December 4, 1971.

Chern classes of a flat bundle are torsion:

Let E be a complex f.d. vector space on which a group Γ acts. Consider the complex $\Omega^{\cdot}$ of holomorphic differential forms on $\mathbb{P}E$, which resolves $\mathbb{C}$ by the Poincare lemma (holomorphic style). The bundle $\mathcal{O}(-1)$ is classified by an element of

$$u \in H^1_\Gamma(\mathbb{P}E, \mathcal{O}^*).$$

Recall that its Chern class is determined by the coboundary for the upper exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, "e^{2\pi i(\,)}"] \arrow[d] & \mathcal{O}^* \arrow[r] \arrow[d, "\frac{1}{2\pi i} d\log"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O} \arrow[r] & \Omega^1_{cl} \arrow[r] & 0.
\end{tikzcd}$$

Therefore if $v \in H^1_\Gamma(\mathbb{P}E, \Omega^1_{cl})$ is the image of u under the map $\frac{1}{2\pi i}\log : \mathcal{O}^* \longrightarrow \Omega^1_{cl}$ we have that under the map

$$\begin{tikzcd}
H^1_\Gamma(\mathbb{P}E, \Omega^1_{cl}) \arrow[r, "\sim"] & H^2_\Gamma(\mathbb{P}E, 0 \to \Omega^1_{cl} \to 0 \to \cdots) \arrow[d] \\
H^2_\Gamma(\mathbb{P}E, \mathbb{C}) \arrow[r, phantom, "\simeq"] & H^2_\Gamma(\mathbb{P}E, \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots)
\end{tikzcd}$$

u goes into the first Chern class ξ of $\mathcal{O}(-1)$. But if $\dim E = m$, then ξ^m is in the image of u^m under

$$H^m_\Gamma(\mathbb{P}E, \Omega^m_{cl}) \longrightarrow H^{2m}_\Gamma(\mathbb{P}E, \mathbb{C}).$$

which is zero, as $\Omega^m_d = 0$, since $\mathbb{P}E$ is of complex dimension $m-1$. Thus $\xi^m = 0$, so taking into account the definition of $c_i(E)$ as the coefficients of the relation, we see all these classes are zero.

Triviality of the spectral sequence

$$(*) \qquad E_1^{pq} = H^q_\Gamma(\mathbb{P}E, \Omega^p) \Longrightarrow H^{p+q}_\Gamma(\mathbb{P}E, \mathbb{C}).$$

$$E_2^{pq} = H^p_\Gamma(pt, H^q(\mathbb{P}E, \Omega^m)) \Longrightarrow H^{p+q}_\Gamma(\mathbb{P}E, \Omega^m)$$

$$H^q(\mathbb{P}E, \Omega^m) = \begin{cases} 0 & q \neq m \\ \mathbb{C} & q = m \quad \text{generated by } c_1(\mathcal{O}(-1))^m. \end{cases}$$

This spec. seq. degenerates showing that

$$H^q_\Gamma(\mathbb{P}E, \Omega^p) \xleftarrow{\sim} H^{q-p}_\Gamma$$

cupping with $c_1(\mathcal{O}(-1))^p = \xi^p$.

Thus in $(*)$ we have that E_1 is generated by powers of ξ and H^*_Γ, hence it degenerates.

On Bott's theorem: Let X be a manifold endowed with a foliation of codim q, and let E be a complex vector bundle over X stratified w.r.t. the foliation (integrable connection over the leaves). Then the Chern subring of $H^*(X,\mathbb{C})$ vanishes in dimension $> 2q$.

(In this form, it generalizes the result that the Chern classes of a flat bundle are torsion. Observe that if N is the normal bundle to an integrable foliation, then $E = N \otimes \mathbb{C}$ is a complex bundle stratified w.r.t. the foliation*, hence the ring

$$\mathrm{Pont}(N) = \mathrm{Chern}(N \otimes \mathbb{C})$$

vanishes in dimensions $> 2q$, $q = \dim_{\mathbb{R}} N$.)

* locally N is the tangent bundle of the quotient manifold.

Proof: We consider the complex of differential forms on X which are constant along the leaves, i.e. the pull-back to X of the De Rham complex of the orbit manifold:

$$a^0 \xrightarrow{d} a^1 \xrightarrow{d} a^2 \longrightarrow \cdots\cdots$$

This is a resolution of $\mathbb{C}$, hence

$$H^*(X,\mathbb{C}) = H^*(X, a^\cdot)$$

the right side being hypercohomology. Now if E is a line bundle stratified with respect to the foliation its isom. class is an element of

$$H^1(X, a^*).$$

Using the mapping $f \longmapsto \frac{1}{2\pi i} d\log f$ from a^* to $a^1_{cl} = \mathrm{Ker}(d|a^1)$, one obtains an element of

$$H^1(X, a^1_{cl}) = H^2(X, 0 \to a^1 \xrightarrow{d} a^2 \to \cdots\cdots)$$

which goes to the first Chern class of E under the evident map to $H^2(X, \mathbb{C})$. What we do is to prove that in general the Chern classes of E may be refined to elements of

$$b_i(E) \in H^{i}(X, a^i_{cl}) = H^{2i}(X, 0 \to 0 \to a^i \to a^{i+1} \to \cdots).$$

Now $a^i = 0$ if $i > q$, so this proves Bott's theorem.

Bott does this using curvature. He extends the integrable connection in the directions of the leaves to a connection D. Then

$$K(D) \in \mathrm{End}(E) \otimes (N^* \cdot T^*) \subset \mathrm{End}(E) \otimes \Lambda^2 T^*.$$

It follows that if φ is an invariant polynomial on $\mathfrak{gl}_n$ of degree i

$$\varphi(K(D)) \in \Lambda^i N^* \cdot \Lambda^i T^*$$

which vanishes if $i > q$.

Next I want to show how this method yields the classes b_i. Observe that to any bundle stratified with respect to the foliation admits a de Rham complex

$$E \longrightarrow E \otimes \Theta^* \longrightarrow E \otimes \Lambda^2 \Theta^* \longrightarrow \cdots\cdots$$

which by Poincaré resolves the germs of horizontal sections of E, where $\Theta \subset T$ is the tangent bundle to the leaves. Applying this to the bundle $\Lambda^j N^*$ of forms ~~[illegible]~~ ω satisfying $i(X)\,\omega = 0$ if X is a section of Θ we obtain a ~~[illegible] complex~~

resolution of $\mathcal{a}^j$:

$$0 \longrightarrow \mathcal{a}^j \longrightarrow \Lambda^j N^* \longrightarrow \Lambda^j N^* \otimes \Theta^* \longrightarrow \cdots\cdots$$

Now recall that

$$0 \longrightarrow N^* \longrightarrow T^* \longrightarrow \Theta^* \longrightarrow 0$$

(perhaps should think of this as Y = orbit space

$$0 \longrightarrow \pi^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0\,)$$

leads to a filtration of $\Lambda^{\cdot} T^*$

$$\mathrm{Filt}_p\, \Lambda^n T^* = \Lambda^p N^* \cdot \Lambda^{n-p} T^*$$

(the pth power of the kernel of $\Lambda T^* \longrightarrow \Lambda \Theta^*$) by subcomplexes such that

$$gr_p\, \Lambda^n T^* = \Lambda^p N^* \otimes \Lambda^{n-p} \Theta^*$$

is the above-mentioned resolution of $\mathcal{a}^p$. It follows that the inclusion

$$(0 \cdots 0 \longrightarrow \mathcal{A}^p \longrightarrow \mathcal{A}^{p+1} \longrightarrow \cdots\cdots) \longrightarrow \mathrm{Filt}_p(\Lambda^\cdot T^*)$$

is a quasi-isomorphism. Thus Bott's curvature procedure does indeed define classes

$$b_i(E) \in H^{2i}(X, 0 \cdots 0 - \mathcal{A}^i - \mathcal{A}^{i+1} \cdots).$$

~~Alternative approach using projective bundle [illegible].~~
~~Regard X as a ringed space with $\mathcal{O}_X = \mathcal{A}$,~~
~~[illegible] and endow $\mathbb{P}E$ with $\mathcal{O}_{\mathbb{P}E}$ [illegible] holomorphic~~
~~[illegible] so that E gives rise to a vector~~
~~bundle on this ringed space, hence to a relative~~
~~scheme $\mathbb{P}E$. Actually I prefer~~

Alternative approach using the projective bundle. Regard X as a ringed space with $\mathcal{O}_X = \mathcal{A}$, so that E is a vector bundle over this ringed space. Let Y be the orbit "space" of the foliation, or better take Y to be the ringed space X with the constant sheaf $\mathbb{C}$ for its functions. Then $\mathcal{A}^\cdot = \Omega^\cdot_{X/Y}$. Now we consider $\mathbb{P}E$ endowed with the sheaf of functions loc. constant along the leaves and holomorphic on the fibres of $\pi : \mathbb{P}E \longrightarrow X$. (Perhaps it might be the same to consider the relative scheme $\mathbb{P}E$.)

To compute cohomology we need

$$R^q\pi_*\left(\Omega^p_{\mathbb{P}E/X}\right) = \begin{cases} 0 & q \neq p \\ \mathcal{O}_X \cdot \xi^p & q = p \end{cases}$$

where $\xi \in H^1(\mathbb{P}E, \Omega^1_{\mathbb{P}E/Y,cl})$ is the Chern class of $\mathcal{O}(-1)$. This formula is well known if we use relative schemes and should be OKAY in a mixed holomorphic-C^∞ situation.

Granted this, the exact sequence

$$0 \longrightarrow \pi^*\Omega^1_{X/Y} \longrightarrow \Omega^1_{\mathbb{P}E/Y} \longrightarrow \Omega^1_{\mathbb{P}E/X} \longrightarrow 0$$

leads to a filtration with

$$\mathrm{Filt}_p\, \Omega^m_{\mathbb{P}E/Y} = \pi^*\Omega^p_{X/Y} \cdot \Omega^{m-p}_{\mathbb{P}E/Y}$$

$$gr_p\, \Omega^m_{\mathbb{P}E/Y} = \pi^*\Omega^p_{X/Y} \otimes \Omega^{m-p}_{\mathbb{P}E/X}$$

$$R^q\pi_*\left(gr_p\,\Omega^m_{\mathbb{P}E/Y}\right) = \begin{cases} 0 & q \neq m-p \text{ or if } q > n \\ \Omega^p_{X/Y} & \text{if } q = m-p < n \end{cases}$$

from which one obtains the formula

$$\boxed{\begin{array}{llll} R^j\pi_*\left(\Omega^m_{\mathbb{P}E/Y}\right) & \xleftarrow[\sim]{\cdot\xi^j} & \Omega^{m-j}_{X/Y} & 0 \leq j \leq n-1 \\ & = & 0 & j \geq n \end{array}}$$

where $n = \dim E$

Using Leray s.s. for π

$$\begin{array}{ccc} E_2^{pq} = H^p(X, R^q\pi_*(\Omega^m_{\mathbb{P}E/Y})) & \Longrightarrow & H^{p+q}(\mathbb{P}E, \Omega^m_{\mathbb{P}E/Y}) \\ \cong\Big\uparrow\cdot\xi^q \quad q<n & & \\ H^p(X, \Omega^{m-q}_{X/Y}) & & \end{array}$$

so the spectral sequence degenerates showing that

$$H^*(\mathbb{P}E, \Omega^m_{\mathbb{P}E/Y}) \xleftarrow{\sim} \bigoplus_{q=0}^{n-1} H^{*-q}(X, \Omega^{m-q}_{X/Y})\cdot \xi^q$$

$$H^*(\mathbb{P}E, \Omega^m_{\mathbb{P}E/Y} \to \Omega^{m+1}_{\mathbb{P}E/Y} \to \cdots) \xleftarrow{\sim} \bigoplus_{q=0}^{n-1} H^{*-q}(X, \Omega^{m-q}_X \to \Omega^{m-q+1}_X \to \cdots)\,\xi^q$$

Therefore if we take $m = n = \dim(E)$ we have unique classes

$$b_i \in H^i(X, \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} \to \cdots)$$

such that

$$\xi^n - \pi^* b_1 \cdot \xi^{n-1} + \cdots + (-1)^n \pi^* b_n = 0$$

This defines the refined Chern classes as desired.

$$\underline{\underline{R}}\pi_*(\Omega^m_{\mathbb{P}E/Y}[-m]) \simeq \bigoplus_{q=0}^{n-1} \Omega^{m-q}_{X/Y}[-m-q] \qquad n = \dim E$$

$$\underline{\underline{R}}\pi_*(F_m \Omega^\bullet_{\mathbb{P}E/Y}) \simeq \bigoplus_{q=0}^{n-1} F_{m-q}\Omega^\bullet_{X/Y}[-2q]$$

where we use the notation that $M[1]$ shifts M to the left so that

$$H^j(X, M[q]) = H^{j+q}(X, M).$$

December 5, 1971: Odd classes.

So far given a complex bundle E over X stratified with respect to a foliation of codim q we have two kinds of Chern classes

$$\begin{tikzcd} & H^{2i}(X, \mathcal{a}^i_{cl}) \ni b_i(E) \arrow[d] \\ c_i(E) \in H^{2i}(X, \mathbb{Z}) \arrow[r] & H^{2i}(X, \mathbb{C}) \end{tikzcd}$$

What I want to do is fill in the square. Let a complex of sheaves on X be defined as pull-back

$$\begin{tikzcd} W^{(i)} \arrow[r] \arrow[d] & F_i\mathcal{a}^\cdot = (0 \cdots 0 \to \mathcal{a}^i \to \mathcal{a}^{i+1} \to \cdots) \arrow[d] \\ \mathbb{Z}[0] \arrow[r] & \mathcal{a}^\cdot \end{tikzcd}$$

Thus a map of a complex Q into $W^{(i)}$ is a triple (u, v, h) where

$$u : Q \longrightarrow \mathbb{Z}[0]$$
$$v : Q \longrightarrow F_i\mathcal{a}^\cdot$$

and h is a homotopy joining u, v as maps from $W^{(i)}$ to $\mathcal{a}^\cdot$. Thinking of u, v and h as elements of the function complex we have

$$du = dv = 0 \qquad dh = u - v \quad \text{in } \mathrm{Hom}^\cdot(Q, W^{(i)})$$

There are product maps

$$W^{(i)} \otimes_{\mathbb{Z}} W^{(j)} \longrightarrow W^{(i+j)}$$

defined as follows. Suppose given maps

$$Q' \xrightarrow{u'} \mathbb{Z}[0] \qquad Q' \xrightarrow{v'} F_i a^\cdot$$

$$Q'' \xrightarrow{u''} \mathbb{Z}[0] \qquad Q'' \xrightarrow{v''} F_j a^\cdot$$

and $dh' = u' - v'$, $dh'' = u'' - v''$. Denote by $u'.u''$ and $v'.v''$ the maps

$$Q' \otimes_{\mathbb{Z}} Q'' \xrightarrow{u' \otimes u''} \mathbb{Z}[0] \otimes \mathbb{Z}[0] \longrightarrow \mathbb{Z}[0]$$

$$Q' \otimes_{\mathbb{Z}} Q'' \xrightarrow{v' \otimes v''} F_i a^\cdot \otimes F_j a^\cdot \longrightarrow F_{i+j} a^\cdot$$

the latter maps being the canon. products. Then as the evident map

$$\mathrm{Hom}^\cdot(K, L) \otimes \mathrm{Hom}^\cdot(A, B) \longrightarrow \mathrm{Hom}^\cdot(K \otimes A, L \otimes B)$$

is a map of complexes

$$\begin{aligned} u' \cdot u'' - v' \cdot v'' &= (u' - v') \cdot u'' + v' \cdot (u'' - v'') \\ &= dh' \cdot u'' + v' \cdot dh'' \\ &= d(h' \cdot u'' + v' \cdot h'') \end{aligned}$$

so $(u' \cdot u'', v' \cdot v'', h' \cdot u'' + v' \cdot h'')$ defines a map

$$Q' \otimes_{\mathbb{Z}} Q'' \longrightarrow W^{(i+j)}$$

The associativity of this product results from

$$(h'u''+v'h'')u''' + v'v''h''' = h'u''u''' + v'(h''u'''+v''h''').$$

Commutativity up to homotopy results from

$$(u'h''+h'v'')-(h'u''+v'h'') = dh'.h'' - h'dh'' = d(h'.h'').$$

It follows that

$$\bigoplus_{i\geq 0} H^*(X, W^{(i)})$$

is a graded ring, anti-commutative w.r.t. $*$.

Clearly $W^{(i)}$ is quasi-isomorphic with the complex

$$\underset{0}{\mathbb{Z}} \longrightarrow \underset{1}{a^0} \longrightarrow a^1 \longrightarrow \cdots\cdots \longrightarrow \underset{i}{a^{i-1}} \longrightarrow 0$$

although its product structure is not clear from this description. In virtue of Poincaré's lemma

$$\mathcal{H}^q(W^{(i)}) = \begin{cases} a^i_{cl} & q = i \\ \mathbb{C}^* & q = 1 \\ 0 & q \neq 1, i \end{cases}$$

for $i \geq 2$, while

$$W^{(1)} \sim a^*[1] \quad , \quad W^{(0)} \sim \mathbb{Z}[0].$$

What I want to prove is that for a foliated bundle the i-th Chern class may be defined in

$$H^{2i}(X, W^{(i)}),$$

this being clear for line bundles, in fact, the group $H^2(X, W^{(1)})$ is precisely the group of iso. classes of foliated line bundles.

The case of flat bundles: Here $W^{(i)} = \mathbb{C}^*[1]$ for all $i \geq 1$, so we want classes in $H^{2i-1}(X, \mathbb{C}^*)$. But if $\xi \in H^2(\mathbb{P}E, W^{(1)}_{\mathbb{P}E})$ ($W_{\mathbb{P}E}$ done holomorphically) is the first Chern class of $\mathcal{O}(-1)$, then

$$\xi^n \in H^{2n}(\mathbb{P}E, W^{(n)}_{\mathbb{P}E}) = H^{2n-1}(\mathbb{P}E, \mathbb{C}^*)$$

where $n = \dim E$. Thus we get the desired classes.

Assume for the moment that the usual formal properties of these Chern classes hold. Thus for any flat bundle E over X we have Chern classes

$$c_i(E) \in H^{2i}(X, \mathbb{Z} \to \mathbb{C})$$

satisfying a product formula where the product comes from the pairing of complexes

$$\begin{array}{ccccc} (\mathbb{Z} \to \mathbb{C}) & \otimes & (\mathbb{Z} \longrightarrow \mathbb{C}) & \longrightarrow & (\mathbb{Z} \longrightarrow \mathbb{C}) \\ (a + b) & \otimes & (a' + b') & \longmapsto & (aa', ab') \end{array}$$

More precisely an element of $H^i(X, \mathbb{Z} \to \mathbb{C})$ is represented by a pair (f, λ) where f is an integral i-cocycle

and λ is a complex $(i-1)$-cochain such that $f = d\lambda$.
The product is

$$(f,\lambda)\cdot(g,\mu) = \left(fg, (-1)^{\deg f} f\mu\right)$$

Note that

$$(-1)^{\deg f} f\cdot\mu - \lambda g = \left(+d(\lambda\mu)\right)(-1)^{\deg f}$$

so the arbitrary choice doesn't matter. Better to define the pairing using ba' so that

$$(f,\lambda)(g,\mu) = (fg, \lambda g).$$

This amounts to defining the product

$$\begin{tikzcd}
H^{i-1}(X,\mathbb{C}^*) \otimes H^{j-1}(X,\mathbb{C}^*) \arrow[r] \arrow[d, "id\otimes\delta"] & H^{i+j-1}(X,\mathbb{C}^*) \\
H^{i-1}(X,\mathbb{C}) \otimes H^{j}(X,\mathbb{Z}) \arrow[ur, "\text{usual cup}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
e^{2\pi i\lambda} \otimes e^{2\pi i\mu} \arrow[d, mapsto] & e^{2\pi i \lambda d\mu} \\
e^{2\pi i\lambda} \otimes d\mu \arrow[ur, mapsto] &
\end{tikzcd}$$

Note that the class $\delta(\cdot) \in H^j(X,\mathbb{Z})$ is a torsion class as it dies "canonically in $\mathbb{C}$. This means if you write $\mathbb{C}^* = \exp(2\pi i\, \mathbb{Q}/\mathbb{Z}) \circledast$ $\mathbb{Q}$-vector space, then the $\mathbb{Q}$-vector space part has zero products.

<u>Construction of the refined Chern classes.</u> Let X be a manifold endowed with a quasi-foliation (e.g. a foliation or a complex analytic structure) and let $\Omega^{\cdot}_X$ be the de Rham complex of complex-valued C^∞-functions which are loc. constant along the leaves. Let E be a complex vector bundle stratified with respect to the foliation. Then we form $\mathbb{P}E$ and endow it with the de Rham complex $\Omega^{\cdot}_{\mathbb{P}E}$ of C^∞ complex valued forms which are holomorphic vertically and constant horizontally.

Let $\pi : \mathbb{P}E \longrightarrow X$ be the projection so that we have

$$0 \longrightarrow \pi^*\Omega^1_X \longrightarrow \Omega^1_{\mathbb{P}E} \longrightarrow \Omega^1_{\mathbb{P}E/X} \longrightarrow 0$$

Then one has isomorphisms in $D(\mathbb{P}E)$

$$\bigoplus_{q=0}^{n-1} \Omega_X^{m-q}[m-q] \simeq \underline{\underline{R}}\pi_*(\Omega^m_{\mathbb{P}E}[m])$$

we the isomorphism is defined as follows. The first Chern class of $\mathcal{O}(+1)$ defines a class $\xi \in H^2(\mathbb{P}E, W^{(1)}_{\mathbb{P}E})$ hence a class in $H^1(\mathbb{P}E, \Omega^1_{\mathbb{P}E})$ which may be viewed as a map

$$\mathbb{Z} \longrightarrow \Omega^1_{\mathbb{P}E}[1]$$

ξ^q may be viewed as a map

$$\mathbb{Z} \longrightarrow \Omega^q_{\mathbb{P}E}[q]$$

So taking product with $\Omega^{m-q}_{\mathbb{P}E}[m-q]$ we have a map

$$\Omega^{m-q}_{\mathbb{P}E}[m-q] \longrightarrow \Omega^m_{\mathbb{P}E}[m]$$

Composing with the map

$$\pi^* \Omega_X^{m-q}[m-q] \longrightarrow \Omega_{\mathbb{P}E}^{m-q}[m-q]$$

we obtain a map whose adjoint is the map

$$\Omega_X^{m-q}[m-q] \longrightarrow \underline{\underline{R}}\pi_*\left(\Omega^m_{\mathbb{P}E}[m]\right)$$

we want.

By filtering we obtain an isomorphism

(1) $$\bigoplus_{q=0}^{n-1} F_{m-q}\Omega_X^{\cdot}[-2q] \xrightarrow{\sim} \underline{\underline{R}}\pi_*\left(F_m \Omega^{\cdot}_{\mathbb{P}E}\right).$$

One knows that

$$\underline{\underline{R}}\pi_*\left(\mathbb{C}^*\right) \xleftarrow{\sim} \bigoplus_{q=0}^{n-1} \mathbb{C}^*$$

so using the triangle

$$W^{(m)} \longrightarrow F_m\Omega^{\cdot} \longrightarrow \mathbb{C}^*$$

on both $\mathbb{P}E$ and X, we see by "5 lemma" that

$$\boxed{\underline{\underline{R}}\pi_*\left(W^{(m)}_{\mathbb{P}E}\right) \xleftarrow{\sim} \bigoplus_{q=0}^{n-1} W_X^{(m-q)}[-2q]}$$

This shows that the Deligne-Griffiths cohomology satisfies the projective bundle theorem permitting definition of Chern classes as usual.

December 12, 1971: odd classes in p-adic case.

Suppose given morphisms

$$\begin{tikzcd} X_0 \arrow[r, hook] & X \arrow[d] \\ & S \end{tikzcd}$$

where $X_0 \to X$ is an immersion defined by an ideal I with a nilpotent system of divided powers. If L is a line bundle on X, we can consider its first Chern class in the De Rham cohomology of X rel S:

$$c_1(L) \in H^2(X, F_1\Omega^\bullet_{X/S}).$$

Recall this is defined as the image of $cl(L) \in H^1(X, \mathcal{O}_X^*)$ under the homomorphism

$$\begin{aligned} \mathcal{O}_X^*[-1] &\longrightarrow F_1\Omega^\bullet_{X/S} \\ f &\longmapsto \frac{df}{f} \end{aligned}$$

Now we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (1+I)^\times \arrow[r] \arrow[d, "\log", "\simeq"'] & \mathcal{O}_X^* \arrow[r] \arrow[d, "f \mapsto \frac{df}{f}"] & \mathcal{O}_{X_0}^* \arrow[r] & 0 \\ & I^\bullet \arrow[r, "d"] & \Omega^1_{X/S} & & \end{tikzcd}$$

with an exact top row. Now suppose given two line bundles L, L' on X and an isomorphism $L_0 \simeq L'_0$ over X_0. Then one obtains an element of

$$H^1(X, 1+I)$$

which goes into $cl(L) - cl(L')$ in $H^1(X, \mathcal{O}_X^*)$, and

$$c_1(L) - c_1(L') \in H^2(X, F_1\Omega^{\bullet}_{X/S})$$

comes from an element of

$$H^1(X, I).$$

Let $V^{(m)}$ be the complex defined by

$$\begin{array}{ccc}
V^{(m)} & \longrightarrow & F_m\Omega^{\bullet}_{X/S} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \left(I^{[m]} \to I^{[m-1]}\Omega^1 \to \cdots \to \Omega^m \to \Omega^{m+1} \to \cdots\right)
\end{array}$$

so that $V^{(1)}$ is quasi-isomorphic to $I[-1]$. I have associated to $(L, L', L_0 \simeq L'_0)$ an element of

$$H^2(X, V^{(1)}).$$

I would like to associate to $(E, E', E_0 \simeq E'_0)$ Chern classes in

$$H^{2m}(X, V^{(m)}),$$

and more generally do this for any K-element over X with a trivialization over X_0.

Given E over X, form $\pi : \mathbb{P}E \longrightarrow X$ which is smooth so

$$0 \longrightarrow \pi^* \Omega^1_{X/S} \longrightarrow \Omega^1_{P/S} \longrightarrow \Omega^1_{P/X} \longrightarrow 0$$

will be exact and locally split. The same is true of

$$0 \longrightarrow F_p \Omega^m_{P/S} \longrightarrow F_{p+1} \Omega^m_{P/S} \longrightarrow \pi^* \Omega^p_{X/S} \otimes \Omega^{m-p}_{P/X} \longrightarrow 0.$$

As $\mathcal{O}_P$ is flat over $\mathcal{O}_X$ we have

$$(\pi^* \mathfrak{a}) \otimes_{\mathcal{O}_P} F = \mathfrak{a} \cdot F$$

for any ideal $\mathfrak{a} \subset \mathcal{O}_X$. Thus

$$\pi^* \mathfrak{a} \otimes F_p \Omega^m_{P/S} = \mathfrak{a} \cdot F_p \Omega^m_{P/S} \qquad \text{and}$$

$$\mathfrak{a} F_p \Omega^m_{P/S} \big/ \mathfrak{a} F_{p+1} \Omega^m_{P/S} = \pi^* (\mathfrak{a} \Omega^p_{X/S}) \otimes \Omega^{m-p}_{P/X}$$

On the other hand one checks that

$$R^j \pi_* \left(\pi^* M \otimes \Omega^q_{P/X} \right) = \begin{cases} 0 & j \neq q \\ M & j = q < n \end{cases}$$

by descending induction on j. Consequently

$$\underline{\underline{R}} \pi_* \left(\mathfrak{a} \Omega^m_{P/S} \right) = \bigoplus_{q=0}^{n-1} \mathfrak{a} \Omega^{m-q}_{X/S} [q]$$

where the isomorphism comes from multiplication by powers of the canonical section of

$$\underline{\underline{R}}\pi_*(\Omega^1_{P/S})\,[1].$$

Consequently multiplying by powers of the canonical section of

$$\underline{\underline{R}}\pi_*(F_1\Omega^{\bullet}_{P/S})\,[2]$$

will give an isomorphism

$$\underline{\underline{R}}\pi_*\left(I^m\mathcal{O}_P \longrightarrow I^{m-1}\Omega^1_{P/S} \longrightarrow \cdots\cdots\right)$$

$$\cong \bigoplus_{q=0}^{n-1}\left(I^{m-q}\mathcal{O}_X \longrightarrow I^{m-q-1}\Omega^1_{X/S} \longrightarrow \cdots\right)[-2q]$$

and so we obtain the projective bundle theorem.

So now suppose that in addition to E we give an E' over X and an isomorphism $E_0 \simeq E_0'$ over X_0. Then I want to produce elements $\lambda_m \in H^{2m}(X, V^{(m)})$ with image $c_m(E) - c_m(E') \in H^{2m}(X, F_m\Omega^{\bullet}_{X/S})$. The point is that these classes will coincide in $H^{2m}(X, \Omega^{\bullet}_{X/S})$ because the ~~latter~~ Chern classes are "crystalline", hence independent of the lifting from X_0 to X.

December 16, 1971.

Let G be a finite [redacted] group and let [redacted] $X(G)$ be the following simplicial complex. It's vertices are the p-subgroups of G and a simplex is a chain

$$P_0 < \cdots < P_q$$

such that P_q/P_0 is elementary p-abelian. Claim $X(G)$ is contractible.

This is clear by the old retraction argument if G is a p-group. In general

$$X(G) = \bigcup X(P)$$

where P runs over the Sylow subgroups and

$$X(P_0) \cap \cdots \cap X(P_q) = X(P_0 \cap \cdots \cap P_q)$$

is contractible. Thus by the lemma in Folkman's paper $X(G)$ has the homotopy type as the simplex on the Sylow subgroups, which is contractible.

December 16, 1971:

Have defined canonical elements

$$e_i \in H^{2i-1}(GL_n(\mathbb{C}), \mathbb{C}^*)$$

suppose now that I am given a continuous family of representations of Γ, i.e. a vector bundle E over a manifold X and a Γ-action on the fibres. (everything C^∞). Then for each x we have a homomorphism

$$H^{2i-1}(GL_n(\mathbb{C}), \mathbb{C}^*) \longrightarrow H^{2i-1}(\Gamma, \mathbb{C}^*)$$

obtained from the representation of Γ on E_x. Denote by $e_i(E_x)$ the image of e_i.

<u>Theorem:</u> $e_i(E_x)$ is a locally constant function of x for $i \geq 2$, but not necessarily for $i = 1$.

For $i=1$, take $\Gamma = \mathbb{Z}$, $X = \mathbb{C}^*$, E = trivial line bundle over X. Let Γ act on E_x by $n(v) = x^n v$. Then $e_1(E_x) \in H^1(\Gamma, \mathbb{C}^*) = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is this homomorphism, and it varies with x.

~~We can suppose that $X = \mathbb{R}$. Form the projective bundle of E, $\pi: PE \to X$ and let Ω_p be the de Rham complex of forms which are holom. along the fibres. Again we form $W_p^{(m)}$~~

We can clearly suppose $X = \mathbb{R}$. There is a Chern class defined in

$$c_i \in H^{2i}_{\Gamma}(X, W_X^{(i)})$$

which restricts to the classes $c_i(E_x)$. Since $\dim X = 1$ $W_X^{(i)} \simeq \mathbb{C}^*[-1]$ for $i \geq 2$. Thus

$$H^{2i}_{\Gamma}(X, W_X^{(i)}) = H^{2i-1}_{\Gamma}(X, \mathbb{C}^*) = H^{2i-1}_{\Gamma}(pt, \mathbb{C}^*)$$

so it's all clear.

December 24, 1971. Summary: "functional equation for ζ".

I. Complete non-singular curve X over $\mathbb{F}_q$

$$\zeta(s) = \prod_{\substack{x \text{ closed} \\ \text{point}}} \frac{1}{1 - Nx^{-s}} \qquad Nx = \text{card } k(x) = q^{\deg(x)}$$

$$= \sum_{D \geq 0} \left(q^{\deg D}\right)^{-s}$$

Rewrite as a sum over $\mathrm{Pic}(X)$ using that the number of $D \geq 0$ with $L(D) \simeq L$ is

$$\frac{q^{h^0(L)} - 1}{\mu} \qquad \begin{aligned} h^0(L) &= \dim_{\mathbb{F}_q} H^0(X, L) \\ \mu &= \text{card } H^0(X, \mathcal{O}_X^*) \\ &= q - 1 \quad \text{if } \mathbb{F}_q = H^0(X, \mathcal{O}_X). \end{aligned}$$

$$\boxed{\zeta(s) = \sum_{L \in \mathrm{Pic}\, X} \left(q^{\deg L}\right)^{-s} \cdot \frac{q^{h^0(L)} - 1}{\mu}}$$

To simplify, suppose $\mathbb{F}_q = H^0(X, \mathcal{O}_X)$ and assume known that $\exists$ an L of degree 1 (according to Artin–Tate notes this follows from R-R). Put $z = q^{-s}$ and

$$Z(z) = \zeta(s).$$

Using the Riemann formula

$$h^0(L) = \deg L + 1 - g$$

for $(\deg L) > 2g - 2$ we have

$$Z(z) = \underbrace{\sum_{\deg L \le 2g-2} z^{\deg L} \frac{q^{h^0(L)}-1}{\mu}}_{\substack{\text{poly of degree } \le 2g-2 \\ \text{with } \mathbb{Z} \text{ coeffs.}}} + \underbrace{\sum_{\deg L > 2g-2} z^{\deg L} \frac{q^{\deg L + 1 - g} - 1}{\mu}}_{\frac{h}{\mu}\left[\frac{q^g z^{2g-1}}{1-qz} - \frac{z^{2g-1}}{1-z}\right]}$$

$(h = \text{card } Pic^0(X))$

This shows that

$$Z(z) = \frac{P(z)}{(1-z)(1-qz)}$$

where $P(z)$ is a polynomial with integral coeffs. of degree $2g$. Moreover

$$\operatorname{Res}_{z=1} Z(z) = \frac{h}{\mu} \qquad \operatorname{Res}_{z=\frac{1}{q}} Z(z) = -\frac{h}{\mu} \cdot q^{-g}$$

where $h = \text{card } Pic^0(X)$.

Using now the analytic continuation formulae

$$\sum_{n \ge N} \omega^n = \frac{\omega^N}{1-\omega} = -\frac{\omega^{N-1}}{1-\omega^{-1}} = -\sum_{n<N} \omega^n$$

convgt. $|\omega|<1$ convgt. $|\omega|>1$

we see the second term analytically continues to

$$-\sum_{\deg L \le 2g-2} z^{\deg L} \frac{q^{\deg L + 1 - g} - 1}{\mu} \qquad |z| > 1$$

so

$$Z(z) = \sum_{L} z^{\deg L} \frac{q^{h^0(L)} - q^{\deg L + 1 - g}}{\mu}$$

Now use Riemann-Roch

$$h^0(L) = \deg L + 1 - g + h^0(K-L)$$

$$Z(z) = \sum_L (qz)^{\deg L} \cdot \frac{q^{h^0(K-L)} - 1}{\mu} \cdot q^{1-g}$$

so changing L to $K-L$ and using $\deg(K) = 2g-2$ we find

$$\boxed{Z(z) = q^{g-1} z^{2g-2} Z\left(\frac{1}{qz}\right)}$$

II. Number fields

<u>Poisson sum formula</u>: Let N be a lattice in a vector space V f.d. over $\mathbb{R}$, let M be the "dual" lattice in V^*. If f is a rapidly decreasing fn. on V

$$\sum_{n \in N} f(x+n) = \sum_{m \in M^*} e^{2\pi i \langle m, x \rangle} \int_V e^{-2\pi i \langle m, x \rangle} f(x)\, dx$$

where dx is the Haar measure on V, normalized so that V/N has measure 1. (This is just Fourier exp. of $\sum f(x+n)$).

<u>Special case</u>: take $f(x) = e^{-\pi Q(x)}$ where Q is a positive-definite quad. form on V.

$$\sum_{n \in N} e^{-\pi Q(n)} = \frac{1}{a} \sum_{m \in M} e^{-\pi Q(m^*)}$$

where

a = volume of V/N with respect to Q

m^* is determined by

$$\forall x \in V \qquad \langle m, x \rangle = B(m^*, x)$$

B bilinear form $\ni Q(x) = B(x,x)$.

Divisors: Let k be a number field of degree $N = r_1 + 2r_2$. By a divisor of k (of $\hat{\mathcal{O}}$ completed at ∞) I will mean a pair $D = (\mathfrak{a}, \{t_\sigma\})$ where $\mathfrak{a}$ is a divisor for $\mathcal{O}$ (i.e. a fractionary ideal) and where $t_\sigma \in \mathbb{R}^+$ for each $\sigma | \infty$. Thus

$$\mathrm{Div}(k) = \mathrm{Div}(\mathcal{O}) \times (\mathbb{R}^+)^{r_1+r_2}.$$

Let

$$d(\mathfrak{a}, \{t_\sigma\}) = N\mathfrak{a} \cdot \left(\prod t_\sigma^{N_\sigma}\right)^{-1}$$

where $N_\sigma = 1$ if σ real, $N_\sigma = 2$ if σ complex. Define ~~[illegible]~~ the divisor of $x \in k^*$ to be

$$(x) = (\mathcal{O}.x, \{|x|_\sigma\})$$

so that $d(x) = 1$. Let

$$P = \mathrm{Div}(k) / \mathrm{Im}\, k^*$$

so that

$$0 \longrightarrow P^\circ \longrightarrow P \xrightarrow{d} \mathbb{R}^+ \longrightarrow 0$$

where

$$0 \longrightarrow (\mathbb{R}^+)^{r_1+r_2-1} / \mathrm{Im}\, \mathcal{O}^* \longrightarrow P^\circ \longrightarrow \mathrm{Pic}(\mathcal{O}) \longrightarrow 0.$$

Given $D = (\mathfrak{a}, \{t_\sigma\})$ ~~[illegible]~~ with image $L \in P$, set

$$\Theta_{L} = \sum_{x \in \mathfrak{a}^{-1}} e^{-\pi \sum t_\sigma^2 |x|_\sigma^2}$$

~~[illegible]~~ This is well-defined since

$$\sum_{x \in y^{-1}\mathfrak{a}^{-1}} e^{-\pi \sum (t_\sigma |y|_\sigma)^2 |x|_\sigma^2} = \sum_{z \in \mathfrak{a}^{-1}} e^{-\pi \sum t_\sigma^2 |z|_\sigma^2}$$

Let $\mathfrak{d}$ be the different ideal of k, defined by

$$\mathfrak{d}^{-1} = \{ x \in k \mid \mathrm{Tr}(x\mathcal{O}) \subset \mathbb{Z} \}$$

and K be the class of the <u>canonical divisor</u> $(\mathfrak{d}, \{N_\sigma\})$

so that

$$d(K) = 2^{-2r_2} \cdot d_k$$

where d_k is the absolute value of the discriminant of k.

<u>Riemann-Roch formula</u>:

$$\theta_L = \left(2^{-2r_2} d_k\right)^{-1/2} \cdot dL \cdot \theta_{K-L}$$

<u>Proof</u>: This will be a consequence of the Poisson sum formula with $V = \mathfrak{o} \otimes \mathbb{R}$, N = image of $\mathfrak{a}^{-1}$. We use the form

$$\langle x, y \rangle = \mathrm{Tr}(xy) = \sum_\sigma (\sigma x\, \sigma y) \qquad \sigma \text{ embeddings of } k \text{ into } \mathbb{C}$$

$$= \sum_{i=1}^{r_1} \sigma_i x \cdot \sigma_i y + \sum_{j=1}^{r_2} \tau_j x \cdot \tau_j y + \overline{\tau_j x} \cdot \overline{\tau_j y}$$

to identify V with its dual. Observe then

$$M = \{ x \in \mathfrak{o} \otimes \mathbb{R} \mid \mathrm{tr}(x\mathfrak{a}^{-1}) \subset \mathbb{Z} \} = \{ x \in \mathfrak{o} \mid x\mathfrak{a}^{-1} \subset \mathfrak{d}^{-1} \}$$

$$= \mathfrak{d}^{-1}\mathfrak{a} = (\mathfrak{d}\mathfrak{a}^{-1})^{-1}.$$

If $Q(x) = \sum' t_v^2 |x|_v^2$ $x \in \mathfrak{o}\otimes\mathbb{R}$, then bilinear form is

$$B(x,y^*) = \sum_{i=1}^{r_1} t_v^2\, \sigma_i x \cdot \sigma_i y^* + \sum_{j=1}^{r_2} \tfrac{1}{2} t_j^2 \left(\tau_j x \cdot \overline{\tau_j y^*} + \overline{\tau_j x} \cdot \tau_j y^*\right)$$

So if $\langle x,y\rangle = B(x,y^*)$ for all $x \in \mathfrak{o}\otimes\mathbb{R}$, we have

$$\sigma_i y^* = \frac{\sigma_i y}{t_i^2} \qquad \tau_j y^* = 2\frac{\tau_j y}{t_j^2}$$

so

$$Q(y^*) = \sum_v' \left(\frac{N_v}{t_v}\right)^2 |y|_v^2 .$$

So P.S. gives

$$\underbrace{\sum_{x\in\mathfrak{a}^{-1}} e^{-\pi\sum' t_v |x|_v^2}}_{=\theta_L} = \frac{1}{vol_Q(\mathfrak{a}^{-1})} \underbrace{\sum_{y\in\mathfrak{d}^{-1}\mathfrak{a}} e^{-\pi\sum\left(\frac{N_v}{t_v}\right)^2 |y|_v^2}}_{=\theta_{K-L}}$$

and

$$vol_Q(\mathfrak{a}^{-1}) = \frac{1}{N\mathfrak{a}} vol_Q(\mathfrak{o}) = \frac{1}{N\mathfrak{a}} \cdot \prod t_v^{N_v} \cdot vol_{Q_0}(\mathfrak{o})$$

where $Q_0(x) = \sum' |x|_v^2$ is the canonical metric on $\mathfrak{o}\otimes\mathbb{R}$.
See Lang's book p. 74 for proof that

$$vol_{Q_0}(\mathfrak{o}) = 2^{-r_2} d_k^{1/2} .$$

QED.

Recall

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s} \frac{dt}{t} = 2\int_0^\infty e^{-\pi t^2} (\pi t^2)^s \frac{dt}{t}$$

Set

$$\overline{\Gamma}(s) = \tfrac{1}{2}\pi^{-s}\Gamma(s) = \int_0^\infty e^{-\pi t^2} t^{2s} \frac{dt}{t}$$

Then

$$\Gamma(s/2)^{r_1}\,\Gamma(s)^{r_2}\,\zeta_k(s) = \int\!\!\int_0^\infty\!\!\cdots\!\int e^{-\pi\sum t_v^2}\left(\prod t_v^{N_v}\right)^{s}\prod\frac{dt_v}{t_v}\cdot\sum_{\mathfrak{a}\geq 0}\frac{1}{(N\mathfrak{a})^s}$$

$$= \int_{\substack{D=(\mathfrak{a},\{t_v\})\in \mathrm{Div} \\ \text{with } \mathfrak{a}\geq 0}} e^{-\pi\sum t_v^2}\,(dD)^{-s}$$

and writing this integral over the map $\mathrm{Div} \longrightarrow P$ we get

$$\boxed{\Gamma(s/2)^{r_1}\,\Gamma(s)^{r_2}\,\zeta_k(s) = \int_{L\in P}(dL)^{-s}\cdot\frac{\theta_L - 1}{\mu}}\,.$$

(In this integral, P is endowed with the Haar measure induced by that of Div. μ is the order of the group of roots of unity.)

Clearly $\theta_L - 1 \longrightarrow 0$ rapidly as $dL \to 0$ while by R-R formula

$$\theta_L - (dK)^{-1/2}\cdot dL = (dK)^{-1/2}\,dL\,(\theta_{K-L} - 1)$$

goes to zero rapidly as $dL \longrightarrow \infty$. Thus

$$\Gamma(s/2)^{r_1}\,\Gamma(s)^{r_2}\,\zeta_k(s) = \int_{dL\leq 1}(dL)^{-s}\,\frac{\theta_L - 1}{\mu} + \int_{dL\geq 1}(dL)^{-s}\,\frac{\theta_L - (dK)^{-1/2}dL}{\mu}$$

$$+ \int_{dL\geq 1}(dL)^{-s}\,\frac{(dK)^{-1/2}dL - 1}{\mu}$$

The first two terms are entire, and the last is

$$\frac{(dK)^{-1/2}}{\mu}\int_{dL\geq 1}(dL)^{-s+1} - \frac{1}{\mu}\int_{dL\geq 1}(dL)^{-s} = \frac{(dK)^{-1/2}}{\mu}R\int_1^\infty t^{-s+1}\frac{dt}{t} - \frac{R}{\mu}\int_1^\infty t^{-s}\frac{dt}{t}$$

where $R = \int_{P^0} 1$, P^0 being endowed with the measure compatible with the one on P and the $\frac{dt}{t}$ measure on $\mathbb{R}^+$.
$R=$ regulator

$$= \frac{(dK)^{-1/2}R}{\mu}\,\frac{1}{s-1} - \frac{R}{\mu}\,\frac{1}{s}.$$

Since we have the analytic continuation

$$\int_{dL\geq 1}(dL)^{-s}\,\frac{(dK)^{-1/2}dL-1}{\mu} = -\int_{dL\leq 1}(dL)^{-s}\,\frac{(dK)^{-1/2}dL-1}{\mu}$$

we get

RR used here

$$\bar{\Gamma}(s/2)^{r_1}\bar{\Gamma}(s)^{r_2}\zeta_k(s) = \int (dL)^{-s}\,\frac{\theta_L-(dK)^{-1/2}dL}{\mu} = (dK)^{-1/2}\int (dL)^{1-s}\,\frac{\theta_{K\bar{L}}-1}{\mu}$$

$$= (dK)^{1/2-s}\int (dL)^{-(1-s)}\,\frac{\theta_L-1}{\mu}$$

or

$$(dK)^{s/2}\,\bar{\Gamma}(s/2)^{r_1}\bar{\Gamma}(s)^{r_2}\zeta_k(s) = \Lambda_0(s)$$

satisfies $\Lambda_0(s) = \Lambda_0(1-s)$.

(Observe that R as defined above is $h\cdot$ classical regulator $\cdot\, 2^{-r_2}$). In fact we have

$$\frac{1}{2^{r_1}}\cancel{\pi^{-r_1/2}}\,\cancel{\Gamma(\tfrac{1}{2})^{r_1}}\,\frac{1}{2^{r_2}}\,\pi^{-r_2}\cancel{\Gamma(1)^{r_2}}\,\zeta(s) \sim \frac{(dK)^{-1/2}R}{\mu}\,\frac{1}{s-1}$$

$$\zeta(s) \sim \frac{2^{r_1}(2\pi)^{r_2} R \cdot 2^{r_2}}{\mu\, d_k^{1/2}} \frac{1}{s-1} = \frac{2^{r_1}(2\pi)^{r_2}\, h R^{\text{class}}}{\mu\, d_k^{1/2}} \frac{1}{s-1}$$

and for s near 0

$$\frac{1}{2^{r_1}} \left(\frac{s}{2}\right)^{-r_1} \frac{1}{2^{r_2}} s^{-r_2} \zeta_k(s) \sim -\frac{R}{\mu}\frac{1}{s}$$

$$\boxed{s^{r_1+r_2-1} \cdot \zeta_k(s) \sim -\frac{2^{r_2} R}{\mu} = -\frac{h \cdot R^{\text{class}}}{\mu}}$$

For $\mathbb{Z}$

$$\Gamma(s/2)\,\zeta(s) = \frac{1}{2}\int_0^\infty (\theta(t)-1)\, t^s \frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty (\theta(t)-1)(t^s + t^{1-s}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

where $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}$

satisfies $\theta(t) = \frac{1}{t}\,\theta\!\left(\frac{1}{t}\right)$