December 4, 1971.

Chern classes of a flat bundle are torsion:

Let $E$ be a complex flat vector space on which a group $\Gamma$ acts. Consider the complex $\Omega^\cdot$ of holomorphic differential forms on $P_E$, which resolves $C_*$ by the Poincaré lemma (holomorphic style). The bundle $O(-1)$ is classified by an element of

$$\nu \in H^1_\Gamma(P_E, O^\times).$$

Recall that its Chern class is determined by the coboundary for the upper exact sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^\times \rightarrow \mathcal{O}$$

$$\text{ker} \frac{d\log}{2\pi i} \rightarrow \mathcal{O} \rightarrow \Omega^1_{\text{hol}} \rightarrow 0.$$

Therefore if $\nu \in H^1_\Gamma(P_E, \Omega^1_{\text{hol}})$ is the image of $\xi$ under the map $\frac{1}{2\pi i} \log: \mathcal{O}^\times \rightarrow \Omega^1_{\text{hol}}$, we have that under the map

$$H^1_\Gamma(P_E, \Omega^1_{\text{hol}}) \sim H^2_\Gamma(P_E, O \rightarrow \Omega^1_{\text{hol}} \rightarrow 0 \rightarrow \ldots)$$

$$H^2_\Gamma(P_E, \mathcal{O}) \sim H^2_\Gamma(P_E, \mathcal{O}^\times \rightarrow \Omega^2 \rightarrow \ldots)$$

$\xi$ goes into the first Chern class of $O(-1)$. But if $\dim E = m$, then $\xi^m$ is in the image of $\mu^m$ under

$$H^m_\Gamma(P_E, \Omega^m_{\text{hol}}) \rightarrow H^2m_\Gamma(P_E, \mathcal{O}).$$
which is zero, as $Q_m = 0$, since $PE$ is $pf$
complex dimension $m-1$. Thus $\xi = 0$, so
taking into account the definition of $c_i(E)$ as
the coefficient of the relation, we see all these
classes are zero.

Triviality of the spectral sequence

$$E_2^{p,q} = H^p_i(PE, \Omega^q) \implies H^{p+q}(PE, C).$$

This spectral sequence degenerates showing that

$$H^0_i(PE, \Omega^p) \leq H^{p-p}_i$$
cupping with $c_i(\mathcal{O}_{-1}) = \xi_i^p$.

Thus in (x) we have that $E_1$ is generated
by powers of $\xi_i$ and $H^*_i$, hence it degenerates.
On Betti's theorem: Let $X$ be a manifold endowed with a foliation of codimension $g$, and let $E$ be a complex vector bundle over $X$ stratified with respect to the foliation (integrable connection over the leaves). Then the Chern subring of $H^*(X, \mathbb{C})$ vanishes in dimension $> 2g$.

(In this form, it generalizes the result that Chern classes of a flat bundle are torsion. Observe that if $N$ is the normal bundle to an integrable foliation, then $E = N \otimes \mathcal{C}$ is a complex bundle stratified with respect to the foliation, hence the Chern ring $\text{Poin} \ (N) = \text{Chern} \ (N \otimes \mathcal{C})$ vanishes in dimensions $> 2g$, $g = \text{dim}_R N$.)

**Proof:** We consider the complex of differential forms on $X$ which are constant along the leaves, i.e., the pull-back to $X$ of the De Rham complex of the orbit manifold:

$$
\mathcal{C}^0 \xrightarrow{d} \mathcal{C}^1 \xrightarrow{d} \mathcal{C}^2 \rightarrow \cdots
$$

This is a resolution of $\mathcal{C}$, hence

$$H^*(X, \mathbb{C}) = H^*(X, \mathcal{C})$$

the right side being hypercohomology. Now, if $E$ is a line bundle stratified with respect to the foliation, its class is an element of $H^1(X, \mathcal{A}^*)$. Using the mapping $f \mapsto \frac{1}{2\pi i} \log f$, one obtains an
element of
\[ H^4(X, a^0_{\text{cl}}) = H^2(X, 0 \to a^1 \to a^2 \to \ldots) \]
which goes to the first Chern class of \( E \) under the evident map to \( H^2(X, \mathbb{C}) \). What we do is to prove that in general the Chern classes of \( E \) may be refined to elements of
\[ b^i(E) \in H^{2i}(X, a^i_{\text{cl}}) = H^{2i}(X, 0 \to a^1 \to a^2 \to \ldots). \]
Now \( a^i = 0 \) if \( i > q \), so this proves Bott's theorem.

Bott does this using curvature. He extends the integrable connection in the direction of the leaves to a connection \( D \).

Then
\[ K(D) \in \text{End}(E) \otimes \text{End}(N^* \otimes T^*) \subset \text{End}(E) \otimes \Lambda^2 \otimes \Lambda^* \]
It follows that if \( \psi \) is an invariant polynomial on \( \text{End}(E) \) of degree \( i \),
\[ \psi(K(D)) \in \Lambda^i N^* \otimes \Lambda^i T^* \]
which vanishes if \( i > q \).

Next I want to show how this method yields the classes \( b^i \). Observe that any bundle stratified with respect to the \( \theta \) foliation admits a de Rham complex
\[ E \to E \otimes \Theta^* \to E \otimes \Lambda^2 \Theta^* \to \ldots \]
which by Poincaré resolves the space of horizontal sections of \( E \), where \( \Theta \subset T \) is the tangent
bundle to the leaves. Applying this to the bundle $\Lambda^j N^*$ of forms $i(X) \omega = 0$ if $X$ is a section of $\Theta$ we obtain a

resolution of $A_j$:

$$0 \rightarrow A_j \rightarrow \Lambda^j N^* \rightarrow \Lambda^j N^* \otimes \Theta^* \rightarrow 0.$$

Now recall that

$$0 \rightarrow N^* \rightarrow T^* \rightarrow \Theta^* \rightarrow 0$$

(possibly should think of this as $Y$ = orbit space

$$0 \rightarrow \tau^* \Omega^1_y \rightarrow \Omega^1_x \rightarrow \Omega^1_{x/y} \rightarrow 0$$

leads to a filtration of $\Lambda^* T^*$

$$\text{Filt}_p \Lambda^* T^* = \Lambda^p N^* . \Lambda^{n-p} T^*$$

(the $p$th power of the kernel of $\Lambda T^* \rightarrow \Lambda \Theta^*$) by subcomplexes such that

$$q_p \Lambda^n T^* = \Lambda^p N^* \otimes \Lambda^{n-p} \Theta^*$$

is the above-mentioned resolution of $A^p$. It follows
that the inclusion
\[(0 \to a^{p} \to a^{p+1} \to \cdots) \to \text{Filt}_{p}(\Lambda^* T^*)\]
is a quasi-isomorphism. Thus Bott's curvature procedure does indeed define classes
\[b_{i}(E) \in H^{2i}(X, 0: 0 - a^{p} - a^{p+1}).\]

Alternative approach using projective bundle.

Regard \(X\) as a ringed space with \(\mathcal{O}_{X} = a\), so that \(E\) is a vector bundle over this ringed space. Let \(Y\) be the orbit "space" of the foliation, or better take \(Y\) to be the ringed space \(X\) with the constant sheaf \(\mathcal{O}\) for its functions. Then \(\mathcal{A} = \mathcal{O}^{-1}Y\). Now let \(\Pi : \mathcal{P}E \to X\). (Perhaps it might be the same to consider the relative scheme \(\mathcal{P}E\).

To compute cohomology we need
\[R^{k}_{\Pi_{X}}(\Omega^{p}_{\mathcal{P}E/X}) = \begin{cases} 0 & \mathcal{P} \neq \mathcal{P} \\ \mathcal{O}_{X} & \mathcal{P} = \mathcal{P} \end{cases}\]
where \( \xi \in H^1(\mathcal{PE}, \Omega^1_{\mathcal{PE}/Y, c}) \) is the Chern class of \( O(-1) \). This formula is well-known in the relative schemes and should be OKAY in a mixed holomorphic-C^\infty situation.

Granted this, the exact sequence

\[
0 \rightarrow \pi^* \Omega^1_{X/Y} \rightarrow \Omega^1_{\mathcal{PE}/Y} \rightarrow \Omega^1_{\mathcal{PE}/X} \rightarrow 0
\]

leads to a filtration with

\[
\text{Filt}_p \Omega^m_{\mathcal{PE}/Y} = \pi^* \Omega^p_{X/Y} \wedge \Omega^{m-p}_{\mathcal{PE}/X}
\]

\[
g^p_p \Omega^m_{\mathcal{PE}/Y} = \pi^* \Omega^p_{X/Y} \wedge \Omega^{m-p}_{\mathcal{PE}/X}
\]

\[
R^6 \pi^* (g^p_p \Omega^m_{\mathcal{PE}/Y}) = \begin{cases} 
\Omega^p_{X/Y} & \text{if } q = m-p < n \\
0 & \text{else}
\end{cases}
\]

from which one obtains the formula

\[
R^4 \pi^* (\Omega^m_{\mathcal{PE}/Y}) \cong \Omega^{m-j}_{X/Y} 0 \leq j < n-1
\]

where \( m = d + \mathbb{E} \).

Using Leray s.g. for \( \pi^* \)

\[
E_2^{pq} = H^p(X, R^q \pi^* \Omega^m_{\mathcal{PE}/Y}) \Rightarrow H^{p+q}(\mathcal{PE}, \Omega^m_{\mathcal{PE}/Y})
\]

\[
\cong \begin{cases} 
\Omega^p_{X/Y} & p < n \\
0 & p \geq n
\end{cases}
\]

\[
H^p(X, \Omega^{m-q}_{X/Y})
\]

so the spectral sequence degenerates showing that
\[ H^*(\mathcal{P}E, \Omega^m_{\mathcal{P}E/Y}) \cong \bigoplus_{\delta = 0}^{n-1} H^*_{\delta}(X, \Omega^m_{X/Y}) \]

\[ H^*(\mathcal{P}E, \Omega^m_{\mathcal{P}E/Y} \to \Omega^{m+1}_{\mathcal{P}E/Y} \to \cdots) \cong \bigoplus_{\delta = 0}^{n-1} H^*_{\delta}(X, \Omega^m_{X/Y} \to \Omega^{m+1}_{X/Y} \to \cdots) \]

Therefore if we take \( m = \dim(E) \) we have unique classes \( b_i \in H^i(X, \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} \to \cdots) \) such that

\[ \xi^n = \pi^* b_1 \xi^{n-1} + \cdots + (-1)^n \pi^* b_n = 0 \]

This defines the refined Chern classes as desired.

\[ \mathbb{R}_{\pi^*}(\Omega^m_{\mathcal{P}E/Y}) = \bigoplus_{\delta = 0}^{n-1} \Omega^m_{X/Y} \]

\[ \mathbb{R}_{\pi^*}(F_m \cdot \Omega^m_{\mathcal{P}E/Y}) = \bigoplus_{\delta = 0}^{n-1} F_m \cdot \Omega^m_{X/Y} \]

where we use the notation that \( M[1] \) shifts \( M \) to the left so that

\[ H^i(X, M[1]) = H^{i+1}(X, M) \]
December 5, 1971: Odd classes.

So far given a complex bundle $E$ over $X$ stratified with respect to a foliation of codim $g$, we have two kinds of Chern classes:

$$H^{2i}(X, C) \ni b_i(\xi)$$

$$c_i(E) \in H^{2i}(X, \mathbb{Z}) \to H^{2i}(X, \mathbb{C})$$

What I want to do is fill in the squares. Let a complex of sheaves on $X$ be defined as pull-back

$$W^{(i)} \to F_i a^* = (\cdots \to a^i \to a^{i+1} \to \cdots)$$

Thus a map of a complex $Q$ into $W^{(i)}$ is a triple $(u, v, h)$ where

$$u: \bigcirc \to \mathbb{Z}[o]$$

$$v: \bigcirc \to F_i a^*$$

and $h$ is a homotopy joining $u, v$ as maps from $W^{(i)}$ to $a^*$. Thinking of $u, v$ and $h$ as elements of the function complex we have

$$du = dv = 0 \quad dh = u - v \quad \text{in } \text{Hom}^*(Q, W^{(i)})$$
There are product maps
\[ W^i \otimes W^j \longrightarrow W^{i+j} \]
defined as follows. Suppose given maps
\[ Q' \longrightarrow \mathbb{Z}[\theta] \quad Q' \longrightarrow F_j A' \]
\[ Q'' \longrightarrow \mathbb{Z}[\theta] \quad Q'' \longrightarrow F_j A' \]
and
\[ dh' = u' - v' \quad dh'' = u'' - v'' \]
Denote by \( u' \cdot u'' \) and \( v' \cdot v'' \) the maps
\[ Q' \otimes_{\mathbb{Z}} Q'' \longrightarrow \mathbb{Z}[\theta] \otimes \mathbb{Z}[\theta] \longrightarrow \mathbb{Z}[\theta] \]
\[ Q' \otimes_{\mathbb{Z}} Q'' \longrightarrow F_j A' \otimes F_j A' \longrightarrow F_{i+j} A' \]
the latter maps being the cokernel products. Then as the evident map
\[ \text{Hom}^*(K, L) \otimes \text{Hom}^*(A, B) \longrightarrow \text{Hom}^*(K \otimes A, L \otimes B) \]
is a map of complexes
\[ u' \cdot u'' - v' \cdot v'' = (u' - v') \cdot u'' + v' (u'' - v'') \]
\[ = dh' \cdot u'' + v' \cdot dh'' \]
\[ = d (h' \cdot u'' + v' \cdot h'') \]
so \( (u' \cdot u'', v' \cdot v'', h' \cdot u'' + v' \cdot h'') \) defines a map
\[ Q' \otimes_{\mathbb{Z}} Q'' \longrightarrow W^{i+j} \]
The associativity of this product results from
\[(h'u'' + v'h') u''' + v'v''h'' = h'u''u''' + v'v''(h''u'' + v''h'').\]

Commutativity up to homotopy results from
\[(u'h'' + h'v'')(h'u'' + v'h'') = dh'. h'' - h'dh' = d(h'h').\]

It follows that
\[\bigoplus_{i \geq 0} H^*(X, W^{(i)})\]
is a graded ring, anti-commutative w.r.t. \(\ast\).

Clearly \(W^{(i)}\) is quasi-isomorphic with the complex
\[\mathbb{Z} \longrightarrow a^0 \longrightarrow a^1 \longrightarrow \cdots \longrightarrow a^{i-1} \longrightarrow 0\]

although its product structure is not clear from this description. In virtue of Poincaré's lemma
\[H^i(W^{(i)}) = \begin{cases} a^i_{cl} & i = 0 \\ C^* & i = 1 \\ 0 & i \neq 1, 0 \end{cases}\]

for \(i \geq 2\), while
\[W^{(1)} \sim \hat{A}^*[1] , \quad W^{(0)} \sim \mathbb{Z} [0].\]
What I want to prove is that for a foliated bundle the $i$-th Chern class may be defined in $H^{2i}(X, W^{(i)})$, this being clear for line bundles, in fact, the group $H^2(X, W^{(i)})$ is precisely the group of iso. classes of foliated line bundles.

The case of flat bundles: Here $W^{(i)} = \mathbb{C}^*[i]$ for all $i > 1$, so we want classes in $H^{2i+1}(X, \mathbb{C}^*)$. But if $\xi \in H^i(\mathcal{E}(E), W_{\mathcal{E}(E)}^{(i)})$ (where $W_{\mathcal{E}(E)}^{(i)}$ is the first Chern class of $\mathcal{E}(E)$), then $\xi^n \in H^{2n}(\mathcal{E}(E), W_{\mathcal{E}(E)}^{(n)}) = H^{2n-1}(\mathcal{E}(E), \mathbb{C}^*)$.

Where $n = \text{dim } E$. Thus we get the desired classes.

Assume for the moment that the usual formal properties of these Chern classes holds. Thus for any flat bundle $E$ over $X$ we have Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z} \to \mathbb{C})$ satisfying a product formula where the product comes from the pairing of complexes:

$$(\mathbb{Z} \to \mathbb{C}) \otimes (\mathbb{Z} \to \mathbb{C}) \longrightarrow (\mathbb{Z} \to \mathbb{C})$$

$$(a + b) \otimes (a' + b') \longrightarrow (aa', ab')$$

More precisely, an element of $H^2(X, \mathbb{Z} \to \mathbb{C})$ is represented by a pair $(f, \lambda)$ where $f$ is an integral $i$-cocycle.
and \( \lambda \) is a complex \((n-1)\)-cochain such that \( f = d\lambda \). The product is
\[
(f, \lambda) \cdot (g, \mu) = (fg, f\mu)
\]
(Note that \((-1)^{\text{deg } f} f\mu = \lambda g = (+d(\lambda\mu))(-1)^{\text{deg } f}\) so the arbitrary choice doesn't matter. Better to define the pairing using \(ba^t\) so that
\[
(f, \lambda)(g, \mu) = (fg, \lambda g).
\]
This amounts to defining the product
\[
\begin{array}{ccc}
H^{i-1}(-, \mathbb{C}^*) \otimes H^{j-1}(-, \mathbb{C}^*) & \longrightarrow & H^{i+j-1}(-, \mathbb{C}^*) \\
\text{ind} \otimes \delta & \downarrow \text{ind} \otimes \delta & \text{usual cop} \\
H^i(-, \mathbb{C}) \otimes H^j(-, \mathbb{Z}) & \longrightarrow & \text{usual cop}
\end{array}
\]
Note that the class \( \delta(\cdot) \in H^n(X, \mathbb{Z}) \) is a torsion class as it dies "canonically in \( C \). This means if you write \( C^* = \exp(2\pi i \mathbb{Q}/\mathbb{Z}) \oplus Q\text{-vector space} \), then the \( \mathbb{Q}\text{-vector space} \) part has zero products.
Construction of the refined Chern classes. Let $X$ be a manifold endowed with a quasi-foliation (e.g., a foliation or a complex structure) and let $\Omega^*$ be the de Rham complex of complex-valued $C^\infty$-functions which are locally constant along the leaves. Let $E$ be a complex vector bundle stratified with respect to the foliation. Then we form $\mathcal{P}E$ and endow it with the de Rham complex $\Omega^*_\mathcal{P}E$ of $C^\infty$ complex-valued forms which are holomorphic vertically and constant horizontally.

Let $\pi: \mathcal{P}E \to X$ be the projection so that we have

$$0 \to \pi^* \Omega^*_X \to \Omega^*_\mathcal{P}E \to \Omega^*_\mathcal{P}E/X \to 0$$

Then one has isomorphisms in $\mathcal{D}(\mathcal{P}E)$

$$\bigoplus_{g=0}^{n-1} \Omega^{m-g}_X \cong \mathbb{R}_\pi^* (\Omega^m_{\mathcal{P}E})$$

We the isomorphism is defined as follows. The first Chern class of $\mathcal{O}(1)$ defines a class $\xi \in H^2(\mathcal{P}E)_W^{\mathcal{O}}$ which may be viewed as a map

$$\mathbb{Z} \to \Omega^1_{\mathcal{P}E}$$

$\xi$ may be viewed as a map

$$\mathbb{Z} \to \Omega^m_{\mathcal{P}E}$$

So taking product with $\Omega^m_{\mathcal{P}E}$ we have a map

$$\Omega^{m-6}_{\mathcal{P}E} \to \Omega^m_{\mathcal{P}E}$$
Composing with the map

\[ \pi^* \Omega^m \mathcal{O}_X \rightarrow \Omega^m \mathcal{O}_{PE} \]

we obtain a map whose adjoint is the map

\[ \Omega^m \mathcal{O}_X \rightarrow R_{\pi*}(\Omega^m \mathcal{O}_{PE}) \]

we want.

By filtering we obtain an isomorphism

\[ \bigoplus_{g=0}^{n-1} F_{m-g} \Omega^r \mathcal{O}_X \cong R_{\pi*}(\mathcal{O}_{PE}) \]

One knows that

\[ R_{\pi*}(\mathcal{O}_X^*) \cong \bigoplus_{g=0}^{n-1} \mathcal{O}_X^* \]

so using the triangle

\[ W^{(m)} \rightarrow F_m \Omega^r \rightarrow \mathcal{O}_X^* \]

on both PE and X, we see by "5 lemma" that

\[ R_{\pi*}(W^{(m)}_{PE}) \cong \bigoplus_{g=0}^{n-1} W^{(m-g)}_X \mathcal{O}_X^* \]

This shows that the Deligne-Griffiths cohomology satisfies the projective bundle theorem permitting definition of Chern classes as usual.
December 12, 1971:

odd classes in p-adic case.

Suppose given morphisms

$$X_0 \rightarrow X \rightarrow S$$

where $X_0 \rightarrow X$ is an immersion defined by an ideal $I$ with $X$ divided powers. If $L$ is a line bundle on $X$, we can consider its first Chern class in the Deligne cohomology of $X$ rel $S$:

$$c_1(L) \in H^2(X, F_1 \Omega^*_X \otimes L).$$

Recall this is defined as the image of $\text{cl}(L) \in H^1(X, \Omega^*_X)$ under the isomorphism

$$\Omega^*_X[-1] \rightarrow F_1 \Omega^*_X \otimes L$$

$$f \mapsto \frac{df}{f}.$$

Now we have a commutative diagram

$$0 \rightarrow (1+I)^x \rightarrow \Omega^*_X \rightarrow \Omega^*_X \rightarrow 0$$

with an exact top row. Now suppose given
two line bundles $L, L'$ on $X$ and an isomorphism
$L_0 \cong L'_0$ over $X_0$. Then one obtains an element
of
$H^1(X, 1 + E)$
which goes into $c(L) - c(L')$ in $H^1(X, \mathcal{O}_X^*)$, and
$c_1(L) - c_1(L') \in H^2(X, F_1 \mathcal{L}_{X/1S}^*)$
comes from an element of
$H^4(X, \mathcal{I})$.

Let $V^{(m)}$ be the complex defined by

$$
\begin{array}{c}
V^{(m)} \\
\downarrow \\
o \\
\downarrow \\
0 \rightarrow (I_1^{[m]} \rightarrow I_1^{[m-1]} \rightarrow \ldots \rightarrow I_1 \rightarrow I_1^{[1]} \rightarrow \ldots)
\end{array}
$$

so that $V^{(1)}$ is quasi-isomorphic to $I_1[1]$. I have associated to $(L, L', L_0 \cong L'_0)$ an element of
$H^2(X, V^{(1)})$.

I would like to associate to $(E, E', E_0 \cong E'_0)$
Chern classes in
$H^{2m}(X, V^{(m)})$,
and more generally do this for any K-element over
$X$ with a trivialization
over $X_0$. 
Given $E$ over $X$, form $\pi : PE \to X$ which is smooth so

$$0 \to \pi^* \Omega^1_{X/S} \to \Omega^1_{P/S} \to \Omega^1_{P/X} \to 0$$

will be exact and locally split. The same is true of

$$0 \to F_\pi \Omega^m_{P/S} \to F_{P+1} \Omega^m_{P/S} \to \pi^* \Omega^p_{X/S} \otimes \Omega^{m-p}_{P/X} \to 0.$$

As $O_P$ is flat over $O_X$ we have

$$\left(\pi^* \alpha \right) \otimes F = \alpha \cdot F$$

for any ideal $\alpha \subset O_X$. Thus

$$\pi^* \alpha \otimes F_\pi \Omega^m_{P/S} = \alpha \cdot F_\pi \Omega^m_{P/S}$$

and

$$\alpha \cdot F_\pi \Omega^m_{P/S} / \alpha \cdot F_{P+1} \Omega^m_{P/S} = \pi^* (\alpha \Omega^p_{X/S}) \otimes \Omega^{m-p}_{P/X}$$

On the other hand one checks that

$$R^j \pi_* (\pi^* \Omega^b_{P/X}) = \begin{cases} 0 & j \neq 0 \\ M & j = 0 < n \end{cases}$$

by descending induction on $j$. Consequently

$$R^j \pi_* (\alpha \cdot \Omega^m_{P/S}) = \bigoplus_{q=0}^{n-1} \alpha \cdot \Omega^{m-q}_{X/S} \text{ Eq. 1}$$
where the isomorphism comes from multiplication by the
powers of the canonical section of
\[ R^1 \text{Tor}_X(\Omega^1_{\mathcal{L}/\mathcal{P}_S}) \] [1].

Consequently multiplying by the canonical section of
\[ R^1 \text{Tor}_X(F_{\mathcal{L}/\mathcal{P}_S}) \] [2]
will give
\[ \bigoplus_{g=0}^{n-1} (I^{m-g} \mathcal{O}_X \rightarrow I^{m-g-1} \Omega^1_{\mathcal{L}/\mathcal{P}_S} \rightarrow \cdots) [-2g] \]
and so we obtain the projective bundle theorem.

So now suppose that in addition to \( E \) we give an \( E' \) over \( X \) and an isomorphism \( E_0 \cong E'_0 \) over \( X_0 \). Then I want to produce elements \( \lambda_m \in H^{2m}(X, V^{(m)}) \) with image \( c_m(E) - c_m(E') \in H^{2m}(X, \mathcal{L}_{X/S}) \). The point is that these classes will coincide in \( H^{2m}(X, \mathcal{L}_{X/S}) \) because their classes are "crystalline"; hence independent of the lifting from \( X_0 \) to \( X \).
December 16, 1971

Let $G$ be a finite group and let $X(G)$ be the following simplicial complex. Its vertices are the $p$-subgroups of $G$ and a simplex is a chain

$$P_0 < ... < P_k$$

such that $P_k / P_0$ is elementary $p$-abelian. Claim $X(G)$ is contractible.

This is clear by the old retraction argument if $G$ is a $p$-group. In general

$$X(G) = \bigcup X(P)$$

where $P$ runs over the Sylow subgroups and

$$X(P_0) \cap ... \cap X(P_k) = X(P_0 \cap ... \cap P_k)$$

is contractible. Thus by the lemma in Folksman's paper $X(G)$ has the homotopy type as the simplex on the Sylow subgroups, which is contractible.
December 16, 1971:

Have defined canonical elements
\[ e_i \in H^{2i-1}(\text{GL}_n(\mathbb{C}), \mathbb{C}^*) \]

Suppose now that I am given a continuous family of representations of \( \Gamma \), i.e. a vector bundle \( E \) over a manifold \( X \) and a \( \Gamma \)-action on the fibres. (everything \( C^\infty \)). Then for each \( x \) we have a homomorphism

\[ H^{2i-1}(\text{GL}_n(\mathbb{C}), \mathbb{C}^*) \rightarrow H^{2i-1}(\Gamma, \mathbb{C}^*) \]

obtained from the representation of \( \Gamma \) on \( E_x \). Denote by \( e_i(E_x) \) the image of \( e_i \).

**Theorem:** \( e_i(E_x) \) is a locally constant function of \( x \) for \( i \geq 2 \), but not necessarily for \( i = 1 \).

For \( i = 1 \), take \( \Gamma = \mathbb{Z}, \quad x = \mathbb{C}^* \), \( E \) = trivial line bundle over \( X \). Let \( \Gamma \) act on \( E_x \) by \( n(w) = x^n \cdot w \). Then \( e_1(E_x) \in H^1(\Gamma, \mathbb{C}^*) = \text{Hom}(\Gamma, \mathbb{C}^*) \) is this homomorphism, and it varies with \( x \).

We can suppose that \( X \cong \mathbb{R} \). From the projective bundle of \( E \), \( \pi : PE \rightarrow X \), and let \( \mathbb{C}^* \) be the de Rham complex of forms which are \( \Gamma \)-invariant along the fibres. Again we find \( W^{(1)} \).

We can clearly suppose \( X = \mathbb{R} \). There is
a Chern class defined in
\[ c_i \in H^{2i}_\Gamma(X, W^c_X) \]
which restricts to the classes \( c_i(\mathcal{E}_X) \). Since \( \dim X = 1 \)
\[ W^c_X = \mathbb{C}^*[-1] \] for \( i \geq 2 \). Thus
\[ H^{2i}_\Gamma(X, W^c_X) = H^{2i-1}_\Gamma(X, \mathbb{C}^*) = H^{2i-1}_\Gamma(pt, \mathbb{C}^*) \]
so it's all clear.
December 24, 1971. Summary: "Functional equation for $f$",

I. Complete non-singular curve $X$ over $\mathbb{F}_p$

\[ f(s) = \prod_{x \text{ closed point}} \frac{1}{1 - Nx^{-s}} \quad \text{where} \quad Nx = \text{card } k(x) = q^{\deg(x)} \]

\[ = \sum_{D \geq 0} \left( q^{\deg D} \right)^{-s} \]

Rewrite as a sum over Pic($X$) using that the number of $D \geq 0$ with $L(D) \cong L$ is

\[ \frac{q^{h^0(L)} - 1}{\mu} \]

\[ h^0(L) = \dim_{\mathbb{F}_p} H^0(X, L) \]

\[ \mu = \text{card } H^0(X, O_X) = g - 1 \quad \text{if} \quad F_p = H^0(X, O_X) \]

\[ f(s) = \sum_{[L] \in \text{Pic } X} (q^{\deg L})^{-s} \cdot \frac{q^{h^0(L)} - 1}{\mu} \]

To simplify, suppose $F_p = H^0(X, O_X)$ and assume known that $f$ an $L$ of degree 1 (according to Artin-Tate notes this follows from RR). Put $z = q^{-s}$ and

\[ Z(z) = f(s) \]

Using the Riemann formula

\[ h^0(L) = \deg L + 1 - g \]

for $(\deg L) > 2g - 2$ we have
\[ Z(z) = \sum_{\text{deg } L \leq 2g-2} z^{\text{deg } L} \frac{g^{h_0(L)}}{\mu} - 1 + \sum_{\text{deg } L > 2g-2} z^{\text{deg } L} \frac{g^{\text{deg } L + 1 - g}}{\mu} \]

This shows that

\[ Z(z) = \frac{P(z)}{(1-z)(1-qz)} \]

where \( P(z) \) is a polynomial with integral coefficients of degree \( 2g \). Moreover,

\[ \text{Res}_{z=1} Z(z) = \frac{h}{\mu} \quad \text{Res}_{z=\frac{1}{b}} Z(z) = -\frac{h}{\mu b} \]

where \( h = \text{card } \text{Pic}^0(X) \).

Using now the analytic continuation formulae

\[ \sum_{n \geq N} \omega^n = \frac{\omega^N}{1-\omega} = -\frac{\omega^{-N-1}}{1-\omega^{-1}} = -\sum_{n < N} \omega^{-n} \]

\[ \text{const. } |\omega| < 1 \]

\[ \text{const. } |\omega| > 1 \]

we see the second term analytically continues to

\[ -\sum_{\text{deg } L \leq 2g-2} z^{\text{deg } L} \frac{g^{\text{deg } L + 1 - g}}{\mu} \quad |z| > 1 \]

so

\[ Z(z) = \sum_{g} z^{\text{deg } L} \frac{g^{h_0(L)} - g^{\text{deg } L + 1 - g}}{\mu} \]

Now use Riemann-Roch

\[ h_0(L) = \text{deg } L + 1 - g + h_0(K - L) \]
\[ Z(z) = \sum_{L} (qz)^{d_g L} \cdot \frac{q^{h_0(K-L)}}{\mu} - 1 \cdot q^{1-g} \]

So changing \( L \) to \( K-L \) and using \( d_g(K) = 2g-2 \)

we find

\[ Z(z) = q^{g-1} z^{2g-2} \frac{Z(\frac{L}{qz})}{qz} \]

\[ \text{(II). Number fields.} \]

Poisson sum formula: Let \( N \) be a lattice in a vector space \( V \) f.d. over \( \mathbb{R} \), let \( M^* \) be the "dual" lattice in \( V^* \). If \( f \) is a rapidly decreasing fn. on \( V \)

\[ \sum_{n \in N} f(x+n) = \sum_{m \in M^*} e^{2\pi i \langle m, x \rangle} \int_{V} e^{-2\pi i \langle m, x \rangle} f(x) dx \]

where \( dx \) is the Haar measure on \( V \), normalized so that \( V/N \) has measure 1. (This is just Fourier exp. of \( \sum f(x+n) \).)

Special case: take \( f(x) = e^{-\pi Q(x)} \) where \( Q \) is a positive-definite quad. form on \( V \).

\[ \sum_{n \in N} e^{-\pi Q(n)} = \frac{1}{a} \sum_{m \in M^*} e^{-\pi Q(m^*)} \]

where \( a = \text{volume of} \ V/N \) with respect to \( Q \)

\( m^* \) is determined by

\[ \forall x \in V \quad \langle m^*, x \rangle = B(m^*, x) \]

\( B \) bilinear form

\( Q(x) = B(m^*, x) \)
Divisors. Let $k$ be a number field of degree $N = r_1 + 2r_2$. By a divisor of $k$, $\sigma_0$ or completed at $\infty$, I will mean a pair $D = (\sigma, t_{\sigma})$ where $t_{\sigma}$ is a divisor for $\sigma$ (i.e. a fractional ideal) and where $t_{\sigma} \in R^+$ for each $\sigma \neq \infty$. Thus

$$\text{Div}(k) = \text{Div}(0) \times (R^+)^{r_1 + r_2}.$$ 

Let

$$d(\sigma, t_{\sigma}) = \text{N} \sigma \cdot (TT_{\sigma})^{-1}$$

where $N_\sigma = 1$ if $\sigma$ real, $N_\sigma = 2$ if $\sigma$ complex. Define the divisor of $x \in k^*$ to be

$$(x) = (0, x, \{|x|_\sigma\})$$

so that $d(x) = 1$. Let

$$P = \text{Div}(k)/\text{Im} k^*$$

so that

$$\xrightarrow{\cdot} P^0 \xrightarrow{d} P \xrightarrow{d} R^+ \xrightarrow{\cdot} 0.$$

where

$$\xrightarrow{\cdot} (R^+)^{r_1 + r_2} \xrightarrow{d} P^0 \xrightarrow{d} \text{Pic}(0) \xrightarrow{d} 0.$$

Given $D = (\sigma, t_{\sigma})$ with image $L \in P$, set

$$\Theta_L = \sum_{\chi \in \sigma^{-1}} e^{-\pi \Sigma t_{\sigma}^2 |x|_\sigma^2}$$

This is well-defined since

$$\sum_{\chi \in \sigma^{-1}} e^{-\pi \Sigma (t_{\sigma} |x|_\sigma)^2} = \sum_{\Sigma \in \sigma^{-1}} e^{-\pi \Sigma t_{\sigma}^2 |x|_\sigma^2}$$
Let \( \mathcal{O} \) be the different of \( k \), defined by

\[
\mathcal{O}^{-1} = \{ x \in k \mid \text{Tr}(x \mathcal{O}) \in \mathbb{Z} \}
\]

and \( K \) be the class of the canonical divisor

\[
(\mathcal{O}, \{N_{\varphi}\})
\]

so that

\[
d(K) = 2^{-r_2} d_k
\]

where \( d_k \) is the absolute value of the discriminant of \( k \).

Riemann-Roch formula:

\[
\Theta_L = (2^{-r_2} d_k)^{-\frac{1}{2}} \cdot dL \cdot \Theta_{K-L}
\]

Proof: This will be a consequence of the Poisson sum formula with \( V = \sigma \otimes \mathbb{R}, \ N = \text{image of } \sigma^{-1} \). We use the form

\[
\langle x, y \rangle = \text{Tr}(xy) = \sum_{\sigma} (\sigma(xy))
\]

\[
= \sum_{i=1}^{r_1} \sigma_i x \cdot \sigma_i y + \sum_{j=1}^{r_2} \tau_j x \cdot \tau_j y + \sum_{j=1}^{r_2} \tau_j y \cdot \tau_j x
\]

to identify \( V \) with its dual. Observe then

\[
M = \{ x \in \sigma \otimes \mathbb{R} \mid \text{Tr}(x \sigma) \in \mathbb{Z} \} = \{ x \in \sigma \mid x \sigma^{-1} \in \mathcal{O}^{-1} \}
\]

\[
= \mathcal{O}^{-1} \sigma = (\sigma \mathcal{O}^{-1})^{-1}.
\]
If 

\[ Q(x) = \sum_{i=1}^{n} t_i \frac{|x_i|}{t_i} \quad x \in \sigma \circ \mathbb{R}, \]

then bilinear form 

\[ B(x, y^*) = \sum_{i=1}^{n} t_i^2 \sigma_i \frac{x_i y_i^*}{t_i} + \sum_{j=1}^{n} \frac{1}{2} t_j^2 (\bar{y}_j x_i \bar{y}_j y_i^* + \bar{x}_j \bar{y}_j y_i^*) \]

do if 

\[ \langle x, y \rangle = B(x, y^*) \quad \text{for all} \quad x \in \sigma \circ \mathbb{R}, \]

we have 

\[ \sigma_i y_i^* = \frac{\sigma_i y_i}{t_i^2} \quad \text{and} \quad \bar{y}_j y_i^* = \frac{\bar{y}_j y_i}{t_j^2} \]

so 

\[ Q(y^*) = \sum_{i=1}^{n} \left( \frac{N_{\sigma_i}}{t_i^2} \right)^2 |y_i|^2 \]

so P.S. gives 

\[ \sum_{x \in \sigma \circ \mathbb{R}} e^{-\pi \sum_{i} \frac{|x_i|^2}{t_i}} \frac{1}{\text{vol}_Q(\sigma^{-1})} \sum_{y \in S^{-1}_{\mathbb{R}}} e^{-\pi \sum_{i} \frac{|y_i|^2}{t_i}} \frac{1}{\text{vol}_Q(\sigma^{-1})} \]

and 

\[ \text{vol}_Q(\sigma^{-1}) = \frac{1}{N_{\sigma}} \text{vol}_Q(\sigma) = \frac{1}{N_{\sigma}} N_{t_{\sigma}}^N \cdot \text{vol}_{Q_0}(0) \]

where \( Q_0(x) = \sum |x_i|^2 \) is the canonical metric on \( \sigma \circ \mathbb{R} \).

See Lang's book p.79 for proof that 

\[ \text{vol}_{Q_0}(0) = 2^{-r_2} d_{\mathbb{R}}^{\frac{1}{2}} \]

Q.E.D.

Recall 

\[ \Gamma(\alpha) = \int_0^{\infty} e^{-t x^2} \frac{dt}{t} = \int_0^{\infty} e^{-\pi \alpha t^2} \frac{dt}{t} \]

Let 

\[ \overline{\Gamma}(\alpha) = \frac{1}{2} \pi^{-\frac{1}{2}} \Gamma(\alpha) = \int_0^{\infty} e^{-\pi \alpha t^2} \frac{t^{\frac{1}{2}} dt}{t} \]

Then
\[ \Gamma(n/2) \Gamma(n/2) \sum_i \gamma_i(a) = \int \int e^{-\pi \sum t_i^2} (\prod_{s_0} N_{s_0})^{1/2} \prod_{t_0} \sum (N_{t_0})^{1/2} (dD)^{-n} \]

\[ D = (\alpha, \{t_i\}) \in \text{Div} \] with \( n > 0 \)

and writing this integral over the map \( \text{Div} \to \mathbb{P} \) we get

\[ \Gamma(n/2) \Gamma(n/2) \sum_i \gamma_i(a) = \int \int (dL)^{-\mu} \frac{\Theta_L - 1}{\mu} \]

\( \leq \mathbb{P} \)

(In this integral, \( \mathbb{P} \) is endowed with the Haar measure induced by that of \( \text{Div} \). \( \mu \) is the order of the group of roots of unity.)

Clearly \( \Theta_L - 1 \to 0 \) rapidly as \( dL \to 0 \)

while by R-R formula

\[ \Theta_L = (dk)^{-1/2} dL = (dk)^{-1/2} dL (\Theta_{K-L} - 1) \]

goes to zero rapidly as \( dL \to \infty \). Thus

\[ \Gamma(n/2) \Gamma(n/2) \sum_i \gamma_i(a) = \int (dL)^{-\mu} \frac{\Theta_L - 1}{\mu} + \int (dL)^{-\mu} \frac{\Theta_L - (dk)^{-1/2} dL}{\mu} \]

\( \leq 1 \)

\( dL \geq 1 \)

\[ + \int (dL)^{-\mu} \frac{(dk)^{-1/2} dL - 1}{\mu} \]

\( dL \geq 1 \)
The first two terms are entire, and the last is

\[ \frac{(dk)^{1/2}}{\mu} \int_{dL \geq 1} (dL)^{-\sigma+1} - \frac{1}{\mu} \int_{dL \geq 1} (dL)^{-\sigma} = \frac{(dk)^{1/2}}{\mu} R \int_{1}^{\infty} \frac{dt}{t^{\sigma+1}} - \frac{R}{\mu} \int_{1}^{\infty} \frac{dt}{t^{\sigma}} \]

where \( R = \int_{1}^{\infty} \frac{dt}{t^{\sigma}} \) being endowed with the measure \( \rho \) compatible with the one on \( P \) and the \( d\ell \). 

\[ R = \text{regulator} \]

\[ = \frac{(dk)^{1/2} R}{\mu} \frac{1}{\sigma-1} - \frac{R}{\mu} \frac{1}{\sigma} \]

Since we have the analytic continuation

\[ \int_{dL \geq 1} (dL)^{-\sigma} \frac{(dk)^{1/2} dL}{\mu} - \int_{dL \leq 1} (dL)^{-\sigma} \frac{(dk)^{1/2} dL}{\mu} \]

we get

\[ \Gamma(a/2)^{\mu} \Gamma(a)^{\mu} / \Gamma(a) = \int (dL)^{-\sigma} \frac{\Theta_L - (dk)^{1/2} dL}{\mu} = (dk)^{1/2} (dL)^{-1-\sigma} \frac{\Theta_L}{\mu} \]

and

\[ \frac{(dk)^{1/2}}{\mu} \Gamma(a/2)^{\mu} \Gamma(a)^{\mu} / \Gamma(a) \frac{\Theta_L}{\mu} \]

satisfies

\[ \Lambda (a) = \Lambda (1-a) \]

(Observe that \( R \) as defined above is \( h \)-classical regulator. \( 2^{-\sigma} \)). In fact

\[ \frac{1}{2^n} \pi^{-n/2} \Gamma (\frac{n}{2})^{\mu} \frac{\Gamma (\frac{1}{2})^{1-\sigma}}{\Gamma (\frac{1}{2} - \sigma)^{\mu}} \gamma (a) \sim \frac{(dk)^{1/2} R}{\mu} \frac{1}{\sigma-1} \]
\[ y(s) \sim \frac{2^{\frac{r_1}{2}}(2\pi)^{\frac{r_2}{2}} R \cdot 2^{\frac{r_2}{2}}}{\mu d_k^{r_2}} \frac{1}{s-1} = \frac{2^{\frac{r_1}{2}}(2\pi)^{\frac{r_2}{2}} \cdot h \cdot R^{\text{class}}}{\mu d_k^{r_2}} \frac{1}{s-1} \]

and for \( a \) near \( 0 \)

\[ \frac{1}{2\pi} \left( \frac{a}{s} \right)^{r_1} \frac{1}{2^{r_2}} a^{r_2} \cdot \int_{h(a)} \sim -\frac{R}{\mu} \frac{1}{s} \]

\[ a^{r_1 + r_2 - 1} \cdot \int_{h(\omega)} \sim -\frac{2^{r_2}R}{\mu} = -\frac{h \cdot R^{\text{class}}}{\mu} \]

For \( Z \)

\[ \overline{y}(s) = \frac{1}{2} \int_{0}^{\infty} (\Theta(t) - 1) \frac{t^s}{t} dt \]

\[ = \frac{1}{2} \int_{0}^{\infty} (\Theta(t) - 1)(t^s + t^{-s}) \frac{dt}{t} = \frac{1}{s} - \frac{1}{1-s} \]

where \( \Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} \)

\[ \Theta(t) = \frac{1}{t} \Theta \left( \frac{1}{t} \right) \]