September 3, 1970

cancellation

Suppose \( P \oplus A \sim Q \oplus A \) and we want to prove \( P \cong Q \). Look at this as follows.
We have a bundle \( E \) and two non-vanishing (everywhere) sections \( s_1 \) and \( s_2 \) with \( E/A_{s_1} = P, \ E/A_{s_2} = Q \). Suppose \( s_1, s_2 \) are everywhere independent, i.e. \( (s_1, s_2) : A^2 \to E \) is a direct injection. Then we have exact sequences

\[
0 \to A \overset{\alpha_1}{\to} E/A_{s_1} \to E/A_{s_1} + A_{s_2} \to 0
\]

\[
0 \to A \overset{\alpha_2}{\to} E/A_{s_2} \to E/A_{s_1} + A_{s_2} \to 0
\]

so

\[ P \cong A \oplus R \cong Q \]

where \( R = E/A_{s_1} + A_{s_2} \). In fact this isomorphism of \( P \) and \( Q \) is unique up to an automorphism of \( P \) inducing the identity on \( A_{s_2} \) and \( R \). (\( \Theta \) is an elementary automorphism of \( P \).)

Thus if \( s_1 \) and \( s_2 \) are connected in the unimodular complex of \( E \), i.e. in the same component, then by choosing a path one obtains an isomorphism of \( E/A_{s_1} \) and \( E/A_{s_2} \) unique up to elementary auto of \( E/A_{s_1} \).
The document contains mathematical expressions and diagrams involving groups and homology groups. Here is a transcription:

\[ \text{GL}_2(\mathbb{F}_2) \text{ begins dim 1} \]
\[ \text{GL}_3(\mathbb{F}_2) \text{ begins dim 2} \]

\[ \Sigma_2 \]

\[ H_2(\Sigma_3) \leq H_2(\Sigma_4) \]
\[ H_n(\Sigma_{2n-1}) < H_n(\Sigma_{2n}) \]

Fibre has dimension \( n-1 \).

\[ \text{GL}_4(\mathbb{F}_2) \]

\[ \Sigma_{15} \]

Regular repn of \( A \)

Thus have group acting on \( \mathbb{P}^3 \)

and have repn on \( k^3 \)

\[ H^4(\Sigma_{15}) \]
\[ H^5(\Sigma_{31}) \]
\[ H^6(\Sigma_{63}) \]

Class degree 4.

\[ 2 \]
\[ \text{GL}_{n-1} \cdot \text{GL}_n \]

\[ 3 \cdot 4 \]

\[ 2(n-1) \text{ GL}_{n-1} \text{ GL}_n \]

\[ 2n-1 \]

$E$, projective $A$-module of rank larger than the dimension of the maximum spectrum of $A$. To show $E$ has an everywhere non-vanishing section.

$\text{rank } E \geq 1$. Let $X = \bigcup X_i$ be the irreducible components and choose $x_i \in X_i \setminus \bigcup X_j$. Then by Chinese remainder theorem at $x_i$, can find a section $s$ of $E$ with prescribed values at $x_i$. Since $E(x_i) \neq 0$ for all $i$, can find $s \equiv s(x_i) \neq 0$ for all $i$. Then the dependency set $D(s)$ is closed in $X$ and doesn't contain any $x_i$, hence is of codim $\geq 1$.

$\text{rank } E \geq 2$. Let $s_1$ be such that $D(s_1)$ has codim $\geq 1$. Choose $s_2$ independent of $s_1$ at the points $x_i$ and at a similarly selected set of points in $D(s_1)$. Then $D(s_1, s_2)$ has codim $\geq 1$. Choose $g$ to be non-zero on the irreducible component of $D(s_1)$ and to be zero-somewhere on the irreducible component of $D(s_1, s_2)$ not in $D(s_1)$. Then $D(s_1 + gs_2)$ has codim $\geq 2$. Indel OKAY over $D(s_1, s_2)$ and on $D(s_1, s_2) - D(s_1)$ OKAY by vanishing of $g$, while in $D(s_1)$ OKAY as $g, s_2 \neq 0$.

$\text{rank } E \geq 3$. Suppose $s_1, s_2$ chosen so that $D(s_1, s_2)$ has codim $\geq 1$, $D(s_1)$ codim $\geq 2$ (possible by preceding). Choose $s_3$ ind of $(s_1, s_2)$ at some interior point of the irreducible components of $X$ ind of $(s_1, s_2)$ at some interior point of each irreducible component of $D(s_1, s_2)$.
and indep of $s_1$ at some interior point of each

irred component of $D(s_1)$. Choose $g$ to be
gen. non-zero in $D(s_1, s_2)$ and $D(s_1)$ and
zero at some interior point of irred. components of

$D(s_1, s_2, s_3)$ outside of $D(s_1, s_2)$. Then

\[ \text{cod } D(s_1, s_2 + g s_3) \geq 2. \]

and $s_1, s_2 + g s_3$ never vanish simultaneously, because

$s_1 = 0 \Rightarrow g s_3 \neq 0$ and $s_2$.

So can assume $D(s_1, s_2)$ codim $> 2$ and

$s_1, s_2 \neq 0$ never simultaneously vanish. Now-
choose $g = 0$ at "gen" points of $D(s_1, s_2)$ where $s_1 \neq 0$
and $g = 1$ at $s_1 + g s_2$. Thus

\[ \text{cod } D(s_1 + g s_2) \geq 3. \]
September 5, 1971:

Recall that a map is called acyclic if its homotopy-theoretic fibres are acyclic, hence acyclic maps are closed under composition and base change. Moreover, given

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \]

then \( gf, f \) acyclic \( \Rightarrow \ g \) acyclic.

Question: In the homotopy category, acyclic maps permit calculus by right fractions. No:

So we want to consider all maps \( Y \rightarrow X \) with \( X \) fixed. For the equalization axiom we would need to know that \( Y \rightarrow Y \times_X Y \) was acyclic, however, when \( X = \text{pt} \) this isn't the case as the map in fundamental groups is not surjective.
Killing perfect subgroup of fundamental group

Pointed spaces $[X,Y]$ homotopy classes of basepoint-preserving maps

$X$ (pointed) CW complex

$\pi_0 X = 0$

$\pi_1 X$ perfect $H_1 X = 0$

$\bigvee_{S^1 \times I} f = \Sigma_i f_i$ $\quad \rightarrow \quad X \quad \rightarrow \quad X'$

Assume class $f_i$ generate $\pi_1 X$

Van Kampen $\Rightarrow \quad \pi_1 X' = 0$

$H_3 X \rightarrow H_3 X'$

$0 \rightarrow H_2 X \rightarrow H_2 X'$

$\oplus \mathbb{Z} \rightarrow 0$

$H_6 X \overset{\sim}{\rightarrow} H_6 X'$ if $g \neq 2$

and

$0 \rightarrow H_2 X \rightarrow H_2 X' \rightarrow \oplus \mathbb{Z} \rightarrow 0$

Choose a splitting and maps $g_i : S^2 \rightarrow X'$ with $\partial(\text{class } g_i) = e_i$
\[ \sqrt{S^2} \xrightarrow{I} X' \xrightarrow{I} X'' \]

van Kampen \Rightarrow \Pi_1 X'' = 0

\[ 0 \xrightarrow{I} H_3 X' \xrightarrow{I} H_3 X'' \]

\[ \oplus \mathbb{Z} \xrightarrow{I} H_2 X' \xrightarrow{I} H_2 X'' \]

Thus \[ H_2 X' = \text{Im} H_2 X + \text{Im} \lambda \]

and \[ \text{Ker} \{ H_2 X' \to H_2 X'' \} = \text{Im} \lambda \]

\[ H_2 X \xrightarrow{\sim} H_2 X' \quad \text{and we obtain} \]

**Proposition 1.** If \( \Pi_0 X = 0 \), \( \Pi_1 X \) perfect, then there is an embedding \( i : X \to X^+ \)

with \( X^+ \) obtained by attaching \( 2+3 \) cells, such that \( \Pi_1(X^+) = 0 \) and

\[ H_* X \xrightarrow{\sim} H_* X^+ \]

**Proposition 2.** (Universal property of \( X^+ \)) Given \( f \\
X \xrightarrow{f} Y \)

There is a unique \( g : X^+ \to Y \) such that \( \pi_1(f) = 0 \) and \( \pi_1(g) = 0 \). Moreover \( g \)

unique up to homotopy.
Remark A: \[ X^+ = \mathbb{Z}_\infty (X) \quad \text{because} \]
\[
\begin{array}{c}
X \\
\downarrow \\
X^+ \\
\cong \\
\pi_1 X^+ = 0
\end{array}
\xrightarrow{\pi_0} \begin{array}{c}
\mathbb{Z}_\infty (X) \\
\cong \\
\mathbb{Z}_\infty (X^+)
\end{array}
\]
(ask Ken why \( X \rightarrow \mathbb{Z}_\infty (X) \) induces ses. on homology.)

Remark B: Let \( AX = \text{Fibre of } X \rightarrow X^+ \)
\[ (*) \quad AX \rightarrow X \rightarrow X^+ \]
\[
\begin{array}{c}
\pi_0 AX = 0 \\
\pi_1 AX = \pi_1 X
\end{array}
\]

spectral sequence
\[
E^2 = H_p (X^+, H_q AX) \Rightarrow H_{p+q} (X)
\]
\[
H_p X^+ \otimes H_q AX \quad \text{if field coeffs.}
\]

one sees from s.c. that \( H_1 (AX, k) = 0 \) all fields \( k \)
hence \( AX \) is acyclic.

\[
AX \xrightarrow{\hat{\imath}} X \rightarrow \text{Con}(j) \xrightarrow{+} X^+
\]
\[
\pi_1 \text{Con}(j) = 0 \quad \text{van Kampen}
\]
\[
H_* \text{Con}(j) \cong H_* X^+ \Rightarrow \text{Con}(j) \cong X^+.
\]
Suppose $E \subset \pi_1 X$ is a perfect subgroup. Let $p: X' \to X$ be the covering space with $\pi_1 X = E$.

Thus $X^+$ obtained by attaching 2+3 cells to $X$.

van Kampen $\Rightarrow$ $\pi_1 X^+ = \pi_1 X / N$ where $N$ is the normal subgroup generated by $E$.

If $L$ is any $\pi_1 X$-module, it defines local coefficient systems on the above four spaces and have

$$
\begin{array}{c}
\cdots \to H_*(X', L) \xrightarrow{i'_*} H_*(X'^+ L) \to H_*(X'^+, x'; L) \\
\to \cdots \to H_*(X, L) \to H_*(X^+ L) \to H_*(X^+, x; L) \\
\end{array}
$$

$L_*$ is an iso, hence

$$i_*: H_*(X, L) \xrightarrow{\sim} H_*(X^+, L).$$
for all $\pi_1 X^+$-modules L.

**Proposition 1**: Let $E \in \pi_1 X$ be perfect and $N$ the normal subgroup gen. by $E$. Then

1. embedding $i : X \to X^+$ such that
2. $\pi_1 X^+ = \pi_1 X/N$
3. $H_*(X, L) \to H_*(X^+, L)$ for all $\pi_1 X^+$-modules L.

**Corollary**: Given $f : X \to Y$, $\pi_1 f(E) = 0$ implies

$g : X^+ \to Y$ s.t. $gi = f$. Moreover

$k^* : [X^+, Y] \to \{x \in [X, Y] \mid \pi_1(x) \text{ kills } E\}$

Obstructions lie in $H^*(X^+, X; \pi_1 Y)$, $n \geq 2$

\[
\begin{array}{c}
X \\ \downarrow \\
X^+ \longrightarrow K(\pi_1 Y, 1)
\end{array}
\]

\(\pi_2 Y\) is a $\pi_1 X^+$ module by hypothesis.

(Corollary implies that any two spaces $X^+$ are homotopy equivalent.)
Notation: $X/E$ for the space of Prop. 1. Reason:

$$\text{Hom}(Z, X) \rightarrow \text{Hom}(Z \cup pt, X) \rightarrow X$$

$\pi_1 X$ acts on $[Z, X]$ for any $Z$.
Hence $\pi_1 X$ acts on $X$ as an objects of $\mathcal{P}$. Homotopy category.

$$X \xrightarrow{f} Y$$

Claim: $\pi_1(f)$ kills $E$ $\iff$ $f \tilde{f} = f$ all $x \in E$

$\Rightarrow$ because $f \cdot \tilde{f} = \pi_1(f) \tilde{f}$, $f = f$

$\Leftarrow$ because $\pi_1(\tilde{f}) (\cdot x) = \tilde{x} \cdot g^{-1}$ so

$\forall x \in f \tilde{f} = f \Rightarrow \pi_1(f)(x \cdot g^{-1}) = \pi_1(f)$

for all $x$ $\Rightarrow$ $\pi_1(f)$ kills $[E, E] = E$.

Remark: $X/E \rightarrow X/N$ is a hom.

Corollary: $X_1$, $X_2$, pointed connected $E_1 \subset \pi_1 X_i$ perfect. Then the canonical hom.

$$(X_1 \times X_2)/(E_1 \times E_2) \rightarrow (X_1/E_1) \times (X_2/E_2)$$
Let $V$ be a vector space over a field $k$. Given a subspace $W$ of $V$ and a subset $S$ of $V$, let $L(S,W)$ be the simplicial complex whose simplices are finite subsets $\{s_1, \ldots, s_m\}$ of $S$ which are independent of $W$, i.e., $\dim(W+ks_1+\cdots+ks_m) = \dim W + m$. Observe that if $\sigma = \{s_1, \ldots, s_m\}$ is a simplex of $L(S,W)$, then

$$\text{Link}(L(S,W), \sigma) = L(S, W + ks_1 + \cdots + ks_m).$$

(\text{If } \tau = \{t_1, \ldots, t_n\} \text{ belongs to the links if } \tau, \sigma, W \text{ independent, i.e. if } \tau \text{ independent of } W + k \omega.)

Observe also that if $v \in L(S,W)$, then

$$L(S,W) = L(S-\{v\}, W) \cup \text{Cone}(L(S, W + k\omega)) - L(S, W + k\omega).$$

(In general, given a simplicial complex $K$, we have

$$K = (K-\bullet \cup \omega) \cup \text{Cone}(\text{Link}(K, \omega)).$$

(Open star of $\omega$)
More generally have for \( \sigma \) a simplex in \( L(S, W) \)

\[
L(S, W) = L(S - \sigma, W) \cup \left( L(S, W + k\sigma) \times \sigma \right)_{\text{join}}
\]

**Proposition (R. Reid):** Assume that \( \dim L(S, W) \geq n \geq 0 \). Then \( L(S, W) \) is \((n-1)\)-connected (meaning \( \neq \emptyset \) for \( n-1 = -1 \)).

**Proof:** We may suppose \( S \) finite. Can argue by induction on \( n \), starting from \( n = 1 \), so assume \( n > 1 \) and the result true for smaller values of \( n \). Suppose \( S \) minimal counterexample.

**Proof:** Arguing by induction on \( n \), starting from \( n = 0 \) which is trivial, we can assume \( n > 0 \) and that the result is true for all \((S, W)\) and smaller values of \( n \). We can assume \( S \) finite. Let \( (S, W) \) be a counterexample with \( S \) having fewest elements.

**Case 1:** There is more than one \( n \)-simplex in \( L(S, W) \). If this is so, we can find a vertex \( v \) belonging to one \( n \)-simplex, so that \( L(S, W + k\sigma) \) has \( \dim \geq n - 1 \), and also not belonging to some \( n \)-simplex, so that \( L(S - \{v\}, W) \) has \( \dim \geq n - 1 \).

By induction hypothesis, \( L(S, W + k\sigma) \) is \((n-1)\)-connected.
and by minimality of $S$

$L(S \setminus \{w\}, W)$ is $(n-1)$-connected

so it follows from the formula

$L(S, W) = L(S \setminus \{v\}, W) \cup \frac{1}{L(S, W + kv)}$

that $L(S, W)$ is $(n-1)$-connected.

**Case 2:** There is only one $n$-simplex in $L(S, W).$

Then we will show that $L(S, W)$ is this simplex.

Denote the simplex by $S = \{s_0, \ldots, s_n\},$ and $v$ be another vertex of $L(S, W).$ Then $v \in W + kv,$ so let $j$ be least such that $v \in W + k s_0 + \ldots + k s_j.$

By exchange condition

$$W + k s_0 + \ldots + k s_j = W + k s_0 + \ldots + k s_{j-1} + kv$$

hence $\{s_0, \ldots, s_{j-1}, v, s_{j+1}, \ldots, s_n\}$ is an $n$-simplex.

By uniqueness of the $n$-simplex, we have $v = s_j \in S$ as claimed. In this case it is clear that $L(S, W)$ is $\infty$-connected. q.e.d.

(Observe: lemma at stake here is that if $\{s_0, \ldots, s_n\}$ is independent in a vector space (V/W here) and if $v \neq 0$ and $v \in \text{span} \{s_0, \ldots, s_n\},$

then for some $j \{s_0, \ldots, s_{j-1}, v, s_{j+1}, \ldots, s_n\}$ is independent.)
Consider now the situation where $V$ is a projective $A$-module, $W$ is a direct summand of $V$, and $A$ is a local ring with residue field $k$. For any subset $S$ of $V$ define $L(S,W)$ to be the simplicial complex whose simplices are subsets $\{s_0, \ldots, s_m\}$ of $S$ such that the images of $s_0, \ldots, s_m$ in $(V/W) \otimes_A k$ are independent. Then

$$\text{link} \left( L(S,W), \sigma \right) = L(S, W + A\sigma);$$

in effect $\tau$ is in the link iff $\tau$ independent of $W + A\sigma$.

I claim the preceding proposition holds. Case 1 clear, so check case 2. Suppose $\sigma = \{s_0, \ldots, s_n\}$ is the only $n$-simplex of $L(S,W)$. However given $\sigma$ in $L(S,W)$ then there is a $j$ such that $s_0, \ldots, s_j, \ldots, s_n, \sigma$ is again an $n$-simplex in $L(S,W)$. The point is that independence is measured within $(V/W) \otimes_A k$, so that it is enough to note that given an independent set over a field, and something in its span, then for some $j$ $\{s_0, \ldots, s_j, \ldots, s_n, \sigma\}$ is independent. So in the case that $L(S,W)$ has a unique $n$-simplex, we see $L(S,W)$ is an $n$-simplex.

So Reid's proposition holds for a local ring.
September 16, 1971: Serre's theorem

Mike wants to prove Serre's theorem as follows:

Let \( E \) be a \( \mathbb{P} \) projective \( A \)-module which is a quotient of \( A^n \). Then to split off a trivial bundle of \( E \) means we must find a section of

\[
\text{Spec } A[x_1, \ldots, x_n] \overset{\sim}{\longrightarrow} \text{Spec } A
\]

\( (x_i) \mapsto a_i \)

which does not meet the closed subschemes of affine \( n \)-space defined by the kernel of \( A^n \rightarrow E \). This idea consists in choosing \( a_1, a_2, \ldots \) inductively so that the dimension of

\[
\mathbb{Z} \cap \{ x_1 = a_1, \ldots, x_n = a_n \}
\]

goes down each time. If \( d = \dim (A) \), and \( r = \text{rank}(E) > d \), then \( \mathbb{Z} \) has dimension \( d + (n - r) \).

So if one can do this \( v = n - d + 1 \) steps one is done for the intersection is empty. Now by Serre's theorem, Mike's program has to work.
Mike's problem: Let $A$ be a noetherian ring and $C$ a closed subset of $\text{Spec } A[X_1, \ldots, X_n]$. Assume that for every closed point of $\text{Spec } A$ one can find a rational point of the affine space over the fibre $\text{Spec } A/m [X_1, \ldots, X_n]$ which does not lie in $C$. Assume also that $\dim C < n$.

Then one can find a section of the affine space $X_i = q_i$ not meeting $C$.

Special case: Suppose $A$ semi-local and let $J = (f_1(X), \ldots, f_m(X))$ be the ideal of polynomials vanishing on $C$. If $m$ is a maximal ideal of $A$, then there exists, by hypothesis, a $\lambda \in A^m$ such that some $f_j(\lambda) \neq 0$ in $A/m$. Since $A$ has finitely many maximal ideals, Chinese R.T. says we can find $a \in A^n$ such that for each $m$ there exists a $j$ with $f_j(a) \notin m$.

But

$$f_j(a) = -\sum_{i=1}^n g_{ji}(x)(X_i - a_i) + f_j(x) \in (x - a) + (f_1(x), \ldots, f_m(x)) \subset A[x]$$
Thus
\[ J + (x-a) \cap A \supset (f_1(a), \ldots, f_m(a)) \]

and the latter = A because it does so at each closed point of $\text{Spec}(A)$. Thus $J + (x-a)$ is the unit ideal, so we have a section not meeting C.

Remark: The preceding shows that every section which does not

Remark: The preceding amounts to the fact that given a section $s$, its bad set is closed in $\text{Spec}(A)$, hence it meets $\text{Max}(A)$.

When $A$ is semi-local, CRT guarantees we can find a section good on $\text{Max}(A)$, hence good everywhere.
Suppose all of the residue fields of $A$ are infinite. We claim $\exists q_i \in A$ such that $x_i = a_i$ doesn't contain any irreducible component of $C$. Indeed, if $p_i$ are the prime ideals belonging to the irreducible components of $C$, then we want $a_i$ such that

$$x_i - a_i \notin \bigcup p_i$$

But

$$x_i - a_i \in p_i \iff a_i \in x_i + p_i$$

so if $a_i$ doesn't exist we have

$$A = \bigcup (x_i + p_i) \cap A$$

But $(x_i + p_i) \cap A$ is a torus for $p_i \cap A$, so we have

$$(x)\quad A = \bigcup (a_i + q_i)$$

for $a_i \in A$ and $q_i$ prime ideals in $A$. Claim (x)

Impossible: Can assume $q_i$ maximal, if $q_i \in \{q_i\}$ then $A/q_i$ is infinite by hypothesis, so $\exists q_i$ distinct mod $q_i$ from all $q_i$ with $q_i = q_i$. By CRT $\exists a_i \equiv a_i \mod q_i$,

Then $a \notin (x + q_i)$ for all $i$.

Thus by induction we can find $a_1, \ldots, a_n$ such that

$$\dim C \cap \{x_1 = a_1, \ldots, x_n = a_n\} \leq \dim C - n$$
and hence a section not meeting $C$ if \( \dim C < n \).
Cor: Let $G$ be a nilpotent group. Then the group of automorphisms of $G$, inducing the identity on $G^{ab}$ is nilpotent.

Proof. If $\theta$ induces $id$ on $G^{ab}$, then it induces the identity on $gr(G)$ which is generated as a Lie algebra by $G$. Then $\theta$ stabilizes the lower central series of $G$, and the group of these is nilpotent by the Kurokawa theorem.

Remark:

$$[[A,B],C] \leq \text{normal subgroup generated by } [[A,C],B] \text{ and } [A,[B,C]].$$
Suppose $A$ is a ring of characteristic $p$ such that $F: A \rightarrow A$ is a finite free map of rank $p^d$. For example, if $A$ is an imperfect field such that $[A:A^p]$ is finite. Then on the $K$-groups $K_n A$ we have maps

$$ K_n A \xrightarrow{V} K_n A \xleftarrow{F} $$

where $V$ is the transfer or trace with respect to $F$. Now

$$ VF = p^d $$

because as an $A$-module $A_F \cong A^{p^d}$. On the other hand, $FV$ is determined by the map

$$ E \rightarrow A \otimes_{A^{(p)}} A = (A \otimes_{A^{(p)}} A) \otimes_{A} E. $$

What should be true is that

$$ \text{gr} (A \otimes_{A^{(p)}} A) = \text{Sym}^A (\Omega^1_{A/F_P})/(x^p = 0) $$

(restricted symmetric algebra). Thus if I filter

$$ A \otimes_{A^{(p)}} A \supset I \supset I^2 \supset \cdots \supset I^{d+1} = 0 $$

this will be a filtration by $(A \otimes A)$-modules and

$$ \text{gr} (A \otimes_{A^{(p)}} E) = \bigoplus I^k \otimes_A E / I^{k+1} \otimes_A E = \text{gr} (A \otimes_{A^{(p)}} A) \otimes E $$
will be multiplication by the element
\[ p^d = [A \otimes A] \in K_0 A \]
(Note: tensoring with an \( A \otimes A \)-bimodule as an operation from \( K_* A \) to \( K_* A \) is not usually the same as multiplying by an element of \( K_* A \), e.g. in the case of a Galois extension it amounts to taking the sum of the conjugates.) Thus have

\[ FV = p^d \]

Now the idea I have is to try to use this together with that \( K^n A \) should have high \( \mathcal{F} \)-filtration for \( n \) large. So consider the essential case.

**Lemma:** Let \( M \) be an abelian group endowed with two endos, \( F, V \) such that
\[ FV = VF = p^d \]
and
\[ F(x) = p^{d+n} x, \quad r > 0. \]
Then
\[ M = M' \oplus M'' \]
where
\[ M' = p^d M \] is the largest \( p \)-divisible subgroup
\[ M'' = (p^d)^* M \] is the \( p \)-torsion subgroup.

Thus \( M \) is the direct sum of a \( \mathbb{Z}p^d \)-module and a group of exp. \( p^d \).
Proof: \( p^d x = F V x = p^{d+r} V x \)
so \( p^d (x - p^r V x) = 0. \)
Thus \( \text{Im} \ (1-p^r V) \subseteq (p^d)^M \).

But \( 1-p^r V \) is an automorphism of \( (p^d)^M \), hence
\[
M = \text{Ker} (1-p^r V) \oplus (p^d)^M.
\]

Clearly \( p^d M \subseteq \text{Ker} (1-p^r V) \)

But \( x = p^r V x = p^{r+1} V^2 x = \ldots \in p^d M \),
so they are equal and \( p^d M = p^s M \) for all \( s > d. \)
Thus \( p^d M \) is the divisible subgroup. Also-
\[
p^{d+r} x = 0 \implies VF x = 0 \implies p^d x = 0
\]
so \( p^d M \) is the \( p \)-torsion subgroup. Thus the lemma is proved.

So if I suppose that \( K_n A \) is of \( \Gamma \)-filtration \( > d \), so that it admits a filtration
\[
o \subseteq \text{Ker} (F - p^{d+1}) \subseteq \text{Ker} (F - p^{d+2}) (F - p^{d+1}) \subseteq \ldots \subseteq K_n A
\]
it follows that \( K_n A \) is direct sum of a \( \mathbb{Z} [p^{-1}] \)-module and group of exponent \( p^d \).
If \( \text{rank}(P) \geq k \), there exist \( x_1, \ldots, x_r \in P \) such that
\[
\text{Codim } D_j(x_1, \ldots, x_r) \geq r-j
\]
where
\[
D_j(x_1, \ldots, x_r) = \{ x | \text{rank } \{ x_1(x), \ldots, x_r(x) \} \leq j \}.
\]

Proof: Assume true for \( n-1 \), whence \( \exists x_1, \ldots, x_{n-1} \)
\[
\text{codim } D_j(x_1, \ldots, x_{n-1}) \geq r-1-j \quad \forall j \geq 0
\]

Let \( C \) be an irreducible component of \( D_j(x_1, \ldots, x_{n-1}) \) of codimension \( r-1-j \). Then \( C \neq D_{j-1}(x_1, \ldots, x_{n-1}) \) can find a finite set \( S_j \subset D_j(x_1, \ldots, x_{n-1}) \) not meeting \( D_{j-1}(x_1, \ldots, x_{n-1}) \) and meeting each irreducible component of codimension \( r-1-j \). Now arrange \( x_r \) to be independent of \( x_1, \ldots, x_{n-1} \) at each point of \( U_{S_j} \). Then possible because multiplicity
\[
D_j(x_1, \ldots, x_r) \subset D_j(x_1, \ldots, x_{n-1}) - S_j
\]
because at a point of \( S_j \) the rank of \( x_1, \ldots, x_{n-1} \) is \( j \), hence \( x_1, \ldots, x_r \) has rank \( j+1 \) there. Thus
\[
\text{codim } D_j(x_1, \ldots, x_r) \geq r-j
\]

Suppose \( \text{Codim } D_j(x_1, \ldots, x_r) \geq k-j \) \( \forall j, 0 \leq j < n \)
\[
\Rightarrow \exists \beta_i = x_i + a_i x_r \quad 1 \leq i < n \quad \exists \epsilon > 0
\]
\[
\text{Codim } D_j(\beta_1, \ldots, \beta_{n-1}) \geq k-j \quad \forall j, 0 \leq j < n-1.
\]
Proof: Assume $0 \leq j < n$.

$$D_{j-1}(x_1, \ldots, x_n) \subset D_j(x_1, \ldots, x_n)$$

so no irreducible component $C$ of $D_j(x_1, \ldots, x_n)$ contained in $D_{j-1}$, hence $\exists S_j$ meeting each $C$ not meeting $D_j$. At each point of $S_j$, $x_1, \ldots, x_n$ have rank $j < r$ hence can find $a_i$ at $m$ so that

$$\beta_i = x_i + a_i x_r \quad 1 \leq i < n$$

have rank $j$ at $m$. Then

$$D_{j-1}(\beta_1, \ldots, \beta_{j-1}) \subset D_j(x_1, \ldots, x_n)$$

and doesn't meet $S_j$ because the $\beta$'s have rank $j$ there. Thus

$$\text{Codim } D_{j-1}(\beta_1, \ldots, \beta_{j-1}) \geq k-j+1 \quad 0 \leq j < n$$

$$\text{Codim } D_j(\beta_1, \ldots, \beta_{j-1}) \geq k-j \quad 0 \leq j < n-1.$$
September 25, 1971: Burnside ring

If $G$ is a finite group, then the Burnside ring $B(G)$ is the Grothendieck group of finite $G$-sets. This is the naive $K$-functor associated to the family of symmetric groups.

$B(G)$ is a free $\mathbb{Z}$-module with basis the iso. classes of transitive $G$-sets, which may be identified with conjugacy classes of subgroups of $G$.

Given a subgroup $H$ of $G$, the map

$$X \mapsto \text{card}(X^H)$$

transforms sums to sums and products to products, hence it induces a ring homomorphism

$$\varphi_H : B(G) \longrightarrow \mathbb{Z}$$

which clearly depends only on the conjugacy class of $H$.

Let $k$ be a field and consider the composite homomorphism

$$B(G) \longrightarrow \mathbb{Z} \longrightarrow k$$

In fact, take $k$ to

Let $l$ be a prime number not dividing $|G|$, and suppose $H, H'$ are two subgroups such that

$$\varphi_H \equiv \varphi_{H'} \pmod{l}.$$ 

Then as
\[(G/H)^H = N/H \quad N = \text{normalizer of } H \text{ in } G\]

has cardinality prime to \(l\), we have

\[\varphi_H(G/H) \equiv 0 \pmod{l}\]

so

\[(G/H)^{H'} \neq \emptyset \quad H' \times H = xH\]

i.e.

\[H' \subset xHx^{-1} \quad \text{for some } x \in G.\]

Similarly, \(H\) is conjugate to a subgroup of \(H'\) and so \(H\) and \(H'\) are conjugate subgroups.

This shows that the homomorphisms \(\varphi_H\) mod \(l\) are distinct, and so by comm. algebra we know that

\[\left(\varphi_H\right)_{H \in I} : \mathbb{B}(G) \rightarrow \prod_{H \in I} \mathbb{Z}\]

(I = reps. for conjugacy classes of subgroups) becomes an isomorphism after inverting order of \(G\).

Next suppose \(G\) is a \(p\)-group. In this case \(\text{card}(X) \equiv \text{card}(X^H) \pmod{p}\) for all subgroups, so all the \(\varphi_H\) mod \(p\) coincide with the mod \(p\) augmentation. I claim the augmentation ideal of \(\mathbb{B}(G) \otimes \mathbb{Z}/p\mathbb{Z}\) is nilpotent. This ideal is generated by \([G/H]\) with \(H < G\). Assume we know \([G/H]\) belong to the nilideal for all \(H'\) with \(|H'| < |H|\). Then by double coset formula...
\[
\left[\frac{G}{H}\right] \times \left[\frac{G}{H}\right] = \left[\frac{N}{H}\right] \left[\frac{G}{H}\right] + \sum_i \left[\frac{G}{H_i}\right]
\]

where \( H_i = H \cap g_i H g_i^{-1} < H \). In a p-group \( N > H \) so this formula shows that \([G/H]^2\) belongs to the nil-ideal, and so have

**Prop.** If \( G \) is a p-group, then the augmentation ideal of \( B(G) \otimes \mathbb{Z}/p\mathbb{Z} \) is nilpotent.

So this shows that \( B(G) \) is split etale off \( p \) and totally ramified at \( p \). It also implies that

\[ I^N < pI \]

\( I \) = aug. ideal of \( B(G) \). Note quite generally that

\[ [G] \cdot [S] = [G] \cdot \text{card}(S) \]

Whence \( [G] \cdot I = 0 \) so

\[ |G| \cdot I = (|G| - [G])I < I^2 \]

Thus for a p-group, the \( I \)-adic topology on \( I \) and the \( p \)-adic topology on \( I \) coincide implying that

\[ \lim_{n \to \infty} \frac{I}{I^n} = I \otimes_{\mathbb{Z}} \mathbb{Z}_p \]
Suppose now that \( p \) divides the order of \( G \). If \( H \triangleleft H' \) and \( H'/H \) is a \( p \)-group, then
\[
\text{card } X^{H'} = \text{card } (X^H)_{H'/H} \equiv \text{card } X^H \pmod{p}
\]
so \( \phi_H \equiv \phi_{H'} \pmod{p} \). Now starting with \( H \) we can form groups
\[
H_0 \subset H \subset H_1
\]
where \( H_0 \) is the char. subgroup of \( H \) gen. by the \( p' \)-elements and where \( H_1 \) is a Sylow \( p \)-subgroup of the normalizer of \( H_1 \). The indices are powers of \( p \) and \( H_1 \) is of index prime to \( p \) in its normalizer. Any subgroup \( H' \) of \( G \) conjugate to a subgroup between \( H_0 \) and \( H_1 \) satisfies
\[
\phi_{H'} \equiv \phi_{H_0} \equiv \phi_H \pmod{p}
\]
On the other hand, if \( H \) is already of \( p' \)-index in its normalizer, then
\[
\phi_H \equiv \phi_{H'} \pmod{p}
\]
\[
\Rightarrow \phi_{H'}(G/H) \equiv \text{card } N/H \not\equiv 0 \pmod{p}
\]
so \( (G/H)^{H'} \not= \emptyset \) and \( H' \rightarrow H \). Similarly if \( H' = H_1 \), then we have \( H' \approx H \). It follows that we get all the different \( \phi_H \), when we let \( H \) range over a set \( \mathcal{J} \) of representatives for subgroups \( H' \) of \( H \) up to conjugacy (or of \( p' \)-index in their normalizers).
**Conjecture:**

\[ (\varphi_H) : \mathbb{B}(G) \otimes \mathbb{Z}/p\mathbb{Z} / \text{rad} \rightarrow \prod_{H \in \mathcal{F}} \mathbb{Z}/p\mathbb{Z} \]

We know this map is onto. It only remains to produce enough nilpotent elements.

**Lemma:** If the index of \( H \) in its normalizer is divisible by \( p \), then \([G/H]\) is nilpotent.

First we show for any subgroup \( K \) that
\[ \varphi_K([G/H]) = 0 \mod p. \]
We may assume that \( K \) is generated by its \( p' \)-elements. Then consider the principal \( H/H_0 \)-bundle

\[ G/H_0 \rightarrow G/H \]

where \( H/H_0 \) acts on the right, hence commutes with the action of \( K \) on the left. If \( xH \in (G/H)^K \) then
\[ x^{-1}Kx < H \]
and as \( K^p \) is gen. by its \( p' \)-elements
\[ x^{-1}Kx < H_0. \]
Thus

\[ (G/H_0)^K \rightarrow (G/H)^K \]

is a principal \( H/H_0 \)-bundle, so

\[ \varphi_K[G/H_0] = [H : H_0] \varphi_K[G/H] \]

Taking \( H = H_1 \) gives

\[ \varphi_K[G/H_0] = [H_1 : H_0] \varphi_K[G/H_1] \]
\[ \varphi_K[G/H] = [H_1:H] \varphi_K[G/H_1] \]

for these special \( K \). Therefore if \( H \triangleleft H_1 \), \( \varphi_K[G/H] \equiv 0 \pmod{p} \) for all subgroups \( K \).

It follows that for any subgroup \( K \),

the number of elements equal to \( K \) in \( G/H \) with isotropy group \( \equiv 0 \pmod{p} \). This is a

Möbius inversion type formula: specifically one has

\[
X^K = \prod_{K' \supseteq K} X^{K'}
\]

\( X^{\{K\}} \) points with

isot grp. \( K \)

and so if one knows that \( X^K \) for \( K' \supset K \) have card \( \equiv 0 \pmod{p} \), it follows \( X^{\{K\}} \) has card \( \equiv 0 \pmod{p} \).

Now

\[
[G/H] \bullet [G/H] = \sum_{H \times H} [G/H \cap H \times H^{-1}]
\]

The number of double cosets are the orbits of \( H \) on \( G/H \). We break these orbits up into orbit types.

Suppose that any orbit \( H/K \) occurs \( d \) times. Then the number of elements of \( G/H \) with isotropy group \( K \)

is

\[ d \cdot \text{card } (H/K)^K = d \cdot |N/K| \]

where \( N \) is the normalizer of \( K \) in \( H \). Since this is \( \equiv 0 \pmod{p} \), either \( d \equiv 0 \pmod{p} \) whence the orbit type \( H/K \) contribution is 0, or \( |N/K| \equiv 0 \pmod{p} \) whence we know by an induction hypothesis that \( [G/K] \) is nilpotent.
This proves the lemma, and with it the conjecture at the top of page 5.
September 29, 1971: Theorem of Kaloujnine

Thm. Let \( G = G_0 \supset G_1 \supset G_2 \supset \cdots \) be a sequence of normal subgroups of a group \( G \), and let \( A_{r \cdot} \), \( r \geq 0 \), be the group of automorphisms of \( G \) which induce the identity on \( G_i / G_{i+r} \) for each \( i \geq 0 \). Then \( A_0 \supset A_1 \supset A_2 \supset \cdots \) is a filtration of the group \( A_0 \), i.e., \([A_0, A_0] < A_0\).

\[ \text{Proof.} \text{ In the semi-direct product } A_0 \rtimes G, \text{ the subgroups } G_i \text{ are normal, hence } \]

\[ [A_0, G_i], A_0 \] is a subgroup of \( A_0 \).

\[ [G_i, A_{r \cdot}], A_{r \cdot}] \] is a subgroup of \( G_i \rtimes G_{i+r} \) for all \( i \).

So working in the group \( A_0 \rtimes G / G_{i+r} \) and applying the three subgroup lemma, we have

\[ [A_{r \cdot}, A_{r \cdot}], G_i] \] is a subgroup of \( G_i \rtimes G_{i+r} \) for all \( i \).

Hence \([A_0, A_0] < A_0\), q.e.d.
Tate’s theorem: Let $G$ be a finite group such that
$$H^1(G, \mathbb{Z}/p) \cong H^1(P, \mathbb{Z}/p),$$
where $P$ is a Sylow $p$-subgroup. Then $G$ is $p$-nilpotent.

Proof. Let $G'$ be the $p$-completion of $G$. Then $P$ maps onto $G'$, as all Sylow groups map onto Sylow groups for surjective homomorphisms. One has in general
$$H^1(G) \cong H^1(G'),$$
$$H^2(G) \subset H^2(G').$$
Indeed if an extension of $G$ by $\mathbb{Z}/p$ lifts to a trivial extension over $G$, then one has a homomorphism $G \to E$ which factors through $G'$ as $E$ is a $p$-group. Let $N = \text{Ker } \{ P \to G' \}$, so that we have an exact sequence
$$0 \to H^1(G') \to H^1(P) \to H^1(N) \to H^2(G) \to H^2(P).$$
Since $H^2(G) \to H^2(P)$ by transfer, the last map is injective, so we conclude $H^1(N) = 0$.

Thus $H^i(N) = 0$ and $N = 0$, so $G$ is $p$-nilpotent.
October 3, 1971: Cohomology theories and $\Sigma n$

I want to understand why

$$\Omega B \left( \prod_{n \geq 0} P \Sigma_n \times \Sigma^n X^n \right) \cong \Omega^\infty S^\infty (X \cup \infty).$$

The important thing seems to be to understand why the $\Sigma^n$ on the left transforms cofibrations to fibrations.

Work semi-simplicially. Given a set $X$ form simplicial monoid

$$M(X) = \prod_{n \geq 0} P \Sigma_n \times \Sigma^n X^n$$

which is the nerve of the category whose objects are finite sequences $(X_1, \ldots, X_n)$ and permutations for morphisms. Basic fact:

$$P \Sigma_n \times \Sigma^n (A \amalg B)^n \cong \prod_{i \leq n} (P \Sigma_i \times \Sigma_i A^i) \times (P \Sigma_{n-i} \times \Sigma_{n-i} B^{n-i})$$

because the category defined by $\Sigma^n$ acting on $(A \amalg B)^n$ is equivalent to the disjoint union of the full subcategory with objects $A^i \times B^{n-i}$ and the latter is the category defined by $\Sigma_i \times \Sigma_{n-i}$ acting on $A^i \times B^{n-i}$. Consequence:

$$M(A \amalg B) \leftrightarrow M(A) \times M(B)$$

$$\text{in}_1(X) \cdot \text{in}_2(Y) \quad (X, Y)$$

is a weak equivalence (note: it is not a monoid homomorphism).
Now I denote by $\Gamma(X)$ an intelligent group completion of $M(X)$. Thus we can choose a functorial free monoid resolution

$$P(M(X)) \longrightarrow M(X)$$

\[\Gamma(X) = \overline{P(M(X))} \quad (\text{group completion})\]

Then we have a commutative diagram

$$\begin{array}{ccc}
M(A) \times M(B) & \longrightarrow & M(A \cup B) \\
\uparrow & & \uparrow \\
PM(A) \times PM(B) & \longrightarrow & PM(A \cup B) \\
\downarrow & & \downarrow \\
\Gamma(A) \times \Gamma(B) & \longrightarrow & \Gamma(A \cup B)
\end{array}$$

Because $M(A)$ is homotopy commutative, it follows that the group completion theorem applies to it, hence the map

$$H_\ast(M(A)) \leftarrow H_\ast(\overline{P(M(A)}) \longrightarrow H_\ast(\Gamma(A))$$

is localization with $\Pi_0 M(A) = \amalg \bigwedge SP^n(A) = \text{free commutative monoid gen. by } A$. Consequently, we see that

$$\Gamma(A) \times \Gamma(B) \longrightarrow \Gamma(A \cup B)$$

is a homotopy equivalence of s. sets by Whitehead thm.
because both s-sets are simple.

More generally, suppose we have a functor \( \Gamma \colon (\text{sets}) \to \text{s.groups} \) :

i) \[\Gamma(A) \times \Gamma(B) \to \Gamma(A \sqcup B) \]

ii) \( \Gamma \) commutes with filtered lim. ind.

Then given a simplicial set \( X \) I get a bisimplicial group \( \Gamma(X) \) and the claim is that

\[ \pi_\ast(\Delta \Gamma(X)) = h_\ast(X; \Gamma) \]

is a generalized homology theory. To prove this we prove the Mayer-Vietoris axiom:

**Lemma:** Let \( A, B \subset C \) be sets, and let \( \Gamma(A \sqcup B) \) act on the right of \( \Gamma(A) \times \Gamma(B) \) by

\[ (x, \beta) \cdot y = (x \cdot y, y^{-1} \beta) \]

(Observe that \( A \leq A' \Rightarrow \Gamma(A) \subset \Gamma(A') \) because \( \exists \chi : A' \to A \) retraction.) Then

\[ \Gamma(A) \times \Gamma(B) / \Gamma(A \cap B) \to \Gamma(A \cup B) \]

\[ (x, \beta) \quad \mapsto \quad x \beta \]

is a homotopy equivalence of simplicial sets.
This lemma shows that if $A, B$ are simplicial subsets of $X$, then we have a principal fibration

$$\Gamma(A \cap B) \rightarrow \Gamma(A) \times \Gamma(B) \rightarrow \Gamma(A \cup B)$$

and hence a long exact Mayer-Vietoris sequence.

**Proof.** By hypothesis the vertical maps in

$$\Gamma(A) \times \Gamma(B) \xrightarrow{(x, y)} \Gamma(A \cup B)$$

are leg's. The bottom arrow is a principal right $\Gamma(A \cap B)$-bundle with action

$$(x, z', z''y) \cdot y = (x, z'z''y^{-1}z'', y)$$

so done.

The relation with Anderson's chain functors. Suppose $X, Y$ are pointed. Then we claim canonical map

$$\Gamma(X \vee Y) \rightarrow \Gamma(X) \times \Gamma(Y)$$

is a leg. It suffices to show that
\[
\Gamma(X) \times \Gamma(Y) / \Gamma(\text{pt}) \to \Gamma(X) \times \Gamma(Y)
\]

\[
(a, \beta) \quad \mapsto \quad (a \circ \varepsilon(\beta), \varepsilon(a)\beta)
\]

is bijective, where \(\varepsilon: \Gamma(\text{pt}) \to \Gamma(\text{pt})\) is the augmentation. Clear.

Thus if we set

\[
\overline{\Gamma}(X) = \text{Ker} \left\{ \Gamma(X) \to \Gamma(\text{pt}) \right\}
\]

for a pointed set, we have that

\[
\overline{\Gamma}(X \times Y) \to \overline{\Gamma}(X) \times \overline{\Gamma}(Y)
\]

is a breq., hence a chain functor à la Anderson.
Observation: Segal introduces category $Γ$ consisting of finite sets, where a map $T_1 \rightarrow T_2$ is a partition of a subset of $T_2$ indexed by $T_1$, i.e., a family of disjoint subsets of $T_2$ indexed by $T_1$. Such a thing is the same as a function

$$\overline{T}_1 \leftarrow \overline{T}_2$$

where $\overline{T} = T \cup \text{pt}$. Thus $Γ$ is dual to the category of finite pointed sets. Hence Segal's special $Γ$-spaces and Anderson's chain functors are essentially the same thing.
Recall that Lang's theorem: $G/G(F_0) \to G$ for the general linear group amounts to the following. Let $k$ be a separably closed field of char. $p$ and let $V$ be a finite-dim. vector space over $k$ endowed with a semi-linear automorphism $F: V \to V$ satisfying $F(\lambda v) = \lambda^p F(v)$. Then

\[ k \otimes V^F \to V. \]

I want now to understand this result over a general ring $A$ of characteristic $p$.

Proposition 1: Let $M$ be a finitely presented $A$-module provided with an isomorphism

\[ F: M \otimes \overset{\circ}{A} \to M, \]

where $M \otimes \overset{\circ}{A} = \begin{array}{c} A \otimes M \end{array} \overset{\sigma(a)}{\to} A$.

Then $M$ is a projective $A$-module.

Proof. Can suppose $A$ finitely generated over $\mathbb{Z}/p$, hence can suppose $A$ noetherian. We can suppose $A$ local, let

\[ A^i \to A^i \to M \to 0 \]

be a minimal resolution. To prove $j = 0$, we can
enlarge $A$ by a faithfully flat extension $A'$. This as in appendix to spectrum paper, Part II, can suppose $A$ with algebraically closed residue field $k$. Now one has that $M \otimes_k k' = k' \otimes_k (M \otimes_k k')^F$ should observe that $A$ is canonically a $k$-algebra since Teichmüller section is an isomorphism. Suppose we find $m \in M$ such that

$$F(m) - m \in \mathfrak{m}^e M$$

where $\mathfrak{m} = \text{max. ideal of } A$. Then

$$F(F(m)) - F(m) \in \mathfrak{m} \otimes M$$

hence sequence $F^m \in M$ converges to an element $m^* \in M^F$. This shows that $M^F$ generates $M$ over $A$, because of the theorem for a field. Also $M^F \otimes \mathfrak{m}M = 0$; thus we have a minimal surjection

$$A \otimes M^F \xrightarrow{\pi} M \xrightarrow{\pi} 0$$

Now apply same argument to the kernel of $\pi$; it will be generated by elements of $K^F \subseteq M^F \otimes \mathfrak{m}M = 0$.

Thus $\pi$ is an isomorphism and we are finished.
General idea now is given $M$ over $A$ with an $F$ as in the proposition, it is locally free. To obtain a generating subspace $M^F$, it is necessary to make an etale covering of $A$. Thus first make covering

$$A \otimes_{\overline{F}} F \leftarrow A$$

and assuming $A$ over $\overline{F}$, we have a map

$$Sp \ A \longrightarrow GL_n \leftarrow GL_n / GL_n(F_{\overline{F}})$$

which gives us a principal covering of $A$ with group $GL_n(F_{\overline{F}})$.

Point from June 3, 1971 omitted above: Given $A \in GL_n$, to write $A = B(B^\sigma)$, define $F$ on $V = k^n$ by

$$F(e_i) = \sum_j a_{ji} e_j$$

Then if $\{e_i\}$ is a basis for $V$,

$$e_i = \sum_j a_{ji} e_j$$

we have

$$\sum k_{ij} a_{ji} = \sum e_j a_{ji} = F(e_i) = \sum w_{ji} b_{ji}$$

so $BA = B^\sigma$ as desired.
Summary of problems.

1. Group-completion theorem for topological monoids; what is a torsor for a top. monoid?
2. Good point of view simultaneously explaining the group-completion theorem, quasi-fibrations
   Segal's lemma on when $X \to Y$ is a h-fibration, and Sullivan's theory of rational h-type,
   and Friedlander's problem. One produces a cohomology theory over the space $B$, and the
   point is to check the homotopy axiom, and this can be done locally.
3. Stable splitting of exact sequences theorem, nice model for $EGL(n)^+$, stability,
   exterior power the descent problem and the $\mathbb{Z}_2$ exact sequence, products and $\Sigma$ operations in
   exact sequence $K$-theory. Models for the gamma-filtration (Segal's suggestion).
   The Moore theorem-reduction of problem to representation of a cyclic group $C$; nice
   formula for the gamma j-element of $\pi_{2n-1}(\Omega^c)$. Steinberg homology; any relation
   between this and gamma filtration $K$-theory of the dual numbers, and of curves
4. Configurations and iterated loop spaces, braid groups, Barratt theorem, Tornehave
   problem, why $\overline{\Omega}(\text{loop} S^2)^n$ has the homotopy type of $BG(k^{-1})$ roughly.
A ring with 1
\[ \text{GL}(A) = \bigcup_n \text{GL}_n(A) \]
\[ E_n A \subset \text{GL}_n(A) \quad \text{gen. by} \]
\[ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad a \in A. \]
\[ E(A) = \bigcup E_n(A). \]

One knows
\[ E(A) = (E(A), E(A)) = (\text{GL}(A), \text{GL}(A)). \]

Defn:
- \( K_0 A = \) Groth group proj f.g. \( A \)-modules
- \( K_1 A = \) rel \( \text{GL}(A)/E(A) = H_1(\text{GL}(A), \mathbb{Z}) \)
- \( K_2 A = H_2(E(A), \mathbb{Z}) \).

Work of Bass, Tate + others shows there are interesting invariants
General definition of \( K_n A \), \( n \geq 0 \):
Acyclic maps
Will work only with pointed conn. sp. ~ CW comp.

Defn: $X$ acyclic if

$$H_i(X, Z) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

Poincaré: $H_1(X, Z) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$

so $\pi_1 X = (\pi_1(X), \pi_1(X))$. Such groups called perfect.

$\pi_1 X = 0$ means $X \simeq \mathbb{C}$ Whitehead.

Given map $f: X \to Y$,

\[\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\overline{f} & & \overline{f}
\end{array}\]

Defn: $f$ acyclic if fibres of $f$

are acyclic spaces.

$F \to X \xrightarrow{f} Y$
\[ \pi_1 F \rightarrow \pi_1 X \rightarrow \pi_1 Y \rightarrow \pi_0 F \]

so \( \pi_1(f) \) surjective +

Ker \( \pi_1(f) \) perfect.
Prop 1: (i) \( f : X \to Y \) acyclic
(ii) for all \( \ell \) in \( Y \), \( f^* \to \)
(iii) commutes \( \Rightarrow \)
\[
\begin{array}{ccc}
X \times Y & \to & Y \\
\downarrow & & \downarrow p \\
X & \to & Y
\end{array}
\]
universal covering
is cart., then \( f' \) commutes induces iso. in homology.

(i) \( \Rightarrow \) (ii) \( F \to X \to Y \). Then
\[
E_2^{p,b} = H^p(Y, H^b(\#_Y f^* L)) \Rightarrow H_{p+b}(X, \#_Y f^* L)
\]
\[
\begin{cases}
0 & b > 0 \\
\#_Y & b = 0.
\end{cases}
\]

(ii) \( \Rightarrow \) (iii)
\[
\begin{array}{ccc}
H^*(Y, \mathbb{Z}) & \to & H^*(X \times Y, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^*(Y, p_* \mathbb{Z}) & \to & H^*(X, f_* p_* \mathbb{Z})
\end{array}
\]
(iii) $\Rightarrow$ (i): $\pi_1 \tilde{Y} = 0$

fiber of $f'$ must be acyclic.

fiber of $f$.

**Cor. 1:** If

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

is cocartesian with $f$ cofibrant, and $f'$ acyclic, then $f'$ is acyclic.

**Proof:** For any $L$ on $Y'$

\[H^*(Y';X'; L) \Rightarrow H^*(y, x; L) = 0\]

so done from.

\[H^*(x, L) \Rightarrow H^*(y, L) \Rightarrow H^*(x, L)\]
Cor 2: Given

\[ X \rightarrow Y \rightarrow Z \]
\[ f \uparrow \hspace{1cm} \uparrow \]
\[ g \hspace{1cm} h \]

If acyclic \( f \), then \( \pi_1 g (\text{Ker } \pi_1 f) = 0 \)

then \( \exists \ h \) unique up to

\[ h f \sim g \]

Proof: May assume \( f \) cofibration.

\[ X \rightarrow Y \rightarrow Z \]
\[ f \uparrow \hspace{1cm} \uparrow f' \]
\[ g \hspace{1cm} h \rightarrow Z' \]

Then \( f' \) acyclic. Van Kampen

\[ H(Z) \cong \mathbb{Z} \]

\[ \pi_1 Z \sim \pi_1 (Y) \times_{\pi_1 X} \pi_1 Z \sim \pi_1 Z' \]

\( f' \) hes by Whitehead
Theorem: Given $X$ and $N \subset \pi_1 X$ a perfect normal subgroup, there exists a $f: X \to Y$ acyclic if $\pi_1(f) = N$. $f$ unique up to $\sim$.

- Uniqueness clear from Cor 2.
- Existence: Let $p': X' \to X$ be covering with group $\pi_1(X') = N$. If can find $f: X' \to Y'$ acyclic, $\pi_1 Y' = 0$ done. So can assume $N = \pi_1 X$.

(Ded)

\[ \begin{array}{c}
X_3 \\
\downarrow \\
X_2 \to K(H_3 X_2, 3) \\
\downarrow \\
X = X_1 \to K(H_2 X_1, 2)
\end{array} \]
Assume have constructed \( X_n \to X \) \( n \geq 1 \)

\[
\begin{align*}
&\text{(i)} \quad \pi_1 X_n \to \pi_1 X \\
&\text{(ii)} \quad H_i X_n = 0 \quad 1 \leq i \leq n \\
\end{align*}
\]

\( \check{H}^{n+1}(X_n, A) \cong \check{H}_n(H_{n+1} X_n, A) \)

\( \chi \in \check{H}^{n+1}(X_n, H_{n+1} X_n) = [X_n, K(H_{n+1} X_n, n+1)] \)

**Def.:**

\[
\begin{align*}
X_{n+1} & \to X_n \to X \to K(H_{n+1} X_n, n+1) \\
H_{n+2} X_n & \to 0 \\
\end{align*}
\]

\[
\begin{align*}
\leftarrow H_{n+1} X_{n+1} & \to H_{n+1} X_n \overset{\sim}{\to} H_{n+1} X_n \\
0 & \to
\end{align*}
\]

\( X_\infty = \lim X_n. \quad X_\infty \text{ acyclic} \)

\( \pi_1 X_\infty \to \pi_1 X \)

set \( X_\infty \to X \overset{f}{\to} X/X_\infty = Y \)

\( f \text{ acyclic} \quad \pi_1 Y = 0 \quad \text{van Kampen} \)
\[ BGL(A) = K(GL(A), 1) \]

Classifying space

\[ \pi_1 BGL(A) = GL(A) \to E(A) \]
Perfect normal

\[ \exists! \text{ acyclic map} \]

\[ f : BGL(A) \to BGL(A)^+ \]

\[ \exists \ker \pi_1(f) = E(A). \]

**Defn:** \[ K_i(A) = \pi_i BGL(A)^+ \quad i \geq 1. \]

Recall \[ K_1(A) = \frac{GL(A)}{E(A)} \]

\[ BE(A) \xrightarrow{f'} BGL(A)^+ \]

\[ \text{homology} \quad \xrightarrow{\text{homology}} \quad \text{cover} \]

\[ BGL(A) \xrightarrow{f} BGL(A)^+ \]

\[ K_2 A = \pi_2(BGL(A)^+) = \pi_2(BGL(A)^+\sim) \]

\[ \text{Hurewicz} \quad \text{Hurewicz} \]

\[ H_2(BE(A)) \xrightarrow{\text{OKAY}} H_2(BGL(A)^+\sim) \]

Okay with Milnor.
\[ k = \overline{F_p} \quad k_d < k \text{ subfield of degree } d \]

\[ \bigcup \text{BGL}_n(k_d) = \text{BGL}(k) \]

Let \( G \) be a finite group acting on a finite-dimensional \( k \)-vector space \( V \) via \( \rho : G \to \text{Aut}_k(V) \).

Choose \( \varphi : k^* \to \mathbb{C}^* \).

Define \( \varphi_V : G \to \mathbb{C} \) by:

\[ \varphi_V(g) = \sum \varphi(\lambda_i) \]

where \( \{\lambda_1, \ldots, \lambda_{\dim V}\} \) are the eigenvalues of \( \rho(g) \).

Thm. (Brauer): \( \varphi_V \) is a \( \mathbb{Z} \)-combination of characters of \( G/\mathbb{F}_p \).

\[ \varphi_V \in R(G) \]

\[ \varphi_V : BG \to BU. \]
do for $GL_n(k_n!) < GL_n(k)$ acts on $k^n$

$\bigcup BGL_n(k_n!) \xrightarrow{\phi} BU$

$\phi^\#: BGL(k) \to BU$

**Theorem**

Adams conjecture paper

$\phi^#$ induces isom. on $H_* \big( \mathbb{Z}/\ell \mathbb{Z} \big)$

\[ \ell \text{ prime } \neq p \]

2. $H_* (BGL(k), \mathbb{Z}/p \mathbb{Z}) = 0 \quad \ast > 0.$

[Diagram]

$\psi \quad F$

$\downarrow$

$BGL(k)$

$\to$

$BU$

$\leftarrow$

$BU \otimes \mathbb{Q} \xrightarrow{\text{ch. i} > 0} \prod K(\mathbb{Q}, 2i)$

$\psi$ induces iso over $\mathbb{Z}[p^{-1}]$. by 1.
Serre - Whitehead

\[ \pi_i \text{BGL}(k)^+ \xrightarrow{\sim} \pi_i F \quad \text{except for } p \text{-torsion} \]

\[ i = 1 \Rightarrow \text{no } p \text{-torsion in } \pi_i \text{BGL}(k)^+ \]

Thm: \[ K_i^*(k) \cong \begin{cases} \bigoplus_{\ell \neq p} \mathbb{Q}_\ell / \mathbb{Z}_\ell & i \text{ odd } \geq 1 \\ 0 & i \text{ even } \geq 1 \end{cases} \]
October 27, 1971

Let \( k \) be a field and suppose we consider the problem of proving the homotopy axiom: \( k \to k[\mathbb{Z}] \) induces an isomorphism on \( k \)-groups. Start with a representation of \( G \) on a f.t. proj. \( k[\mathbb{Z}] \)-module \( E \). We want to show that \( E \) comes from \( k \). Ideally we would like to produce a f.t. \( k \)-submodule \( L \) of \( E \) which is invariant under \( G \) and which generates. Replacing \( L \) by

\[
L + zL + \ldots + z^nL
\]

we can suppose that \( L \) is "involutive", i.e. that the conditions of the following hold.

**Lemma:** \( E \) finite type \( A[\mathbb{Z}] \)-module, \( A \) noetherian

(i) \( \frac{z^{-1}L}{L} \subset L \)

(ii) \( L \to A[\mathbb{Z}] \otimes_A (z^{-1}L) \to A[\mathbb{Z}] \otimes_A L \to E \to 0 \)

is exact.

(iii) \( L \) generates \( E \) over \( A[\mathbb{Z}] \) and \( A[\mathbb{Z},z^{-1}] \)-submodule of \( A[\mathbb{Z},z^{-1}] \otimes_{A[\mathbb{Z}]} E \) and \( \varphi(e) = 1 \otimes e \).

(iv) Set \( L^{(n)} = L + zL + \ldots + z^nL \). Then

\[
z^n: L^{(n-1)} \to L^{(n)} / L^{(n-1)} \quad n \geq 1
\]

where \( L^{(1)} = z^{-1}L \cap L \).
Assume that $z^{-1}L \subseteq L$. The map
\[ L/zL \rightarrow L^{(n)}/(L^{(n-1)}) \quad n \geq 1. \]
is clearly surjective always. Suppose
\[ z^n x = \sum_{i<n} a_i z^i \quad x, a_i \in L \]
Then
\[ z \left( z^{n-1} x - \sum_{i<n} a_i z^{i-1} \right) = a_0 \]
so as $z^{-1}L \subseteq L$
\[ z^{n-1} x \in \sum_{i<n} a_i z^{i-1} + L \subseteq L^{(n-2)} \]
so by induction $x \in z^{-1}L$. Thus (i) $\Rightarrow$ (iv).

Conversely, given $z x \in L$, let $n$ be least $\geq x \in L^{(n)}$, so that
\[ x = \sum_{i<n} a_i z^i. \]
If $n \geq 1$, then because $L^{(n)}/L^{(n-1)} \overset{z}{\rightarrow} L^{(n+1)}/L^{(n)}$ follows that $x \in L^{(n-1)}$. \quad n = 0. Thus (iv) $\Rightarrow$ (i).

Clearly (iii) $\Rightarrow$ (i). Conversely, let $M = A[z^{-1}] q(L)$. If $q(x) \in M$, then $z^N x \in L$ for some $N$, so $x \in L$, hence $q_0(M) = L$.

(iv) $\Rightarrow$ (ii) by filtering

(ii) $\Rightarrow$ (iv) ?
If we can find an $L$ f.t. over $A$, generating $E$ and $G$-invariant, then we can suppose $L$ involutive, and so by (ii)

$$0 \rightarrow A[z] \otimes_A (z^{-1}L) \rightarrow A[z] \otimes_A L \rightarrow E \rightarrow 0$$

which shows that $E$ comes from $L$.

In general, we are going to have to choose an involutive $L$. Suppose $L' \leq L$. We have an $z^{-1}L \sim L' \leq L$. Then $z^{-1}$ acts on $L'/L''$ and kills $L/L''$, we have an isomorphism.

Suppose $L'$ and $L$ are both involutive and that $L \cap L' \leq L + zL$. Then

$$L' \otimes_A z \rightarrow L'/zL' = L' + zL'$$

(onto because $(L + zL) + zL' \supset L' + zL'$, it's injective because $L' \cap z(L + zL) = L' \cap L = L$.)


This is nice because it shows that the inclusion of pairs
\[(L + zL, L) \rightarrow (L' + zL', L')\]
has contractible cokernel.

**Conjecture 1:** Consider the simplicial complex whose vertices are involutive \(L\) generating \(E\) and in which a \(q\)-simplex is a chain
\[L_0 \subset L_1 \subset \ldots \subset L_q\]
such that \(L_q \subset L_0 + zL_0\). I conjecture this complex is contractible.

**Evidence from Bruhat - Tits:** They show that if one identifies \(L\) with \(\text{L}_i\) for all \(i > 0\), then one obtains a contractible complex.

**Conjecture 2:** Let \(I\) be an ideal in \(A\) with \(I = 0\). If \(M\) is an \(A\)-module, consider the simplicial complex whose vertices are submodules \(M_q \subset M\) and whose simplices are chains
\[M_0 \subset M_1 \subset \ldots \subset M_q\]
of submodules such that \(IM_q \subset M_0\). Then this simplicial complex is contractible.
Reduction of conj. 1 to conj. 2: Let $X_1$ denote the complex of conj. 1. It may be identified with the subcomplex of f.g. $A[z^{-1}]$-modules in $A[z_0^+ z^{-1}] \otimes A[z]$ $E$ such that $z^{-1} M$ generates $M$. Thus $X_1$ is the subcomplex of the complex of conjecture 2 associated to $A[z^{-1}]$ and the ideal generated by $z^{-1}$. Enough to show $X_2$ contractible. Indeed, given a finite subcomplex $K$ of $X_1$, it contracts in $X_2$, so $z^n K$ contracts in $X_1$ for large enough $n$. Since $K \sim z^n K$ ??
Theorem: Let \( I \) be an ideal in a ring \( A \) and \( M \) an \( A \)-module such that \( IM^n = 0 \) for some \( n \). Let \( X(M) \) be the simplicial complex whose simplices are chains of \( A \)-submodules of \( M \)

\[ M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_8 \]

such that \( IM_8 \subset M_0 \). Then \( X(M) \) is collapsible to a point.

Proof: We collapse \( X(M) \) to \( X(IM) \). Given a submodule \( L \subset M \) send it to \( L \cap IM \). Claim it sends simplices to simplices:

\[ L_0 \subset \ldots \subset L_8 \]

\[ L_0 \cap IM \subset \ldots \subset L_8 \cap IM \]

and

\[ L_8 \cap IM / L_0 \cap IM \rightarrow L_8 / L_0 \cap IM \]

and

\[ L_8 / L_0 \cap IM \rightarrow L_8 / L_0 \times M / IM \]

Thus we have a simplicial map \( X(M) \rightarrow X(IM) \) which is a retraction, i.e. \( f \circ i = id \) where \( i: X(IM) \rightarrow X(M) \) is the inclusion. But actually we have a homotopy

\[ X(M) \times \Delta(1) \rightarrow X(M) \]

\[ L \xrightarrow{0} \begin{array}{c} L \cap IM \\downarrow i \\downarrow L_0 \cap IM \end{array} \]

\[ L \subset 0 \subset L \cap IM \subset L \]
Remark: Proof shows more generally that $X_{A, I}(M)$ collapses to $X_{A, I}(M')$ where $M/M'$ is killed by a power of $I$. 