January 1, 1971:

Complements to the preceding:

1) One might hope to use it to get a hold on the cohomology of p-groups. Thus if G is a p-group and Z is a cyclic order p central subgroup one chooses an representation \lambda of G whose restriction to Z has no trivial component. If S is the sphere, then e(V) is a non-zero divisor in \( H^*_G \) and

\[
H^*_G(e(V)) = H^*_G(S) = H^*_{G/Z}(S/Z)
\]

Now if we choose a cyclic order p central subgroup Z' of G/Z and denotes by B_1, ..., B_k those cyclic [p]-subgroups mapping isomorphically onto Z', we have exact sequence:

\[
0 \xrightarrow{i} \oplus H^*_G(S^{B_i}) \xrightarrow{i*} H^*_G(S) 
\]

Next we must lift this exact sequence back up to an exact sequence for \( H^*(G) \). (This is what corresponds to \((S/Z)^Z\) in \( S/Z \).) But the problem is to lift this exact sequence back up to an exact sequence for \( H^*(G) \).
2) Example of a $p$-group $G$ such that $J(G)$ as defined by Thompson or Gorenstein differ. Take $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$, $x^p = y^p = 1$, $xyx^{-1} = y^{1+p}$. Then every subgroup is abelian and the maximal ones are of order $p^2$, so Gorenstein’s $J(G)$ = subgroup generated by $A$ of maximal order is all of $G$. But $\langle x, y^p \rangle$ is the unique abelian subgroup with two generators, so it is the Thompson $J(G)$.

Computation of the cohomology of this group. Use Hochschild-Serre for extension

$$1 \rightarrow \langle y \rangle \rightarrow G \rightarrow \langle x \rangle \rightarrow 1$$

which splits

<table>
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<tr>
<th>$u^p$</th>
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i) $u^p$ is an infinite cycle because if one induces a character of $\langle y \rangle$ non-trivial in $u^p$ one get a repn. $V$ whose Euler class restricts to $u$.  
ii) No differently one into bottom row as extension splits, hence $d_2 v = 0$.  
iii) $d_3 u = 0$ impossible, because then $d_3 u = 0$ by ii)
and so \( u \) is an infinite cycle. This means that

There is a central extension

\[ 1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \]

where \( Z \) cyclic order \( p \) such that \( \pi^{-1}\langle y^p \rangle \) is cyclic order \( p^2 \). If \( \pi \bar{y} = y \), then \( \bar{y} p^3 = 1 \); if \( \pi \bar{x} = x \)

\[ \bar{x} \bar{x} = \bar{y} p^3 \]

\[ \bar{x} \bar{y} \bar{x}^{-1} = \bar{y} 1 + p + \beta p^2 \]

But

\[ 1 = \bar{x} \bar{y} \bar{x}^{-1} \]

and

\[ (1 + p + \beta p^2)^p \equiv 1 + p(1 + \beta p^2 + p(\frac{p-1}{2})(1 + \beta p^2)^2 \mod p^3 \]

so we reach a contradiction.

iv) Thus \( d_2 u = \lambda v \), \( \lambda \neq 0 \), and so row \( t = 2 \) kills the \( t = 1 \) row except for \( v \) and \( bv \).

v) \( d_2 u^i = \bar{v} u^{i-1} \lambda v = (i!) \bar{v} u^{i-1} \)

so the \( t = 2i \) row kills all of the \( t = 2i-1 \) row except for \( \bar{v} u^{i-1} \) and \( bv u^{i-1} \), for \( 1 \leq i < p \).

vi) All other differentials have nowhere to go so are zero.

(continued)
vii) A way of checking this computation is as follows. The spectral sequence we are looking at, is that of

$$H^*_0(SV) = H^*_{<x>}(SV/<y>)$$

where \( V \) is the representation induced from \( X: <y> \to \mathbb{C}^* \) which is non-trivial on \( <y^p> \). This procedure gives an irreducible representation as the degree is \( p \) and one gets at least \( p-1 \) different ones; these are all the non-abelian reps as

\[
\begin{align*}
\text{deg} & : 1 \\
\text{deg} & : 1
\end{align*}
\]

for \( G/F^* \).

Now \( SV/<y> \) has \( <x> \) fixed points; hence contains two copies of \( H^*_<x> \) corresponding to the images of \( f^* \) and \( i^* \):

\[
SV/<y> \xrightarrow{f} \text{ft}
\]

These are the elements corresponding to the two infinite rows. Now the fixed point set \( (SV/<y>)^{<x>} \) is covered by the fixed point sets of the complementary subgroups \( <xy^a> \) where \( a = 0, 1, \ldots, p-1 \). Some notation
\[ \pi: SV \rightarrow SV/\langle y \rangle. \quad \text{Then} \]
\[ \pi^{-1}\left\{ (SV/\langle y \rangle)^{\langle x \rangle} \right\} = \bigoplus_{a=0}^{\infty} SV^{\langle xy^a \rangle} \]

These are conjugate under \( y \), so

\[ H^*_G((SV/\langle y \rangle)^{\langle x \rangle}) = H^*_G\left( \bigoplus_{a=0}^{\infty} SV^{\langle xy^a \rangle} \right) \]
\[ = H^*_{\langle xy, y \rangle}(SV^{\langle x \rangle}) \]

Now \( V \) is the regular representation of \( \langle x \rangle \) so \( V^{\langle x \rangle} \) is one-dimensional. Hence

\[ H^*_{\langle xy, y \rangle}(SV^{\langle x \rangle}) = H^*_{\langle x \rangle} \otimes H^*(SV/\langle y \rangle)^{\langle x \rangle} \]

This also shows \( d_2 u = 0 \) is impossible because one would otherwise get the wrong rank for \( H^*_{\langle x \rangle}(SV/\langle y \rangle) \).

So at this point I am satisfied that my computations are correct and want to see if the cohomology is detected by primary subgroups.

Looking at the formula for \( E^\infty \) we see that \( H^{2p-1}(G) \) is of rank 2 with

\[ 0 \subset E^\infty_{2p-1} H^{2p-1} = F_1 H^{2p-1} \subset H^{2p-1} \]
and where $F_{2^p - 1} H^{2p-1}$ is generated by $ba^p$.

Precisely we have an isomorphism

$$H^{2p-1}(G) \cong H^{2p-1}(\langle x \rangle) \oplus H^{2p-1}(\langle y \rangle).$$

Let $\gamma \in H^{2p-1}(G)$ be the element such that $\gamma|_{\langle x \rangle} = 0$ and $\gamma|_{\langle y \rangle} = \nu u^{p-1}$. Then

(i). Multiplication by $\gamma$

$$H^*(\langle x \rangle) \xrightarrow{\gamma} H^*(G)$$

is injective. In other words the $\lambda \in H^{2p-1}(G)$ all non-zero in $H^*(G)$.

(ii). What is $\gamma|_{\langle x, y \rangle}$?

Let us consider the analogue of the representation $V$ over a finite field $\mathbb{F}_q$ having $\nu q = p^2$, i.e. $q \equiv 1 \mod p^2$ using Dirichlet's theorem. Then one takes a faithful character $\chi: \langle y \rangle \rightarrow \mathbb{F}^\times_p$, and makes $V$ the induced representation. Now I know that

$$c_{\chi}(\chi) = u \otimes \chi,$$

and

$$V|_{\langle y \rangle} = \bigoplus_{\alpha=0}^{p^2-1} \chi^{(1+p)^\alpha}.$$  But $(1+p)^{p^2} \equiv 1 \mod p^2$.
\[ V | <y> = \bigoplus_{a=0}^{p-1} x^{(1 + a \rho)} \]

\[ c_1(x^{\rho d}) = \frac{1}{d} c_1(x) \]

so

\[ c(V) | x = \prod_{a=0}^{p-1} 1 + (1 + a \rho) c_1(x) \]

\[ = (1 + c_1(x))^p \]

\[ = 1 + (u + v \xi)^p \]

and so we don't get the desired element of degree \(2p-1\).
January 2, 1971:

The group $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$ is an example of a $p$-group such that $H^*(G)$ has embedded components. Indeed, the subgroup $A = \langle x, y^p \rangle$ is the only maximal $[p]$-subgroup, so represents the minimal prime. On the other hand, $A$ is its own centralizer so the kernel of $H^*(G) \to H^*(A)$ is the intersection of the other primary ideals. But $H^*(G) \to H^*(A)$ is not injective, so one must have other associated primes than $A$.

I want now to compute the restriction homomorphism from $G$ to $A$. To find degrees $< 2p$ we can look at the map

$$H^*_G(SV) \to H^*_A(SV)$$

$$H^*_{\langle x \rangle} (SV/<y>) \to H^*_{\langle x \rangle} (SV/<y^p>)$$

For general reasons this should be injective after inverting the generator $a \in H^2_{\langle x \rangle}$. Precisely we have a commutative square
where the vertical maps become isomorphisms after inverting $q$. So the next thing to do is compute $H^*_A(SV)$. So $V$ as a rep. of $A = \langle x, y^p \rangle$ is the regular rep of $\langle x \rangle$ tensor product with the character $\chi$ on $y^p$ whose first Chern class is the generator $u$ of $H^2(\langle y^p \rangle) \cong H^2(\langle y \rangle)$. So

$$e(V)|_A = \prod_{i=0}^{p-1} e(id + u) = u^p - a^p u.$$

Consequently

$$H^*_A(SV) = H^*_A/\langle e(V) \rangle = \mathbb{Z}/p \otimes \mathbb{Z}[b, a, w, u]/(u^p - a^p u) = \wedge [b, w] \otimes S[a, u]/(u^p - a^p u)$$
January 3, 1971:

Let \( G = \langle x, y \rangle \), \( x^p = y^p = 1 \), \( xyx^{-1}y^{-1} = y^p \). Then \( H^1(G) \) has for basis the elements \( b, v \) defined by:

\[
\begin{align*}
  b(x) &= 1 \\
  b(y) &= 0 \\
  v(x) &= 0 \\
  v(y) &= 1.
\end{align*}
\]

Then \( bv \in H^2(G) \) is non-zero by the spectral sequence (see p. 17) yet it vanishes on any cyclic subgroup \( C \) as the multiplicative \( H^1(C) \otimes H^1(C) \rightarrow H^2(C) \) is identically zero. In addition it vanishes on the subgroup \( \langle x, y^p \rangle \) because \( v \) does. Consequently \( G \) is an example of a \( p \)-group whose mod \( p \) cohomology is not detected by primary subgroups.

One might hope to be able to prove something by the following inductive method. Suppose \( G \) is a \( p \)-group and

\[
0 \leq Z_1 \leq Z_2 \leq \cdots \leq Z_n = G
\]
is a chief series, i.e. $Z_i \triangleleft G$ and $Z_i/Z_{i-1}$ cyclic of order $p$. Then

$$H^*_G(X) \cong H^*_G(X/Z_1) \oplus H^*_G(X/Z_1)$$

needs a shift depending on codimension of $X/Z_1$ in $X$.

The second factor reduces to a smaller group

$$H^*_G(X-Z_1) = H^*_G((X-Z_1)/Z_1)$$

while the first we have that $e(V)$ is a non-zero divisor and

$$H^*_G(X/Z_1) \cong H^*_G(X/Z_1 \times SV)$$

where $V$ is any representation of $G$ whose restriction to $Z_1$ has no trivial components.

For example we can prove the dimension theorem this way because

$$\dim \{H^*_G(X)\} = \max \{\dim \{H^*_G(X/Z_1)\}, \dim \{H^*_G(X/Z_1)\}\} + 1$$

and hence we get to a smaller group $G/Z_1$. 
The problem with using this method stems from ignorance of \( X - X^2 \), even when \( X \) is nice. Thus in the example before we had to contend with

\[
SV = \prod_{i=0}^n SV \langle y^p_i \rangle
\]

\[
= \bigcup_{i=0}^{p-1} \left[ SV \langle y^p_i \rangle \times S \oplus V \langle y^p_i \rangle \right]
\]

(here \( SV = V = 0 \)). Perhaps this kind of space is not too unreasonable, e.g. for \( p = 2 \) it is just the product

\[
SW \times SW
\]

where \( x \) flips the two factors and \( y \) acts by the characters \( f \) and \( f^3 \).
February 12, 1971

$K$-groups for a curve over a finite field.

Let $X$ be a complete non-singular curve over a finite field $k$. Assume that $k = \Gamma(X, \mathcal{O}_X)$. In the classical picture the function field of $X$ is a finitely generated extension of $k$ of tr.d. 1, and $k$ is the set of elements algebraic over the prime field $\mathbb{F}_p$. The points of $X$ correspond to valuations $v$ on $\mathbb{F}_p$, $\mathcal{O}_v$ is the valuation ring. Writing $F$ as a finite extension of $k([z])$ corresponds to viewing $X$ as a ramified covering:

$$X \longrightarrow \mathbb{P}^1_k$$

finite flat

given by the rational function $z$.

I want to determine all characteristic classes for representations of groups acting on vector bundles over $X$, with coeffs. mod $l$. Given $E$ over $X$

$$\text{End}(E) = \Gamma(X, \text{Hom}(E, E))$$

is finite dimensional over $k$, hence $\text{End}(E)$ and $\text{Aut}(E)$ are finite. Thus we can restrict to considering only reps. of finite groups. (Quite generally this argument shows that when $X$ is proper over $\mathbb{A}$ a field $k$, we can restrict attention to algebraic groups over $k$, for which one may eventually have a good hold on the cohomology of its underlying discrete group.)
Denote by $\mathcal{R}_X(G)$ the Grothendieck group of the $G$-bundles on $X$, and by $\mathcal{R}_X(G)_c$ the Grothendieck group of coherent $G$-sheaves. Then

$$\mathcal{R}_X(G) \rightarrow \mathcal{R}_X(G)_c$$

Indeed, if $F$ is coherent and $L$ is ample on $X$, then

$$\Gamma(X, F \otimes L^\otimes N) \otimes_k L^{-N} \rightarrow F$$

is surjective for $N$ sufficiently large. If $G$ acts on $F$, $\Gamma(X, F \otimes L^\otimes N)$ is a representation of $G$ and the above map is equivariant. This shows $F$ is the quotient of a $G$-bundle, hence we can resolve $F$ by bundles. Thus we get an exact sequence

$$0 \rightarrow E' \rightarrow E \rightarrow F \rightarrow 0$$

where $E', E$ are $G$-bundles, and the rest of the argument should go through. (This should work when $X$ non-singular and either $G$ finite or $X$ proper.)

Next, there should be an exact sequence for $G$ finite

$$\bigoplus_{v \in X} \mathcal{R}_{k(v)}(G) \rightarrow \mathcal{R}_X(G) \rightarrow \mathcal{R}_F(G) \rightarrow 0$$

defined in usual way: Given a repn. over $k(v)$ view it as a coherent $G$-sheaf supported at on $\{v\} \subset X$; this together
with the above gives $\mathfrak{m}_v$. $j^*E = \text{stalk at generic point}$.
(Will assume above OK; surjectivity of $j^*$ clear: given a repn $V$ of $G$ over $F$, choose a coherent sheaf $M$ (ugly notation) over $X$ with $j^*M = V$, i.e. $M = \{M_v\} \vdash M_v \subseteq V$ is an $G_v$ module, then as $G$ finite, we can make $M$ stable under $G$ by averaging.)

Lemma: $R_k(G) \xrightarrow{\sim} R_{F_k}(G)$

Proof: (Need to recall following fact from Galois theory: Given fields

\[
\begin{array}{ccc}
F & \longrightarrow & F_{k_1} \\
\downarrow & & \downarrow \\
k & \longrightarrow & k_1
\end{array}
\]

with $k_1/k$ Galoisian and $F \cap k_1 = k$, then $F_{k_1}/F$ is Galoisian and

$\text{Gal}(F_{k_1}/F) \xrightarrow{\sim} \text{Gal}(k_1/k)$.

If $F/k$ sep. alg., follows as $\text{Gal}(F/k)$ gen by $\text{Gal}(F_{k_1}/k)$ and $\text{Gal}(k_1/F)$. OKAY if $F/k$ purely trans; OKAY if $F/k$ purely insept; i.e. done)

Let $E$ be an representation of $G$ over $k$, i.e. $\text{End}(E)$ skew-field, hence ($k$ finite) a finite extension $k_1$ of $k$. As $k_1/k$ Galoisian, above fact recalled shows that

$\text{End}_k(E)$ finite.
\[
\text{End}_{F[G]}(F \otimes_k E) = F \otimes_k \text{End}_{k[G]}(E)
\]

is a field, hence \(F \otimes_k E\) irreducible. So have
\[
k[G]/\text{rad} = \prod_{i=1}^{n} M_{n_i}(k_i)
\]
\[
F \otimes (k[G]/\text{rad}) = \prod_{i=1}^{n} M_{n_i}(k_i \otimes_k F)
\]
and \(k_i \otimes_k F\) is a field. Thus \(\text{rad } F[G] = F \otimes_k \text{rad } k[G]\) and so \(E_i \mapsto F \otimes_k E_i\) bijective maps of iso classes of irrept. \(k[G]\)-modules \(\approx \) irrept \(F[G]\)-modules. As \(R_k(G) \approx R_F(G)\) are free abelian groups in these iso classes resp., the lemma follows.

The lemma implies that \(j^*\) has a section
\[
\begin{array}{ccc}
R_k(G) & \xrightarrow{j} & R_F(G) \\
\downarrow{S} & & \downarrow{S} \\
R_X(G) & \xrightarrow{j^*} & 0
\end{array}
\]
where \(S(V) = V \otimes_k O_X\).

Proposition: Let \(\Theta: R_X(\bullet) \longrightarrow H^0(\bullet, S)\) be an exponential chart class. Assume \(k|q-1\). If \(\Theta = 1\) on the following representations, then \(\Theta = 1\):

- \(k(v)^*\) acting by multiplication on \(k(v), \quad v \in X\)
- \(k^*\) acting by multiplication on \(O_X\)
Proof. I have to recall my theorem that an exponential class $\Theta: R^s_k(\varphi) \to H^0(\varphi, S)$ with $k_0$ a finite field and $\mu^n \subset k_0^\times$ is determined by the element of $H^0(k_0^\times, S)$ which is $\Theta$ applied to the multiplication map of $k_0^\times$ on $k_0$. Denote this by $\theta(m_k)$; then the map $\Theta \mapsto \theta(m_k)$ is a bijection of the set of exp. classes $\Theta$ and elements of $H^0(k_0^\times, S)$ with augmentation 1. This bijection preserves the group structures also.

Using this, the proposition follows from the fact that

$$R^s_k(G) \oplus \bigoplus_{v \in X} R^s_{k(v)}(G) \xrightarrow{s+i} R^s_X(G)$$

is surjective. Also this surjectivity implies:

**Proposition.** There are no non-trivial $p$-character classes for $R^s_X(\varphi)$.

In effect there are none for $R^s_{k_0}(G)$, $k_0$ finite. In more concrete terms, suppose $\Theta$ an exp. class mod $p$ for $R^s_X$. To show $\Theta = 1$ enough to consider $G$ a $p$-group. If a $p$-group $G$ acts on a vector bundle $E$, then there is a filtration on the generic stalk of $E$ stable under $G$ such that $G$ acts trivially in the quotients. This filtration extends uniquely to a flag $0 \subset E \subset \cdots \subset E$ of $E$ (evident to Carin for separation of $p$-torsion of flag bundles) and $G$ acts trivially on the quotient line bundles, so $\Theta(E) = \prod \Theta(E/E_i) = 1$. 

Let $\Lambda$ be a complete d.v.r. finite over $\mathbb{Z}_p$ and such that $\Lambda \supset \mathbb{Q}_p$. Let an elementary abelian $p$-group $A$ act on a free $\Lambda$-module $E$ in such a way that the action is trivial on $E/(p-1)E$.

I claim that then $E$ is the direct sum of its eigenspaces.

$$E = \bigoplus_{x \in A} E_x$$

$E_x = \{e \mid a.e = x(a)e\}$

It is obviously enough to consider the case where $A$ is cyclic, since the eigenspaces of $A$, a cyclic subgroup of $A$, are stable under $A$. Write

$$E \otimes \Lambda^* F = \bigoplus_{i=0}^{p-1} V_i$$

where a fixed generator $\sigma$ of $A$ has eigenvalue $j^i$ on $V_i$.

Let $j$ be least such that

$$v_j + \cdots + v_{p-1} \in E \Rightarrow v_j, v_{j+1}, \ldots, v_{p-1} \in E.$$  

(then $j \leq p-1$).

Let $e = v_{j-1} + \cdots + v_{p-1} \in E$.

By hypothesis $\sigma$ acts trivially on $E/(j-1)E$, i.e.

$$\sigma e - e \in (j-1)E,$$

so

$$\frac{j-1}{j-1} e = \frac{j-1}{j-1} v_{j-1} + \frac{j-1}{j-1} v_j + \cdots + \frac{j-1}{j-1} v_{p-1}.$$
belongs to $E$. Thus
\[
\frac{y_{j-1}}{y-1} e - \frac{y_{j+1}}{y-1} e = \frac{y_{j-1}}{y-1} v_j + \frac{y_{j-1} - 1}{y-1} v_{j+1} + \ldots + \frac{y_{p-1} - 1}{y-1} v_{p-1} \\
= y_{j-1} \left\{ v_j + \frac{y_{j-1}}{y-1} v_{j+1} + \ldots + \frac{y_{p-1} - 1}{y-1} v_{p-1} \right\} \in E
\]
By choice of $j$ we have
\[
v_j, \frac{y_{j-1}}{y-1} v_{j+1}, \ldots, \frac{y_{p-1} - 1}{y-1} v_{p-1} \in E.
\]
but $\frac{y_{j-1}}{y-1}$ is a unit for $0 < i < p$, hence
\[
v_j, v_{j+1}, \ldots, v_{p-1} \in E, \text{ hence also } v_{j-1} \in E. \text{ This contradicts } j \text{ least unless } j = 0. \text{ Thus}
\]
\[
E = \bigoplus_{i=0}^{p-1} E_i
\]
a claimed.

Remark: The above argument is completely general and reversible.

Theorem: Let $A$ be an integral domain containing $\mathbb{Z}_p$, and let an elementary abelian $p$-group $A$ act on a projective $A$-module $E$. Then
\[
E = \bigoplus_{\chi \in \hat{A}} E_{\chi} \iff A \text{ acts trivially on } E/(y-1)E.
\]
The hope might be that we can compute the cohomology of $\text{GL}_n(\Lambda, (\mathcal{Q}-1)\Lambda) = \text{subgroup of } \text{GL}_n(\Lambda)$ of $A \equiv 1 \mod (\mathcal{Q}-1)\Lambda$, because this group has such a nice $[p]$-subgroup structure. The conjecture, of course, is that this group has no embedded components, whence the cohomology is detected on the centralizer.

$$\Lambda = \mathbb{Z}_p[t^p], \quad \text{SL}_2(\Lambda)$$ is a special case to consider.

Thus $\text{SL}_2(\Lambda, (\mathcal{Q}-1)\Lambda)$ consists of matrices

$$\begin{pmatrix}
1 + \alpha \pi & \beta \pi \\
\gamma \pi & 1 + \delta \pi
\end{pmatrix}, \quad \pi = \mathcal{Q} - 1
$$

such that

$$(1 + \alpha \pi)(1 + \delta \pi) - \beta \pi^2 = 1$$

$$\alpha + \delta \pi + (\delta - \beta \pi) \pi = 0$$

Remark 1: $\text{SL}_2(\Lambda, (\mathcal{Q}-1)\Lambda) = \Gamma$ has at least 3 generators as a pro-$p$-group since

$$\begin{pmatrix}
1 + \alpha \pi & \beta \pi \\
\gamma \pi & 1 + \delta \pi
\end{pmatrix} \mapsto \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \quad \text{(mod } \pi)$$

is a homomorphism onto $$(\Lambda/\pi\Lambda)^3 = (\mathbb{Z}/p\mathbb{Z})^3.$$ 

Remark 2: The subgroup $\Gamma_0 \subset \Gamma$ of index $p$ for which $\alpha \equiv \delta \equiv 0 \pmod{\pi}$ has no
torsion. In effect we've seen that one cyclic subgroup of order \( p \) up to conjugacy and that it is generated by \( (0,1,0) \) which has \( \delta = 1 \). Consequently \( \Gamma_0 \) is a torsion-free analytic \( p \)-group of dimension \( 3(p-1) \), hence by Serre-Lazard is a Poincaré pro \( p \)-group of dimension \( 3(p-1) \).

Remark 3: If you have an analytic \( p \)-group \( G \) with a filtration \( G_i, i \geq 1 \) such that \((G_i, G_j) \subset G_{i+j}, \; G_i \subset G_{i+1} \) and \( p \)th power

\[
G_i \rightarrow G_{i+1}^{G_i} \rightarrow G_{i+1}/G_{i+2}
\]

is an isomorphism, then its cohomology is an exterior algebra on the dual of \( G_1/G_2 \). This applies to

\[
G_1 = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 + \delta \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 + \alpha^p & \beta^p \\ \gamma^p & 1 + \delta^p \end{pmatrix}
\]

because

\[
(1 + \alpha^p \Delta)^p = 1 + \alpha^{p_1} \Delta + \ldots + \alpha^{p_1 \Delta^p}
\]

Thus \( H^*(SL_2(\Lambda, \Lambda)) \) is an exterior algebra on \( 3(p-1) \) generators.
Remark 4: The cohomology \( \Gamma = \{1 + \pi \Delta\} \) is not detected by the centralizer of \((\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix})\) since that centralizer is \((\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix})\) and the homomorphisms

\[
\begin{pmatrix} 1 + \alpha \pi & \beta \pi \\ \gamma \pi & 1 + \delta \pi \end{pmatrix} \rightarrow \beta, \gamma \mod \pi
\]

vanish on this centralizer. Thus either the basic Chern class \( c_2 \in H^4(\Gamma) \) is a zero-divisor, or the primary support for maximal \([p]\)-subgroups doesn't extend to pro-\(p\)-groups.

The building $X$ of $SL_2(F)$ gives amalgamated product representation

$$SL_2(F) = G_0 \times G_1$$

where $G_0$ is stabilizer of $\Lambda$-lattice $L_0 = \Lambda \oplus \Lambda \subset F^2$ and $G_1$ is the stabilizer of $L_1 = \Lambda \oplus \pi \Lambda$ and $G_0 \cap G_1 = G_{01}$. Consequently the Mayer-Vietoris sequence shows that

$$H^*(SL_2(F)) \rightarrow H^*(G_0) \times_{H^*(G_{01})} H^*(G_1)$$

is an $F$-isomorphism. Now this formula tells us what the spectrum of $H^*(SL_2(F))$ is, hence we can check agreement with our conjecture.

i) every cyclic $A$ in $SL_2 F$ conjugate to $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ hence comes from a cyclic subgroup of $G_{01}$.

ii) every $A < G_0$ conjugate to a subgroup of $G_{01}$. Clear as $G_{01}$ contains the Sylow group.

Thus

$$H(G_{01})(\Omega) \rightarrow H(SL_2 F)(\Omega) \rightarrow H(G)(\Omega)$$
iii) Given \( A < G_{01} \) either \( L_0 \) or \( L_1 \) is good

\[ i.e. \ A \text{ acts trivially on } L/\pi L, \ \theta \text{ whence } L \cong \Lambda \oplus \Lambda \]

with \( A \) standard diagonal form. (In effect given \( A < SL_2(F) \) one sees that the lattices fixed by
\( A \) are of two types: either good or bad and that every
neighbor of a good lattice is fixed by \( A \) while, every bad lattice
is joined directly to a good one

\[
good \quad A + \pi \Lambda, \quad \text{bad} \quad \Lambda + \pi \Lambda + \epsilon
\]

This means that \( X^A \) is a "halo" around \( X^H \), \( H = \text{the max. compact subgroup of the centralizer of } A \)

By iii) if \( A < G_{01} \) then \( A \) is conjugate in \( G_0 
\) or \( G_1 \) to standard \( A \). This shows variety of \( H(SL_2(F)) \)

has at most one non-trivial stratum. The Weyl group
for the standard \( A \) in \( G_0 \) being same as for \( SL_2(F) 
\) one sees the conjecture is true.
Computations for the prime 2. The hope is that working with \( SL_2^c(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = \pm 1 \right\} \) avoids the symplectic pathology at 2. Hence

\[
\tilde{G} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_2) \mid \begin{array}{l}
\equiv 1 \mod 2 \\
det \pm 1
\end{array} \right\}
\]

perhaps should be computable. More generally one should look at the subgroup of \( GL_n(\mathbb{Z}_2) \) consisting of matrices \( \equiv 1 \mod 2 \), as this has a nice spectrum. Now \( GL_n(\mathbb{Z}_2;\mathbb{Z}_2) \) is susceptible to Langard theory.

\[
\xrightarrow{\cdot} SL_2(\mathbb{Z}_2;\mathbb{Z}_2) \xrightarrow{\Gamma} (\mathbb{Z}/2\mathbb{Z})^4 \xrightarrow{\cdot} ^
\]

\[
\begin{pmatrix} 1 & 2x \\ 2y & 1 + 2z \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod 2
\]

onto because of \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

By Langard \( H^*(SL_2(\mathbb{Z}_2;\mathbb{Z}_2)) \) should be an exterior algebra on three generators of dimension 1, dual to

\[
\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 8 & -3 \end{pmatrix}
\]

(We can check this by showing that \( 0 \) for each \( n \) we get matrices congruent to \( 1 + \)

\[
2^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 2^n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 2^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mod 2^n \)
but we have the identity

\[
\begin{pmatrix}
1 & \pi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\alpha & 1
\end{pmatrix}
\begin{pmatrix}
1 & \pi \\
\alpha & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\alpha & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1+\pi\alpha & \pi \\
\alpha & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1+\pi\alpha \pi & -\pi^2\alpha \\
\alpha & 1-\pi\alpha
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1+\pi^2a+\pi^2a^2 & -\pi^2a \\
\pi a^2 & 1-\pi a
\end{pmatrix}
\]

take $\pi = 2^i, \alpha = 2^j$ we see that

\[
\begin{pmatrix}
1+2^{i+j} & 0 \\
0 & 1-2^{i+j}
\end{pmatrix} \mod 2^{i+j+1}
\]

so we get the right diagonal terms.

Conclusion: The subgroup $\Gamma = \begin{pmatrix}
1+4\alpha & 2\beta \\
2\beta & 1+4\beta
\end{pmatrix}$ of $\Gamma$

has no torsion and has 2 generators, namely
\[
\begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
2 & 1 \\
\end{pmatrix}
\]

Thus by Poincaré duality the Betti nos. of \( \Gamma_0 \) are 1, 2, 2, 1. Generally, if one considers the spectral sequence of

\[
1 \rightarrow \mathbb{Z}_2 \langle 1 \rangle \rightarrow \Gamma_0 \rightarrow (\mathbb{Z}/2)^2 \rightarrow 1
\]

one has

\[
\begin{align*}
1 & \quad 2 \\
\downarrow & \quad \downarrow \\
3 & \quad 3 \\
\downarrow & \quad \downarrow \\
1 & \quad 3 \\
\end{align*}
\]

where the indicated \( d_2 \) is an isomorphism as \( H^1 \) has 2 generators. This shows that the product

\[
H^1(\Gamma_0) \otimes H^2(\Gamma_0) \rightarrow H^2(\Gamma_0)
\]

is zero.

Now let \( \Gamma_1 = \begin{pmatrix}
1 + 4x & 2x \\
2x & 1 + 2x
\end{pmatrix} \) be the subgroup generated by \( \Gamma_0 \) and \( A = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \), i.e., the semi-direct product. Then \( A \) acts trivially on \( H^1(\Gamma_0) \), since the commutator:

\[
\left\{ \{1+2\Delta\}, \{1+2\Delta\} \right\} \subset \{1+4\Delta\},
\]

is trivialize \( H^3(\Gamma_0) \) and preserve Poincaré duality it follows
that A acts trivially on $H^*(\Gamma_0)$, hence the $E_2$ of the spectral sequence for

$$1 \rightarrow \Gamma_0 \rightarrow \Gamma_1 \rightarrow A \rightarrow 1$$

is of the form of a tensor product

Now $\Gamma_0$ has 3 generators so $d_2(E_2^{01}) = 0$.

Now consider the conjugation action of the matrix

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad (0 -1)^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

of order 3. (Finished Feb 3: I computed that $\Gamma_1$ is stable under this and that, one gets the irreducible decomposition for $H^4(\Gamma_0)$ and $H^3(\Gamma_0)$ while $\omega$ acts trivially on $H^3(\Gamma_0)$. This shows that $d_2, d_3$ vanish on $E_2^{03}$ in above spectral sequence. Only problem left is whether $d_2 E_2^{02} \neq 0$, which is probably the case from past experience: (centre of A in $\Gamma_1$ is $\frac{a}{2} \pm \frac{b}{2}$ a = 1, b = 1)

Computation inconclusive.
January 24, 1971  (David is 7)

Notes on trip to Atlantic City & Institute.

Theorem of Borel: Let $\Lambda$ be a Dedekind domain in a number field $F$ with $r_1$ real places and $r_2$ complex places. Then

$$K_{2i} \otimes \mathbb{Q} = 0 \quad i > 1$$

$$\dim K_{4i+1} \otimes \mathbb{Q} = r_1 + r_2$$

$$\dim K_{4i-1} \otimes \mathbb{Q} = r_2$$

(If $\Lambda$ is the ring of $S$-units in $F$, then $K_{\Lambda} \otimes \mathbb{Q}$ by Dirichlet has rank $r_1 + r_2 - 1 + \text{no. of non-arch. places in } S$)

Borel's theorem amounts to a computation of the homology $H_*(SL(\Lambda), \mathbb{Q})$. Here are the essential steps in his argument.

Now $\Lambda$ is the ring of $S$-units for some possibly infinite set of places in $\mathbb{Q}_F$, because an integral closed domain is the intersection of the discrete valuation rings associated to its minimal primes. By passage to the limit can assume $S$ finite.
The case where $\Lambda$ is the ring of integers:
Then one makes $\Gamma = \text{SL}_n(\Lambda)$ act on the symmetric space
$$X = G/K = \prod_{\text{real places}} \text{SL}_n(\mathbb{R})/\text{SO}_n \times \prod_{\text{cx places}} \text{SL}_n(\mathbb{C})/\text{SU}_n$$

Consider real cohomology. If $\Gamma_0 \triangleleft \Gamma$ is torsion-free then we have maps
$$H^*_\Gamma \to (H^*_\Gamma_0)^{\Gamma_0 \backslash \Gamma} \to H^*_\Gamma_0 \to H^*_\Gamma_0 (G/K) = H^*(\Gamma_0 \backslash G/K)$$

and on the other hand we have a map
$$\text{Complex of } G\text{-invar forms on } X \to DR \text{ ex of } \Gamma_0 \backslash G/K$$

$$\left(\Lambda(g/K)^*\right)^k = I^*$$

and one knows that $d=0$ on $I^*$ (this is standard, symmetric space version of fact that for reductive $G$, $\Lambda^*G$ has zero differentials). Thus we get a map
$$I^* \to H^*(\Gamma_0 \backslash G/K)$$

and it's clear that the image lies in $\Gamma \backslash \Gamma_0$-invariants since $I^*$ forms are $G$-invariant. So we obtain
a map

(*) \[ I^* \rightarrow H^*_\Gamma. \]

Borel proves this map is an isomorphism in a range which increases with \( n \). The method is to use the coreresed extension \( X \) to show that cohomology class of \( \Gamma_0 \backslash X \) can be represented by \( L^2 \)-forms in the good range, then one can generalize the Matsushima treatment when \( \Gamma_0 \backslash X \) is compact. (The surjectivity part comes by an integrating by parts)

\[
(\Delta \omega, \omega) = \sum a_{ij} (\chi_i \omega, \chi_j \omega) + b ||\omega||^2
\]

where the constants are such that these things imply a harmonic form on \( \Gamma_0 \backslash X \) is actually \( G \)-invariant.

I should be able to understand the above map (*) in concrete representation terms. Thus it should be possible to construct additive characteristic classes using the real and complex places.
Let $G$ be a connected real Lie group and $K$ a maximal compact subgroup. Look at the de Rham complex:

$$0 \to \mathbb{R} \to A^0(X) \to A^1(X) \to A^2(X) \to \cdots$$

as a resolution of $\mathbb{R}$ in the category of $G$-modules. Here $X = G/K$ which is contractible; by $G$-modules we mean $G$ with discrete topology, although it makes sense to work with continuous $G$-modules and then, so it seems, the classes to be defined live in the continuous EM coh. of $G$ coefficients in $\mathbb{R}$. This resolution gives rise therefore to a spectral sequence

$$E_1^{p,q} = H^p(G, A^q(X)) \Rightarrow H^{p+q}(G, \mathbb{R})$$

with edge homomorphism

$$E_2^{p,0} \to H^p$$

and use that the invariant differential forms

$$H^0(G, A^1(X)) = \Gamma'(X)$$
on the symmetric space $X$ has zero differentials. Hence the edge homomorphism gives map

$$\text{I}^p(X) \longrightarrow H^p(G_d, \mathbb{R})$$

$$(N^p / X)^k$$

(As mentioned these classes lie in the Eilenberg-MacLane cent. coh. of $G$ and I think it's known that the above map is an isomorphism with this cohomology.)
February 16, 1971

Let \( X \) be a complete non-singular curve over an alg. closed field \( k \). Then for \( G \) finite we have

\[ R_k(G) \otimes K(X) \longrightarrow R_X(G). \]

Indeed let \( G \) act on a vector bundle \( E \). Let

\[ E = \sum V_i \otimes \text{Hom}_G(V_i, E) \]

as in the compact group case. Then one decomposes \( E \) according to the reps. of \( G \).

This being so, it follows that an exponential class

\[ R_X(G) \longrightarrow H^0(G, S) \]

where \( S \) is a graded anti-commutative algebra over \( \mathbb{Z} \), will be trivial in the divisible part of \( \text{Pic}(X) \).

Recall that

\[ K(X) = \mathbb{Z} \oplus \text{Pic} X \]

\[ \cong \mathbb{Z} \oplus \text{Pic}^0 X \oplus \mathbb{Z}. \]

The point is that \( R_k(G) \otimes \text{Pic}^0 X \) is divisible, hence there can be no homomorphism of it into an \( l \)-complete group.
For any finite group $G$, say of order prime to $p$, let $V_1, \ldots, V_m$ be the distinct irreducible representations of $G$ over $k$, and let $L_i = \text{End}_{k[G]}(V_i)$. Then $L_i$ is a finite extension of $k$. Now

$$\text{End}_{k[G]}(V_i) \cong \overline{k} \otimes_k L_i \cong \overline{k}^{d_i},$$

where $d_i = \frac{[L_i : k]}{[k : k]}$. These $\overline{k} \otimes_k V_i$ are a sum of $d_i$ inequivalent representations over $\overline{k}$, in fact an orbit under the Galois group.

$$\overline{k} \otimes_k V_i \cong \bigoplus_{a = 0}^{d_i - 1} \overline{E}^a(W_i),$$

where $W_i$ is irreducible over $\overline{k}$. Now, by Wedderburn

$$k[G] \cong \prod_{i=1}^m M_{n_i}(L_i) \quad |G| = \sum n_i^2 d_i.$$ 

$$\overline{k}[G] \cong \prod_{i=1}^m M_{n_i}(\overline{k} \otimes_k L_i)$$

so it's clear that $\{\otimes a W_i, \text{ for } a < d_i, \overline{a} \leq m\}$ is the complete...
set of irreducible representations of G over k. Thus

\[ R_k(G) \xrightarrow{\sim} R_k^G(\text{Gal}(\overline{k}/k)) \]

Now given a G-vector bundle E over X, we have

\[(x) \quad E \cong \bigoplus_{i=1}^m \text{Hom}_{k[G]}(V_i, E) \otimes L_i \]

where \(\text{Hom}_{k[G]}(V_i, E)\) is the G-invariant subbundle (recall \(G\) is (1) prime to \(p\)) of \(\text{Hom}_X(\mathcal{O}_X \otimes_k V_i, E)\). Now set

\[ E_i = \text{Hom}_{k[G]}(V_i, E). \]

It has \(L_i\) as endomorphisms, hence is the restriction of a bundle on \(L_i \otimes_k X\). Thus we obtain an iso:

\[ R_X(G) \xleftarrow{\sim} \bigoplus_{i=1}^m \sum_{i=1}^m E_i \otimes L_i \otimes V_i \xrightarrow{\sim} K(L_i \otimes_k X) \]

On the other hand we have

\[ R_X(G) \xleftarrow{\sim} \bigoplus_{i=1}^m \bigoplus_{a=0}^{d_i-1} K(X) \]

\[ \sum_{i=1}^m E_i \otimes k^{d_i} W_i \xrightarrow{\sim} K(X) \]
Since
\[ \overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} (E_i \otimes_{L_i} V_i) = (E_i \otimes_{L_i} \overline{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} (\overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} V_i) \]
\[= (E_i \otimes_{L_i} \overline{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} (\overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathcal{E}_i W_i) \]

all we have to show is that
\[ K(L_i \times_k X) \rightarrow K(X) \text{ Gal}(\overline{\mathbb{F}_p}/L_i) \]
\[ E_i \rightarrow E_i \otimes_{L_i} \overline{\mathbb{F}_p} \]
is an isomorphism.

So we are reduced to showing that
\[ K(X) \rightarrow K(X) \text{ Gal}(\overline{\mathbb{F}_p}/k) \]

and recall that
\[ K(X) = \mathbb{Z} \oplus \text{Pic}^0(X) \text{ canonical} \]
\[ \text{Pic}(X) = \mathbb{Z} \oplus \text{Pic}^0(X) \text{ dependence on a divisor of degree 1} \]

also for \( \overline{X} \), hence want to show
\[ \text{Pic}(X) \rightarrow \text{Pic}(\overline{X}) \text{ Gal} \]

but we have
\[ 0 \rightarrow \overline{\mathbb{F}_p} \rightarrow \mathbb{F}_p \rightarrow \text{Div}(X) \rightarrow \text{Pic}(X) \rightarrow 0 \]
and \( H^1(\text{Gal}, \overline{k}^x) = 0 \)

\[ \begin{array}{c}
\text{F}^x = 0 \\
H^2(\text{Gal}, \overline{k}^x) = 0
\end{array} \]  

Hence \( H^4(\text{Gal}, I) = 0 \).

So

\[ F^x \rightarrow \text{Div}(\overline{x})^\text{Gal} \rightarrow \text{Pic}(\overline{x})^\text{Gal} \rightarrow 0 \]

exact. Since \( \text{Div}(\overline{x})^\text{Gal} = \text{Div}(x) \)

one sees that \( \text{Pic}(x) \rightarrow \text{Pic}(\overline{x})^\text{Gal} \), q.e.d.

(As a check we expect 5 terms

\[ \begin{array}{c}
0 \\
H^4(\text{Gal}, H^1(X, G_m)) \rightarrow H^1(X, G_m) \rightarrow H^0(\text{Gal}, H^1(X, G_m)) \rightarrow H^2(\text{Gal}, H^1(X, G_m)) \\
H^4(\text{Gal}, \overline{k}^x) \rightarrow H^2(\text{Gal}, \overline{k}^x)
\end{array} \]

Next problem: Compute natural transf. from

\[ R_X(G) = R_k(G) \oplus R_k(G) \oplus \left[ R_k(G) \otimes \text{Pic}^0(x) \right]_{\text{Gal}(k/\overline{k})} \]

to \( H^0(G, S.) \).

It is now clear that your earlier version of the stability theorem has a mistake due to the fact that a non-degenerate simplex in the singular semi-simplicial set of a simplicial complex may have repeated vertices, e.g. (x y x).

\[ \Lambda \text{ is an alg. over } \mathbb{Z}[e^{-1}] \]

**Special case:** Assume that \( e \) for some prime \( e \) and that the coefficients of cohomology is \( \mathbb{Q} \). Then we know that \( \text{the }\Lambda\text{-homomorphism} \[ GL_i \rightarrow \begin{bmatrix} I_{n-i} & 0 \\ \Lambda^{n-i} & GL_i \end{bmatrix} \]

induces an isomorphism on cohomology. (Recall: Let the group \( GL_i \times \text{Hom}(\Lambda^{n-i}, \Lambda^i) \) be denoted \( G \times M \) and consider \( H-S \) for extension

\[ 0 \rightarrow M \rightarrow G \times M \rightarrow G \rightarrow 0 \]

\[ E_2^{ij} = H^i(G, H^j(M)) \Rightarrow H^{i+j}(G \times M). \]

Consider the automorphism of the spectral sequence produced by conjugating by \( \phi \in G \). Recall

\[ H^i(M) = \text{Hom}_\Lambda(\Lambda^i M, \mathbb{Q}) \]
This isomorphism is functorial in the abelian group $M$. As

$$\Theta \left[ \begin{array}{c|c} I & I \\ \hline m & I \end{array} \right] = \left[ \begin{array}{c|c} I & I \\ \hline \lambda I & m \end{array} \right] \left[ \begin{array}{c|c} I & \lambda I \\ \hline m & I \end{array} \right] = \left[ \begin{array}{c|c} I & I \\ \hline \lambda m & I \end{array} \right]$$

we have that $\Theta$ acts on $H^q(M)$ by multiplying by $\lambda^q$, i.e., $\Theta^*(u) = \lambda^q u$. On the other hand, $\Theta$ acts trivially on $G$ since $\lambda I$ is in the center of $G$. Thus $\Theta$ acts on $E_2^{q,s}$ by multiplying by $\lambda^q$, i.e., $\Theta^* u = \lambda^q u$ if $u \in E_2^{q,s}$. Naturality of the spectral sequence and fact that $\lambda + 1 \Rightarrow$ all differentials are zero. But $\Theta$ acts trivially on abutment, hence $E_2^{q,s} = 0$ for $s > 0$ and $E_2^{0,0} = H^n(G) \Rightarrow H^q(G \times M)$.

Now we consider the following simplicial complex denoted $X^n$. A vertex of $X^n$ is a unimodular vector $v$ in $\Lambda^n$, i.e., a direct injection $v : \Lambda \rightarrow \Lambda^n$. An $i$-simplex of $X^n$ is by definition a subset $\{v_0, \ldots, v_i\}$ such that

$$\sum_{\sigma \subseteq i} \sigma^i v_\sigma = 0$$

is a direct injection. We assume for some integer $d$
(i) $\text{GL}_n \Lambda$ acts transitively on the set of direct injections from $\Lambda^j$ to $\Lambda^n$ for $1 \leq j < n$.

(ii) $\tilde{H}_j(X^n, \mathbb{Q}) = 0$ for $1 \leq j < n$.

For example, if $\Lambda$ is a field, or more generally a local ring, then (i) holds with $d = 0$. If $\Lambda$ is an infinite field, then (ii) holds with $d = 0$, since I know that $X^n$ has the homotopy type of a wedge of $S^{n-1}$'s.

In virtue of (ii), the complex of rational chains

(*): $C_{n-d-1}(X^n) \rightarrow C_{n-d}(X^n) \rightarrow \cdots \rightarrow C_1(X^n) \rightarrow C_0(X^n) \rightarrow \mathbb{Q} \rightarrow 0$

has homology beginning in dimension $n-d-1$.

In virtue of (i) we have with $\Delta_j = \left[ \begin{array}{c} \text{Im} \delta_j \delta_j^\text{tr} \end{array} \right]_{\Lambda^n}$ that

$$\text{set of } (j-1)\text{-simplices} = \text{GL}_n \Lambda \backslash \left[ \begin{array}{c} \Sigma_j \delta_j^\text{tr} \\ \Sigma_j \delta_j \end{array} \right]_{\text{GL}_{n-d}}$$

for $1 \leq j < n$, hence

$$C_{j-1}(X^n) = \text{Ind}_{\text{GL}_{n-j}^+}^{\text{GL}_n} \left( \text{sgn} \left[ \begin{array}{c} \Sigma_j \delta_j \delta_j^\text{tr} \\ \Sigma_j \delta_j \end{array} \right]_{\text{GL}_n} \right)$$

where sgn denotes the sign representation of $\left[ \begin{array}{c} \Sigma_j \delta_j^\text{tr} \\ \Sigma_j \delta_j \end{array} \right]_{\text{GL}_{n-j}}$ on $\mathbb{Q}$.

Thus

$$H_q \left( \text{GL}_n, C_{j-1}(X^n) \right) = H_q \left( \left[ \begin{array}{c} \Sigma_j \delta_j^\text{tr} \\ \Sigma_j \delta_j \end{array} \right]_{\text{GL}_{n-j}}, \text{sgn} \right)$$
\[ H_0 \left( \left[ \begin{array}{c|c} I & 0 \\ \hline x & GL_{n-j} \end{array} \right], \text{sgn} \right) \sum_j \quad (\text{over } \mathbb{Q} \text{ used here})
\]

\[ = \left( H_0 \left( \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & GL_{n-j} \end{array} \right] \otimes \text{sgn} \right) \right) \sum_j
\]

\[ = 0 \quad \text{if} \quad j \geq 2
\]

Now use acyclicity of

\[ 0 \longrightarrow Z_{n-d-1}(X^n) \longrightarrow C_{n-d-1}(X^n) \longrightarrow \cdots \longrightarrow C_1(X^n) \longrightarrow Z_0(X^n) \longrightarrow 0
\]

and you have

\[ H_0^G(Z_0(X^n)) \sim H_{-n+d+1}(GL_n, Z_{n-d-1}(X^n))
\]

i.e.

\[ H_0^G(Z_0(X^n)) = 0 \quad \text{if} \quad g < n-d-1
\]

so as

\[ \cdots \rightarrow H_0^G(Z_0(X^n)) \rightarrow H_0^G \left( \left[ \begin{array}{c|c} I & 0 \\ \hline x & GL_{n-j} \end{array} \right] \right) \rightarrow H_0^G(GL_n)
\]

we see that

\[ H_0^G(GL_{n-1}) \rightarrow H_0^G(GL_n) \quad \text{subjective } g = n-d-1
\]

\[ H_0^G(GL_{n-1}) \rightarrow H_0^G(GL_n) \quad \text{isomorphism } g = n-d-1
\]
Remarks concerning the range:

Topologically one obtains the same estimate

\[ H_b(S^0) \to H_b(S^n) \]

because of the Gysin sequence.

\[ H_b(S^{n-1}) \to H_b(S^n) \to H_b(S^n, S^0) \to H_b(S^n, S^0) \to \cdots \]

Thus, this range is what we expect.

Actually it appears that we get a more general result. Let \( M \) be a \( \mathbb{Q}[GL_n] \)-module.

\[ H_b(GL_n, C_{j-n}(x^n) \otimes \mathbb{Q} M) \]

\[ = H_b \left( \left[ \begin{array}{c|c} \Sigma_j & 0 \\ \hline \ast & GL_{n-j} \end{array} \right], \text{sgn} \otimes M \right) \]

\[ \Rightarrow H_b \left( \left[ \begin{array}{c|c} I & 0 \\ \hline \ast & GL_{n-j} \end{array} \right], \text{sgn} \otimes M \right) \Sigma_j \]

Now assume \( E_n \Lambda \).
acts trivially on $M$ set the following module in Bass-stable range

as well as the permutation matrices and the diagonal matrices with entries $e^n$, $n \in \mathbb{Z}$. Then the above argument goes through and shows

$$H_8 \left( \frac{I}{x} \mathbf{GL}_{n-j}, \text{sgn} \otimes M \right) \cong 0 \quad \text{for } g \geq 2$$

hence

$$H_8 \left( \mathbf{GL}_{n-1}, M \right) \rightarrow H_8 \left( \mathbf{GL}_n, M \right) \quad g < n-d-1$$

What one needs now,

(i) argument for $\mathbb{Z}/p\mathbb{Z}$-cohomology

(ii) argument for $\pi_1$
acyclicity of standard complex for a finite field.
 Chern classes using geometric things.

One method of proving acyclicity is to reduce to building
other possibility is to make a base extension
$n-1 \to n$

$H^*(BO_n(F_8)) = \mathbb{Z}/2 \mathbb{Z} \left[ \omega_1', \ldots, \omega_n' \right]$

and $\omega_n', \omega_n''$ disappear on $BO_n(F_8)$.

So the map

\[ H_{n-1}(BO_{n-1}(F_8)) \to H_{n-1}(BO_n(F_8)) \]

is injective but not surjective.

$O_2(F_8) = \mathbb{Z} \times F_8^{*}$

\[ \mathbb{Z}_2[t_1, t_2, e] / (t_1^2 + t_2) \quad e \in H^2 \quad \omega_2 \]

$t_1, t_2 \in H^1$

$O_1(F_8) = \mathbb{Z}/\mathbb{Z}$ (±1)

And only single generator.

Thus

\[ H^2(BO_2(F_8)) \to H^2(BO_1(F_8)) \]

not surjective
not true that what you want holds unstably.

question: given finite field k when is

\[ H^*(\text{GL}(k)) \] 

Z/pZ coeff

first non-trivial?

for the symmetric groups one knows the basic classes

\[ H(\Sigma_{2^n}) \]

\[ H^s(\Sigma_{2^n-1}) \rightarrow H(\Sigma_{2^n}) \]

not onto in dim 2^{n-1}

\[ H_i(BO_{n-1}) \xrightarrow{\text{inj always}} H_i(BO_n) \]

\[ \xleftarrow{\text{inj always}} H^i(BO_n) \xleftarrow{\text{inj always}} H^i(BO_{n-1}) \]
and so it is now clear that $\Theta$ can be any automorphism of the exact sequence

$$0 \to P \xrightarrow{j} Q$$

Thus $\Theta$ is essentially an arbitrary auto of $\text{Ker}(j)$.

so go back to preceding page.

Also

$$\begin{align*}
\tilde{i}''(p,r) &= \tilde{i}''(p,0) + \tilde{i}''(0,r) \\
&= (p,0) + (x,y)
\end{align*}$$

$\tilde{x} = \tilde{j}''(x,y) = \tilde{j}\tilde{j}''(x,y) = \tilde{j}\tilde{j}''(0,r) = \tilde{j}(0,r) = 0$.

$\therefore \tilde{i}''(R) \subset R'$.

Then

$\tilde{j}\tilde{j}''(0,r') = \tilde{j}(0,r') = 0$.

$\therefore \tilde{j}''(R') \subset R$.

$\tilde{j}''(P) = \tilde{j}''(c^*P) = P.$
to the full subcategory whose objects are diagrams
\[(P, Q) \xrightarrow{\alpha} (P', Q') \xleftarrow{\text{id}_{P'}} (0, 0).\]

Another possibility would be to associate to a pair consisting of a module \(R\) and an isomorphism \(\alpha : P \oplus R \cong Q \oplus R\) the object of \(\text{End}(P, Q)\) consisting of the diagram
\[\langle R, \alpha \rangle : (P, Q) \xrightarrow{(\alpha, \text{id}_P)} (P, Q) \xrightarrow{\alpha} (0, 0)\]
\[\langle (m_1, m_2), (n_1, n_2) \rangle : (m_1, m_2, n_1, n_2) \xrightarrow{\alpha} (0, 0, 0, 0).\]

Clearly, every object of \(\text{End}(P, Q)\) is isomorphic to such a one.

So what I want now is to calculate the set of maps from \(\langle R, \alpha \rangle\) to \(\langle R', \alpha' \rangle\).

\[\begin{align*}
\alpha(p \oplus 0) &= (p, 0), \\
\alpha(p \oplus r) &= (p, 0, 0) + \alpha(p \oplus r) \\
\beta(p \oplus 0) &= (p, 0), \\
\beta(p \oplus r) &= (p, 0, 0) + \beta(p \oplus r).
\end{align*}\]
Suppose \((P, Q)\) is in the identity component of \(SD\). I propose to consider the category of stable trivializations of \((P, Q)\) or of stable isomorphisms of \(P\) and \(Q\). Its objects should be commutative diagrams in \(SD\) with morphisms given by comm. diagrams.

\[
\begin{array}{ccc}
(P, Q) & \rightarrow & (P', Q') \\
\downarrow & & \downarrow \\
(0, 0) & \leftarrow & (0, 0)
\end{array}
\]

\[
\begin{array}{ccc}
(P, Q) & \rightarrow & (P'', Q'') \\
\downarrow & & \downarrow \\
(P', Q') & \leftarrow & (0, 0)
\end{array}
\]

\[
\begin{array}{ccc}
(P', Q') & \rightarrow & (P'', Q'') \\
\downarrow & & \downarrow \\
(0, 0) & \leftarrow & (0, 0)
\end{array}
\]

As a start I should try to determine the components of this category. Observe it is non-empty iff \(\text{cl}(P) = \text{cl}(Q)\) in \(K_0\).

**NOTATION:** \(ST(PQ)\)

Notice that the map \((*)\) is an isomorphism iff

\[
(P', Q') \rightarrow (P'', Q'')
\]

is a pair of isomorphisms \(P' \cong P'', Q' \cong Q''\). Thus I may replace \(ST(PQ)\) by the equivalent subcategory whose objects are diagrams of the form

\[
\begin{array}{ccc}
(P, Q) & \rightarrow & (P', P') \\
\downarrow & & \downarrow \\
(0, 0) & \leftarrow & (0, 0)
\end{array}
\]

Assuming the second arrow is given by an automorphism \(\alpha: P' \cong P'\), if I have

\[
(P', P') \xrightarrow{\theta_1 \theta_2} (P', P')
\]

\(\alpha\) gets changed to \(\theta_2 \times \theta_1^{-1}\). Therefore \(ST(PQ)\) is equivalent...
will send \((\Theta_1\Theta_2) \mapsto \text{id}\) if \(\exists x \in \text{Aut}(P)\) such that
\[
\alpha = \Theta_2 \times \Theta_1^{-1}
\]
i.e.
\[
\alpha \Theta_1 = \Theta_2 \alpha
\]
i.e.
\[
\alpha \Theta_1 x^{-1} = \Theta_2
\]
In particular
\(g_2 g_1\) is conjugate to \(g_2 g_1\), so we see that
\[
(g_1 g_2, g_2 g_1) \mapsto \text{id}
\]
\[
-(x, x) \mapsto \text{id}
\]
and therefore \(\exists\) factorization:
\[
\text{Aut}(P) \times \text{Aut}(P) \rightarrow \text{Aut}(F(P))
\]
\[
\text{Aut}(P) \rightarrow \text{Aut}(F(P))
\]
\[
\Theta_1 \Theta_2
\]
\[
\Theta_1 \Theta_2^{-1}
\]
so let us compute the fundamental group, which must be abelian as we have an h-space.

\[ F(P) \text{ morph. inv. functor} \]

\[ \text{can suppose the pair } (o \leq P, id_P) \]

gives \[ F(0) \to F(P). \]

Then \[ \text{given } P \alpha \in \text{Aut}(P) \]

\[ (o \leq P, \alpha) : F(0) \to F(P) = F(0) \]

so we get an element \[ [P\alpha] \in \text{Aut}(F(0)). \]

Observe it depends only on the conj. class?

\[ (P, \alpha) \in \text{Aut}(F(P)). \]

\[ \text{Aut}_G(P) = \text{Aut}(P) \times \text{Aut}(P). \]

\[ \text{Aut}(P)^2 = \text{Aut}_G(P) \to \text{Aut}(F(P)) \cong \text{Aut}(F(0)). \]
The diagram shows a mapping between points $(P, Q)$ and $(P', Q)$ with transformations $\theta_2 \theta_1^{-1}$ and $\theta_2 \alpha \theta_1^{-1}$. The inverse mapping $\beta^{-1} : P \to P$ is also indicated, along with the transformation $(\theta_2 \alpha \theta_1^{-1}) \theta_2 \beta \theta_1^{-1}$.
Actually, it seems possible to use polar notation.

This is a commutative diagram where the diagonal and inverse of line are got a commu.

so what I must want do to see that there is

\[ p \xrightarrow{\theta} p \]

\[ \phi \]

\[ \theta \]

\[ p \xrightarrow{\theta} p \]

\[ \phi \]

\[ \theta \]

\[ p \xrightarrow{\theta} p \]

\[ \phi \]

\[ \theta \]

\[ p \xrightarrow{\theta} p \]

\[ \phi \]

\[ \theta \]

\[ p \xrightarrow{\theta} p \]

\[ \phi \]

\[ \theta \]

\[ p \xrightarrow{\theta} p \]

\[ \phi \]

\[ \theta \]
\[ P \xrightarrow{(\theta, \theta)} P, \theta \] 

\[ (0 \leq P, \alpha) \quad (0 \leq P, \beta \theta \alpha \theta^{-1}) \]

\[ (P, P) \xrightarrow{(\theta, \theta)} (P, P) \]

\[ (0, 0) \]

\[ P \xrightarrow{\theta \alpha \theta^{-1}} P \]

Observe that the \( \Delta \)

\[ (P, P) \xrightarrow{(\theta, \theta)} (P, P) \]

\[ (0, 0) \]

Shows that

\[ (\theta, \theta) : F(P) \longrightarrow F(P) \]

will be the identity iff \( \exists \alpha \in \mathbb{G} \) such that \( \theta_2 \alpha \theta_1^{-1} = \alpha \)

i.e., \( \theta_2 \alpha = \alpha \theta_1 \)

Example, this implies it is the identity if \( \alpha = \text{id} \) and \( \theta_2 = \theta_1 \)

or if \( \theta_2 = \alpha \theta_1 \theta_2^{-1} \).

are conjugate. i.e., the quotient is abelian

\[ \delta_1 (g_1 g_2 g_1) = g_2 g_1 \]

\[ (g_1 g_2 g_1)^* = 1. \]
The point to observe probably is that if we have
\[ 0 \to P \to E \to V \to 0 \]
and an auto-\( \Theta \) of \( E \) inducing the identity on \( P \) and on \( V \), then I can conclude that by means of the isomorphism \( E \times E = E \times P \)
that \( \Theta \times \Theta \) is conjugate to \( \Theta \circ \text{id}_P \).

\[
\begin{array}{ccc}
(E,E) & \rightarrow & (E \times E, E \oplus P) \\
\Theta & \nearrow & \leftarrow \\
(0,0) & & \\
\end{array}
\]

\[
\begin{array}{ccc}
(E,E) & \rightarrow & E \oplus P, E \\
\Theta & \nearrow & \\
(0,0) & \rightarrow & (E,E) \\
\end{array}
\]

Basic fact: a map \( \gamma : P \to Q \)

\[
\gamma = (i \circ \Theta \circ i, \Theta) \in Aut(Q)
\]

and we observe that any auto. of \( Q \)
maybe we should try the functor
\[ R \rightarrow \text{Iso}(P \oplus_R Q \oplus R, Q \oplus R) \times \text{Aut}(Q \oplus R) K \]

**Question:** Why is an elem. auto trivial?

\[ \theta \xrightarrow{\text{BP}} \begin{array}{c} \text{id} \\ (0,0) \end{array} \]

\[ P = P_1 \oplus P_2 \]
\[ \theta = \text{id} + \iota P_2 \quad \iota : P_2 \rightarrow P_1 \]

Thus the basic problem is clear. Why in the diagonal category does an elementary auto go to zero?

In any case what happens is clear.
The category consists of \((P, x)\) with a map \((P, x) \rightarrow (P', x')\).

Brian \((P, Q)\) with \(clP = clQ\) in \(K_0\).

I was interested in paths in \(\mathbb{D}\) of the form

\[
(P, Q) \rightarrow (P', Q') \rightarrow (0, 0)
\]

These form the objects of a category in which the maps are

\[
(P, Q) \xrightarrow{f} (0)
\]

\[
\xrightarrow{g}
\]

Comm. \text{ alternately objects are pairs} \((R, P \oplus R \Rightarrow Q\Theta)\), \text{and a morphism} \((R, \Theta) \rightarrow (R', \Theta')\) is an isomorphism.

\[
R \oplus S \rightarrow R'
\]

\[
P \oplus R \oplus S \rightarrow P \oplus R'
\]

\[
S \Theta \oplus S' \Theta'
\]

\[
Q \oplus R \oplus S \rightarrow Q \oplus R'
\]

Thus we have the cofibred category over \(I\) with discrete fibres defined by the functor

\[
R \rightarrow I_{\mathcal{O}}(P \oplus R, Q \oplus R).
\]

If \(P, Q\) are \(0\). Then we have the functor

\[
R \rightarrow \text{Aut}(R)
\]

not reasonable.
Thus before I can get my hands on the universal covering of $\mathcal{H}_0(G)$, I need to select a maximal tree, unless of course there is a natural candidate for the universal covering. It might be associated to a pair $(P, Q)$ the set of sections of the equivalence classes of stable isos. of $P$ and $Q$.

$$(P, Q) \rightarrow \text{Iso}(P \oplus R, Q \oplus R)$$

Thus we will consider paths in $\mathcal{I}$ of the form:

$$(0, 0) \rightarrow (, ) \rightarrow (P, Q) \rightarrow (0, 0)$$

Is it possible to put a reasonable equivalence relation on these? We are considering pairs of objects of $\mathcal{I}$ which are isomorphic under $(P, Q)$ and $(0, 0)$.

**Question:** Is a component of this category what we want?

**Example:** Take $$(P, P) \rightarrow (, ) \leftarrow (0, 0)$$

Suppose we have $\alpha: P \oplus R \sim P \oplus R$.

**Better Suppose:** We have $\alpha: P \rightarrow P$ becoming $0$ in the K.A.

Already a reasonable question to ask:

$$(0, 0) \rightarrow (P, Q) \rightarrow (0, 0)$$

$\beta \circ \alpha \in \text{Aut}(P).$$
We have defined $\mathcal{S}$ as a set of pairs $(P,Q)$ and determined its homology. I want now to compute its homotopy groups with basepoint at $0$.

Since the $O$-camp is an $H$-space, we know

$$\pi_1(O) = H_1(\mathcal{S}) \cong \varinjlim \mathbb{Z} \cong H_1(\text{Aut } P) = K_1P.$$ 

I should give the isomorphism $\mathcal{S}$ enough to give a functor

$$\mathcal{S} \rightarrow K_1P$$

$$(P \oplus Q, R) = (P, Q) \rightarrow$$

so you would have to first of all give a maximal tree, i.e. join $(P, Q)$ to $(Q, Q)$, and this amounts to giving a stable isomorphism $P \oplus S \cong Q \oplus S$.

$$H_*(\mathcal{S}) = \bigoplus_{S} H_*(\text{Aut } P_S)$$

$$H_*(\mathcal{S})[S^{-1}] = \mathbb{Z}[S] \otimes H_*(\mathcal{S}) / S$$

$$a/s \quad a \in H_*(\text{Aut } P_S)$$

$$s \in S$$

$$t - s = a.$$
Let \( G \) be the functor from \( J \) to groups such that
\[
G(P) = \text{Aut}(P)
\]
and such that if \( (i; p) \) denotes the morphism from \( P \)
to \( P' \) given by the pair
\[
P \xrightarrow{i} P'
\]
then \( (i; p)_* : G(P) \to G(P') \) is the homomorphism
given by
\[
(i; p)_* i = i \\
(i; p)_* \alpha = \text{id on } \ker(p).
\]
Associated to this functor is a cofibered category over \( J \), which will be denoted \( \mathcal{G}_J \), whose
fibers over \( P \) is the group \( G(P) \) regarded as a category.
The category \( \mathcal{G}_J \) has the same objects as \( P \), and
a morphism from \( P \) to \( P' \) in \( \mathcal{G}_J \) is a pair
\[
(\alpha; i; p) \text{ consisting of a morphism } (i; p) : P \to P' \text{ in } J \text{ and an element } \alpha \text{ of } G(P).
\]
Composition is defined by the formula
\[
(\alpha; i; p)(\beta; i; p') = (\alpha \circ \beta; i; p) \circ (i; p).
\]
We may identify \( \mathcal{G}_J \) with the full
subcategory of \( J \) consisting of the pairs \( (P; P) \), as follows:
Associate to \( P \) the pair \( (P; P) \) and to \( (i; p) : P \to P' \)
the triple consisting of the pair
\[
p \xleftarrow{p \theta^{-1}} p' \xrightarrow{i \theta} p \xrightarrow{i} p'
\]
and the isomorphism
\[
\text{Ker } (p \theta^{-1}) \cong \text{Ker } p
\]
induced by \( \theta \). We note that this second description shows that \( G \delta \) is a connected \( h \)-space.

We now wish to determine the fundamental group \( \pi_1(G \delta, o) \).
Let \( G: I \to \text{Groups} \) be the functor with

\[ G(P) = \text{Aut}(P) \]

and where \( \theta: P \oplus R \to Q \) is a morphism in \( I \), then \( \alpha: G(P) \to G(Q) \) is the composition

\[ \text{Aut}(P) \to \text{Aut}(P \oplus R) \to \text{Aut}(Q) \]

\[ \alpha \to \alpha \oplus \text{id}_R \to B(\alpha \oplus \text{id}_R) \]

Associated to this functor \( \times \) is a cofibred cat. over \( I \) denoted \( JG \)

We let \( JG \) denote the cofibred category over \( I \) with fibre the group \( G(P) \) over \( P \).

The objects of \( JG \) are the same as the objects of \( I \) and a morphism from \( P \) to \( Q \) in \( JG \) is a pair consisting of a morphism \( \alpha: P \to Q \) in \( I \) and \( \beta \in \text{Aut}(P) \) with composition defined by

\[ (v, \beta)(u, \alpha) = (vu, \beta(\alpha)) \]

We may identify \( JG \) with the full subcat of \( IG \) containing all the pairs \((P, P)\) as follows: associate to \( P \) the pair \((P, P)\) and to \( \theta: P \to Q \) where \( \theta \) is det. by \( \theta: P \oplus R \to Q \), the pair of \( \theta \).

\[ \theta: P \oplus R \to Q, \quad \alpha \theta: P \oplus R \to Q. \]

This second description shows us that \( JG \) has a \textit{assoc. comm. unitary operation}. Thus \( JG \) is a converted \( H \)-space.
Lichtenbaum letter:

**GOAL:**

**Theorem:** $A = \text{ring of integers in a number field } F \\
\Rightarrow K_i A$ finitely generated for all $i \geq 0$.

Recall new definition of groups $K_i A$, $A$ arbitrary:

$P_A = \text{proj. f.g. } A\text{-modules}$

$Q(P_A)$

$BQ(P_A)$ a classifying space for $Q(P_A)$

**Defn:** $K_i A = \pi_{i+1}(BQ(P_A))$.

**Thm:** Equivalent to the old one.

By classical results of Serre, one only has to show that $H_*(BQ(P_A), \mathbb{Z})$ is finitely gen. for $i$. I should mention that Serre's results apply even though $BQ(P_A)$ is not simply-connected, because it is a connected H-space. In fact, the direct sum operation on projective modules makes $Q(P_A)$ into a "permutative" category, hence by the Segal-Anderson theory, $BQ(P_A)$ is an infinite loop space.

$\therefore H_*(BQ(P_A), \mathbb{Z}) = \text{derived functors of } \lim_+ \to \lim_-$ for $\text{Hom}(Q(P_A), \mathbb{Z})$.
Fix n, set \( C = F_n Q(P_A) \)
\( C' = F_{n-1} Q(P_A) \)
and let \( f: C' \to C \) be the inclusion.
We consider the spectral sequence

\[
E^{2}_{pq} = H_p(C, L^q f(\mathbb{Z})) \Rightarrow H_{p+q}(C', \mathbb{Z}).
\]

\[
(L^q f(\mathbb{Z}))(M) = H_q(f/M, \mathbb{Z})
\]

where \( f/M \) is the category over \( C' \) whose objects are pairs \((N, u), N \in C' \) \( u: fN \to M \).

Identify \( f/M \) with the ordered set of admissible layers \((M_0, M)\) of \( M \) such that \( M/M_0 \) has rank \( \leq n \).
Question: To understand L-conjectures, curves over finite fields, how to relate homotopy groups with Euler characteristics.

Possibility: Deligne's idea of recovering the $\psi$-function using the symmetric product.

\[
\frac{1}{\det(1 - A)} = 1 + \text{tr}A + \text{tr}(A^2) + \cdots = e^{\sum \frac{1}{k} \text{tr}(A^k)}
\]
Acylicity of the map $f : J \rightarrow \mathcal{F}$.

Given a topological category $\mathcal{C}$ such that the source map $\text{Ar} \mathcal{C} \rightarrow \text{Ob} \mathcal{C}$ is stale, we can form the standard resolution of the final object of $\mathcal{C}$ (a = a roof, $\mathcal{C}^\#$ is the category of sheaves over $\text{Ob} \mathcal{C}$ with right $\text{Ar} \mathcal{C}$-action, right $\mathcal{C}$-objects):

$$
\text{Ar} \mathcal{C}^\# \xrightarrow{\text{Fr}_2} \text{Ar} \mathcal{C} \xrightarrow{i} \text{Ob} \mathcal{C}
$$

Alternative description: Consider the (functor) from the subcategory of $\mathcal{C}$ with same objects and only identity morphisms; then the above simplicial object is the standard resolution associated to the pair of adjoint functors $(i^!, i^*)$:

$$
\text{Ar} \mathcal{C}^\# = (i_1^!, i^*)^\mathbb{Z} \text{Ob} \mathcal{C}.
$$

Claim this resolution can be used to compute cohomology of $f^*L$ for any $\mathcal{F}$-sheaf $L$ in the sense that $Z^\mathbb{F}_{\text{Ar} \mathcal{C}^\#}$ is.

Claim for any injective $\mathcal{F}$-sheaf $L$ that

$\text{Ext}^n_{\mathcal{F}}(f^*L) = H^n(\mathcal{F} \xrightarrow{i} \text{Hom}_{\mathcal{C}}((i_1^! \text{lab}^*)^{\mathbb{Z}}; f^*L))$

To establish the first isomorphism we must know that

$\text{Ext}^n_{\mathcal{F}}(i_1^! \text{lab}^*; \mathbb{Z}, f^*L) = 0$

But $i_1^! \text{lab}$ is exact, hence $i^*$ carries injectives to injectives, so

$\text{Ext}^n_{\mathcal{F}}(i_1^! \text{lab}^*; \mathbb{Z}, f^*L) = 0$

and in the case where $B = f^*L$, we have that $i^*f^*L$ is just the pullback of $L$ over $\text{Ob} \mathcal{F}$ to $L$ over $\text{Ob} \mathcal{F} J$; as this pullback is evidently injective, done.

Now $f_1^! \text{lab}^* = Z^\mathbb{F}_{\text{Ar} \mathcal{C}^\# \rightarrow \text{Ar} \mathcal{C}^\#}$, of the sort that $\text{ixxxyxx}$ has for its stalk $\text{ixxxyxx}$ the chains on the nerve of the category of pairs $(y, u)$ where $x$ is an object of $J$ and $u : x \rightarrow f(x)$.

Now Mather's axiom implies that this category is equivalent to the fibre category over the nerve of the $y$, so we are reduced to proving the fibre category $\text{ixxxyxx}$ has trivial homology.
nerve of the fibre category.

The homology of the nerve is the same as the homology of the category, defined as derived functors of \( \text{ind}\text{-}\lim \). \( \text{Hom}(\mathcal{O}, \text{Ab}) \):

\[
\lim_{x} \text{ind.} \quad \mathcal{F} = H_{p}^{\infty}(n \leftarrow \mathcal{F} \otimes \mathbb{Z}_{\mathcal{A}}^n+iC)
\]

Suffices by universal coefficients formulas to show this is zero when \( \mathcal{F} \) is the constant functor with value a field \( k \).

Because \( \mathcal{F} \) is fibred over \( J^{-} \), we have a spectral sequence

\[
E_{pq}^{2} = H_{p}(J^{-}, n \leftarrow H_{q}(J_{x}^{n}, \mathcal{F})) \Rightarrow H_{p+q}(J_{x}^{n} \mathcal{F})
\]

Take \( \mathcal{F} = k \).

\( J_{x}^{n} \) is the category with objects \((x, a_{0}, \ldots, a_{n})/\mathbb{Z}_{\mathcal{A}}^{n}\) where a morphism from this to another with primes is a diffeomorphism of \(/x, a_{n}/ \leftarrow /x, a_{n}'/\) which is the identity near \( x \) and \( x \) translation carrying \( a_{j} \) to \( a_{j}' \). It is clear that \( J_{x}^{n} \) is isomorphic to the product

\[
I_{1}^{*} \times I_{1} \times \ldots \times I_{1}
\]

where \( I_{1}^{*} \) is the full subcategory of \( I_{1} \) consisting of the non-degenerate intervals. The map sends \((0, b), (0, a_{1}), \ldots, (0, a_{n})\) into

\[
(-1, b-1, b-1+a_{1}, \ldots)
\]

assuming \( x = -1 \).

Thus \( H(I_{x}^{n}) = H(I_{1}^{*}) \otimes \mathbb{E}^{n} \). Identify the \( E^{1} \) term with the bar resolution of \( k \) over \( R \) tensored over \( R \) with \( H(I_{1}^{*}) \). Then getting

\[
E^{2} = \text{Tor}_{p}^{R}(H(I_{1}^{*}), k)_{q}
\]

Next recall that

\[
R = k \otimes H(\mathcal{O}) \quad \pi_{0}(I_{1}) = (\mathfrak{1}, e)
\]

where \( e \) is in the center, and is an idempotent. Thus \( H(I_{1}^{*}) = \mathcal{R} \) is a projective \( R \)-module, and we see the Tor is zero except for \( k \) in degree 00.
Outline of the proof of the cohomological Nather theorem.

Simplicial/categories $\mathcal{F}$ and $\mathcal{K}$, and the maps

$$
\mathcal{F} \xrightarrow{\gamma} \mathcal{K} \xrightarrow{\delta} \Gamma
$$

Acyclicity of the first map for constant coefficients: \textbf{Need $\mathcal{F}$ to be constant.}

$$
E_2 = H^n - H^n(\mathcal{F}) = H^n(\mathcal{K}, \mathcal{F})
$$

and corresponding one for $\mathcal{K}$ to reduce to showing that $\mathcal{F}$ is acyclic for constant coefficients. Using equivalent subcategories one sees that $\mathcal{F}$ is equivalent to $\mathcal{K} \times \mathcal{K}$ where $\mathcal{K}$ is the \textit{everywhere} with objects $(x,0)$. Need some kind of argument to reduce to showing $\mathcal{K}$ acyclic, and another argument $\textit{mix}$ to show this follows from the contractibility of the $\mathcal{A}_n[Q]$.

Acyclicity of the second map for all $\wedge$-sheaves: \textbf{Denote second map by $f$.}

To prove $\mathcal{K} \xrightarrow{f} \mathcal{K} = 0$ enough to do so after pulling back over $\mathcal{A}_n[\wedge]$, i.e. after forgetting the $\wedge$-action. \textbf{Identify the pullback: $\mathcal{G}$-sheaves/Ar $\wedge \rightarrow \wedge$-sheaves/Ar/()} with $\mathcal{G}$-sheaves sheaves over $\mathcal{R}$. $\mathcal{G}$-sheaves same as $\mathcal{G}$-sheaves for a suitable category $\mathcal{G}$-category in the topos of sheaves over $\mathcal{R}$, hence by your notes it suffices to show i.e. have trivial homology.

that the fibres categories are acyclic! One identifies the fibre category with the simplicial category $\mathcal{F}(\mathcal{N}', \mathcal{N})$ where $\mathcal{N} = \mathcal{W}$ and $\mathcal{N}'$ is $\mathcal{N}$ minus the degenerate object.

Then one uses the \textit{笋校} descent spectral sequence in homology plus computation of the Tor term to prove the acyclicity of the fibres.

\textbf{Use $\tilde{I}$ to denote $\wedge$-sheaves.}
New notation: \( \mathfrak{I} \) will be the simplicial category with objects \((a_0, a_1, \ldots, a_n)\) monic sequences in \( \mathbb{N} \times \mathbb{N} \) with \( a_0 = 0 \). Thus \( \mathfrak{I} \) is the \textit{classifying} nerve of the monoid category \( \mathfrak{A} \) with objects the intervals \( /0,n/ \) for each \( n \in \mathbb{N} \).

Let \( \mathfrak{J} \) be the simplicial topological \textit{ext} groupoid with objects \((x, a_0, \ldots, a_n)\).

\( \mathfrak{J} \) will be as before.

Use letters \( u, v \) for monic maps.

\[ f : \mathfrak{I} \rightarrow \mathfrak{J} \] functor sending \((x, a_0, \ldots, a_n)\) to \( x \), \( f_n \) its restriction to \( \mathfrak{I}_n \).

\( \mathfrak{I}' \) the subcategory of \( \mathfrak{I} \) with same \textit{ext} whose morphisms are the morphisms in \( \mathfrak{I} \) which become identity morphisms in \( \mathfrak{J} \). Then \( f' : \mathfrak{I}' \rightarrow \mathfrak{J} \) the \textit{ext} morphism induced by \( f \) (\( \mathfrak{J} \) viewed as a category with only the identity morphisms).

\[ g : \mathfrak{I} \rightarrow \mathfrak{J} \] functor sending \((x, a_0, \ldots, a_n)\) to \((0, a_1 - a_0, \ldots, a_n - a_0)\).

\( g_n : \mathfrak{I}_n \rightarrow \mathfrak{J}_n \) its effect in degree \( n \).

This seems to include all of the data in the proof. Next we need to understand the \textit{proof} details. First identify the sheaves, which are \textit{essentially} contravariant functors on the categories. Thus a \( \mathfrak{J} \)-sheaf consists of a family of sheaves \( \mathcal{F}_{a_0, \ldots, a_n} \) for each monotone sequence in \( \mathbb{N} \) with action data expressible as follows. Denote by \( \mathcal{F}_{a_0, \ldots, a_n} \) the stalk of the sheaf \( \mathcal{F}_{a_0, \ldots, a_n} \) at \( x \); then given a monotone map \( u : (0, \ldots, m) \rightarrow (0, \ldots, n) \) and a diffeo. germ \( h : (x, a_0(0), \ldots, a_0(m)) \rightarrow (x', a_0', \ldots, a_m') \), one has a map

\[ (u, h) : \mathcal{F}_{x, a_0', \ldots, a_m'} \rightarrow \mathcal{F}_{x, a_0', \ldots, a_m'} \]
I recall we have defined the topological groupoid \( G^n \) as follows. Its objects are sequences \((z,a_0,...,a_n)\) where \( z \) is a real number less than \( 0 \) and \( a_0,a_1,...,a_n \) are in \( N \). Morphisms are \( \text{diffeomorphisms of } h:z,a_0,...,a_n \rightarrow z',a'_0,...,a'_n \) which each \( j \) of \( 0,j,n \), coincide in a neighborhood of \( a_j \) with the translation of \( a_j \) by \( a_j' \). We take sequences \((z,a_0,...,a_n)\) to \((z',a_0,...,a_n)\) at \( h(z,a_0,...,a_n) \) consists of the points \((z,0,a_0,...,a_n)\) with \( z' \) running over a neighborhood \( z \). We take morphisms \( h:z,a_0,...,a_n \rightarrow z',a_0,...,a_n \) to consist of the germ \( \tilde{h} \) restricted to \((z,a_0,...,a_n)\) where \( h \) is a diffeomorphism representing \( h \) and \( z \) runs over a neighborhood of \( z \). Given \( h \) etc., let \( 0; \tilde{h}(x) \) be a diffeomorphism of an open interval containing \((z,a_0,...,a_n)\) with an open interval containing \((z',a_0,...,a_n)\) representing the germ \( h \). Then the germ \( \tilde{h} \) with \( z \) running over a nbhd of \( z \) in the domain of \( \tilde{h} \) represented by \( \tilde{h} \) form a basic neighborhood for \( h \) in \( A_n(G^n) \).

Claim \( A_n(G^n) = \mathcal{W}_n \) is a topological groupoid and that the source and target maps are etale.

Next we have the functor from \( G^n \) to \( \mathcal{W}_n \) sending \((z,a_0,...,a_n)\) to \((z,a_0,...,a'_n)\).

I want to show this map is induces isomorphisms on cohomology with \( \mathcal{W}_n \) its locally constant coefficients. \( \mathcal{W}_n \) is equivalent to the full subcategory consisting of the objects \((0,a_0,a_n)\) with \( \mathcal{W}_n \). \( G^n \) is equivalent to the full subcategory consisting of objects \((z,a_0,...,a_n)\) with \( z=0 \), and this subcategory is the product of the category with objects \((z,0)\) and \( \mathcal{W}_n \). \( \mathcal{W}_n \) is in turn equivalent to its full subcategory consisting of the objects \((0,a_0,...,a_n)\) with \( a_n \) and this subcategory is the direct product of the category \( pt \times (G^n)^n \) denoting the category defined by \( G \) the group \( G \) of diffeomorphisms of \((0,1)\) with support in the interior. Thus we have equivalences

\[
\begin{align*}
\mathcal{W}_n &\cong \mathcal{W} \times (pt \times (G^n)^n) \\
G^n &\cong (pt \times (G^n)^n)
\end{align*}
\]

hence to prove the claim it suffices to show that \( \mathcal{W} \) has no cohomology with constant coefficients.
We do this by showing each of the spaces \( \text{Ar}_n \) are contractible and using the spectral sequence (of Čech type)

\[
\Sigma^q_2 = H^n(\text{Ar}_n^\text{V}_n) \Rightarrow H^{p+q}(\text{Ar}_n^\text{V}_n)
\]

Given a sequence \((h_i)\) with source \(h_i = \text{target of } h_i\), we define a real number \(r(h_i)\) to be the least \(r\) such that \(h_i(z) = z\) for all \(z\) and \(|z| > r\). Thus \(r(h_i)\) is the upper bound for the support of the family \(h_i\).

Claim that \(r(h_i)\) is continuous in \(h\). Indeed let \(\Theta_i\) be different. Let an \(r(h_i)\)

representing the germs \(h_i\). Then if \(h_i: \mathcal{X}_{i-1,0} \to \mathcal{X}_i,0\) want \(\Theta_i\) to be a diffeo. Set \(r(h_i)\) of open intervals containing these closed intervals.

a sequence \((h_i')\) near \(h\) consists of the germs \(h_i': \mathcal{X}_{i-1,0} \to \mathcal{X}_i,0\) represented by \(\Theta_i\) where \(y_i = \text{near } x_i\). There are two cases. First of all suppose \(r(h)\) is greater than all of the \(x_i\). Then \(r(h') = r(h)\) provided \(r(h)\) max all \(y_i\) are less than \(r(h)\). Secondly, can have \(x_i = r(h_i)\).

Then have \(\max y_i = r(h')\) less than \(r(h_i)\).

Let \(h = (h_i, \ldots, h_n)\) be an element of \(\text{Ar}_n^\text{V}\) consisting of germs \(h_i: \mathcal{X}_{i-1,0} \to \mathcal{X}_i,0\). Let \(r(h)\) be the least \(x_i\) such that \(h_i(z) = z\) for all \(z > x_i\) and all \(i\); \(r(h)\) is the upper bound of the support of the family \(h_i\).

Claim \(r(h)\) is a continuous function of \(h\). Indeed, let \(\Theta_i\) be a diffeomorphism from an open interval containing \(\mathcal{X}_{i-1,0}\) to an open interval containing \(\mathcal{X}_i,0\) which represents the germ \(h_i\).

Then any \(h' = (h_i', \ldots, h_n')\) where \(h_i': \mathcal{X}_{i-1,0} \to \mathcal{X}_i,0\) is the germ represented by \(\Theta_i\) and \(y_i\) is near \(x_i\). For simplicity, take a neighborhood of \(h\). Ar\(\text{Ar}_n^\text{V}\) consists of germs \(\text{Ar}_n^\text{V}\). For each \(h\), \(r(h)\) is defined as the germ represented by \(\Theta_i\) where \(x_i\) is near \(y_i\), that is \(\max y_i < r(h)\), then \(r(h') = r(h)\) provided \(\max y_i < r(h)\). On the other hand, if \(\max x_i = r(h)\), then \(\max y_i = r(h) = \max y_i, r(h)\), so in either case we have that \(r(h')\) tends to \(r(h)\) as \(h'\) tends to \(h\).
Set $F(h, t) = /d(h), 0/ \text{ restriction of } h$.

Given $h$ in $\text{Ar}_n V$, let $d(h)$ be the source of $h$, i.e., $A$. If $0 \times d(h)$, define the restriction of $h$ to the interval $/x, 0/$, denoted $h' /x, 0/$, to be the sequence of germs $h'_i : /x_i, 0/ \rightarrow /x_{i-1}, 0/$ (where $x_0 = x$, $x_{i-1} = h'_i(x_i)$) induced by $h_i$ on the smaller intervals. Set

$$F(h, t) = \begin{cases} 0 \leq t \leq \frac{1}{2} \\ /h' / x, 0/ + (z(h'))d(h), 0/ \\ 1/2 < t \leq 1 \\ (id_{I(t)} \ldots , id_{I(t)}) /0 < t < 1/ \\ /0 < t < 1/ \\ (1-t)r(h') + (2t-1)d(h'), 0/ \\ \end{cases}$$

where $I(t) = (1-t)r(h') + (2t-1)d(h')$. 0/.

Claim $F : \text{Ar}_n V \times I \rightarrow \text{Ar}_n V$ is continuous. Clear.

This gives a homotopy of the composite

$$\text{Ar}_n V \xrightarrow{id} \xrightarrow{id} \text{Ar}_n V$$

with the identity map of $\text{Ar}_n V$, showing that $\text{Ar}_n V$ is contractible.
Nather's theorem: Logical structure.

If \( n \in \mathbb{N} \), let \( \mathcal{G}_{m} \) be the topological groupoid with objects \((x,a_{0},...,a_{n})\) where \( x \) is a real no. less than 0 and \( a_{0} - \cdots - a_{n} \) are in \( \mathbb{N} \). Make these into the space

\[
\theta_{m} = \text{disjoint union } \mathbb{R} \cup (a_{0} - \cdots - a_{n})
\]

A morphism in \( \mathcal{G}_{m} \) from \((x,a_{0},...,a_{n})\) to \((x',a_{0}',...,a_{n}')\) consists of a germ of diffeomorphism from \( /x,a_{n}/ \) to \( /x',a_{n}'\) such that

\[
u(z) = z - a_{1} + a_{1}' \quad \text{for } z \text{ near } a_{1} \quad 0 \leq n.
\]

One topologizes the morphisms \( \mathbb{R} \) so as to be an etale \( \mathbb{R} \) space over the objects.

This is a simplicial topological groupoid: Given a monotone map \( \beta: \mathbb{R} \rightarrow 0,\ldots,n \) one defines a functor \( \beta^{*}: \mathcal{G}_{m} \rightarrow \mathcal{G}_{n} \) by

\[
\beta^{*}(x,a_{0},...,a_{n}) = (x,\beta(x),\beta(a_{0}),\ldots,\beta(a_{n}))
\]

and in the obvious way for the morphisms.

Evident \( \mathbb{R} \) augmentation to \( \beta^{*} \): sending \((x,a_{0},...,a_{n})\) to \( x \). (In this one \( \mathbb{R} \) that takes the objects of \( \beta^{*} \) to be \( \mathbb{R} \).)

Functor to the simplicial category \( A_{\mathbb{N}} = (\mathbb{A})^{\mathbb{N}} \). Here \( \mathbb{A} \) is the category with objects \( \mathbb{A}^{0,n} \) for each \( n \in \mathbb{N} \) and \( \mathbb{A}^{0,n} \) translating diffeos. in the evident way: \( u: (x,a_{0},...,a_{n}) \rightarrow (x',a_{0}',...,a_{n}') \) goes to the diffeo from \((a_{1}-a_{0},...,a_{n}-a_{0})\) to \((a_{1}'-a_{0}',...,a_{n}'-a_{0}')\) given by

\[
z - (z-a_{0})_{\mathbb{A}} - a_{0}'.
\]

Claim: \( \mathcal{G}_{m} \) equivalent to the disjoint union of the full subcategories

consisting of \((x,a_{0},...,a_{n})\) with \( a_{n} = 0 \) i.e. \( \mathcal{G}_{m} \) has \( 2^{m} \) components. Enough to consider the case \( n = 0 \) to get the idea.

The problem is to assign to \((x,a_{0})\) an isomorphism with \((x,0)\), \( \mathbb{R} \) continuously for all \( x \). Therefore I ask for a diffeom. \( u_{x}: \mathbb{R} \rightarrow \mathbb{R} \)

\[
u_{x} : \mathbb{R} \rightarrow \mathbb{R}
\]
Lemma (?): Let $P$ be a $G$-torsor over $X$ such that the map $P \to \text{Ob}G$ is acyclic $(H^q_t, f^* = 0 \text{ for } q < 0)$, in particular if this map is a fibration homotopy equivalent to the identity map of $\text{Ob}G$. Then $$H^*(g^*, F) = H^*(B, P, F).$$ (Everything here involves sheaf cohomology so there should be no problem.)

Lemma (?): There exists a $\mathcal{F}$-torsor $P$ over a CW complex $B$ satisfying the conditions of the above lemma. Moreover $G$ can be constructed functorially from $\mathcal{F}$. (Denote by $R^*G$)

**Necessary**

Lemma (?): (Ehresmann's lemma) Let $P \to X$ be a map. Assume that there is a covering of $X$ such that the map when restricted to each finite intersection of members of the covering is a homotopy equivalence. Then the map is a homotopy equivalence.

(This should have the consequence that if $P \to \text{Ob}G$ is a fibre equivalence, then $B$ classifies bundles over numerable coverings, hence over all paracompact spaces.)

**Theorem** Consequences of the first two lemmas:

i) There is a universal CW complex $BG$ which is universal among all paracompact spaces.

ii) The cohomology of $BG$ is the same as the $G$-cohomology with coefficients in any $G$-sheaf.

iii) One has a homotopy equivalence of $BBG$ and $B^\infty$ in the Eilenberg situation, because we can take our basic diagram of topological groupoids and make it into a diagram of CW complexes, and then the isomorphism on cohomology which have been established by sheaf theory will give by the Whitehead theorem actual homotopy equivalences.

The only thing that might not be true is the second lemma; one must use something special about the groupoid $G$, such as the fact that $\text{Ob}G$ is a nice space.
A G-sheaf consists of a family of sheaves $F_{a_0 \ldots a_n}$ on $\mathbb{R}_+$ for each/sequence of elements of $\mathbb{N}$, together with for each monotone map $u: 0 \ldots m \to 0 \ldots n$

\begin{equation}
(x, a_{u(0)} \ldots a_{u(n)}) \mapsto (x', a_{u(0)}' \ldots a_{u(n)}')
\end{equation}

\[(u, h)_* : F_{xa_0 \ldots a_n} \to F_{xa_0' \ldots a_n'}
\]

Given a monotone map $u: 0 \ldots m \to 0 \ldots n$, there is a functor from $G_m$ to $G_n$ sending $(x, a_0 \ldots a_n)$ to $(x, a_{u(0)} \ldots a_{u(n)})$. And this leads us to know to a map

\[(u, h?) : F_{xa_{u(0)} \ldots a_{u(n)}} \to F_{xa_0 \ldots a_n}
\]

combined with a morphism $h : \mathbb{R} x_{a_0' \ldots a_n'} \to \mathbb{R} x_{u(0)} \ldots a_{u(n)}$, one obtains a map

\[(u, h)_* : F_{xa_0' \ldots a_n'} \to F_{xa_0 \ldots a_n}
\]

(Thus it seems we want to consider contravariant functors on categories.) On one hand so that the category of G-sheaves appears as a functor on a suitable topological category, on the other hand to include things like the sheaves of functions and forms which are naturally contravariant functors with respect to diffeomorphisms.

Thus $G$ is the topological category with objects $x_0 \ldots a_n$ and morphisms $x_{a_0 \ldots a_n}$ in which a morphism from $x_{a_0 \ldots a_n}$ to $x_{a_0' \ldots a_n'}$ consists of a monotone map $u : (0 \ldots m) \to (0 \ldots n)$ and a germ of diffeomorphisms $h : /x, a_{u(m)} / \to /x', a_{u(n)} /$ near $a_j$ such that for $j = 0 \ldots m$ $h$ coincides with the translation carrying $a_{u(j)}$ to $a_j$.

G-sheaves are like contravariant functors on this category.

Use the Deligne descent spectral sequence

\[
E_2^{pq} = H^p(\mathbb{R} - H^q(G_n, F_n)) = H^{p+q}(G, F)
\]

Actually we use the map of this spectral sequence to the one for the category $W$, because the contractibility argument will show that $G_n \to W_n$ induces an isomorphism on cohomology with constant coefficients.

I also want the Deligne spectral sequence in homology for the fibres of $G$ over $x$

\[
E_2^{pq} = H_p(\mathbb{R} - H_q(G_n)) = H_p(W)
\]
Problem: Suppose $\mathbb{A} = \text{disjoint union of the nerves of } G = n \times H$. $\mathbb{A}$ is a simplicial monoid and we can consider the simplicial category $(\mathbb{A}, \mathbb{A})$ of $\mathbb{A}$ acting on itself to the right. I have a desire to think of this simplicial category as being the same as the category with objects $\text{free module}$ and where a morphism is a direct injection $f : E \to E'$ together with a choice of complement, i.e. the category of $\text{free pairs}$ $(Q, e), Q : E \to Q$ is equivalent to a category with only identity morphisms.

Similarly the simplicial category $(\mathbb{A}, \mathbb{A})$ I would like to think of as being the same as the category with objects $E$ and with morphism from $E$ to $E'$ consisting of an isomorphism class of triples $(Q, u_1, u_2)$ where $u_1 : E \to Q = E'$ and $u_2 : E \to Q = E'$ are two isomorphisms. Again the category of such triples is equivalent to the category consisting of those triples with $Q$ a complement for the image of $u_1$, and only identity morphisms.

This category looks suspiciously like the cofibered category constructed over the category of pairs $(i, q)$ with direct injection $i : Q \to E$ and identity morphism for the functor $E' \to \text{Aut}(E)$. In fact they are the same. Therefore it would be nice to show that the simplicial category $(\mathbb{A}, \mathbb{A})$ and the category of $(i, q)$ have the same homotopy type.

First a functor is needed. So I recall that in simplicial degree $n$, we are considering the category of sequences $(V_i, \cdots, V_n)$ of projective modules, i.e. $\mathbb{A} = \mathbb{A}$.

Suppose that I have an $n$-simplex in the category of $(i, q)$, i.e. I have a vector space $\mathbb{A}$ written as a direct sum $\mathbb{A}_n + \mathbb{A}_{n-1} + \cdots + \mathbb{A}_0$.

So back to the original category $\mathbb{A}$ with objects $E$ and morphisms pairs consisting of a direct injection $i : E \to E'$ and a choice $Q$ of complement for the image. Then exists an equivalent full subcategory consisting of the free modules $E = \mathbb{R}^n$. So a morphism from $\mathbb{R}^n$ to $\mathbb{R}^m$ consists of a direct injection $i : \mathbb{R}^n \to \mathbb{R}^m$ and a choice for the complement, i.e. a projection operator backwards. Now take a simplex in the nerve of this category; it consists of a diagram

$$
\begin{array}{ccc}
\mathbb{R}^0 & \to & \mathbb{R}^n \\
\mathbb{R}^1 & \to & \mathbb{R}^n \\
\mathbb{R}^n & \to & \mathbb{R}^n
\end{array}
$$

with direct injection and projection operators. Now it is clear how to obtain an element of the $(\mathbb{A}, \mathbb{A})$ simplicial category, namely, take the kernels of the various projection operators.

The point somehow is this: Take the category consisting of the n-simplices in
There are many things to be understood. $\mathbb{R}^n \cong \sigma_n$. $SE = \text{sphere bundle of } E$.

at the moment I don't understand why $\mathbb{R}^n \cong \sigma_n$. $SE = \text{sphere bundle of } E$.

should have anything to do with the loop spaces of the spheres. There is a basic $\text{K}$-action, and one can look at the orbit of a critical point.

Back to Mather's theorem: I have defined now a simplicial topological groupoid $G_\sigma$ with augmentation to $\sigma$ and a map $G_\sigma \to A$, where $A_\sigma$ is the nerve of the monoid category $\text{Mon}$ equivalent to the union of a point to the category $B\Sigma$.

Now there are three things to prove.

1) $G_\sigma \to \sigma$ is acyclic, i.e. $\text{Ext}(\sigma, A_\sigma) = H_\sigma^k(F, F)$ for all $\sigma$-sheaves $F$ (locally?)

2) $G_\sigma \to A_\sigma$ is acyclic for constant coefficients.

The $\mathbb{R}^n$ above two give me an isomorphism of $H^*(\sigma, A) = H^*(B\Sigma, A)$ for all abelian groups $A$.

Now to finish Mather's theorem I need

3) $H^*(\sigma, A) = H^*(B/\sigma, A)$ for all abelian groups $A$.

In fact what I want to prove is that for the Milnor classifying spaces the singular cohomology equals the sheaf cohomology. $\text{Ext}$ is precisely the Milnor weak $\text{Ext}$.

It is not difficult to show this space has the correct/homotopy type, torsors $\text{Ext}$ isomorphically classify homotopy classes of $G$-torsors over paracompact spaces. Now I want to know that singular and sheaf cohomology for $B\Sigma$ are the same, because if I then replace $B\Sigma$ by the realization of its singular complex, then I have not changed the singular cohomology, hence the sheaf cohomology remains the same.

The logical point is that because the space $B\Sigma$ is so nasty, it is not a priori clear that it doesn't have cohomology classes which die when pulled back to any finite complex, or that there are no characteristic classes for $G$-torsors over finite complexes that do not come from classes $\text{Ext}$ computed on the category of $\sigma$-sheaves. So a more precise version of what you want is:

We have defined a map $H^*(\sigma, F) \to H^*(X, \mathbb{R}^n \sigma, F)$ for any $\sigma$-sheaf $F$. Show that this map is an isomorphism when $X$ is a CW complex which is $B\sigma$. It suffices to show therefore that there is a CW complex $B$ endowed with a $G$-structure.