April 2, 1970 exterior generators for $H^*(BG_\sigma)$

Let σ be an endomorphism of the connected algebraic group G. ~~We suppose the hypotheses of the first theorem hold and would like to produce the exterior generators~~

Let $H^*(BG)^\sigma$ denote the subring of invariants for the action of σ^*. I propose to define a map

$$(*) \qquad \Phi : H^i(BG)^\sigma \longrightarrow \mathrm{Coker}\left\{H^{i-1}(BG) \to H^{i-1}(BG^\sigma)\right\}$$

which will in fact be a derivation of $H^*(BG)$ modules.

Given $x \in H^*(BG)$ with $\sigma^* x = x$, choose a cocycle $\dot{x}$ representing x. Then there is a cochain c such that

$$\sigma^* \dot{x} - \dot{x} = dc.$$

Since σ^* is the identity on BG^σ it follows that $c|BG^\sigma$ is a cocycle. We define $\Phi(x)$ to be the class represented by $c|BG^\sigma$.

Note that upon changing $\dot{x}$ by dy we alter c to $c + \sigma^* y - y$, which has the same restriction to BG^σ as c. c is unique up to a cocycle on BG, which is why one must take the cokernel in $(*)$.

Given another invariant class y and $\sigma^* \dot{y} - \dot{y} = dc'$ we have

$$\sigma^*(\dot{x}\dot{y}) - \dot{x}\dot{y} = (\sigma^*\dot{x} - \dot{x})\,\sigma^*\dot{y} + \dot{x}\,(\sigma^*\dot{y} - \dot{y})$$
$$= d(c \cdot \sigma^* \dot{y}) + (-1)^{\deg x}\, d(\dot{x} \cdot c')$$

so $\boxed{\Phi(xy) = \Phi(x) \cdot y + (-1)^{\deg x}\, x \cdot \Phi(y)}$.

In the applications $H^*(BG)$ will often be concentrated in even dimensions, hence there will be no indeterminacy on the left side of $(*)$.

~~Example: Take $G = \mathbb{G}_m$ over $\mathbb{F}_q$ with σ = Frobenius and work with mod l cohomology where l divides $q-1$. Let $x \in H^2(B\mathbb{G}_m)$ be the Euler class of the identity character of $\mathbb{G}_m$ (suppose $\mathbb{Z}_l \cong \mu_l$ is given). Then $\sigma^* x = qx = x$. Recall that x is defined~~

$$
\begin{tikzcd}
H^1(B\mathbb{G}_m, \mathbb{G}_m) \arrow[r] & H^1(B\mathbb{G}_m, \mathbb{G}_m) \arrow[r, "\delta"] & H^2(B\mathbb{G}_m, \mathbb{Z}_l) \\
\arrow[r, "l"] & \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \arrow[u] &
\end{tikzcd}
$$

$$\mathrm{id} \mapsto x$$

~~as $\delta(\mathrm{id}) = x$. For c we take the character $q-1: \mathbb{G}_m \to \mathbb{G}_m$ and when we restrict this to $\mu_{q-1} = \mathbb{F}_q^*$ we get~~

?

Example: Take $G = \mathbb{G}_m$ defined over $\mathbb{F}_q$ with σ = Frobenius and work with mod l cohomology, $l \mid q-1$. Let $x \in H^2(B\mathbb{G}_m)$ be the Euler class of the identity character (suppose $\mathbb{Z}_l \simeq \mu_l$ given). By definition of Euler class as boundary in the Kummer sequence one sees that x represents the extension

(think of the extension as $\dot{x}$)

$$0 \to \mu_l \to \mathbb{G}_m \xrightarrow{l} \mathbb{G}_m \to 0$$

of $\mathbb{G}_m$ by $\mu_l = \mathbb{Z}_l$. $\sigma: \mathbb{G}_m \to \mathbb{G}_m$ is multiplication by q, and as $\ell \mid q-1$, this extension is isomorphic to its pull-back by σ. The class c is essentially an isomorphism of these extensions; choose the obvious one

$$\begin{tikzcd}
0 \arrow[r] & \mu_\ell \arrow[r] \arrow[d, "id"] & \mathbb{G}_m \arrow[r, "\ell"] \arrow[d, "q"] & \mathbb{G}_m \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \mu_\ell \arrow[r] & \mathbb{G}_m \arrow[r, "\ell"] & \mathbb{G}_m \arrow[r] & 0
\end{tikzcd}$$

and then restrict to $\mathbb{F}_q^*$

$$\begin{tikzcd}
0 \arrow[r] & \mu_\ell \arrow[r] \arrow[d, "id"] & \mu_{\ell(q-1)} \arrow[r] \arrow[d, "q"] & \mathbb{F}_q^* \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mu_\ell \arrow[r] & \mu_{\ell(q-1)} \arrow[r] & \mathbb{F}_q^* \arrow[r] & 0
\end{tikzcd}$$

thus getting for $c \mid \mathbb{F}_q^*$ the homomorphism $\mathbb{F}_q^* \longrightarrow \mu_\ell$ ~~which arises~~ which sends $x \longmapsto x^{(q-1)/\ell}$

The result of this computation shows that if $x \in H^2(B\mathbb{G}_m, \mu_\ell)$ is the canonical Euler class, then $\Phi(x) \in H^1(B(\mathbb{F}_q^*), \mu_\ell)$ is the surjective homomorphism $x \longmapsto x^{(q-1)/\ell}$.

Example 2: $GL_n(\mathbb{F}_q)$, where $\ell \mid q-1$.

Recall

$$H^*(B\mathbb{F}_q^*) = \mathbb{Z}_\ell[v, u]$$

where u is the Euler class of the standard character, ~~and~~ i.e. $u = x \mid \mathbb{F}_q^*$, x as above and where $v = \Phi(x)$. Note $v^2 = 0$

unless $\ell = 2$ and $v_2(q-1) = 1$. Thus by Kunneth

$$H^*(BT_n(\mathbb{F}_q)) = \mathbb{Z}_\ell[v_1, \dots, v_n, u_1, \dots, u_n].$$

Now by naturality there is a commutative diagram

$$\begin{tikzcd}
H^{2i}(BGL_n) \arrow[r, "\Phi"] \arrow[d] & H^{2i-1}(BGL_n(\mathbb{F}_q)) \arrow[d] \\
H^{2i}(BT_n) \arrow[r, "\Phi"] & H^{2i-1}(BT_n(\mathbb{F}_q))
\end{tikzcd}$$

and

$$\begin{tikzcd}
c_i \arrow[r, mapsto] \arrow[d, mapsto] & \Phi c_i = e_i \arrow[d, mapsto] \\
\sigma_i(x) \ \text{ith elem. sym. fn. of } x \arrow[r, mapsto] & \Phi \sigma_i(x).
\end{tikzcd}$$

where $\Phi \sigma_i(x)$ may be computed in terms of the u's and the v's using the fact Φ is a derivation. Thus

$$\sigma_1(x) = \sum x_i \qquad \Phi\sigma_1 = \sum_i v_i$$

$$\sigma_2(x) = \sum_{i<j} x_i x_j \qquad \Phi\sigma_2 = \sum_{i \neq j} u_i v_j$$

Goal: Complete determination of mod ℓ cohomology of the finite general linear orthogonal, and symplectic groups with entries in a finite field of characteristic p $\neq \ell$.

General cases: ℓ odd, then everything goes easily because on one hand I know that the restriction to the appropriate torus is injective, and on the other I know that the invariants for the Weyl group is an exterior tensored with a plynomial algebra with the correct number of generators. The point is that Galois is either cyclic of order d dividing ℓ-1, which is prime to ℓ or cyclic of order 2 and then the rest is the symmetric group.

ℓ=2 is exceptional for all. ~~For GL the problem is how to spot the Weyl group invariants~~

$$\cancel{H^*(BG(F_q)) = H^*_G(G^t) \overset{u}{---} H_A(G^c) \overset{v}{---} H^*_A(T)}$$

~~Maybe the point is to use the Hopf algebra structure somehow.~~

In the case of $GL_n(\mathbb{F}_q)$ $2 | q-1$ $\ell = 2$ we produce operator

$$\Phi : H^{2i}(BGL_n) \longrightarrow H^{2i-1}(BGL_n(\mathbb{F}_q))$$

and thus exhibit elements $\Phi c_i = e_i$ with desired restriction to the torus. As restriction to torus $T_n(\mathbb{F}_q)$ is injective and as spectral sequence gives Poincaré series one is finished.

The same method works for $Sp_{2n}(\mathbb{F}_q)$, namely one produces the $\Phi(cq_i) \in H^{4i-1}(BSp_{2n}(\mathbb{F}_q))$ and computes them on $Sp_2(\mathbb{F}_q)^n$ by the derivation property. For this it is necessary to know what the situation is for $Sl_2(\mathbb{F}_q)$ where one deduces it from $Gl_2(\mathbb{F}_q)$.

I need the localization thm.

Suppose G finite acting on a scheme X such that X/G exists (i.e. every orbit contained in an open affine of X). Then I would like a spectral sequence

$$E_2^{pq} = H^p(X/G, \mathcal{H}^q) \Longrightarrow H_G^{p+q}(X)$$

where $\mathcal{H}^q$ is the sheaf on X/G associated to the presheaf

$$U \longmapsto H_G^q(\pi^{-1}U).$$

Here $U \longrightarrow X/G$ is a scheme over X/G and $\pi: X \longrightarrow X/G$ is the projection. Moreover I want to know that at a point y of X/G, say y equals the orbit of $x \in X$, then

$$\mathcal{H}_y^q = H_G^q(Gx) \cong H_{G_x}^q(pt)$$

where G_x is the stabilizer of x.

Now in fact the spectral sequence should be the Leray spectral sequence for the map of topi

$$T_G/X \xrightarrow{f} T/(X/G)$$

where $f_*(F) = (\pi_* F)^G$. Now

$$R^q f_*(F) \cong H^q(G, \pi_* F)$$

since π is finite. Also if $y \in X/G$, then as G is finite and $H^q(G, -)$ commutes with limits we have

$$R^q f_*(F)_y = H^q(G, \pi_* F)_y = H^q(G, (\pi_* F)_y)$$

and $\quad (\pi_* F)_y = \prod_{x \in f_*^{-1}\{y\}} F_x$. Thus for $F = \mathbb{Z}_\ell$

$$R^q f_*(F)_y = H^q(G; \mathbb{Z}_\ell^{G/G_x}) \cong H^q(G_x; \mathbb{Z}_\ell)$$

Application: The localization theorem. Suppose that S is a multiplicative system in $H^{ev}(pt, G)$ and that $S^{-1}H^*(G/G_x) = 0$ for every $x \in X$ not in $Y \overset{i}{\subset} X$ (closed). Then

$$(*) \qquad S^{-1} H_G^*(X) \xrightarrow{\sim} S^{-1} H_G^*(Y)$$

In effect compute on the E_2 terms of the spectral sequences +

$$\begin{array}{c} H^*(X/G, R^q f_*(\mathbb{Z}_{\ell,X})) \\ \downarrow i^* \\ H^*(Y/G, R^q \pi_{Y*}(\mathbb{Z}_{\ell,Y})) \end{array}$$

notation! $H^*(pt, G; \cdot\cdot)$

where the natural map

$$(i/G)^* R^q \pi_*(\mathbb{Z}_{\ell,X}) \longrightarrow R^q \pi_*(\mathbb{Z}_{\ell,Y})$$

is an isomorphism as one sees by computing the stalks. But in fact one sees (by stalks) that $R^q \pi_*(\mathbb{Z}_{\ell,X})$ is concentrated on Y so the map is an isomorphism.

§7. <u>The operation</u> Φ. In this section H^*X denotes the singular cohomology of the space X with coefficients in $\mathbb{Z}/\ell$, where ℓ is [redacted] a non-zero integer which needn't be prime. Suppose given a diagram of maps of spaces

ℓ script "l"

$$(7.1)\qquad \begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] \arrow[ur, phantom, "h\Nearrow"] & Y \end{tikzcd}$$

where $h: gf' \Rightarrow fg'$ is a homotopy. In the case where [redacted] h is a constant homotopy and the square commutes, there is a homomorphism of long exact sequences

Greek δ

$$(7.2)\qquad \begin{tikzcd} H^{i-1}Y \arrow[r, "f^*"] \arrow[d, "g^*"] & H^{i-1}X \arrow[r, "\delta"] \arrow[d, "g'^*"] & \tilde{H}^i\mathrm{Cone}(f) \arrow[r, "j^*"] \arrow[d, "g''^*"] & H^iY \arrow[r, "f^*"] \arrow[d, "g^*"] & H^iX \arrow[d, "g'^*"] \\ H^{i-1}Y' \arrow[r, "f'^*"] & H^{i-1}X' \arrow[r, "\delta'"] & \tilde{H}^i\mathrm{Cone}(f') \arrow[r, "j'^*"] & H^iY' \arrow[r, "f'^*"] & H^iX' \end{tikzcd}$$

δ

$\cup$ union sign

$\times$ cross

where g'' is the induced map on mapping cones (recall: $\mathrm{Cone}(f) = pt \cup_X (X \times [0,1]) \cup_f Y$) space and j, j' are the obvious inclusions. If the homotopy h is non-trivial then we still obtain a homomorphism 7.2 [redacted] by replacing the maps f' and g in 7.1 by the maps

$$X' \xrightarrow{i_0} (X' \times [0,1]) \cup_{f'} Y' \xrightarrow{(-h,g)} Y,$$

obtaining a commutative square and then using the [redacted] canonical isomorphism [redacted] between the long exact sequences of i_0 and f'.

We define the operation Φ associated to the diagram 7.1 to be the map

Greek Φ

$$(7.3)\qquad \begin{array}{c} \mathrm{Ker}\left\{(f^*,g^*): H^iY \longrightarrow H^iX \oplus H^iY'\right\} \\ \big\downarrow \Phi \\ \mathrm{Coker}\left\{g'^* + f'^*: H^{i-1}X \oplus H^{i-1}Y' \longrightarrow H^{i-1}X'\right\} \end{array}$$

given by the formulas

α, β

$$(7.4)\qquad \delta'\Phi(\alpha) = g''^*(\beta) \quad \text{[redacted] where [redacted]} \quad j^*(\beta) = \alpha.$$

<u>Proposition</u> 7.5: Φ <u>is a</u> [redacted] (<u>degree</u> -1) <u>homomorphism of</u> H^*Y-<u>modules</u>. $\Phi(\alpha) = 0$ <u>if</u> $\alpha \in (\mathrm{Ker}\, f^*)\cdot(\mathrm{Ker}\, g^*)$.

Greek $\in$

The first assertion follows from the fact that [redacted] the long exact cohomology sequence of f [redacted]

consists of [redacted] homomorphisms of graded H^*Y-modules with δ of degree -1 and similarly for [redacted] f'. If $\alpha = \alpha_1\alpha_2$ with
j [redacted] $g^*\alpha_1 = 0$ and $f^*\alpha_2 = 0$, then in 7.4 we can take $\beta = \alpha_1\beta_2$ where $j^*\beta_2 = \alpha_2$, and then $\delta'\Phi(\alpha) = g''^*(\alpha_1\beta_2) = g^*\alpha_1 \cdot g''^*\beta_2 = 0$, proving that $\Phi(\alpha) = 0$.

Let BG be a [redacted] classifying space for the topological group G and denote by [redacted] $PG \longrightarrow BG$ the universal principal G-bundle over BG. If $b\colon G' \longrightarrow G$ is a homomorphism of topological groups, the space of maps $PG' \longrightarrow PG$ which are equivariant with respect to b is contractible (at least when BG' is a CW complex and this will be true in the [redacted] following). Choosing such an equivariant map we obtain an induced map $(b)\colon BG' \longrightarrow BG$ whose homotopy class depends only on u, hence $(b)^*\colon H^*BG \longrightarrow H^*BG'$ [redacted] also depends only on b. Similarly if

(7.6)
$$\begin{tikzcd} G'_0 \arrow[r, "b'"] \arrow[d, "a'"'] & G_0 \arrow[d, "a"] \\ G' \arrow[r, "b"] & G \end{tikzcd}$$

is a commutative square of homomorphisms and if we choose corresponding equivariant maps

$$\begin{tikzcd} PG'_0 \arrow[r, "(b')^\sim"] \arrow[d, "(a')^\sim"'] & PG_0 \arrow[d, "(a)^\sim"] \\ PG' \arrow[r, "(b)^\sim"] & PG \end{tikzcd}$$

there is a homotopy $h^\sim\colon (b)^\sim(a')^\sim \Longrightarrow (a)^\sim(b')^\sim$, which is equivariant with respect to $ba' = ab'$ and is unique up to 2-homotopy. Passing to classifying spaces we obtain a square

$$\begin{tikzcd} BG'_0 \arrow[r, "(b')"] \arrow[d, "(a')"'] & BG_0 \arrow[d, "(a)"] \\ BG' \arrow[r, "(b)"'] \arrow[ur, phantom, "\Nearrow h"] & BG \end{tikzcd}$$

permitting [redacted] us to [redacted] define the operation Φ from the kernel of $(a)^*$ and $(b)^*$ on H^iBG to the quotient of $H^{i-1}BG'$ by the images of $(a')^*$ and $(b')^*$. It is clear that Φ depends only on the square 7.6.

We [redacted] now [redacted] compute Φ for the square

(7.7)
$$\begin{tikzcd} \mu \arrow[r] \arrow[d, "a"'] & e \arrow[d] \\ S^1 \arrow[r, "\chi"] & S^1 \end{tikzcd}$$

[redacted] where S^1 is the circle group, ℓ denotes the map $z \mapsto z^{\ell}$, and a is the [redacted] inclusion of the ℓ-th roots of unity. Recall that there is a canonical isomorphism

(7.8) $$H^1BG = \mathrm{Hom}_{\text{top.grps.}}(G, \mathbb{Z}/\ell)$$

making correspond to [redacted] a cohomology class u the homomorphism u^+ given by

Greek σ

$u^+(g)\sigma = f_g^*(u)$, [redacted] where σ is the canonical generator of H^1S^1 and $f_g\colon S^1 \longrightarrow BG$ is the map of classifying spaces induced by the homomorphism $\mathbb{Z} \longrightarrow G$ sending 1 to g. Recall also that

(7.9) $$H^*BS^1 = \mathbb{Z}/\ell[e(L)]$$

where L is the complex line bundle [redacted] associated to the universal bundle $PS^1 \longrightarrow BS^1$ and [redacted] $e(L)$ is its Euler (or first Chern) class. Since $(\ell)^*e(L) =$

⊗ tensor sign

$e(L^{\otimes\ell}) = \ell e(L) = 0$ and since $H^{\mathrm{odd}}BS^1 = 0$, the map Φ [redacted] associated to $\underbrace{\text{}}_{\text{the square}}$ 7.7 takes the form

(7.10) $$\Phi : H^{2i}BS^1 \longrightarrow H^{2i-1}B\mu \qquad i > 0.$$

Greek ε

Proposition [redacted] 7.11: Let $v \in H^1B\mu$ be the class corresponding under 7.8 to the homomorphism $\mu \longrightarrow \mathbb{Z}/\ell$ [redacted] sending $\exp(2\pi i/\ell)$ to [redacted] $1 \pmod{\ell}$, and let $u = (a)^*e(L)$. Then $\Phi(e(L)^i) = u^{i-1}v$.

It suffices by 7.5 to prove this formula when $i = 1$. [redacted] Let $f\colon PS^1 \longrightarrow BS^1$ denote the universal bundle and consider the principal S^1-bundle maps

(7.12)
$$\begin{tikzcd} S^1 \arrow[r, "i'"] \arrow[d, "f''"'] & PS^1/\mu \arrow[r, "g'"] \arrow[d, "f'"'] & PS^1 \arrow[d, "f"'] \\ pt \arrow[r, "i"'] & BS^1 \arrow[r, "g"'] & BS^1 \end{tikzcd}$$

where (g, g') is a classifying map for f' and i' is the inclusion of a fibre of f'. The second square with the constant homotopy may be used to compute [redacted] Φ because the canonical map $p\colon PS^1 \longrightarrow PS^1/\mu$ is a universal principal μ-bundle and the square is covered by the equivariant maps

$$\begin{tikzcd} PS^1 \arrow[r, dash, "g'p"] \arrow[d, "id"'] & PS^1 \arrow[d, "id"'] \\ PS^1 \arrow[r, dash, "g'p"] & PS^1 \end{tikzcd}$$

[redacted] of universal bundles for the groups in 7.7.

[redacted] Also i' is the map of classifying spaces induced by the homomorphism $\mathbb{Z} \longrightarrow \mu$ sending 1 to [redacted] $\exp(2\pi i/\ell)$, hence taking into account the definition of 7.8 it suffices to show that $(i')^*\Phi(e(L)) = \sigma$.

Now consider the diagram of long exact cohomology sequences associated to the vertical maps in 7.12 [redacted]

(too crowded expand vertically a bit)

$$\begin{tikzcd}
 & & H^2BS^1 \\
H^2\mathrm{Cone}(f'') & H^2\mathrm{Cone}(f') \arrow[l, "i''^*"'] & H^2\mathrm{Cone}(f) \arrow[l, "g''^*"'] \arrow[u, "j^*"'] \\
H^1S^1 \arrow[u, "\delta''"', "\cong"] & H^1B\mu \arrow[l, "i'^*"'] \arrow[u, "\delta'"']
\end{tikzcd}$$

and use that the cones of f, f', f'' are the Thom [redacted] spaces of these S^1-bundles. By definition $e(L) = j^*U$ where U is the Thom class of f, i.e. $U \in H^2\mathrm{Cone}(f)$ is

$i''^* g''^* U$

the unique element such that $\boxed{\cancel{(g''i'')^* U}} = \delta''\sigma$. By definition [redacted] $\delta'\Phi(e(L)) = g''^*U$, hence [redacted] $i'^*\Phi(e(L)) = \sigma$, completing the proof of the proposition.

§8. <u>Computation of</u> Φ. This section contains a computation of the operation Φ needed in the proof of 10. . We suppose that p is a prime number; all vector spaces, algebras, and tensor products will be taken over $\mathbb{Z}/p$ unless mentioned otherwise.

Let SV be the symmetric algebra of the vector space V and let V', V'' be subspaces of V. Then there is a graded algebra isomorphism

⊗ sign

$$\mathrm{Tor}_*^{SV}(S(V/V'), S(V/V'')) \cong S(V/V'+V'') \otimes \Lambda^*(V' \cap V''). \tag{8.1}$$

Indeed the Koszul sequence

$$SV \otimes \Lambda^2 V' \xrightarrow{d} SV \otimes V' \xrightarrow{d} SV \longrightarrow S(V/V') \longrightarrow 0,$$

where d is the (anti-)derivation of $SV \otimes \Lambda^* V'$ with $d(1 \otimes v') = v' \otimes 1$ and $d(v \otimes 1) = 0$, is a resolution of $S(V/V')$, hence the Tor algebra is the homology of the differential graded algera

$$S(V/V'') \otimes \Lambda^* V' \quad \text{with} \quad d(V/V'' \otimes 1) = 0,\ d(1 \otimes v') = \xi(v') \otimes 1 \tag{8.2}$$

Greek ξ

where $\xi: V' \longrightarrow V/V''$ is the composition of the inclusion of V' in V followed by the canonical map to V/V''. Choosing splittings of the exact sequences

$$0 \longrightarrow V' \cap V'' \longrightarrow V' \longrightarrow \mathrm{Im}\,\xi \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}\,\xi \longrightarrow V/V'' \longrightarrow V/V'+V'' \longrightarrow 0$$

the complex 8.2 becomes isomorphic to the tensor product of the three complexes:

$S(V/V'+V'')$	concentrated in degree 0
$S(\mathrm{Im}\,\xi) \otimes \Lambda^*(\mathrm{Im}\,\xi)$	with differential $d(1 \otimes z) = z \otimes 1$
$\Lambda^*(V' \cap V'')$	with zero differentials.

As the middle complex is acyclic, 8.1 follows from the Kunneth formula.

Let $I' = (SV)V' = \mathrm{Ker}\{SV \longrightarrow S(V/V')\}$ and define I'' similarly. Then the inclusion of $V' \cap V''$ in $I' \cap I''$ induces a homomorphism of $S(V/V'+V'')$-modules

$$S(V/V'+V'') \otimes (V' \cap V'') \xrightarrow{\sim} I' \cap I''/I'I'' \tag{8.3}$$

which we claim is an isomorphism. In view of the canonical isomorphism

$$\mathrm{Tor}_1^{SV}(S(V/V'), S(V/V'')) = I' \cap I''/I'I'' \tag{8.4}$$

this would follow from 8.1 after checking the compatibility of the various maps, however it seems slightly more efficient to argue as follows. For the proof that [redacted] 8.3 is an isomorphism [redacted] we may assume V is finite dimensional, and it will be enough to show that this map is surjective as both sides are free $S(V/V'+V'')$-modules of the same rank by 8.1 and 8.4. Let $g \in I' \cap I''$ and write

$$(8.5) \qquad g = \sum_{j\in J} g_j v_j' \qquad g_j \in SV$$

where $v_j', j \in J$ [redacted] is a basis for V[redacted]$'$ such that $v_j', j\in J' \subset J$ [redacted] spans $V' \cap V''$. Denoting by $x \mapsto \bar{x}$ the map $SV \longrightarrow S(V/V'')$ we have

$$0 = \sum_{j\in J''} \bar{g}_j \bar{v}_j' \qquad J'' = J - J'$$

so as the elements $\bar{v}_j'$ in V/V'' are independent, they form a regular sequence in $S(V/V'')$ hence

$$\bar{g}_j = \sum_{j \in J''} \bar{g}_{jk} \bar{v}_k'$$

with $g_{jk} \in SV$ satisfying $g_{jk} = -g_{kj}$, $g_{jj} = 0$. Thus

$$\sum_{j\in J''} g_j v_j' \equiv \sum_{j,k\in J''} g_{jk} v_k' v_j' \equiv 0 \qquad (\text{mod } I'I'')$$

hence

$$(8.6) \qquad g \equiv \sum_{j\in J'} g_j v_j' \qquad (\text{mod } I'I'')$$

[redacted] proving that 8.3 is surjective and hence an isomorphism. Notice that there is a canonical [redacted] algebra isomorphism of the form [redacted] 8.1 [redacted] agreeing with 8.3 and 8.4 in degree one.

Greek σ

Let W be a vector space endowed with an endomorphism σ and let W^σ (resp. W_σ) be the kernel (resp. cokernel) of [redacted] the endomorphism σ-id. Consider the [redacted] cocartesian square of algebras

$$(8.7) \qquad \begin{array}{ccc} S(W_\sigma) & \xleftarrow{\ p\ } & SW \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle \Delta} \\ SW & \xleftarrow{\ \Gamma\ } & SW \otimes SW \end{array} \qquad \begin{array}{l} \Delta(w\otimes 1) = \Delta(1\otimes w) = w \\ \Gamma(w\otimes 1) = w, \quad \Gamma(1\otimes w) = \sigma(w) \end{array}$$

where p is the extension to symmetric algebras of the canonical [redacted] map $W \to W_\sigma$. Denote by SW_Δ and SW_Γ the ring SW viewed as an algebra over $SW \otimes SW$ by means of Δ and Γ, and let I_Δ and I_Γ be the kernels of Δ and Γ respectively.

Setting $V = W \oplus W$, $V' = \{(w, -w), w \in W\}$, and $V'' = \{(\sigma w, -w), w \in W\}$ and using the canonical isomorphisms $SV = SW \otimes SW$, $V' \cap V'' = W^{\sigma}$, $V/V'+V'' = W_{\sigma}$, we obtain the [redacted] following from the above discussion.

<u>Proposition</u> 8.8: <u>There is a canonical isomorphism of graded algebras</u>

$$\mathrm{Tor}_{*}^{SW \otimes SW}(SW_{\Delta}, SW_{\Gamma}) = S(W_{\sigma}) \otimes \Lambda^{*} W^{\sigma}$$

<u>which in dimension one may be identified (by the isomorphism analogous to</u> 8.4) <u>with the map</u>

(8.9) $$S(W_{\sigma}) \otimes W^{\sigma} \longrightarrow I_{\Delta} \cap I_{\Gamma} / I_{\Delta} I_{\Gamma}$$
$$p(f) \otimes w \longmapsto (f \otimes 1)(w \otimes 1 - 1 \otimes w) \qquad (\mathrm{mod}\ I_{\Delta} I_{\Gamma}).$$

Denote by

(8.10) $$d\colon SW \longrightarrow SW \otimes W$$

the universal derivation of the algebra SW, i.e. the derivation such that $d(w) = 1 \otimes w$. If $f \in (SW)^{\sigma}$ [redacted] is σ-invariant, so is $(p \otimes id)df \in S(W_{\sigma}) \otimes W$, hence we obtain a map

[redacted]
(8.11) $$(SW)^{\sigma} \longrightarrow S(W_{\sigma}) \otimes W^{\sigma}$$
$$f \longmapsto (p \otimes id)df.$$

<u>Proposition</u> 8.12: <u>If</u> [redacted] $f \in (SW)^{\sigma}$ <u>and</u> $(p \otimes id)df = \Sigma p(f_j) \otimes w_j$ <u>with</u> $w_j \in W^{\sigma}$, <u>then</u>

(8.13) $$f \otimes 1 - 1 \otimes f \equiv \Sigma\ (f_j \otimes 1)(w_j \otimes 1 - 1 \otimes w_j) \qquad (\mathrm{mod}\ I_{\Delta} I_{\Gamma}).$$

We may assume that the w_j form part of a basis w_j, $j \in J'$ for W^{σ}, which in turn is part of a basis [redacted] w_j, $j \in J$ for W. Recall that if

[redacted] $$f \otimes 1 - 1 \otimes f = \sum_{j \in J} g_j (w_j \otimes 1 - 1 \otimes w_j)$$

with $g_j \in SW \otimes SW$, then

$$df = \sum_{j \in J} (\Delta g_j) \otimes w_j,$$

this being nothing but the canonical isomorphism of the module of differentials of the algebra SW with $I_{\Delta}/I_{\Delta}^{2}$. Applying $p \otimes id$ to both sides we [redacted] find $p(\Delta g_j) = f_j$ for $j \in J'$, so

(8.14) $$g_j \equiv f_j \otimes 1 \qquad (\mathrm{mod}\ I_{\Delta} + I_{\Gamma}), \quad j \in J'.$$

From
[redacted] 8.5 and 8.6 with $g = f\otimes 1 - 1\otimes f$ [redacted] and $v_j' =$ [redacted] $w_j\otimes 1 - 1\otimes w_j$ we have

$$f\otimes 1 - 1\otimes f \equiv \sum_{j\in J'} g_j(w_j\otimes 1 - 1\otimes w_j) \qquad (\mathrm{mod}\ I_\Delta I_\Gamma),$$

so using 8.14 we deduce formula 8.13, completing the proof.

This proposition will be applied in the following geometric [redacted] situation. [redacted] Suppose given a space Y endowed with [redacted] a map $\sigma : Y \to Y$ and a diagram

(8.15)
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "t"'] & Y \arrow[d, "\Delta_Y"] \\ Y \arrow[r, "\Gamma_\sigma"'] & Y\times Y \end{tikzcd} \qquad \Gamma_\sigma = (id, \sigma)$$

and suppose we want to compute $\Phi(y\otimes 1 - 1\otimes y)$ where $y \in H^*Y$ is invariant under σ^*. Suppose we can find a vector space W endowed with an endomorphism, which will also be denoted σ, and a linear map $\rho : W \to H^*Y$ compatible with the two actions of σ such that $y = \rho(f)$ with $f \in (SW)^\sigma$, where ρ is extended to an algebra homomorphism $SW \to H^*Y$ (if γ is odd we assume $\rho(W) \in H^{ev}Y$). Let [redacted] $\rho\otimes\rho : SW \otimes SW \to H^*(Y\times Y)$ denote[redacted] the map sending $s\otimes s'$ to $pr_1^*\rho(s).pr_2^*\rho(s')$; then $\rho\otimes\rho$ carries I_Δ [redacted] to [redacted] $\mathrm{Ker}\,\Delta_Y^*$ and I_Γ to $\mathrm{Ker}\,\Gamma_\sigma^*$, hence by 7.5

$$\Phi(y\otimes 1 - 1\otimes y) = \Phi(\rho\otimes\rho)(f\otimes 1 - 1\otimes f)$$

depends only on [redacted] $f\otimes 1 - 1\otimes f$ $(\mathrm{mod}\ I_\Delta I_\Gamma)$. [redacted] So by [redacted] 8.12

$$\Phi(y\otimes 1 - 1\otimes y) = \Phi(\rho\otimes\rho) \sum_j (f_j\otimes 1)(w_j\otimes 1 - 1\otimes w_j)$$

(8.16)
$$= \sum_j t^*(\rho f_j).\Phi(\rho w_j\otimes 1 - 1\otimes \rho w_j) \qquad \text{if}$$

[redacted]
$$(p\otimes id)df = \sum_j p(f_j)\otimes w_j \in S(W_\sigma\ \text{[redacted]})\otimes W^\sigma.$$

For example, let σ be an endomorphism of a torus T, [redacted] let $a : T^\sigma \to T$ be the inclusion of the subgroup of fixpoints, and take 8.15 to be the [redacted] diagram of classifying spaces associated to the square of topological groups

(8.17)
$$\begin{tikzcd} T^\sigma \arrow[r, "a"] \arrow[d, "a"'] & T \arrow[d, "\Delta_T"] \\ T \arrow[r, "\Gamma_\sigma"'] & T\times T \end{tikzcd} \qquad \Gamma_\sigma = (id, \sigma)$$

Let $T^\wedge$ be the character [redacted] group of T; there is a canonical isomorphism

[redacted]

Greek ρ

$\wedge$ roof.

$$T^\wedge \otimes_{\mathbb{Z}} \mathbb{Z}/\ell = H^2BT$$

Greek χ "chi" obtained by associating to a character $\chi: T \longrightarrow S^1$ the inverse [redacted] image of $e(L)$ (7.9) under the map induced by χ. We take $W = H^2BT$ and take ρ to be the identity on W so that $\rho: SW \longrightarrow H^*BT$ is an isomorphism. The following proposition/[redacted] computes $\Phi(w\otimes 1 - 1\otimes w)$

$\Phi(y\otimes 1 - 1\otimes y)$ for $w \in W^\sigma$, hence the formula 8.16 tells us what $\Phi(y\otimes 1-1\otimes y)$ is for any $y \in$ [redacted] $H^*(BT)^\sigma$.

[redacted]

Proposition 8.18: Let $w \in W^\sigma$ be represented by the character χ and let χ_0 be the unique character such that $\chi_0(z)^\ell = \chi(z.\sigma(z)^{-1})$. Then $\Phi(w\otimes 1 - 1\otimes w) \in$ [redacted]

Greek ξ $H^1(BT^\sigma)$ is the class corresponding to the homomorphism $\xi: T^\sigma \longrightarrow \mathbb{Z}/\ell$ given by

" $\chi_0(z) = \exp(2\pi i\xi(z)/\ell)$.

The homomorphisms

$$\begin{tikzcd} T \arrow[r, "\Gamma_\sigma"] \arrow[d, "\chi_0"'] & T\times T \arrow[d, "b"] \\ S^1 \arrow[r, "\ell"] & S^1 \end{tikzcd} \qquad b(z,z') = \chi(z.z'^{-1})$$

induce a map of the square 8.17 to the square 7.7 such that $e(L)$ goes to $w\otimes 1 - 1\otimes w$, [redacted] hence by naturality of Φ and 7.11 the proposition follows.

Finally we consider a concrete example which will be used in [redacted] §10. Let σ be the endomorphism of $T = (S^1)^r$ given by

$$\sigma(z_1,\ldots,z_r) = (z_r{}^q, z_1{}^q,\ldots,z_{r-1}{}^q)$$

where q and r are positive integers such that ℓ divides q^r-1. [redacted] Let C be a cyclic group of order q^r-1 and let $i: C \longrightarrow S^1$ be [redacted] a faithful character; then [redacted] the inclusion of T^σ in T may be identified with [redacted] the map $a: C \longrightarrow T$ given by

[redacted]

$$a(c) = (i(c)^q,\ldots,i(c)^{q^{r-1}}, i(c)).$$

Let $e_i = pr_i^* e(L)$ where $pr_i: T \longrightarrow S^1$ is the [redacted] i-th projection. Then

$$\sigma^*(e_i) = \begin{cases} qe_{i-1} & 1 < i \leq r \\ qe_r & i = 1 \end{cases}$$

so as q^r-1 is divisible by ℓ, the product $e_1\ldots e_r$ is an invariant [redacted] element of H^*BT.

Proposition 8.18: $\Phi(e_1\cdots e_r\otimes 1 - 1\otimes e_1\cdots e_r) = (-1)^{r-1}u^{r-1}v \in H^{2r-1}(BC)$ where [redacted] $u = i^*e(L)$ and $v \in H^1BC$ is the class of the homomorphism $\xi\colon C \to \mathbb{Z}/\ell$ given by

$$\exp(2\pi i\,\xi(c)/\ell) = i(c)^{(1-q^r)/\ell}.$$

In this case W^σ is [redacted] one-dimensional with generator [redacted] $w = q^{r-1}e_1 + \cdots + e_r$ which is the Euler [redacted] class of the character

[redacted] $$\chi(z_1,\ldots,z_r) = \prod_1^r z_i^{q^{r-i}}$$

and

$$(\chi\cdot\sigma^*\chi^{-1})(z_1,\ldots,z_r) = \Big(\prod_1^r z_i^{q^{r-i}}\Big)\Big(z_r^{q^r}\prod_1^{r-1} z_i^{q^{r-i}}\Big)^{-1}$$

$$= (\chi_o(z_1,\ldots,z_r))^\ell \qquad \text{where}$$

[redacted] $$\chi_o(z_1,\ldots,z_r) = z_r^{(1-q^r)/\ell}.$$

Using 8.18 we see that $\Phi(w\otimes 1 - 1\otimes w) = v$. Now W_σ is one-dimensional with generator $p(e_r)$ and [redacted] $p(e_i) =$ [redacted] $q^i p(e_r)$, so

$$(p\otimes id)d(e_1\cdots e_r) = \text{[redacted]}\ \sum_1^r p(e_1\cdots\hat{e}_i\cdots e_r)\otimes e_i$$

$$= \Big(\prod_1^r q^i\Big)\, p(e_r)^{r-1}\otimes \sum_1^r q^{r-i}e_i$$

$$= (-1)^{r-1}\, p(e_r)^{r-1}\otimes w.$$

As $a^*(e_r) = u$, the proposition [redacted] follows from formula 8.16.

Lemma: If $f \in (SW)^{\sigma}$ and $(p\otimes id)df = \Sigma\, p(f_i)(w_i\otimes 1-1\otimes w_i)$ with $w_i \in W^{\sigma}$, then

$$f\otimes 1-1\otimes f \equiv \Sigma\,(f_i\otimes 1)(w_i\otimes 1-1\otimes w_i) \pmod{I_{\Delta} I_{\Gamma}}$$

Proof: Choose g_i $SW\otimes SW$ such that

$$(*) \qquad f\otimes 1-1\otimes f = \sum_1^n g_i(w_i\otimes 1-1\otimes w_i)$$

where we assume that w_i $1\leq i\leq n$ is a basis of W such that the w_i with $i\leq m$ for a basis for W°. Then

$$df = \sum_1^n \Delta g_i \otimes w_i$$

hence

$$(**) \qquad p(\Delta g_i) = \begin{cases} 0 & i>m \\ p(f_i) & i\leq m. \end{cases}$$

Applying Γ to both sides of $(*)$ we get

$$0 = \sum_{i>m} \Gamma(g_i)(w_i-\sigma w_i)$$

hence as the elements $w_i-\sigma w_i$ for $i>m$ form a regular sequence in SW we have

$$\Gamma(g_i) = \sum_{i,j>m} \Gamma(g_{ij})(w_i-\sigma w_i)$$

where $g_{ij}\in SW\otimes SW$ satisfies $g_{ij} = -g_{ji}$ and $g_{ii} = 0$. Hence

$$\sum_{i>m} g_i(w_i\otimes 1-1\otimes w_i) \equiv \sum_{i,j>m} g_{ij}(w_j\otimes 1-1\otimes w_j)(w_i\otimes 1-1\otimes w_i) \equiv 0 \pmod{I_{\Delta} I_{\Gamma}},$$

so

$$\begin{aligned} f\otimes 1-1\otimes f &\equiv \sum_{i\leq m} g_i(w_i\otimes 1-1\otimes w_i) && \pmod{I_{\Delta} I_{\Gamma}} \\ &\equiv \sum_{i\leq m} (f_i\otimes 1)(w_i\otimes 1-1\otimes w_i) && \text{"} \end{aligned}$$

since $g_i \equiv f_i\otimes 1 \pmod{I_{\Delta}+I_{\Gamma}}$ by $(**)$.

~~draft for paper:~~ Cohomology of finite groups of rational points.
(first draft: rendered obsolete by decision to use E-M spectral sequence instead of Leray-Serre one.)

k algebraically closed field, G alg. gp defined over k, $\sigma: G \to G$ ~~surjective~~ endomorphism with fixpoints G_σ supposed finite. ~~[illegible]~~ l prime number distinct from char k.

First theorem deals with computing $H^*(BG_\sigma)$

~~[illegible]~~

Lemma 1: $$G/G_\sigma \xrightarrow{\sim} G$$
$$xG_\sigma \longmapsto x(\sigma x)^{-1}$$

~~for this I need to know that $(1-\sigma)G = G$ in Steinberg's notation~~

~~[illegible]~~ To prove this I consider the map $\mathrm{id}-\sigma$: $x \mapsto x(\sigma x)^{-1}$ from G to G. If surjective, then $G/G_\sigma \to G$ is a 1-1 onto map of ~~[illegible]~~ transitive G varieties, so we must also know that $\mathrm{id}-\sigma$ is etale at the origin. Thus I need

(i) $(1-\sigma)G = G$

(ii) $\mathrm{id} - d\sigma : \mathfrak{g} \to \mathfrak{g}$ isomorphism. (equiv. $\sigma \pitchfork \mathrm{id}$ at e)

(From lemma 1 it follows that
$$\tilde{G}/\tilde{G}_\sigma \simeq \tilde{G}$$
as sheaves for gross etale topology.) ~~Hence~~

~~Lemma 2: There is a spectral sequence~~

$$\cancel{E_2 = H(BG) \otimes H(G) \Longrightarrow H(BG_\sigma)}$$

with the usual multiplicative properties.

In fact I need mult. spectral sequences

1) $E_2 = H(BG) \otimes H(G) \Longrightarrow H(pt)$

2) $E_2 = H(BG) \otimes H(G/G_\sigma) \Longrightarrow H(BG_\sigma)$

3) $E_2 = H(B(G\times G)) \otimes H(G\times G/\Delta G) \Longrightarrow H(BG)$

Let's carefully see what we need.

Each spectral sequence is of the type

(*) $E_2 = H(BG) \otimes H(G/H) \Longrightarrow H(BH)$

~~and~~ in the following cases

1) $H = \{1\}$

2) $H = G_\sigma$

3) $H = \Delta$ in $G \times G$

Assume that $\exists$ mult. spectral sequence (*) whenever H is a closed algebraic subgroup of G. ~~and mult for spectral sequence.~~

Recall the transgression is the "relation"

(**) $$E_2^{0q} \supset E_{q+1}^{0q} \xrightarrow{d_{q+1}} E_{q+1}^{q+1,0} \twoheadleftarrow E_2^{q+1,0}$$

Hypothesis: $H^*(G)$ has a simple system c_i $i\in I$ of transgressive generators for 1) $\Longrightarrow$ $H^*(BG) = \mathbb{Z}_\ell[c_i]_{i\in I}$ where c_i

represents τe_i. (Borel's theorem. This should be augmented by implications

$$\begin{array}{c} G \text{ has no } l\text{-torsion} \\ \Updownarrow \\ H^*(G) \text{ exterior alg. on odd degree generators} \\ \Downarrow \\ H^*(G) \text{ has simple system of transg. gens. for } 1). \end{array}$$

and the ~~equival~~ equivalence of these when l is odd).

In spectral sequence 3) then:

$$E_2 = H^*(B(G\times G)) \otimes H^*(G) \Longrightarrow H^*(BG)$$

the elements $1\otimes e_i$ are transgressive with $\tau(1\otimes e_i)$ represented by $pr_1^* c_i - pr_2^* c_i \in H^*(B(G\times G))$.

~~To see this we use that map~~

$$\begin{tikzcd} G\times G/G \arrow[r] & BG \arrow[r] & B(G\times G) \\ G\times G \arrow[r] \arrow[u] & E(G\times G) \arrow[r] \arrow[u] & B(G\times G) \arrow[u, leftrightarrow] \end{tikzcd}$$

For the proof it is necessary to ~~[illegible]~~ have the other interpretation of the transgression

$$\begin{tikzcd} 0 \arrow[r] & H^{q-1}(G) \arrow[r, "\delta"] & H^q(BG, G) \\ & & H^q(B(G\times G)). \arrow[u, leftrightarrow] \end{tikzcd}$$

Thus it requires us to prove ~~that~~ somehow that the transgressin (**) is amenable to the computation

$$E_2^{0,g-1} = H^{g-1}(F) \xrightarrow{\delta} H^{g}(E,F)$$
$$\uparrow \pi^*$$
$$H^g(B,pt) \cong \tilde{H}^g(B)$$

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ pt & \longrightarrow & B \end{array}$$

In the course of my old proof it seems that I must show that ~~if~~ c is transgressive $\overset{\text{in 1)}}{\Longrightarrow}$ c is primitive.

Once 3) is analyzed we can handle 2 by the map

$$\begin{array}{ccc} G/G^\sigma & \xrightarrow{\sim} & G \\ \downarrow & & \downarrow \\ BG^\sigma & \longrightarrow & BG \\ \downarrow & & \downarrow \Delta \\ BG & \xrightarrow{(id,\sigma)} & B(G\times G) \end{array}$$

and its effect on spectral sequences. So we conclude that 2) has fibre with trans. gen. and transgressions are known. It is necessary to be able to calculate the spec. sequence from this information, by mapping an ideal spectral sequence in.

So the theorem then gives

$$\begin{array}{c} E_\infty = \Lambda(P^\sigma) \otimes S(P_\sigma) \\ \parallel \\ gr\, H^*(BG^\sigma) \end{array}$$

with isom if l odd.

possibility to simplify things

Start with X as an object of T_G and form the Cech cohomology of the covering $G\times X \xrightarrow{\mu} X$ which is

$$G\times G\times G\times X \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; G\times G\times X \; \substack{\xrightarrow{\mu\times id}\\ \xleftarrow{id\times 1\times id}\\ \xrightarrow[id\times\mu]{}} \; G\times X \xrightarrow{\mu} X$$

whence we get a spectral sequence

$$E_2^{pq} = \check{H}^p(\nu \longmapsto H^q(G^\nu \times X)) \Longrightarrow H_G^{p+q}(X).$$

It would be nice to know when $E_2 = E_\infty$ for this spectral sequence. ~~Suppose~~ The E_2 term may be identified with

$$\mathrm{Cotor}^p_{H^*(G)}(k, H^*(X))^q = \mathrm{Ext}^p_{H_*(G)}(k, H^*(X))^q$$

this spectral sequence unfortunately seems even less ~~easy to~~ likely to admit a simple proof that $E_2 = E_\infty$. Thus for p odd, $X = pt$ the E_2 term is $\Lambda[e]\otimes S[c]$ for rank 1, so already one can't conclude that $H^*(BG)$ is a polynomial ring

April 4, 1970: Eilenberg-Moore spectral sequence approach to the cohomology of finite groups of rational points

Let G be a group scheme over S, T etale topos of sheaves on schemes over S, T_G classifying topos of G. If X is a scheme on which G acts, then $H^*_G(X)$ is the cohomology (coeffs. $\mathbb{Z}_\ell$) of X as an object of T_G. Then the map of G-sheaves

$$G \times X \longrightarrow X$$

is surjective and this gives rise to a spectral sequence

$$(*) \qquad E_2^{pq} = \check{H}^p(\nu \longmapsto H^q(G^\nu \times X)) \Longrightarrow H^{p+q}_G(X).$$

Here $\check{H}^*$ denotes the cohomology of a cosimplicial abelian group and $\nu \mapsto H^*(G^\nu \times X)$ is the cosimplicial ring obtained by taking the equivariant cohomology of the simplicial G-scheme

$$\mathcal{X} \qquad \cdots \; G \times G \times X \rightrightarrows G \times X \longrightarrow X$$

where

$$\mathcal{X}_\nu = G^{\nu+1} \times X$$

and

$$d_i(g_0, \ldots, g_\nu, x) = \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_\nu, x) & 0 \le i < \nu \\ (g_0, \ldots, g_{\nu-1}, g_\nu x) & i = \nu \end{cases}$$

and

$$s_i(g_0, \ldots, g_\nu, x) = (g_0, \ldots, g_{i-1}, 1, g_i, \ldots, x) \qquad 0 \le i \le \nu$$

etc.

Properties of (*) ~~[illegible]~~ should include multiplicative structure and action of Steenrod algebra.

Suppose G has Kunneth property. ~~[illegible]~~ ~~[illegible]~~ Then the E_2 term of $*$ is the cohomology of the complex

$$0 \longrightarrow H^*(X) \longrightarrow H^*(G) \otimes H^*(X) \longrightarrow H^*(G) \otimes H^*(G) \otimes H^*(X) \longrightarrow \cdots$$

which is nothing other ~~[illegible]~~ than the cohomology of the bigebra $H^*(G)$ with coefficients in $H^*(X)$, which following the standard language is

$$E_2^{pq} = \mathrm{Cotor}^p_{H^*(G)}(k, H^*(X))^q.$$

If $H^*(G)$ is finite dimensional and we let $H_*(G)$ be its dual, then this can also be written in a more familiar way as

$$(**) \qquad E_2^{pq} = \mathrm{Ext}^p_{H_*(G)}(k, H^*(X))^q \Longrightarrow H_G^{p+q}(X).$$

Example 1: Suppose G ~~[illegible]~~ has no l-torsion, which we shall define to mean that $H^*(G)$ is an exterior algebra with odd degree generators. Then if l is odd then by Borel's theorem this ~~[illegible]~~ ~~[illegible]~~ implies that $H^*(G)$ is primitively generated, so $H_*(G)$ is commutative and must be an exterior algebra with primitive odd degree generators. If $l=2$, the hypothesis implies that the squaring operation on $\tilde{H}^*(G)$ is zero, hence $H_*(G)$ is the enveloping algebra of its ~~sub~~ space L of primitive elements. Since L is concentrated in odd dimensions, it follows that ~~this~~ has trivial bracket and squaring operation, so $H_*(G)$ is also an exterior algebra with odd degree primitive generators.

Now

$$\operatorname{Ext}^*_{k[\varepsilon]}(k,k)^* = k[\eta] \qquad \varepsilon^2=0$$

where η is a bidegree $p=1$, $q=\deg\varepsilon$. This implies that the spectral sequence (**) with $X=pt$ has

$$E_2 = \text{poly. ring with generators of degree } (1,q)$$

for each generator of $H_*(G)$ of degree q. Thus E_2 is of even total degree so all differentials are zero, $E_2 = E_\infty$ and $H^*(BG)$ is a polynomial ring, whose generators are detected by the map

$$(***) \qquad H^q(BG) \longrightarrow E_2^{1,q-1} \cong PH^{q-1}(G).$$

__Example 1′:__ Suppose $l=2$ and $H^*(G)$ is primitively generated. Then $H_*(G)$ is commutative killed by the squaring operation, hence is an exterior algebra, so E_2 is again a polynomial ring. Therefore if one knows that $PH^*(G)$ is ~~[illegible]~~ transgressive e.g. (***) is surjective then the same conclusions can be drawn.

__Example 2:__ Now take up the situation of G^σ. Assume the Borel condition: $H^*(G)$ simple system of transgressive generators (which when l is odd is equivalent to G having no l-torsion). Take $X = G/G^\sigma$ which in virtue of the radicial surjective map

$$\begin{aligned} \theta : G/G^\sigma &\longrightarrow G \\ gG^\sigma &\longmapsto g(\sigma g)^{-1} \end{aligned}$$

has the same cohomology as G.

<u>Lemma:</u> If $\mu_X : G\times X \to X$ is the action map, then

$$\begin{tikzcd} H^*(X) \arrow[r, "\mu_X^*"] & H^*(G\times X) \cong H^*(G)\otimes H^*(X) \\ H^*(G) \arrow[u, "\cong"', "\theta^*"] \arrow[r, "\gamma"] & H^*(G)\otimes H^*(G) \arrow[u, "\cong"', "id\otimes\theta^*"] \end{tikzcd}$$

commutes, where $\gamma(x) = (x - \sigma^* x)\otimes 1 + 1\otimes x$ if $x \in P = P H^*(G)$.

<u>Proof:</u> γ is the map on cohomology induced by the composition

$$\begin{array}{ccccccc} G\times G & \longrightarrow & G\times G\times G & \xrightarrow{\mu_G\times \text{inv.}} & G\times G & \xrightarrow{\mu_G} & G \\ (g, g') & \mapsto & (g, g', \sigma g) & \longmapsto & (gg', \sigma g^{-1}) & \longmapsto & gg'(\sigma g)^{-1} \end{array}$$

and by defining of primitive $\mu_G^* x = x\otimes 1 + 1\otimes x \mapsto x\otimes 1\otimes 1 + 1\otimes x\otimes 1 - 1\otimes 1\otimes x \longmapsto x\otimes 1 + 1\otimes x - \sigma^* x\otimes 1$

$$= (x - \sigma^* x)\otimes 1 + 1\otimes x.$$

Here I have used that $(inv)^*(x) = -x$.

~~Our next job is to compute $\mathrm{Ext}_{H_*(G)}(k, H^*(X))$. The method is to construct an injective resolution of $H^*(G)$ over $H_*(G)$ and then to tensor with $H^*(X)$. First I consider the case where $H^*(G) = U(P)$ and $P^{ev}=0$, so that $H^*(G) = \Lambda(P)$ as a Hopf algebras, and hence $H_*(G) = \Lambda(P^*)$. Now form the complex of Koszul~~

~~$\Lambda P \otimes S(P)$ $\qquad d(p\otimes 1) = 1 \otimes p$, d derivation~~

~~where $P^q \otimes 1$ is of bidegree $(0,q)$ and $1 \otimes P^q$ is of bidegree $(1,q)$. This complex is a resolution of k by $H_*(G)$~~

Our next job is to compute $\mathrm{Ext}_{H_*(G)}(k, H^*(X))$. First we compute $H^*(X)$ as a module over $H_*(G)$. Given $z \in H_*(G)$ and $x \in P$, then

$$
\begin{aligned}
z(\theta^* x) &= i(z)(\mu_X^* \theta^* x) \\
(*) \qquad &= i(z)\left[(x - \sigma^* x)\otimes 1 + 1 \otimes \theta^* x\right] \\
&= \langle z, x - \sigma^* x\rangle
\end{aligned}
$$

Let $M \subset H_*(G)$ be a minimal generating subspace; then the pairing $M \otimes P \to k$ is non-degenerate and $\Lambda M \xrightarrow{\sim} H_*(G)$. Write $P = P^\sigma \oplus C$. Then by choosing a basis in

C one obtains an isomorphism of $U(P^\sigma)$-modules

$$(**) \qquad U(P^\sigma) \otimes \Lambda C \xrightarrow{\sim} H^*(G)$$

(One has to be careful in characteristic 2 since the squaring operator on P need not fix C). By $(*)$ the corresponding decomposition

$$U(\theta^* P^\sigma) \otimes \Lambda \theta^* C \xrightarrow{\sim} H^*(X)$$

is stable under the action of $H_*(G)$ and in fact $H_*(G)$ acts trivially on $U(\theta^* P^\sigma)$, i.e. the action on $H^*(X)$ is commutes with the $U(\theta^* P^\sigma)$-module structure. Thus

$$\mathrm{Ext}^p_{H_*(G)}(k, H^*(X)) \simeq \overbrace{\mathrm{Ext}^p_{H_*(G)}(k, \Lambda\theta^* C)}^{U(\theta^* P^\sigma)\otimes} .$$

Now write $M = M_1 \oplus M_2$, $M_1 = ((1-\sigma^*)C)^\perp$ + M_2 complement ~~the corresponding under~~ ~~the pairing to $P = P^\sigma \oplus C$~~. Then ~~$M_1 \perp C$ and so~~ $M_2 \simeq C^*$.

$$H_*(G) = \Lambda M_1 \otimes \Lambda M_2$$

where ΛM_1 acts trivially on $\Lambda\theta^* C$ and where

$$\Lambda \theta^* C \cong \mathrm{Hom}_k(\Lambda M_2, k)$$

as ΛM_2-modules. Consequently

$$\mathrm{Ext}_{H_*(G)}(k, \Lambda\theta^* C) = \mathrm{Ext}_{\Lambda M_1}(k,k) \otimes \mathrm{Ext}_{\Lambda M_2}(k, \Lambda\theta^* C)$$

$$= \mathrm{Ext}_{\Lambda M_1}(k,k)$$
$$\cong S(M_1^*)$$

and $M_1^* \simeq P/(\mathrm{id}-\sigma^*)P \cong P_\sigma$. So we have proved

<u>Lemma:</u> $\mathrm{Ext}^*_{H_*(G)}(k, H^*(X)) \cong U(P^\sigma) \otimes S(P_\sigma)$.

Now $U(P^\sigma) = \mathrm{Ext}^0$ and one ought to be able to realize the map $P \longrightarrow \mathrm{Ext}^1$ by means of the map

$$\begin{array}{ccc} \mathrm{Ext}^1_{H_*(G)}(k,k) & \longrightarrow & \mathrm{Ext}^1_{H_*(G)}(k, H^*(X)) \\ \wr\| & & \\ P & & \end{array}$$

which comes from $f: X \longrightarrow pt$.

The next problem is to show $E_2 = E_\infty$ in the spectral sequence by producing elements of $H^*(BG^\sigma)$ which restrict in $U(P^\sigma) \cong E_2^{0*}$ to P^σ.

Definition of the homomorphism

$$\Phi: H^q(BG)^\sigma \longrightarrow H^{q-1}(BG^\sigma)/j^*H^{q-1}(BG).$$

(notation $H^*(BG) \xrightarrow{j^*} H^*(BG^\sigma) \xrightarrow{i^*} H^*(G/G^\sigma)$)

Let $I^\cdot$ be an injective resolution of the constant sheaf k in the category of G-sheaves on C, the category of schemes. Recall that a G-sheaf I is a sheaf on C endowed with an action $G \times I \longrightarrow I$, i.e. $\forall\, U$ in C a map

$$G(U) \times I(U) \longrightarrow I(U).$$

Moreover

$$\Gamma(e, G; I) = \mathrm{Hom}_{T_G}(e, I) = \{ s \in I(pt) \mid \forall\, U \in C \ \&\ g \in G(U) \quad g \cdot s = s \}.$$

(Note that this is not obviously the same as $I(pt)^{G(pt)}$.) By definition

$$H^q(X; G) = H^q[\Gamma(X, G; I^\cdot)].$$

Suppose $X = pt$, and let $x \in H^q(pt, G) \overset{\text{def}}{=} H^q(BG)$ be invariant under σ, $\sigma^* x = x$. Recall σ^* is defined as follows. Denote by I^σ the sheaf I with new G-action $g \cdot s = \sigma(g) s$. Since $(I^\cdot)^\sigma$ is a resolution of k there is a map of complexes $\varphi : (I^\cdot)^\sigma \longrightarrow I^\cdot$ unique up to homotopy. φ can also be written as a map $\varphi : I^\cdot \to I^\cdot$ of complexes, compatible with augmentation $k \to I^0$, such that

$$\varphi(\sigma(g)s) = g\,\varphi(s) \qquad g \in G(U), \ s \in I^\cdot(U).$$

Let $\dot{x} \in I^q(pt)$ be a G-invariant section representing x. Then $\sigma^* x$ is represented by $\varphi(\dot{x})$, and so there exists a G-invariant $y \in I^{q-1}(pt)$ such that

$$\dot{x} - \varphi^{\bullet}\dot{x} = dy$$

Now consider the restriction of φ to the subgroup G^{σ}. Then φ is an endomorphism of $I^{\cdot}$, hence φ is homotopic to the identity (as $I^{\cdot}$ is an injective complex of G^{σ}-sheaves? ~~...~~ Examine this:

Suppose $H \overset{i}{\subset} G$ are groups in a topos T. Then I have the maps

$$\begin{tikzcd} T_H \arrow[r, shift left=2, "i_!"] \arrow[r, shift right=2, "i_*"'] & T_G \arrow[l, "i^*"'] \end{tikzcd}$$

where i^* is restriction, and where

$$i_*(F) = \underline{Map}_H(G, F)$$

$$i_!(F) = G \times_H F.$$

Here $G \times_H F$ is the cokernel of $\quad G \times H \times F \underset{id \times \mu}{\overset{\mu \times id}{\rightrightarrows}} G \times F$ in the category of sheaves.

For abelian sheaves one must take

$$i_!^{ab}(F) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} F$$

and it is clear that this is an exact functor of F, because it is the sheaf associated to the presheaf

$$U \longmapsto \mathbb{Z}[G(U)] \otimes_{\mathbb{Z}[H(U)]} F(U)$$

and $H(U) \subset G(U)$, while we can realize any exact sequence of sheaves as coming from an exact sequence of presheaves. Therefore $i^*: T_G^{ab} \longrightarrow T_H^{ab}$ preserves injectives.

(It is worthwhile to check that

$$i_!: T_H \longrightarrow T_G/(G/H)$$

is an equivalence of categories, the quasi-inverse functor being base change via $pt \longrightarrow G/H$. Hopefully this will be in SGAD, in any case this kind of argument is clear from the sheaf-theoretic point of view, which after all is just like sets except you have to localize.)

So we've seen that $\exists$ homotopy operator $h: I^\delta \to I^{\delta-1}$ all G^σ-equivariant $\ni$

$$id - \varphi = dh + hd$$

Hence

$$\dot{x} - \varphi\dot{x} = d(h\dot{x})$$

whence

$$d(h\dot{x} - y) = 0.$$

The element $h\dot{x} - y$ defines a cycle of $I^{\delta-1}(pt)$ which is G^σ-equivariant, hence it defines an element $\square$ of $H^{\delta-1}(BG^\sigma)$.

Let us now see how dependent this element is on the various choices made: I, φ, h, $\dot{x}$, y. Given another resolution $\bar{I}^\cdot$ and another $\bar{\varphi}$ there is a map $f: I^\cdot \longrightarrow \bar{I}^\cdot$ and a homotopy h_1 with $\bar{\varphi}f - f\varphi = dh_1 + h_1 d$. Suppose another $\bar{h}$ found $\ni$ G^σ-equiv. and $id - \bar{\varphi} = d\bar{h} + \bar{h}d$. Then

$$f\dot{x} - \bar{\varphi}f\dot{x} = f\dot{x} - f\varphi\dot{x} - dh_1\dot{x} = fdy - dh_1\dot{x}$$

and also

$$f\dot{x} - \bar{\varphi} f \dot{x} = d \bar{h} f \dot{x}$$

so our new cycle is

$$\bar{h} f \dot{x} - (f y - h_1 \dot{x}).$$

Now we have homotopies which are G^σ-invariant joining f to $\bar{\varphi} f$, $\bar{\varphi} f$ to $f\varphi$ and $f\varphi$ to f which are related by a higher homotopy. Thus

$$f - \bar{\varphi} f = [d, \bar{h} f]$$
$$\bar{\varphi} f - f\varphi = [d, h_1]$$
$$- f + f\varphi = [d, -fh]$$

so $[d, \bar{h} f + h_1 - fh] = 0$ where $\bar{h} f + h_1 - fh : I^\bullet \to \bar{I}^\bullet$ is of degree -1, hence

$$\bar{h} f + h_1 - fh = d h_2 - h_2 d \qquad h_2 : I^q \to I^{q-2}$$

and h_2 is G^σ equivariant. Thus

$$\bar{h} f \dot{x} + h_1 \dot{x} - f h \dot{x} = d h_2 \dot{x} \qquad \text{or}$$

$$\boxed{\left[\bar{h} f \dot{x} - (f y - h_1 \dot{x})\right] - f\left[h \dot{x} - y\right] = d h_2 \dot{x}}$$

which shows that the ~~[illegible]~~ element $\square$ of $H^{q-1}(BG^\sigma)$ doesn't depend upon the choice of $I^\bullet, \varphi, h$. ~~[illegible]~~ Also if one modifies $\dot{x}$ by dz, $\bar{\dot{x}} = \dot{x} + dz$ where $z \in I^{q-1}(pt)$ is G-equivariant, then ~~φx~~

$$\bar{\dot{x}} - \varphi\bar{\dot{x}} = \dot{x} - \varphi\dot{x} + d(z - \varphi z)$$
$$= d\{y + (z - \varphi z)\}$$

so can take $\bar{y} = y + (z - \varphi z)$. Then our cycle is

$$h\bar{\dot{x}} - \bar{y} = h(\dot{x} + dz) - y - (z - \varphi z)$$
$$= h\dot{x} - y - dhz$$

where hz is G^σ-equivariant, so the class in $H^{q-1}(BG^\sigma)$ doesn't change. Thus $\square$ doesn't depend on the choices of $\dot{x}$. Finally altering y, changes $\square$ by ~~anything~~ a cycle invariant under G, hence $x \mapsto \square$ is a well-defined map:

$$\boxed{\Phi : H^q(BG)^\sigma \longrightarrow H^{q-1}(BG^\sigma)/j^* H^{q-1}(BG)}$$

<u>Derivation property of Φ:</u> $I^\cdot \otimes I^\cdot$ (tensor over k) is a G-resolution of k, hence there is a map of complexes

$$f : I^\cdot \otimes I^\cdot \longrightarrow I^\cdot$$

unique up to homotopy and whose induced effect on cohomology is the cup product. Suppose $x_1, x_2 \in H^*(BG)^\sigma$ and $\dot{x}_1, \dot{x}_2$ representing cocycles in $I^\cdot(pt)$ invariant under G. By the previous independence proof (which used only that the second $I^\cdot$

was injective to construct f, h_1 and h_2) we get a compatibility condition on the cocycles constructed in the above manner from $\dot{x}_1 \otimes \dot{x}_2$ and $f(\dot{x}_1 \otimes \dot{x}_2)$. More precisely with $\bar{I} = I \otimes I$ and $\bar{\varphi} = \varphi \otimes \varphi$ and $\bar{h} = h \otimes \varphi + id \otimes h$

$$\begin{aligned}[d, \bar{h}] &= [d \otimes id + id \otimes d,\ h \otimes \varphi + id \otimes h] \\ &= (id - \varphi) \otimes \varphi + id \otimes (id - \varphi) \\ &= id \otimes id - \varphi \otimes \varphi .\end{aligned}$$

we know that $\Phi(x_1 x_2)$ is computed thusly: Start with $\dot{x}_1 \otimes \dot{x}_2$ which represents $x_1 x_2$. If

$$\dot{x}_1 - \varphi \dot{x}_1 = dy_1$$
$$\dot{x}_2 - \varphi \dot{x}_2 = dy_2$$

then

$$\begin{aligned}\dot{x}_1 \otimes \dot{x}_2 - \varphi \dot{x}_1 \otimes \varphi \dot{x}_2 &= (\dot{x}_1 - \varphi \dot{x}_1) \otimes \varphi \dot{x}_2 + \dot{x}_1 \otimes (\dot{x}_2 - \varphi \dot{x}_2) \\ &= dy_1 \otimes \varphi \dot{x}_2 + \dot{x}_1 \otimes dy_2 \\ &= d(y_1 \otimes \varphi \dot{x}_2 + (-1)^{\deg x_1} \dot{x}_1 \otimes y_2)\end{aligned}$$

But this is also

$$= d\bar{h}(\dot{x}_1 \otimes \dot{x}_2) = d(h\dot{x}_1 \otimes \varphi \dot{x}_2 + (-1)^{\deg x_1} \dot{x}_1 \otimes h\dot{x}_2)$$

So a representing cocycle for $\Phi(x_1, x_2)$ is

$$f\left[(y_1 - h\dot{x}_1) \otimes \varphi \dot{x}_2 + (-1)^{\deg x_1} \dot{x}_1 \otimes (y_2 - h\dot{x}_2)\right] \quad \text{in } I$$

~~which is the same as the one constructed from $f(x_1 \otimes x_2)$~~
~~by [illegible] independent result. [illegible]~~ and therefore

$$\Phi(x, x_2) = \Phi x_1 \cdot \cancel{\varphi^*} x_2 + (-1)^{\deg x_1} x_1 \cdot \Phi x_2$$

or

$$\boxed{\Phi(x, x_2) = \Phi x_1 \cdot x_2 + (-1)^{\deg x_1} x \cdot \Phi x_2}$$

which proves the desired derivation property.

The next step is to compute the composite $i^* \Phi$ where $i : G/G^\sigma \longrightarrow BG^\sigma$. This consists from the sheaf point of view in taking a G^σ-sheaf lifting it via the map $G \xrightarrow{f} pt$ of G^σ-schemes and then descending to an honest sheaf over G/G^σ. Suppose we have a sheaf I with G^σ action; then we consider the sheaf $G \times I$, and finally the quotient sheaf $G \times_{G^\sigma} I$, which is the sheaf associated to the presheaf $U \mapsto G(U) \times_{G^\sigma(U)} I(U)$. Now I do this with a resolution $I^\cdot$ of k with G^σ action, and then I take sections ~~of~~ over G/G^σ. Thus it seems that I want

$$\operatorname{Hom}_{T/(G/G^\sigma)}\left(G/G^\sigma, G \times_{G^\sigma} I^\cdot\right) = \Gamma\left(G \times_{G^\sigma} I^\cdot / G/G^\sigma\right).$$

One way of obtaining such sections is to have a map

$$\alpha \in \operatorname{Hom}(G, I)$$

such that $\alpha(g \cdot z) = z^{-1} \alpha(g)$ $\qquad z \in G^\sigma(U) \quad g \in G(U)$

any U.

In fact these are all the sections:

$$\Gamma(G\times_{G^\sigma} I / (G/G^\sigma)) = \left\{ \alpha\in I(G) \;\middle|\; \begin{array}{l} R_z^*\alpha = z^{-1}\alpha \\ z\in G^\sigma(U) \end{array} \right\}.$$

So now given $x\in H^*(BG)^\sigma$ represented by $\dot{x}$ and $\Phi(x)$ represented by $h\dot{x}-y \in I^{q-1}(pt)$ G^σ inv.

then $i^*\Phi(x)$ has to be computed as follows. Namely $G\times_{G^\sigma} I^\circ$ is a resolution of k in $T/(G/G^\sigma)$ hence there is a map

$$G\times_{G^\sigma} I^\circ \xrightarrow{\psi} J^\circ$$

where J° is an injective ~~[illegible]~~ resolution of k in $T/(G/G^\sigma)$. Now the cycle $h\dot{x}-y\in I^{q-1}(pt)$ defines a section α of $G\times_{G^\sigma} I^{q-1}$ over G/G^σ, namely the one α sending any $\alpha(g) = h\dot{x}-y$. Then we apply ψ to α to get a cycle in $\Gamma(J^\circ/(G/G^\sigma))$ which represents the element $i^*\Phi x$ in $H^*(G/G^\sigma)$. (~~Is~~ is it clear that y disappears under this process?)

Recall that I am trying to prove commutativity of

$$\begin{array}{ccc} H^q(BG) & \xrightarrow{\Phi} & H^{q-1}(BG^\sigma)/H^{q-1}(BG) \\ \Big\downarrow \text{edge hom.} & & \Big\downarrow i^* \\ H^{q-1}(G) & \xrightarrow{\theta^*} & H^{q-1}(G/G^\sigma)/H^{q-1}(pt) \quad q\geq 1 \end{array}$$

(Therefore it seems reasonable that J° is unnecessary.) (Actually since $G/G^\sigma \xrightarrow{f} pt$ is a map in the site, so f^* should preserve injectives.)

Suppose I is an object of T_G, i.e. a sheaf endowed with a G-action $G(U)\times I(U) \longrightarrow I(U)$. If X is a G-variety

$$\Gamma_G(X,I) \overset{\text{defn}}{=\!=\!=} \mathrm{Hom}_{T_G}(X,I) = \{\alpha : X \to I \mid \alpha \text{ } G\text{-map}\}$$

~~[illegible]~~ For α ~~[illegible]~~ to be a G-map means that

$$\begin{array}{ccc} G\times X & \xrightarrow{\mu_X} & X \\ \downarrow{\scriptstyle id\times\alpha} & & \downarrow{\scriptstyle\alpha} \\ G\times I & \xrightarrow{\mu_I} & I \end{array}$$

must commutes. Now α is the same as $\alpha(id_X) \in I(X)$, which we shall denote α^b; commutativity of the square means that the two elements of $I(G\times X)$ coincide:

$$\mu_X^*(\alpha^b) = pr_1^b \cdot pr_2^* \alpha^b$$

where $pr_1^b \in G(G\times X)$ is the class of the projection. Thus

$$\Gamma_G(X,I) \longrightarrow I(X) \overset{\mu_X^*}{\underset{}{\rightrightarrows}} I(G\times X)$$

$$u \longmapsto pr_1^b \cdot pr_2^* u$$

uses G-action on I.

is exact.

day 8

Derivation of the basic spectral sequence: Start with a G-variety X and the covering $pr_2: G\times X \to X$. Then one can form the iterated fibre products

$$Y_\nu = (G\times X/X)^{\nu+1} \cong G^{\nu+1}\times X$$

$$d_i(g_0,\dots,g_\nu,x) = (g_0,\dots,\hat{g}_i,\dots,g_\nu,x) \qquad 0\le i\le\nu$$

$$s_i(g_0,\dots,g_\nu,x) = (g_0,\dots,g_{i-1},g_i,g_i,\dots,g_\nu,x) \qquad ''$$

all with the diagonal action of G. On the other hand one can form

$$Y'_\nu = G^{\nu+1}\times X \qquad \text{with}$$

$$d_i(g_0,\dots,g_\nu,x) = \begin{cases}(g_0,\dots,g_{i-1},g_ig_{i+1},\dots,g_\nu,x) & 0\le i<\nu\\ (g_0,\dots,g_{\nu-1},g_\nu x) & i=\nu\end{cases}$$

$$s_i(g_0,\dots,g_\nu,x) = (g_0,\dots,g_i,1,g_{i+1},\dots,g_\nu,x) \qquad 0\le i\le\nu$$

which G acting on the first factor:

$$g(g_0,\dots,g_\nu,x) = (gg_0,g_1,\dots,g_\nu,x).$$

These two simplicial G-objects are isomorphic by the map

$$Y'_\nu \longrightarrow Y_\nu$$

$$(g_0,\dots,g_\nu,x) \longmapsto (g_0,g_0g_1,g_0g_1g_2,\dots,g_0g_1\cdots g_\nu,g_0g_1\cdots g_\nu x)$$

For example in low dimensions I get

$$\begin{tikzcd}[column sep=large]
Y'_\bullet : & G\times G\times X \arrow[r, shift left, "d_0=\mu_G\times id_X"] \arrow[r, shift right, "d_1=id_G\times\mu_X"'] \arrow[d, "{(g_0,g_1,x)\mapsto(g_0,g_0g_1,g_0g_1x)}"'] & G\times X \arrow[r, "\mu_X"] \arrow[d, "{(id_G,\mu_X)}"] & X \arrow[d] \\
Y_\bullet : & G\times G\times X \arrow[r, shift left, "d_0=pr_{23}"] \arrow[r, shift right, "d_1=pr_{13}"'] & G\times X \arrow[r, "pr_2"] & X
\end{tikzcd}$$

The reason I want to use Y' is that

<u>Lemma:</u> If G acts trivially on X, then

$$\Gamma_G(G\times X, I) \cong I(X)$$

$$((g,x)\mapsto g\beta(x)) \longleftarrow\!\shortmid\ \beta$$

$$(\alpha) \longmapsto (x\mapsto \alpha(1,x))$$

(Here g, x should be thought of as elements of $G(U)$, $I(U)$ for variable U)

<u>Conclusion:</u> The complex obtained by applying a G-sheaf I to $Y'_\bullet$ is

$$0 \longrightarrow I(\cancel{X}) \xrightarrow{\ \delta\ } I(G\times X) \xrightarrow{\ \delta\ } I(G^2\times X) \longrightarrow \cdots$$

where δ is given by the standard formula

$$(\delta f)(g_1,\ldots,g_{v+1},x) = g_1 f(g_2,\ldots,g_{v+1},x) - f(g_1g_2,\ldots,g_{v+1},x) \cdots \pm f(g_1,\ldots,g_v,g_{v+1}x).$$

This complex is acyclic if I is injective

Therefore if $I^{\cdot}$ is a resolution of k by injective G-sheaves, the spectral sequence ~~[illegible]~~ we are after results from the double complex

$$E_0^{pq} = I^q(G^p \times X) \qquad \begin{array}{l} d_0 = d \text{ vertically } : I^q \to I^{q+1} \\ \delta \text{ horizontally} \end{array}$$

and $$E_1^{pq} = H^q(G^p \times X) \quad \text{with } d_1 \text{ induced by } \delta$$

~~[illegible]~~

and when G satisfies the Kunneth theorem, this is the bar resolution of $H_*(G)$ acting on $H^*(X)$, so

$$E_2^{pq} = \mathrm{Ext}^p_{H_*(G)}(k, H^*(X))^q.$$

From this we can calculate the edge homomorphism (inverse to Borel's transgression) $\tau : H^q(BG) \longrightarrow H^{q-1}(G)$ $(q > 0)$

$$\begin{tikzcd}
\dot{x} \quad I^q(pt) \arrow[r, "\delta"] & I^q(G) \quad 0 \\
y \quad I^{q-1}(pt) \arrow[u, "d"] \arrow[r, "\delta"] & \delta y \quad I^{q-1}(G) \arrow[u, "d"]
\end{tikzcd}$$

Thus if $x \in H^q(BG)$ is represented by $\underbrace{\text{a ~~[illegible]~~ G-invariant}}$ cocycle $\dot{x}$, we know as $H^+(pt) = 0$, that there is a $y \in I^{q-1}(pt)$ with $dy = \dot{x}$, whence δy represents the desired element τx. ~~[illegible]~~ y can be altered by a cocycle, which alters the class of δy by an element of $\delta H^{q-1}(pt)$ but this is zero.

I ~~[illegible]~~ will eventually want to know that τ factors

$$H^{8}(BG) \longrightarrow H_G^{8-1}(G^c) \longrightarrow H^{8-1}(G)$$

where G^c denotes the G-variety given by G itself with the conjugation action. Thus ~~I~~ want to show that δy represents a conjugation-invariant cocycle on G, i.e. that it is in the kernel of

$$I^{8-1}(G) \xrightarrow{\delta^c} I^{8-1}(G \times G^c)$$

$$f \longmapsto (\delta^c f)(g_1, x) = g_1 f(x) - f(g_1 x g_1^{-1})$$

Now

$$(\delta y)(g) = gy - y$$

so

$$(\delta^c \delta y)(g_1, x) = g_1(xy - y) - (g_1 x g_1^{-1} y - y),$$

which doesn't work. Thus argument is non-trivial, and to gain insight we go through the basic geometry:

~~Then~~ The join $G * G = \{ t g_0 + (1-t) g_1 \mid 0 \leq t \leq 1 \}$ has both a left and right action of G of the sort that one gets a left equivariant principal G-bundle over ΣG. Thus one has a classifying map

$$\Sigma G = I \times G / \dot{I} \times G$$

$$EG \times_G (\Sigma G) \longrightarrow BG$$

yielding a definite homomorphism

$$(*) \qquad H^*(BG) \longrightarrow \tilde{H}^*_G(\Sigma G) = H_G^{*-1}(G)$$

(This should eventually be of great interest to me, since it implies that I get a distinguished system of exterior generators in $H^*_G(G^c)$ once I choose generators $P \subset H^*(BG)$.)

One can do this already with BG instead of ΣG. Thus on EG:

(1) $$G\times G\times G \;\overset{pr_{23}}{\underset{pr_{12}}{\overset{pr_{13}}{\rightrightarrows}}}\; G\times G \;\overset{pr_2}{\underset{pr_1}{\rightrightarrows}}\; G$$

one has both a left and right action of G, and so one inherits an action on BG

(2) $$G\times G \;\overset{pr_2}{\underset{pr_1}{\overset{\mu}{\rightrightarrows}}}\; G \rightrightarrows pt$$

The map from (1) to (2) sends $(g_0, g_1, \dots, g_\nu) \longmapsto (g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{\nu-1}^{-1}g_\nu)$

and consequently the left action of G on EG becomes the conjugation action on BG. Consider the diagram

$$\begin{tikzcd}
EG \arrow[r, "i"] \arrow[d] & EG\times_G (EG)^c \arrow[r] \arrow[d, "\pi"] & EG \arrow[d] \\
BG \arrow[r, "i"] \arrow[d] & EG\times_G (BG)^c \arrow[r, "f"] \arrow[d, "p_1"] & BG \\
pt \arrow[r, "i"] & BG &
\end{tikzcd}$$

where the map f classifies π and where the i maps come from forgetting the G-action. Thus $fi \sim id_{BG}$

and the map f furnishes a map

$$H^*(BG) \xrightarrow{f^*} H^*_G(BG)$$

which composed with the G map $\Sigma G \to BG$ gives the map (*) on page 20.

(It's now clear how to proceed in principle; you work out the map f on the cocycle level. Why is it so that the image of τ lies in $PH^*(G)$?)

I now wish to work out the map (*) $H^*(BG) \longrightarrow H^{*-1}_G(G)$ explicitly in the case where G is discrete. Suppose X is a G-set; then we compute its equivariant cohomology using standard cochains

$$0 \longrightarrow I(X) \xrightarrow{\delta} I(G\times X) \xrightarrow{\delta} I(G\times G\times X) \longrightarrow$$

where $I(?)$ denotes functions on $?$. Suppose X is a simplicial G-set; then its equivariant cohomology should be that of the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 \arrow[r] & I(X_1) \arrow[r] \arrow[u] & I(G\times X_1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I(X_0) \arrow[r, "\delta"] \arrow[u, "d"'] & I(G\times X_0) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Now apply this to the case where $X_0 = BG$, and there should, by the map f on page 21, be two ways of mapping $H^*(BG)$ into the total cohomology of this double complex. The first, valid for any $X_\cdot$, should put a cocycle in $I(G^p \times pt)$ into $I(G^p \times X_0)$ using the map $X_0 \to pt$. The second, which produces the effect f, is more subtle; if $z \in I(G^p)$ is a cocycle one must produce a cocycle for the double complex

$$u_i \in I(G^i \times G^{p-i}) \qquad 0 \le i \le p$$

with $u_0 = z$. Suppose $p=1$ so that $z \in I(G)$ is a homomorphism $z : G \to k$. Then δ, which uses the conjugation action of G, is

$$(\delta z)(g_1, g_2) = (g_1 \cdot z)(g_2) - z(g_2)$$

$$= z(g_1^{-1} g_2 g_1) - z(g_2) = 0$$

so that in this case $u_0 = z$, $u_1 = 0$, is the desired cocycle. For higher p what I'm after is apt to be a big mess, since $f^* z$ will produce a cocycle in the complex $EG \times_G (BG)$ which is a diagonal complex, and then the u_i will comes from $f^* z$ by means of an Alexander-Whitney type formulas.

Need double simplicial set $EG \times_G X$:

$$(EG \times_G X_q)_p = (BG)_p \times X_q = G^p \times X_q$$

with
$$d_i^h(g_1, \ldots, g_p, x) = \begin{cases} (g_2, \ldots, g_p, x) & i=0 \\ (\ldots, g_i g_{i+1}, \ldots, x) & 0<i<p \\ (\ldots, g_p x) & i=p \end{cases}$$

Need ~~[scribbled]~~ shuffle-type formula: So A^{pq} is a bicosimplicial abelian group, and if $a \in A^{pp}$, then one associates the family

$$b_i \in A^{i,p-i} \qquad b_i = \sum_{\substack{(\mu,\nu)\text{ runs} \\ \text{over } (i,p-i)\text{-shuffles}}} \varepsilon(\mu,\nu)\, s^h_\nu s^v_\mu a$$

and the collection $(b_0, .., b_p)$ should be a cocycle in simple complex associated to the double complex. (To test things out suppose $A^{pq} = I(X_p \times Y_q)$ where $X_\cdot$ and $Y_\cdot$ are simplicial sets and $I(?)$ is functions on ? with values in k. Then the b_i are the Kunneth components of a; i.e. $b_i \in I(X_i \times Y_{p-i})$. Here the map you are looking at is the transpose of the shuffle map

$$C_i(X) \otimes C_{p-i}(Y) \longrightarrow C_p(X \times Y)$$

which by E-Z is a ~~[scribbled]~~ quasi-isomorphism.)

Need formula for $f: EG \times_G (BG) \longrightarrow BG$: Start with

$$BG \tilde{\times} EG \xleftarrow{\sim} EG \times_G EG \longrightarrow EG$$

$$(g_0', .., g_\nu') \longleftarrow (g_0, .., g_\nu), (g_0', .., g_\nu') \longmapsto (g_0 g_0', .., g_\nu g_\nu')$$

This ~~[scribbled]~~ commutes with the right G-action so induces

$$\begin{array}{c} EG \times_G EG \\ \downarrow \\ EG \times_G (BG) \end{array}$$