R}_{\bar{k}}(G) \otimes A(\bar{k})\right)^{Gal(\bar{k}/k)}.$$

$$0 \longrightarrow {}_{\ell^\nu}A(\bar{k}) \longrightarrow A(\bar{k}) \xrightarrow{\ \ell^\nu\ } A(\bar{k}) \longrightarrow 0$$

$$0 \longrightarrow \tilde{R}_{\bar{k}}(G) \otimes {}_{\ell^\nu}A(\bar{k}) \longrightarrow \tilde{R}_{\bar{k}}(G) \otimes A(\bar{k}) \xrightarrow{\ \ell^\nu\ } \tilde{R}_{\bar{k}}(G) \otimes A(\bar{k}) \to 0$$

Now take cohomology wrt $Gal(\bar{k}/k)$. Claim that

$$H^1(Gal(\bar{k}/k), R_{\bar{k}}(G) \otimes A(\bar{k}))$$

is a p-torsion group. Indeed $Gal(\bar{k}/k) = \hat{\mathbb{Z}}$ with Frob as generator, so the H^1 is the coinvariants, hence is (Pontryagin) dual to the invariants in

$$Hom(R_{\bar{k}}(G), A(\bar{k})^\vee)$$

Now the ℓ-primary part of $A(\bar{k})^\vee$ is $\cong (\mathbb{Z}_\ell)^{2g}$ and Frobenius acts with no eigenvalues a root of 1 since their eigenvalues by Weil are alg. integers of absolute value $\sqrt{q}$. But Frob on $R_{\bar{k}}(G)$ ~~and~~ permutes the basis of irreducible reps. around so the eigenvalues are roots of 1. Conclude there are no invariants.

So the long exact sequence in coh. gives

$$\left(\tilde{R}_{\bar{k}}(G) \otimes \bar{A}(k)\right)^{Gal} \otimes \mathbb{Z}/\ell^\nu\mathbb{Z} \xrightarrow{\sim} \left(\tilde{R}_{\bar{k}}(G) \otimes {}_{\ell^\nu} A(\bar{k})\right)_{Gal}$$

Taking the inverse limit over ν, and using the fact that the above groups are finite, we get

$$\left(\tilde{R}_{\bar{k}}(G) \otimes \bar{A}(k)\right)^{Gal} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \left(\tilde{R}_{\bar{k}}(G) \otimes T_\ell(A)\right)_{Gal}$$

(a finite gp.)

where chez Tate

$$T_\ell(A) = \varprojlim_{\nu} {}_{\ell^\nu}A(\bar{k}).$$

Now suppose that we have an exponential class

$$\theta : \left(\tilde{R}_{\bar{k}}(G) \otimes T_\ell(A)\right)_{Gal} \longrightarrow H^0(G, S_\cdot)^\times$$

In particular for each $t \in T_\ell(A)$ we have an exponential class

$$\begin{aligned} \tilde{R}_{\bar{k}}(G) &\longrightarrow H^0(G, S_\cdot) \\ u &\longmapsto \theta(u \otimes t). \end{aligned}$$

Such a class by our previous work is the same thing as a ~~[illegible]~~ power series

$$\sum_{i \geq 0}' s_i x^i \qquad s_i \in S_{2i} \qquad s_0 = 1.$$

Recall how this series is obtained: Let $\bar{k}^\times$ act on $\bar{k}$; it gives a canonical element u in $R_{\bar{k}}(\bar{k}^\times)$; applying $\theta(u \otimes t)$ gives an element of

$$H^\circ(\bar{k}^\times, S_\cdot) = \prod_{i \geq 0} H^i(\bar{k}^\times, S_i)$$

$$= \prod_{i \geq 0} x^i S_{2i}$$

where $x \in H^2(\bar{k}^\times)$ is the element represented by the extension

$$0 \longrightarrow \mu_\ell \longrightarrow \bar{k}^\times \xrightarrow{\ \ell\ } \bar{k}^\times \longrightarrow 0$$

plus an iso. $\longrightarrow$ $\mu_\ell = \mathbb{Z}/\ell\mathbb{Z}$. $\qquad x = c_1(u)$

Therefore for each $t \in T_\ell(A)$ we get a power series

$$\theta(u \otimes t) = \sum s_i(t) x^i;$$

denote by $\varphi(t)$ this series. Then

$$T_\ell(A) \longrightarrow 1 + \prod_{i \geq 0} x^i S_{2i}$$

$$t \longmapsto \varphi(t)$$

is a homomorphism.

Next we must figure out action of Galois.

First put earlier results in better form: Given

$$\theta : \tilde{R}_{\bar{k}}(G) \otimes T_\ell(A) \longrightarrow H^0(G, S_\cdot)^\times$$

one obtains a homomorphism

$$\varphi^\theta : T_\ell(A) \longrightarrow H^0(\bar{k}^\times, S_\cdot)^\times = \Big(1 + \prod_{i>0} x^i S_{2i}\Big)^\times$$

by applying θ to $u - 1$ where u is standard repn of $\bar{k}^\times$ on $\bar{k}$. Now we have that generator of $Gal(\bar{k}/k)$ acts on $\tilde{R}_{\bar{k}}(G) \otimes T_\ell(A)$ as $\Psi^q \otimes \sigma$ where σ is the auto. of $T_\ell(A)$ produced by Galois action on $Pic(\bar{X})$. (I think this means that σ is ($q^{-1}\cdot$) the inverse of ← the geometric Frob.) Now θ is invariant means commutativity in

$$\begin{array}{ccc}
\tilde{R}_{\bar{k}}(G) \otimes T_\ell(A) & \xrightarrow{\ \theta\ } & \\
\Big\downarrow {\scriptstyle \Psi^q \otimes \sigma} & & H^0(G, S_\cdot) \\
\tilde{R}_{\bar{k}}(G) \otimes T_\ell(A) & \xrightarrow{\ \theta\ } &
\end{array}$$

But for $G = \bar{k}^\times$, $\Psi^q = \lambda^*$ where $\lambda : \bar{k}^* \to \bar{k}^*$ is raising to the q-th power. λ acts trivially on cohomology provided we <u>assume</u> $\mu_\ell \subset k$. Therefore we see that

$$\begin{array}{ccc} T_\ell(A) & \longrightarrow & H^0(\bar{k}^\times, S_\cdot)^\times \\ \downarrow \sigma & & \| \\ T_\ell(A) & \longrightarrow & H^0(\bar{k}^\times, S_\cdot)^\times \end{array}$$

commutes. So

$$\text{Exp classes}\left(\tilde{R}_X, H^0(\,, S_\cdot)\right) = \text{Hom}\left(T_\ell(A)_{Gal}, \left(1 + \prod_{i>0} x^i S_{2i}\right)^\times\right)$$

showing the homology is quite far from being a polynomial ring.

Assume $\exists X$ such that $T_\ell(A)_{Gal} = \mathbb{Z}/\ell\mathbb{Z}$.
Then an exponential class is a series

$$\sum x^i s_i \qquad s_0 = 1 \qquad s_i \in S_{2i}$$

such that

$$1 = \left(\sum x^i s_i\right)^\ell = \sum x^{\ell i} s_i^\ell$$

$$\Longrightarrow \quad s_i^\ell = 0 \quad \text{for all } i \geq 1.$$

Thus the Hopf algebra of homology is

$$\mathbb{Z}/\ell\mathbb{Z}[z_1, z_2, \ldots] \big/ (z_1^\ell, z_2^\ell, \ldots)$$

with

$$\Delta z_n = \sum_{i+j=n} z_i \otimes z_j$$

Does this belong to any space?

February 20, 197[illegible]. Conjectures about $K_a(X)$, X curve over $\mathbb{F}_q$:

1) The K-groups should split into three parts

$$K_a(X) = K_a(k) \oplus K_a^{pr}(X) \oplus K_a(k) \qquad k = \mathbb{F}_q$$

where the outer two summands come from $i^* : K_*(X) \longrightarrow K_*(k)$ $\quad i : \mathrm{Spec}(k) \longrightarrow X$ inclusion of a point and

$$f_* : K_*(X) \longrightarrow K_*(k)$$

$f : X \longrightarrow \mathrm{Spec}(k)$ being the canonical map. The primitive part comes from the Jacobian of $\bar{X}$.

2) Formula for $K_*^{pr}(X)$: This should be a finite group of order prime to the characteristic p, and we consider only the l-primary part. Let T_l be the Tate module of rank $2g$ over $\mathbb{Z}_l$ associated to the Jacobian of $\bar{X}$, and denote by F the Frobenius automorphism of T_l so that

$$\zeta^{pr}(s) = \det\left(1 - q^{-s}F\right)$$

(ζ^{pr} equals part of ζ not involving H^0 and H^2.)

Now form the "space" $U \otimes T_\ell$; it is a product of $2g$ copies of the ℓ-adic completion of U. On $U \otimes T_\ell$ we put the endomorphism $\sigma = \Psi^q \otimes F$ and we form the fibre

$$E(\sigma) \longrightarrow U \otimes T_\ell \xrightarrow{\sigma - 1} U \otimes T_\ell$$

I conjecture that

$$K_a^{pr}(X)_{(\ell)} = \pi_a E(\sigma) \qquad a \geq 0.$$

Can check this:

$$K_0^{pr}(X)_{(\ell)} = T_\ell / (\sigma - 1) = \mathrm{Jac}(\bar{X})^{Gal} = \mathrm{Pic}^\circ(X)$$

Also

$$K_1^{pr}(X)_{(\ell)} = 0$$

and

$$\mathrm{card}\, K_{2i}^{pr}(X)_{(\ell)} = |\det(1 - q^i F)| \qquad |\ | = |\ |_\ell$$

Hence

$$K_{2i+1}^{pr}(X) = 0 \quad \text{and}$$

$$\mathrm{card}\, K_{2i}^{pr}(X) = \det(1 - q^i F)$$

$$= \zeta^{pr}(-i)$$

For $i = 1$, this is compatible with Tate's computation of $K_2(F)$ and the exact sequence

$$0 \longrightarrow K_2(X) \longrightarrow K_2(F) \xrightarrow{\lambda} \bigoplus_{v \in X} k_v^\cdot \longrightarrow K_1(X) \ldots$$

February 21, 1971

3) Behavior under base extension: If $[k_1 : k] = d$, then the endo. for k_1 is σ^d:

$$\begin{array}{ccccc}
E(\sigma) & \longrightarrow & U\otimes T_\ell & \xrightarrow{\sigma-1} & U\otimes T_\ell \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1+\sigma+\cdots+\sigma^{d-1}} \\
E(\sigma^d) & \longrightarrow & U\otimes T_\ell & \xrightarrow{\sigma^d-1} & U\otimes T_\ell
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{2i+1}\, U\otimes T_\ell & \longrightarrow & \pi_{2i+1}\, U\otimes T_\ell & \longrightarrow & K^{pr}_{2i} X_{(\ell)} & \longrightarrow & 0 \\
 & & \Vert{\scriptstyle 1} & & \downarrow{\scriptstyle 1+\sigma+\cdots\sigma^{d-1}} & & \downarrow & & \\
0 & \longrightarrow & \pi_{2i+1}\, U\otimes T_\ell & \longrightarrow & \pi_{2i+1}\, U\otimes T_\ell & \longrightarrow & K_{2i}(X\times_k k_1)_{(\ell)} & \longrightarrow & 0
\end{array}$$

Serpent lemma shows that

$$K^{pr}_{2i}(X) \hookrightarrow K^{pr}_{2i}(X\times_k k_1)$$

provided $1+\sigma+\cdots+\sigma^{d-1}$ has none of its eigenvalues equal to zero, ~~[illegible]~~ which is clear because after tensoring with $\mathbb{Q}$, it becomes an iso.

Therefore as in the case of a finite field we get

$$K_{2i}^{pr}(\overline{X}) \cong \bigoplus_{\ell \neq p} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g} \qquad (\simeq \mathrm{Jac}(\overline{X})(i))$$

that the Frobenius acts as $q^i F$, and that

$$K_{2i}^{pr}(X) \xrightarrow{\sim} K_{2i}^{pr}(\overline{X})^{\mathrm{Gal}(\bar{k}/k)}.$$

Furthermore it is fairly clear that the restriction of scalars homomorphism from k_1 down to k is given by the norm.

<u>Conjecture:</u> Let $\overline{X}$ be of finite type over the algebraic closure $\bar{k}$ of a finite field. Then the K-groups satisfy periodicity: $K_i(\overline{X}) \simeq K_{i+2}(\overline{X}) \quad i \geq 1$. Moreover if $\overline{X}$ is smooth, then $K_+(\bar{X})$ should have no p-torsion and be ℓ-divisible for all ℓ.

Maybe one should look at things this way: Form the "topological" K-groups: $[\Sigma^* \overline{X}_{et}, BU_{et}]_{(\ell)}$ as suggested by Friedlander. Denote them by $K_*^{top}(\overline{X})_{(\ell)}$; they are free $\mathbb{Z}_\ell$-modules if $H^*(\overline{X}_{et}, \mathbb{Z}_\ell)$ is torsion-free which we will assume. Then there should be an ℓ-divisible type procedure for converting free $\mathbb{Z}_\ell$-modules into $\mathbb{Q}_\ell/\mathbb{Z}_\ell$'s in one lower dimension, and this should give the K-groups of $\overline{X}$ over $\bar{k}$. Now one can take invariants under Frobenius to get the K-groups for X over k.

Basic internal consistency of this scheme with ζ-function: Again X is a curve over k. Then we should have a map

$$c_2^{\#} : K_2(X) \longrightarrow \varprojlim_{\nu} H^2(X, \mu_{\ell^\nu}^{\otimes 2})$$

Exact sequence:

$$0 \to H^1(Gal, H^1(\bar{X}, \mu_{\ell}^{\otimes 2})) \to H^2(X, \mu_{\ell^\nu}^{\otimes 2}) \longrightarrow H^0(Gal, H^2(\bar{X}, \mu_{\ell^\nu}^{\otimes 2})) \longrightarrow 0$$

Use.

$$H^1(\bar{X}, \mu_{\ell^\nu}) \xrightarrow{\sim} {}_{\ell^\nu}Pic^\circ(\bar{X})$$

$$\mathbb{Z}/\ell^\nu = Pic(\bar{X})/\ell^\nu Pic(\bar{X}) \xrightarrow{\sim} H^2(\bar{X}, \mu_{\ell^\nu})$$

as $H^2(\bar{X}, \mathbb{G}_m) = 0$ by Tsen.

hence one gets

$$H^1(Gal, T_\ell(Jac(\bar{X}))(1)) \xrightarrow{\sim} \varprojlim_{\nu} H^2(X, \mu_{\ell^\nu}^{\otimes 2})$$

$$\wr|$$

$$[Pic^\circ(\bar{X})(1)]^{Gal(\bar{k}/k)}$$

and this $c_2^{\#}$ by Tate should be an isom:

$$K_2^{pr}(X) = [Pic^\circ(\bar{X})(1)]^{Gal(\bar{k}/k)}$$

Generalizing this conjecturally, we expect that the groups $K_{2i}^{pr}(X)$ and $\varprojlim_{\nu} H^2(X, \mu_{\ell^\nu}^{\otimes(i+1)})$

are isomorphic and that

$$c^{\#}_{i+1} : K^{pr}_{2i}(X) \longrightarrow \varprojlim_{\nu} H^2(X, \mu_{\ell^\nu}^{\otimes(i+1)})$$

is multiplication by $\pm(i)!$ This conjecture agrees with our earlier conjecture and with the ζ-function nonsense.

What is incredibly mystifying is the way the ζ-function enters into the theory. At the moment we relate K to ζ by these steps:

A.) K to $H^*(\ ,T^{\otimes i})$ via Chern classes

B.) $H^*(T^{\otimes i})$ to values of ζ at $-i \pm \varepsilon$ via Lefschetz formula in etale cohomology.

In higher dimensions the relations aren't so easy to decipher. Ideally one might expect the motive X^i of i-th coh. of a non-sing. proj. var. X over k to have the following K-groups:

$$K_*(X^i) = K_*(\bar{X}^i)^{Gal}$$

$$K_a(\bar{X}^i) = \begin{cases} H^i(\bar{X}) \otimes \mathbb{Q}/\mathbb{Z}(j) & a = i - 1 + 2j \\ 0 & a \not\equiv i-1 \mod 2 \end{cases}$$

Hence

$$\text{card } K_{2j-1+i}(X^i) = \det(1 - q^j F) = \zeta_{X^i}(-j)^{(-1)^{i+1}}.$$

Intriguing possibility: Over ~~[illegible]~~ rings of S-integers in number fields one has cohomology and hopefully a good relation between K-theory and cohomology. Empirically one has by Lichtenbaum a relation between the K-groups and the ζ function. So maybe one will eventually establish this relation fulfilling the dream of a cohomological interpretation of ζ-functions.

February 21, 1970

today I removed stuff on equivariant coh. from desk to bookcase and brought back finite groups of rational points paper for writing. Projects:

part III to spectrum paper:

1. localization thm. + (fixpoint fmla maybe)
2. maximal strata given by Centralizers
3. central elementary A + depth – primary spaces
4. structure of an A-space + recovery of Euler chars. for p odd.

remaining topics not much developed

1. Chern subring of H_G^* has same spectrum?
2. Tate cohomology (duality if ∃ any)
3. ~~...~~ multiplicative transfer + Riemann-Roch
4. Euler characteristic for $H_G(X)$
5. characteristic classes in $H^*(X/G)$ for actions. (Sullivan mod 2 ~~...~~ Whitney classes.)

should write a paper on

symmetric groups, h-symmetry operations

cohomology ops.

February 22, 1971:

Problem: Let $G_\bullet = \{G_\nu\}$ be a simplicial gp, and let $E = \{E_\nu\}$ be a simplicial A-module on which G acts. Assume that the normalization $N_\cdot E$ is a bdd complex which has fin. gen. proj. A-modules in each degree. Does then E give rise to an element of $[BG, BGL(A)^+]$?

We want this element to agree with this special case: If G constant, then it should be the alternating sum of the representations $N_i E$ of G.

Question: Given a simplicial set X consider animals $\mathcal{E} = P \times^G E$ where G is a simplicial group, P is a principal G-bundle over X, and E is a simplicial A module which is "perfect" and which has an action of G. Do such things define elements of $K(X; A) = [X, K_0 A \times BGL(A)^+]$ and is every such K-element so realized?

February 22, 1971 (?): Review of Mather's thm.

Definition: topological category $\mathcal{C}$ = category object in (spaces): Thus consists of two spaces $Ar\,\mathcal{C}$ and $Ob\,\mathcal{C}$ and four maps

$$\begin{array}{c} Ar\,\mathcal{C} \\ t\downarrow \ \ \downarrow s \ \ \uparrow e \\ Ob\,\mathcal{C} \end{array} \qquad Ar\,\mathcal{C} \times_{(s,t)} Ar\,\mathcal{C} \xrightarrow{c} Ar\,\mathcal{C}$$

satisfying habitual identities.

Definition: topological groupoid $\mathcal{C}$ = topological category such that $\exists$ continuous inverse $i : Ar\,\mathcal{C} \to Ar\,\mathcal{C}$ (represents a functor from (spaces) to (groupoids).)

Recall convention that arrows are drawn $\leftarrow$. Thus a left $\mathcal{C}$-space (i.e. a space $X \to Ob\,\mathcal{C}$ with an associative unitary action

$$Ar\,\mathcal{C} \times_{Ob\,\mathcal{C}} X \longrightarrow X\,)$$

is the analogue of a covariant functor to sets. A right $\mathcal{C}$-space is analogous to a contravariant functor.

Definition: ($\mathcal{C}$ topological groupoid) $\mathcal{C}$-torsor over (or with base) a space X = a left $\mathcal{C}$-space $P \to X$, $Ar\,\mathcal{C} \times_{Ob\,\mathcal{C}} P \to P$, which locally on X is of the form

$$Ar\,\mathcal{C} \times_{Ob\,\mathcal{C}} X$$

for some map $X \to Ob\,\mathcal{C}$.

<u>Remarks:</u> 1. $\mathcal{C}$-torsors form a stack over (spaces). It is the stack generated by the pre-stack assigning to each space X the groupoid $\mathcal{C}(X)$. Observe two $\mathcal{C}$-torsors are **not** necessarily locally isomorphic in general (they are, if $\mathcal{C}$ is a top. group)

2. To give an isomorphism $P \cong \mathrm{Ar}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} X$ of $\mathcal{C}$-torsors over X is the same as giving a section $X \longrightarrow P$. The difference of two such sections is a well-defined map $X \longrightarrow \mathrm{Ar}\,\mathcal{C}$, i.e. an arrow in $\mathcal{C}(X)$, because

$$\mathrm{Ar}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} P \xrightarrow{\ \sim\ } P \times_{\mathrm{Ob}\,\mathcal{C}} P$$

Consequently isomorphism classes of $\mathcal{C}$-torsors are the same as Čech cohomology $\check{H}^1(X;\mathcal{C})$.

3. Given two $\mathcal{C}$-torsors: P over X, P' over X' we can form their direct product leading to mixing diagram

$$\begin{array}{ccccc}
P & \longrightarrow & P \times_{\mathrm{Ob}\,\mathcal{C}} P' & \longrightarrow & P' \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & (P \times_{\mathrm{Ob}\,\mathcal{C}} P')_{\mathrm{Ar}\,\mathcal{C}} & \longrightarrow & X'
\end{array}$$

This in turn leads to a theory of universal bundles: P over X is universal if the map $P \longrightarrow \mathrm{Ob}\,\mathcal{C}$ is a fiber-homotopy-equivalence over $\mathrm{Ob}\,\mathcal{C}$.

<u>Problem:</u> $\mathcal{C}$-torsors for a topological category.

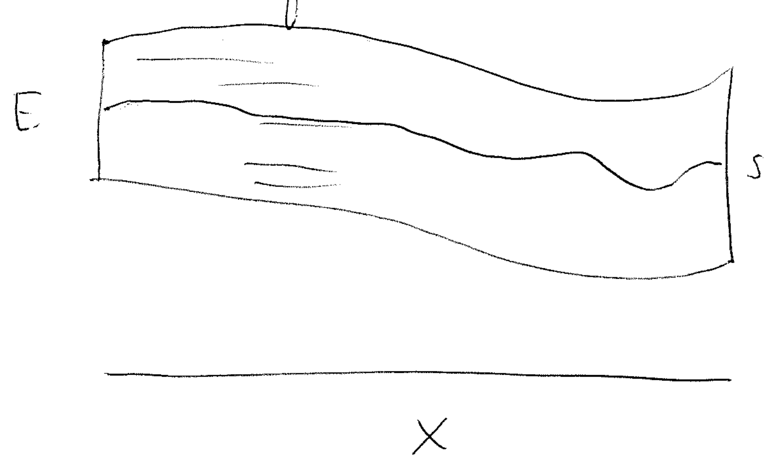
Example: Haefliger structures of codimension q. Let Γ_q denote the top. groupoid with $Ob\,\Gamma_q = \mathbb{R}^q$ and $Ar\,\Gamma_q$ the etale space over $\mathbb{R}^q$ [redacted] whose sheaf of sections is the sheaf of C^∞ maps $\mathbb{R}^q \longrightarrow \mathbb{R}^q$ which are local diffeomorphisms. Γ_q = pseudo-group of local diffeos. of $\mathbb{R}^q$.

Let X be a manifold endowed with a (smooth) foliation of codimension q. [redacted] Locally [redacted] $\exists$ submersions $f: X \to \mathbb{R}^q$ whose fibres are the leaves of the foliation; let P be the sheaf of such submersions. Then P is a Γ_q-torsor, with action

$$(Ar\,\Gamma_q) \times_{Ob\,\Gamma_q} P \longrightarrow P$$

given by composition of $f: X \to \mathbb{R}^q$ with a diffeom. of $\mathbb{R}^q$.

Now in general consider a q-diml vector bundle (smooth) E over a smooth manifold X endowed with a codim q foliation transversal to the fibres and a continuous section s



Then E has a Γ_q-torsor over it which can be pulled back via the section. [redacted] Γ_q-torsors same as Haefliger structures for the pseudo-group Γ_q.

(Now I leave topological categories with "thick" arrow spaces such as topological groups which are not discrete, I want to consider C-sheaves without having to go to gross topoi.)

So now consider a topological category C such that source: $Ar\,C \longrightarrow Ob\,C$ is etale. Then by $C^\wedge$ I mean the category of etale spaces $F \to Ob\,C$ with right C action. Thus if $Ob\,C$ is discrete, C is an ordinary category and $C^\wedge$ is the category of contravariant functors from C to (sets).

Example: $C = \Gamma_q$. any sheaf on R^q intrinsically associated to the differential structure is a Γ_q-sheaf, such as Ω^i, Θ, jets, etc.

$C^\wedge$ is a topos. This is clear (more or less) because of the functor $C^\wedge \longrightarrow Top(Ob\,C)$ forgetting the action, which commutes with everything. Generators of the form

$$Ar\,C \times_{Ob\,C} U \qquad U \subset Ob\,C.$$

Definition: C as above, a C-torsor over a space X is a morphism of topoi

$$f : Top(X) \longrightarrow C^\wedge.$$

(This definition is too virtuous to be immediately understood.)

f as above consider $Ar\,C$ as an object of $C^\wedge$

via the source map. Then $P = f^*(\mathrm{Ar}\,\mathcal{C})$ is a sheaf over X. We have a morphism of induced topoi

$$\begin{array}{ccc} \mathrm{Top}(P) & \longrightarrow & \mathrm{Top}(\mathrm{Ob}\,\mathcal{C}) \\ \| & & \| \\ \mathrm{Top}(X)_{/P} & \longrightarrow & \mathcal{C}^{\wedge}_{/\mathrm{Ar}\,\mathcal{C}} \end{array}.$$

Assuming $\mathrm{Ob}\,\mathcal{C}$ is a <u>sober space</u> we get a map

$$g : P \longrightarrow \mathrm{Ob}\,\mathcal{C}.$$ (need reference here)

It has the property

$$f^*(F \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Ar}\,\mathcal{C}) \simeq g^*F = F \times_{\mathrm{Ob}\,\mathcal{C}} P$$

for any F in $\mathrm{Top}(\mathrm{Ob}\,\mathcal{C})$. In particular, taking $F = \mathrm{Ar}\,\mathcal{C}$ we have

$$\begin{array}{ccc} f^*(\mathrm{Ar}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Ar}\,\mathcal{C}) & = & \mathrm{Ar}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} P \\ \downarrow & & \downarrow \\ f^*(\mathrm{Ar}\,\mathcal{C}) & = & P \end{array}$$

and we get a left action of $\mathrm{Ar}\,\mathcal{C}$ on P. Moreover if F is in $\mathcal{C}^{\wedge}$ we have an exact diagram

$$F \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Ar}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Ar}\,\mathcal{C} \rightrightarrows F \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Ar}\,\mathcal{C} \longrightarrow F$$

so as f^* is a left adjoint we have exact diagram

$$\begin{array}{ccccc} f^*(F \times_{Ob\,C} Ar\,C \times_{Ob\,C} Ar\,C) & \rightrightarrows & f^*(F \times_{Ob\,C} Ar\,C) & \longrightarrow & f^*(F) \\ \| & & \| & & \\ F \times_{Ob\,C} Ar\,C \times_{Ob\,C} P & \rightrightarrows & F \times_{Ob\,C} P & \longrightarrow & F \times^{C} P \end{array}$$

whence (modulo checking the maps) we have

$$f^*(F) = F \times^{C} P$$

in other words, f^* is given by twisting with respect to P.

Conversely given a P etale over X and a left C-action on P we can define f^* by this formula. For f^* to ~~constitute~~ a morphism of topoi it must commute with finite $\varprojlim$'s. Can check this over each $x \in X$. Now

$$(F \times^{C} P)_x = \mathrm{Coker}\left\{ F \times_{Ob\,C} P_x \leftleftarrows F \times_{Ob} Ar \times_{Ob} P_x \right\}$$

(discrete spaces)

$$= \varinjlim_{(p,y)} F_y$$

where (p,y) runs over the category whose objects are pairs with $y \in Ob\,C$ and $p \in P_x$ over y, evident morphisms (i.e. the cofibred category over C determined by the functor $y \mapsto (P_x)(y)$). In order that this be exact it is necessary and sufficient that the category $\{(p,y)\}$

be cofiltering, i.e. that the functor P_x be a pro-object in C. Thus have checked.

<u>Proposition:</u> Let C be a topological category such that source: $\mathrm{Ar}\,C \longrightarrow \mathrm{Ob}\,C$ is etale, and let $C^\wedge$ be the topos of sheaves over $\mathrm{Ob}\,C$ with right C-action. Assume $\mathrm{Ob}\,C$ sober.

(i) A point in C is the same as a pro-object in the underlying discrete category.

(ii) A morphism of topoi $f : \mathrm{Top}(X) \longrightarrow C^\wedge$ is the same as a sheaf P over X with left C-action whose stalks give rise to pro-representable functors on C. The morphism f is given by

$$f^*(F) = F \times^C P .$$

The direct image: Let $f' : P \longrightarrow \mathrm{Ob}\,C$ be the map induced by f. Then

$$f_*(F') = f'_*(F' \times_X P)$$

$$R^q f_*(F') = R^q f'_*(F' \times_X P).$$

(Last assertion results from the fact that f_* is compatible with localization, hence in the cartesian square

$$\begin{array}{ccc} \mathrm{Top}(P) & \xrightarrow{f'} & \mathrm{Top}(\mathrm{Ob}\,C) \\ \downarrow & & \downarrow \\ \mathrm{Top}(X) & \xrightarrow[f]{} & C^\wedge \end{array}$$

we have base change.)

Thus given a C-torsor P over X we have an induced map of cohomology

$$H^i(C; F) \longrightarrow H^i(X, F \times^C P)$$

(it is the morphism belonging to the morphism of topoi

$$f : \mathrm{Top}(X) \longrightarrow C^\wedge .)$$

<u>Corollary</u>: Assume $F \in C^\wedge$ is such that as a sheaf on $\mathrm{Ob}\, C$ it is acyclic w.r.t. the map $f' : P \longrightarrow \mathrm{Ob}\, C$, i.e.

$$R^q f'_*(f^* F) = \begin{cases} 0 & q > 0 \\ F & q = 0. \end{cases}$$

Then

$$H^i(C; F) \xrightarrow{\sim} H^i(X, F \times^C P).$$

<u>Proof</u>: Immediate consequence of the Leray spectral sequence for f. The point is that $R^q f_*(f^* F)$ when "lifted" to $\mathrm{Ob}\, C$ (i.e. you forget the C-action) is the sheaf $R^q f'_*(f^* F)$.

(n-acyclic variation on the preceding).

Remark: If C is an etale groupoid, then pro-representable functors on C are representable, hence ~~[illegible]~~ the above two notions of C-torsors are equivalent.

Problem: Take $C = \Gamma_q$ and construct a Γ_q-torsor P over a CW complex X such that the map $f' : P \longrightarrow Ob\,\Gamma_q = \mathbb{R}^q$ is acyclic for constant sheaves. One wants the map f' ~~to~~ ~~[illegible]~~ admit a fibrewise deformation to a section (i.e. quasi-fibration with contractible fibres, in which case it would be acyclic for all sheaves ~~[illegible]~~ on $\mathbb{R}^q$.

Remarks: If C is an etale groupoid, then obj. of $C^\wedge$ may be identified with characteristic sheaves for C-torsors, i.e. functors F which assign to a C-torsor $P \to X$ a sheaf $F(P,X)$ on X in a functorial way. (cartesian functors from stack of C-torsors to stack of sheaves)

Example. If P comes from a codim q foliation on X, then

$$\Omega^{\bullet}_{\mathbb{R}^q} \times^{\Gamma_q}(P)$$

is the de Rham complex of forms locally constant along the leaves of the foliation

Let $\mathcal{C}$ and $\mathcal{C}'$ be topological categories with étale source maps and $u: \mathcal{C} \to \mathcal{C}'$ a functor. Then have $u^*: \mathcal{C}'^\wedge \longrightarrow \mathcal{C}^\wedge$ given by

$$u^*F' = F' \times_{Ob\,\mathcal{C}'} Ob\,\mathcal{C}.$$

This being compatible with finite proj. limits and arb. ind. lims., it constitutes a morphism of topoi

$$u: \mathcal{C}^\wedge \longrightarrow \mathcal{C}'^\wedge .$$

Suppose u now such that $Ob(u): Ob\,\mathcal{C} \to Ob\,\mathcal{C}'$ is étale whence we have $Ob(u)_!: Top(Ob\,\mathcal{C}) \to Top(Ob\,\mathcal{C}')$. Then we have adjoint functors

$$\mathcal{C}^\wedge \; \begin{array}{c} \xrightarrow{\;u_!\;} \\ \xleftarrow{\;u^*\;} \\ \xrightarrow{\;u_*\;} \end{array} \; \mathcal{C}'^\wedge$$

where

$$(*) \qquad (u_!F) = \mathrm{Coker}\left\{ F \times_{Ob\,\mathcal{C}} Ar\,\mathcal{C} \times_{Ob\,\mathcal{C}'} Ar\,\mathcal{C}' \rightrightarrows F \times_{Ob\,\mathcal{C}'} Ar\,\mathcal{C}' \right\}$$

so

$$(u_!F)_y = \mathrm{Coker}\left\{ \coprod_{\substack{y \to u(x_0) \\ x_0 \to x_1}} F_{x_1} \rightrightarrows \coprod_{y \to u(x)} F_x \right\}$$

$$= \varinjlim_{y \to u(x)} F_x \quad ~~= H^0(y|\mathcal{C}, F)~~.$$

~~where $y|\mathcal{C}$ denotes the category of arrows $y \to u(x)$.~~

Proof of (*):

$$\begin{array}{ccc} \mathrm{Top}(Ob\, \mathcal{C}) & \xrightarrow{Ob(u)} & \mathrm{Top}(Ob\, \mathcal{C}') \\ f \big\downarrow & & \big\downarrow f' \\ \mathcal{C}^\wedge & \xrightarrow[u]{} & \mathcal{C}'^\wedge \end{array}$$

$$f_! F \;=\; F \times_{Ob\, \mathcal{C}} Ar\, \mathcal{C} \qquad \text{(immediate)}$$

so

$$u_! f_! F = f'_! \left(Ob(u)_! F \right)$$

(recall $Ob(u)_! F$ is the composite etale map $F \to Ob\,\mathcal{C} \to Ob\,\mathcal{C}'$)

$$u_! f_! F = F \times_{Ob\, \mathcal{C}'} Ar\, \mathcal{C}'.$$

Now in general we have exact situation

$$(f_! f^*)^2 F \rightrightarrows f_! f^* F \longrightarrow F$$

because f^* is faithfully exact, and when applied to this gadget it becomes homotopically trivial. Thus since $u_!$ is left exact:

$$\begin{array}{ccccc} u_! (f_! f^*)^2 F & \rightrightarrows & u_! f_! f^* F & \longrightarrow & u_! F \\ \parallel & & \parallel & & \\ (F \times_{Ob\,\mathcal{C}} Ar\,\mathcal{C}) \times_{Ob\,\mathcal{C}'} Ar\,\mathcal{C}' & \rightrightarrows & F \times_{Ob\,\mathcal{C}'} Ar\,\mathcal{C}' & & \end{array}$$

q.e.d.

From now on we work with abelian sheaves, and write $u_!$ instead of $u_{!ab}$. Thus

$$(f_! F)_x = \bigoplus_{x \to x'} F_{x'} \qquad \text{exact.}$$

$$(u_! F)_y = \varinjlim_{y \to u(x)} F_x \qquad \text{limit taken as abelian functor}$$

$$= H_0(y\backslash\mathcal{C}, F)$$

where $y\backslash\mathcal{C}$ is the category of arrows $y \to u(x)$, x varying in $\mathcal{C}$.

Existence of derived functor $\mathbb{L}u_!$: Standard resolution

$$(*) \qquad \rightrightarrows (f_! f^*)^2 F \rightrightarrows (f_! f^*) F \longrightarrow F$$

exact functors of F, (compatible with filtered $\varinjlim$s). Set

$\mathbb{L}u_!(F)$: the complex $\nu \mapsto u_!(f_! f^*)^{\nu+1} F$

$L_q u_!(F) =$ q-th homology group.

This is an exact homological functor, effaceable since F quotient of $f_! f^* F$ and the complex $(*)$ splits for $F = f_! M$.

<u>Stalk formulas:</u>

$$L_q u_!(F)_y = L_q \varinjlim_{y \to u(x)} F_x$$

$$= H_q(y\backslash\mathcal{C}, F)$$

Both sides are homological functors, hence need only establish effaceability on the right

$$\begin{array}{ccccc}
\underset{\substack{x \text{ varies} \\ \text{not } x'}}{\{(y \to u(x),\, x \to x')\}} & \xrightarrow{\;p_2\;} & \mathrm{Ob}\,\mathcal{C} & = & \{x',\ \text{no arrows}\} \\
\downarrow p_1 & & \downarrow f & & \\
\underset{x \text{ varies}}{\{(y \to u(x))\}} = y\backslash\mathcal{C} & \xrightarrow{\;j\;} & \mathcal{C} & &
\end{array}$$

$$(j^* f_! M)_{y \to u(x)} = (f_! M)_x = \bigoplus_{x \to x'} M_{x'}$$

$$(p_{1!}\, p_2^* M)_{y \to u(x)} = \varinjlim_{x \to x'} M_{x'} = \bigoplus_{x \to x'} M_{x'}$$

(p_1 is fibred, hence $p_{1!}$ can be computed as the limit over the fibre.) $p_{1!}$ is exact

$$H_q(y\backslash\mathcal{C},\, p_{1!}\, p_2^* M) = H_q(\{(y \to u(x),\, x \to x')\};\, p_2^* M)$$

But the category $\{(y \to u(x),\, x \to x')\}$ is a disjoint sum over the different maps $y \to u(x')$ and different x' of categories with a final object, so the homology is trivial as $p_2^* M$ is constant ~~[illegible]~~ for x' fixed. (Observe - easy to make a mistake here as the $\varinjlim$ functor will be exact for a category with an initial object.)

Another proof of stalk formula: From the explicit construction of $\mathbb{L}u_!$ we have $\mathbb{L}u_!(F)_y$ will be the complex

$$\nu \longmapsto \coprod_{\substack{y \to u(x_\nu) \\ x_\nu \to x_{\nu-1} \to \cdots \to x_0}} F_{x_0}$$

[illegible]

which is the complex of chains for the category $y\backslash C$ with coefficients in the functor $(y \to u(x)) \mapsto F_x$. Thus knowing this chain complex calculates $H_*(y\backslash C, F)$ we win. (already a special case of formula for $\mathbb{L}u_!$ where $u: y\backslash C \to e$).

(both of the above proofs look hard to write down)

Remaining points:

1) Adjunction

$$\operatorname{Hom}_{D(C^\wedge)}(\mathbb{L}u_!(F), F') = \operatorname{Hom}_{D(C'^\wedge)}(F, u^*F')$$

under suitable finiteness (amplitude) conditions on F, F'.

2) Stalk formula when u is pre-fibred:

$$L_q u_!(F)_y = L_q \varinjlim_{u(x)=y} F_x = H_q(C_y, F).$$

Consequence of fact that $\mathcal{C}_y \longrightarrow y\backslash\mathcal{C}$ has the appropriate adjoint.

3) The $(f_!, f^*)$ resolution furnishes a spectral sequence

$$E_1^{pq} = H^q(Ar_p\,\mathcal{C}\,;\,(\text{last vertex})^*F) \Longrightarrow H^{p+q}(\mathcal{C}; F)$$

$$Ar_p\,\mathcal{C} = \underbrace{Ar\,\mathcal{C} \times_{Ob} \cdots \times_{Ob} Ar\,\mathcal{C}}_{p \text{ times}}$$

When $\mathcal{C}$ is a groupoid this is the Čech spectral sequence for the covering $Ar\,\mathcal{C} \xrightarrow{s} Ob\,\mathcal{C}$.