I want to understand if there is any obstruction to your conjecture about cohomology of the group of points with values in the Hensel ring. Let k ~~xhx~~ be an algebraically closed field. Then by means of etale homotopy theory ~~xI~~ I produce a map

$$BGl_n(k) \longrightarrow \hat{BU}_n[p^{-1}]$$

which I would like to know factors through the map of the localization to the completion. Here the completion denotes the Artin-Mazur pro-object, but as $BGl_n(k)$ is a genuine space, we may replace the pro-object by ~~xx~~ its inverse limit. Now Sullivan claims that for a 1-connected space X with f.g. homology groups, the fibre of the map from X to its completion has rational cohomology groups. This reduces to the assertion that the ~~x~~ map

$$A \otimes \mathbb{Z}[p^{-1}] \longrightarrow \hat{A} \otimes [p^{-1}]$$

is injective for a f.g. abelian group A and has a uniquely-divisible abelian ~~xxxxxx~~ group for its cokernel; by additivity it is enough to check this for $\mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}$. The former is clear. Suppose x is an element of the completion off p and that nx is in $\mathbb{Z}\, p^{-1}$; then $p^i nx$ is an integer for some i and so by a straight forward calculation x

$$0 \longrightarrow \mathbb{Z}[p^{-1}] \hookrightarrow \prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell \longrightarrow C \longrightarrow 0$$

$x = (x_\ell)_{\ell \neq p}$ and $nx \in \mathbb{Z}[p^{-1}] \Longrightarrow p^i n x_\ell = m$ for all ℓ for some m.

Choose ℓ equal to all the prime divisors of n one sees that

$$p^j x_\ell = m' \in \mathbb{Z} \text{ for all } \ell \Longrightarrow x = \frac{m'}{p^j}.$$

Thus $x \in \mathbb{Z}[\frac{1}{p}]$. Given $x = (x_\ell)_{\ell \neq p}$ clearly divisible by p. Given $(n,p) = 1$ then by Chinese remainder can find an integer $m \ni m - x_\ell$ divisible by $\ell^{v_\ell(n)}$, whence

$m - x$ divisible by n.

homotopy axiom:

$$M = k[z^{-1}]^n \qquad M \otimes K = k[[z]][z^{-1}]^n$$

L.

so the different homothecy classes of lattices are of the form $z^{-n}L$

so consider the family of k-subspaces

$$V_n = z^{-n}L \cap M$$

~~Remake~~ ~~remark~~ $\quad V_0 \subset V_1 \subset V_2 \subset \cdots$

I claim that $\quad z^{-1}V_n + V_n = V_{n+1}$

once n is large enough i.e.

$$z^{-1} : V_n/V_{n-1} \xrightarrow{\sim} V_{n+1}/V_n$$

and this is of rank n.

~~Pf~~

Proof: Choose m so large that $z^{-m}L \supset k[[z]]^n$

then

z^{-m}

~~Kern~~ question: homothecy classes of lattices are they ~~Z~~ equal to ~~families~~ equivalence classes of families $\{V_n\}_{n \geq 0} \ni z^{-1}: V_n/V_{n-1} \cong V_{n+1}/V_n$?

~~not il~~ Can L be recovered from the V_n?

$$z^{-n}L_1 \cap M = z^{-n}L_2 \cap M \qquad \text{all } n$$

$\overset{?}{\Longrightarrow} L_1 = L_2$. $\qquad$ suppose $x \in L_1$

$z^{-n}L_1 \cap M =$

G acts on M

If I can get it to stabilize a V which generates M I am finished because then

$$0 \longrightarrow k[z^{-1}] \otimes V_{n-1} \longrightarrow k[z^{-1}] \otimes V_n \longrightarrow M \longrightarrow 0$$

is exact for large n.

So the idea will be to let it act on these subspaces and to introduce some kind of combinatorial structure so that it becomes clear that cohomologically I can restrict to k?

First stage:

$$GL_n(k) \longrightarrow GL_n(k[t]) \qquad \text{mod } l \text{ isom?}$$

so we make $\Gamma = GL_n(k[t])$ act on a space X and consider the spectral sequence

$$H^p\left(X/\Gamma,\ \Gamma x \longmapsto H^q_\Gamma(\Gamma x)\right) \Rightarrow H^{p+q}_\Gamma(X)$$

idea is to arrange things so that X/Γ is nice enough so this can be analyzed.

Thus X is contractible so the abutment is the cohomology of Γ.

~~Proposition 3: If X_1 and X_2 are pointed connected CW complexes and $E_i \subset \pi_1 X_i$, $i = 1, 2$, are perfect subgroups, then the~~

Clearly lattices have to be understood from a Spencer point of view!!

$$k[\tfrac{1}{z}]^n = M$$

$$L \subset M \otimes_{k[z^{-1}]} K = k(z)^\wedge = k[[z]][z^{-1}].$$

The idea is that L is completely the same as a v bundle of rank n over $\mathbb{P}_1$ together with a family of n gen. sections over A_1. ~~What becomes important for me to understand is exactly why~~

$$L \supset k[z]^n$$

then can always insist that L be made reasonable!

$$\vee$$

$$L \cap M = \Gamma(\mathcal{L}) \quad \text{integral element}$$

$$\downarrow z^{-1} \qquad\qquad \downarrow \text{canonical section}$$

$$z^{-1}L \cap M = \Gamma(\mathcal{L}(1))$$

thus lattices I now see as families $V_n \ni$ $\underline{z^{-1}V_n + V_n = V_{n+1}}$

equivalent to $\Gamma(\mathcal{O}(1)) \otimes \Gamma(\mathcal{L}) \twoheadrightarrow \Gamma(\mathcal{L}(1))$.

Does the immeuble have an interpretation as a topos? Consider the case of the symmetric groups $\mathfrak{S}_n$.1 or better the case of the maximal compact subgroup.

Let G be a real connected Lie group and K a maximal compact subgroup. Then if \$ is a discretes subgroup of G without torsion, one knows that \$ acts freely on G/K, and in fact that $H^*(\$) = H^*_K(\$/G)$. The immeuble of the symmetric groups.

Question: Can you give a toporific interpretation of the immeuble? One can understand

Thus to get at the classifying topos of \$ (think discrete) one finds a \$-space, so one replaces the category (\$,e) by the category (\$,X).of al . Now what is important is not to find a crude interpretation of things which is always possible in the topos setup byt to find a truly natural category which is equivalent to the immeuble. The first example is that of the covering G/H e, which gives rise to the spectral sequecne

$H^p(v \ H^q \ )$. ?

Think of having $GL_n k$ acting on a vector space V. What is so natural about the ordered set of proper subspaces as a cohomological object of interest? Let G act on P PP P PP

Z I I I Daniel Quillen

Make G act on the set of lines. Then the simplicial complex obtained is the set of all substets of lines

Lattices form an ordered set, with two ends in some sense. So one factors out by the action of the center scalars, and one considers the induced simplicial structure. Why does it remain contractible? Unclear. X a contractible G-space. But the point is this: Anytime you have a contractible ~~simplicial~~ semi-simplicial animal on which G acts you get a spectral sequence of Cech type because such a semi-simplicial animal is a hyper-covering, and given anything you can make it contractible by forming products. So the real question is when do you gain something.

So we seem to have two different techniques for getting at the cohomology of GL(A). On one hand we embed the group inside a continuous group whose cohomology is understood and then we try to understand the homogeneous space or topos.

Now all I have to do is to state Segal's result that for a Γ-space one has ~~the Ω spectrum~~ a spectrum Y_n $n \geq 0$ which $\ni$ $Y_n \simeq \Omega Y_{n+1}$ for $n \geq 1$. Maybe given a formula for Y_n. Point now

$$Y_1 = |B\mathcal{a}_.|$$

<u>Theorem:</u> $\Omega Y_1 = K_0 A \times BGL(A)^+$.

I need the map from from $K_0 A \times BGL(A)^+ \longrightarrow \Omega|B\mathcal{a}_.|$ on the fact that on homology this map

Start with for each projective module P a map $B\,Aut(P) \longrightarrow \Omega|B\mathcal{a}_.|$ and you observe that

$$\begin{tikzcd}
B\,Aut(P) \times B\,Aut(Q) \arrow[r] \arrow[d] & \Omega|B\mathcal{a}_.| \times \Omega|B\mathcal{a}_.| \arrow[d] \\
B\,Aut(P \oplus Q) \arrow[r] & \Omega|B\mathcal{a}_.|
\end{tikzcd}$$

is homotopy commutative because in $\mathcal{a}_2$ one has the object $P, Q, P\oplus Q$ and you have a map

$$
\begin{tikzcd}
BGL_n A \arrow[r] \arrow[d] & BGL_n(A)^+ \arrow[r] & BGL(A)^+ \arrow[d, "\Lambda^i"] \\
BGL_{\binom{n}{i}} A \arrow[rr] & & BGL(A)^+
\end{tikzcd}
$$

$\parallel$ ic

p.

Segal's spectrum

if x ~~dim~~ $\nu(x) \le i$

$\gamma^j(x) = 0 \qquad j > i$

and that implies nilpotence results, for any finitely generated ideal in $\bar{K}(X; A)$.

Program: Homotopy axiom~~x~~ for a field: ~~XXXXXXXXXXXXXXXXXXXXXXXXXXX~~ k. First I consider the Serre method for getting the cohomology of $GL_n k[t]$ reduced to simpler things. So I set $z = t^{-1}$ and let $G = GL_n k[t]$ act on the lattice associayed to the local field of Laurent series in z. I let $M = k[t]^n$ and given a <u>lattice</u> L or equivalently a vector bundle ofer P_1 with trivivaliziton over affine space, I consider $M\ L$ = global section of the bundle. Following Serre it is necessary to make the homothecy class es of lattices into a contractible simplicial complex X, then identify X/G as weal as the ~~xxxxxxxxx~~ sheaf of isotropy group cohomology of X/G and then compute. Now I know that every ~~xxxxxxx~~ lattice is equivalent to the follwoign types

$$L = A \oplus t^{r_1}A \oplus \dots \oplus t^{r_{n-1}}A \quad \text{with } A=k[[z]] \text{ and } 0 \cdot r_1 \quad r_{n-1}. \quad \text{Set}$$

$$s_i = r_i - r_{i-1}$$

then the stabilizer of this lattice is

$$\begin{array}{ccc} k^* & W_{s_1} & W_{s_1+s_2} \\ & k^* & W_{s_2} \\ & & k^* \end{array}$$

where W_s equals the ~~xxxxxxxxxxx~~ k module of polynomials of degrees $\leq s$. Now in addition I must worry about the stabilizer of the various simplices. Now a typical 1-simplex occurs when you augment an r_i by 1. Thus you mest start with a simple kind of thing.What are the simplices? Answer unknwown So I need to find two different kinds of homotopy axiom:

The problem is to compute cohomology of $GL_n(k[t])$. The idea is to make $GL_n(k[t])$ act on a suitable ~~arbre~~ immeuble assoc. with field $k((\frac{1}{t})) = K$

The immeuble is the set of homothecy classes of lattices. Serre's point of view was to consider vector bundles E over $P_{1/k}$ of rank n and then to look at their stalks at ∞

<u>Idea:</u> $X = P_{1/k}$ is a curve over k with a point at ∞. Given a vector bundle E over X of rank n I want to describe it in some nice way; so I consider the quotient field $K = k(t)$ and the completion $\hat{K} = k((\frac{1}{t}))$.

By E_K I mean the stalk at the generic point, i.e.

$$E_K \cong K^n$$

choose such an isomorphism and then consider the lattice generated by E in $\hat{K}^n$ i.e. $\hat{E}$. Thus to each homothecy class

Given a lattice in $L \subset \hat{K}^n$ consider $L \cap K^n$ which is a module over

---

E vector bun

To describe all vector bundles over X.

vector bundles

E vector bundle over X of rang n.
first choose isomorphism

$$E_K \cong K^n \qquad \text{ie } \forall x \in X \text{ get } E_x \subset K^n$$

then $\quad E_\infty^{\wedge} \subset K^n \qquad E_\infty = K^n \cap E_\infty^{\wedge}$
and $E_\infty^{\wedge}$ is a lattice in $\hat{K}^n$.

Given a lattice $L \subset \hat{K}^n \Longrightarrow L \cap K^n$ is a $\mathcal{O}_\infty$ module of rank n.

Idea is to look at homothecy classes of lattices under the action of $GL_n(k[t])$

map given $\mathcal{E} \subset \mathcal{E} \otimes K \cong K^n$. Then

$$\underline{\Gamma(A, \mathcal{E})} \subset K^n$$

is a lattice because free, i.e. $k[t]$ principal ideal domain.

Question: What is stalk at ∞? So the
thus we have a embedding

$$k[t]^n \subset k(t)^n$$

ie a matrix of rational functions. Can you recover what happens at ∞.

then u

the character induced by i. Thus SV (resp. ΛV) is the tensor product of the SL_a (resp. ΛL_a), ~~[illegible]~~ so using the fact that Poincare series multiply for tensor

A finite dimensional algebra over $\mathbb{F}_p$.
~~[illegible]~~ Is $H_*(GL(A)) = 0$?

Have to worry about cohomology of the Borel subgroup.

$$0 \to \mathfrak{m} \to A \to k \to 0$$

which is the ~~[illegible]~~ mini-Borel.

$$P \longrightarrow B(k)$$

According to Serre

$$GL_2(k[t]) = GL_2(k) *_{B(k)} B(k[t])$$

$$H_* \, GL_2(k[t]) \longleftarrow$$

symplectic cobordism

$$\operatorname{Im}\{Sp^*(pt) \to \mathfrak{N}^*(pt)\} = R$$

$$\operatorname{Im}\{Sp^*(B\mathbb{Z}_2) \to \mathfrak{N}^*(B\mathbb{Z}_2)\} = Q \qquad Q \supset R[w^4]$$

$$\begin{tikzcd}
Sp^{q+3}(\mathbb{R}P^3) \arrow[r] & Sp^{q}(B\mathbb{Z}_2) \arrow[r, "v"] \arrow[d] & Sp^{q+4}(B\mathbb{Z}_2) \arrow[r] \arrow[d] & Sp^{q+4}(\mathbb{R}P^3) \arrow[d] & \\
\arrow[r, "0"] & \mathfrak{N}^{q}(B\mathbb{Z}_2) \arrow[r, "w^4"] & \mathfrak{N}^{q+4}(B\mathbb{Z}_2) \arrow[r] & \mathfrak{N}^{q+4}(\mathbb{R}P^3) \arrow[r, "0"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
Sp^{q+3}(\mathbb{R}P^3) \arrow[d] \arrow[r, phantom, "="] & Sp^{q+3}(pt)\cdot 1 \arrow[d] \oplus Sp^{q}(pt)\cdot\mu \oplus Sp^{q+1}(pt;\mathbb{Z}_2)\cdot\nu \\
\mathfrak{N}^{q+3}(\mathbb{R}P^3) \arrow[r, phantom, "="] & \mathfrak{N}^{q+3}(pt) + \mathfrak{N}^{q+2}(pt)\,w + \mathfrak{N}^{q+1}(pt)\,w^2 + \mathfrak{N}^{q}(pt)\,w^3
\end{tikzcd}$$

An element of $Sp^*(pt;\mathbb{Z}_2)$ is a pair $(M_4, W) \ni 2M = \partial W$

$$\widetilde{Sp}(\mathbb{R}P^2) \longrightarrow \widetilde{\mathfrak{N}}(\mathbb{R}P^2) = \mathfrak{N}^*(pt)\,w + \mathfrak{N}^*(pt)\,w^2$$

$$\begin{tikzcd}
Sp^{q+1}(pt)/2 \arrow[r] \arrow[d] & \widetilde{Sp}(\mathbb{R}P^2) \arrow[r] \arrow[d] & {}_2Sp(pt) \arrow[d] \\
\mathfrak{N}(pt) \arrow[r] & \mathfrak{N} \arrow[r] & \mathfrak{N}(pt)
\end{tikzcd}$$

so we have to introduce $\quad K = \operatorname{Im}\left\{ {}_2Sp^*(pt) \longrightarrow \mathfrak{N}^*(pt) \right\}$

Then

$$\operatorname{Im}\left\{ Sp^*(\mathbb{R}P^3) \longrightarrow \mathfrak{N}^*(\mathbb{R}P^3) \right\} = \mathbf{R}\cdot 1 + \mathbf{K}\cdot\omega + R\omega^2 + R\omega^3$$

$$0 \longrightarrow Q \xrightarrow{\omega^4} Q \longrightarrow \left(R + K\omega + R\omega^2 + R\omega^3\right) \longrightarrow 0$$

bit of a mess.

$Sp^0(pt) = \mathbb{Z}$

$Sp^{-1}(pt) = Sp^0(\mathbb{R}_c)$

$$Sp^q(\mathbb{R}P^\infty \times X) \xrightarrow{\omega} Sp^{q+4}(\mathbb{R}P^\infty \times X) \rightarrow Sp^{q+4}(\mathbb{R}P^3 \times X)$$

P_x

~~Second Theorem~~

$\boxed{\text{have to think globally.}}$

To do globally: X base M quaternaionic line bundle over X, then have sphere bundle; Gysin sequence

to say M ~~has~~ lives over $B\mathbb{Z}_2$ is similar to have a line ie. a real subbundle.

M quaternion line bundle over X with real line η

$\ni$ $\eta \otimes \mathbb{H} = M$. why does sphere bundle map to $\mathbb{R}P^3$. because at each point of M get unique quaternion

$$S(M) \longrightarrow \mathbb{R}P^3$$

$v \in S(M)$ ~~$\ni$~~ $v = q \cdot \tau$ where $\tau \in \eta$
$q \in S^3$

then send v to $\pm q^{\#} \in \mathbb{R}P^3$

---

next ~~given~~ note this is smooth ~~&~~ hence

$$\begin{tikzcd} \eta\otimes\mathbb{H} \simeq M \arrow[d] \arrow[dr, phantom, ""] \\ X \end{tikzcd} \qquad \boxed{\begin{tikzcd} SM \arrow[d, "\text{double covering}"] \\ X \times \mathbb{R}P^3 \end{tikzcd}}$$

~~question~~: any case have

$$\begin{tikzcd} \underline{SM} \arrow[r] & \mathbb{R}P^3 \\ S(\eta\otimes\mathbb{C}) \arrow[u] & \mathbb{R}P^1 \arrow[u] \\ S(\eta) \arrow[u] & pt \arrow[u] \end{tikzcd}$$

each inclusion framed.

$$S \longrightarrow MSp \longrightarrow \underline{MSp/S}$$

$$\pi_4$$

$$\pi_3(S) \twoheadrightarrow \pi_3(MSp) \longrightarrow \pi_3(MSp/S) = 0$$

$\mathbb{Z}_2$ $\quad \pi_2 \cong \pi_2$ $\qquad \pi_3(S) = \underline{\mathbb{Z}_{24}}$ / $\mathbb{Z}_3$ is out in $\pi_3(MSp)$

$\mathbb{Z}_2$ $\quad \pi_1 \cong$

$\mathbb{Z}$ $\quad \pi_0 \cong$

$$Sp^{-1}(pt) = \mathbb{Z}_2 \eta$$
$$Sp^{-2}(pt) = \mathbb{Z}_2 \eta^2$$

$$Sp^{\circ}(B\mathbb{Z}_2$$

Conjecture would be that $\pi_3(MSp) = \mathbb{Z}_2$ generated by η^3

$$S^1 \longrightarrow S^3 \longrightarrow S^2 \qquad s_t\, c_1(L)$$
$$S^3 \longrightarrow S^7 \longrightarrow S^4$$

$$\pi_k(S^7) = \pi_k(S^4) + \pi$$

symplectic cobordism

$$e(\eta \otimes L) = \omega + a_1 e(L) + a_2 e(L)^2 + \cdots$$

where $a_i \in Sp^{4-4i}(B\mathbb{Z}_2)$

$m+q-q$
$n+\cancel{1}_f$

$$\omega^m (\omega^{-q} P x - x) = \sum_{\alpha > 0} \omega^{m-\langle\alpha\rangle} a^\alpha s_\alpha(x) \quad \text{in } Sp^{-q+4m}(X)$$

~~now the idea will be maybe to determine the image~~

It's a complicated induction which will ~~show that~~ enable me to produce a formula for elements of $Sp^*(pt)$ and $Sp^*(B\mathbb{Z}_2)$.

start low dimensions

$$SP^0(pt) = \mathbb{Z}$$

$$x \in SP^{-1}(pt) = SP^0_c(R)$$

$$\begin{array}{ccccccc}
\widetilde{SP}^{q+3}(\mathbb{R}P^3 \times X) & \longrightarrow & SP^q(B\mathbb{Z}_2 \times X) & \xrightarrow{\omega} & SP^{q+4}(B\mathbb{Z}_2 \times X) & \longrightarrow & SP^{q+4}(\mathbb{R}P^3 \times X) \\
\wr\| & & & & & & \\
SP^q(X) \oplus SP^{q+1}(X; \mathbb{Z}_2) & & & & & &
\end{array}$$

the most important maps is the ~~one~~ composition

$$\begin{array}{ccccc}
\widetilde{SP}^{q+3}(\mathbb{R}P^3) & \to & SP^q(B\mathbb{Z}_2) & \longrightarrow & SP^q(\mathbb{R}P^3) \\
\wr\| & & & & \\
SP^q(pt) \oplus SP^{q+1}(\cancel{pt}, \mathbb{Z}_2) & & & &
\end{array}$$

module homomorphism. on $SP^q(pt)$ it is multiplication by $[S^3 \to \mathbb{R}P^3]$ but what about the map

$$SP^{q+1}(X; \mathbb{Z}_2) \longrightarrow SP^q(\mathbb{R}P^3 \times X)$$

Let $K = \mathrm{Im}\left\{Sp^*(pt) \xrightarrow{\rho} \mathcal{U}^*(pt)\right\}$. I want to calculate K using my method.

There is nothing except in dimension $\equiv 0 \ (4)$

Let $x \in Sp^{-4g}(pt)$. Then I know that

$$w^m(w^g P_x - x) = \sum_{\alpha > 0} w^{m-\langle\alpha\rangle} a^\alpha \bar{s}_\alpha(x)$$

for m sufficiently large in $Sp^{-4g+4m}(B\mathbb{Z}_2)$. So the idea is to apply ρ which ought to commute with everything yielding

$$w^m(w^g P(\rho x) - \rho x) = \sum_{\alpha>0} w^{m-\langle\alpha\rangle} (\rho a)^\alpha \bar{s}_\alpha(\rho x)$$

We have to distinguish between the symplectic $\bar{s}_\alpha$ & complex s_α.

$\rho\{e(\eta \otimes L)\}$

The idea will be to find

In the unoriented case $\rho: Sp^*(pt) \longrightarrow \mathcal{N}^*(pt)$ I presumably can compute the $\rho(a_k)$ and $\rho(v) = w^4$. ~~Moreover~~

Let us do so

$$\rho\{eg_1(\eta \otimes L)\} = F(w,\tau)^4$$

L universal bundle over $\mathbb{H}P^\infty = BS^3$

Corresponding real proj. bundle is $\mathbb{R}P^\infty = B\mathbb{Z}_2$

and

$$\begin{array}{ccc} \eta^*(B\mathbb{Z}_2) & \longleftarrow & \eta^*(BS^3) \\ & & \parallel \\ \eta^*(pt)[[\omega]] & \longleftarrow & \eta^*(pt)[[e(L)]] \\ \omega^4 & \longleftarrow\!\shortmid & e(L) \end{array}$$

$$e(\eta\otimes L) = \omega + a_1 e(L) + a_2 e(L)^2 + \cdots \text{ in } \eta^*(B\mathbb{Z}_2\times BS^3)$$

$$\downarrow$$

$$\eta^*(B\mathbb{Z}_2\times B\mathbb{Z}_2)$$

$$\omega + a_1(\omega)\tau^4 + a_2(\omega)\tau^8 + \cdots$$

$$e(4\eta\otimes\eta)$$

$$\parallel$$

$$F(\omega,\tau)^4 = \sum_{k,l\geq 0} c^4_{kl}\,\omega^{4k}\,\tau^{4l}$$

$a_k \in \eta^*(B\mathbb{Z}_2)$

Comes from $Sp^*(B\mathbb{Z}_2)$?

Therefore I see that the $\eta^*(pt)^{(4)}$

It more or less follows that $\rho(Sp^*(pt)) \subset \eta^*(pt)^{(4)}$

Precisely Let $R = \eta^*(pt)^{(4)} \subset \eta^*(pt)$.

Assume by induction that $\overline{Sp}^{-4j} \subset R^{-4j}$ $\quad j < g$.

Then we know that $\overline{s}_\alpha(x) \in R$ $\quad \alpha > 0$.

$$\rho a_k \in R[[\omega]]$$

$$\therefore\ \omega^m(\omega^{-8}P(\rho x) - \rho x) \overset{=\psi(\omega)}{\in} R[[\omega]]$$

assume m least ≥ 0 $\ni$ this holds

$\Longrightarrow m = 0$ set $\omega = 0$ $\Longrightarrow \rho x = \psi(0) \in R$.

March 1, 1970:

The following result comes out of your proof of the Adams conjecture

Theorem: ~~Proposition~~ Let E be a complex vector bundle (over X) with associated principal bundle P for U_n. Then if h is any gen. coh. theory with finite coefficient groups the map

$$h(X) \longrightarrow h(P \times_{U_n} U_n/N_n)$$

is injective.

Proof: It is enough to split h into its primary components; so suppose h is l-primary and choose $p \neq l$, and $l \mid p-1$. Then we know by etale homotopy theory that there is a map

$$B\,Gl_n(\mathbb{F}_p) \longrightarrow (BU_n)[p^{-1}]$$

inducing an isomorphism on cohomology mod l. (Recall this is done in the following steps. We have maps of topi

$$BGl_n(\mathbb{F}_{p^\nu}) \longrightarrow B(Gl_n, \mathbb{F}_{p^\nu})_{et} \xrightarrow{heq} B(Gl_n, R_\nu)_{et} \xleftarrow{heq\ off\ p} B(Gl_n, \mathbb{C})_{et}$$

where R_ν is a hensel ring with residue field $\mathbb{F}_{p^\nu}$ contained in $\mathbb{C}$ and all compatible with ν. Then we have a heq.

$$B(Gl_n, \mathbb{C})_{et} \longleftarrow \widehat{BU}_n \quad \text{(Artin-Mazur pro-object)}$$

and hence a map $BGl_n(\mathbb{F}_{p^\nu}) \longrightarrow \widehat{BU}_n[p^{-1}]$ in the homotopy

(It's necessary to be careful you aren't using that $BGl_n(\mathbb{F}_q) \longrightarrow BU_n$ exists in this)

category. Now by Sullivan the map

$$BU_n[p^{-1}] \longrightarrow \varprojlim \widehat{BU_n}[p^{-1}]$$

has a fibre with $\mathbb{Q}$-vector spaces $\hat{A}/A$ for homotopy groups, hence (by obstruction theory) we obtain a God-given map in the homotopy category

$$BGl_n(\mathbb{F}_{p^\nu}) \longrightarrow BU_n[p^{-1}]$$

(obstructions lie in $H^*(BGl_n(\mathbb{F}_{p^\nu}), \hat{A}/A$ trivial action$)$)

which induces an isomorphism on cohomology mod ℓ after one lets $v_\ell(\nu) \longrightarrow \infty$.)

E defines a map $X \longrightarrow BU_n$ and we can form a pull-back diagram in the homotopy category

$$\begin{tikzcd} X_\nu \arrow[r, dashed] \arrow[d, dashed] & BGl_n(\mathbb{F}_{p^\nu})[p^{-1}] \arrow[d] \\ X \arrow[r] & BU_n[p^{-1}]. \end{tikzcd}$$

$\longleftarrow$ meaningless

Let F_ν be the common fibre of the two vertical maps. I want to show that

$$\varprojlim \tilde{H}^*(F_\nu) = 0 \qquad \text{(coh. mod } \ell)$$

for then it will follow that

$$H^*(X) \xrightarrow{\sim} \varprojlim H^*(X_\nu)$$

Now in fact this is so because I know that $F_\nu \overset{\text{off } p}{\simeq} U_n$ and the exterior generators get killed. (a little more precision is desirable here.)

Actually one does have a map $BGl_n(\mathbb{F}_{p^\nu}) \longrightarrow BU_n$ | NO
and one can form the fibre F_ν and we get

$$\begin{tikzcd}
F_\nu \arrow[r] \arrow[d] & (U_n)^\wedge_\ell \arrow[d] \\
BGl_n(\mathbb{F}_{p^\nu}) \arrow[r] \arrow[d] & BGl_n(\mathbb{F}_{p^\nu})^\wedge_\ell \arrow[d] \\
BU_n \arrow[r] & (BU_n)^\wedge_\ell
\end{tikzcd}$$

By use of the Zeeman comparison theorems I see that $(F_\nu)^\wedge_\ell \simeq (U_n)^\wedge_\ell$.
~~[illegible]~~ So now I can ~~compute~~ define X_ν by the pull-back

$$\begin{tikzcd}
F_\nu \arrow[r] \arrow[d] & F_\nu \arrow[d] \\
X_\nu \arrow[r] \arrow[d] & BGl_n(\mathbb{F}_{p^\nu}) \arrow[d] \\
X \arrow[r] & BU_n[p^{-1}]
\end{tikzcd}$$

and it follows that $H^*(X) \xrightarrow{\sim} \varprojlim_\nu H^*(X_\nu)$

In a similar way we try to do the same for N_n.

$$\begin{tikzcd} BN_n(\mathbb{F}_{p^\nu}) \arrow[r] \arrow[d] & BGl_n(\mathbb{F}_{p^\nu}) \arrow[d] \\ BN_n^{[p^{-1}]} \arrow[r] & BU_n[p^{-1}] \end{tikzcd}$$

commutes by etale homotopy theory. Moreover if F'_ν is the fibre of the first vertical arrow, then $F'_\nu = BT_n$ and its easy to check that $\varprojlim H^*(F'_\nu) = 0$. So finally we end up with this situation

$$\begin{tikzcd} Y_\nu \arrow[rr] \arrow[dr] \arrow[dd] & & BN_n(\mathbb{F}_{p^\nu}) \arrow[dr] \arrow[dd] & \\ & X_\nu \arrow[rr] \arrow[dd] & & BGl_n(\mathbb{F}_{p^\nu}) \arrow[dd] \\ P\times_{U_n}(U_n/N_n) = Y \arrow[rr] \arrow[dr] & & BN_n[p^{-1}] \arrow[dr] & \\ & X \arrow[rr] & & BU_n[p^{-1}] \end{tikzcd}$$

where the front, back, and bottom squares are cartesian. Hence the top is also cartesian, so we have

$$H^*(Y) = \varprojlim H^*(Y_\nu)$$
$$H^*(X) = \varprojlim H^*(X_\nu)$$
mod $\ell \neq p$

and $Y_\nu \longrightarrow X_\nu$ is a covering space of degree d prime to ℓ in fact $d \equiv 1$ (ℓ) since $\ell \mid p^\nu - 1$.

We can replace $\{X_\nu\}$ by an inverse system of finite complexes if we wish

Now by Postnikov systems I know that if h has only finite coefficient groups prime to p, then

$$h(Y) \simeq \varprojlim h(Y_\nu)$$
$$h(X) \simeq \varprojlim h(X_\nu).$$

But ~~[illegible]~~ by covering lemma one knows that

$$\mathrm{Ker}\ \{h(X_\nu) \longrightarrow h(Y_\nu)\}$$

is killed by a power of the degree of the covering, hence is zero if ν is large. Thus $h(X) \longrightarrow h(Y)$ is injective by passage to the limit. q.e.d.

Question: Is it injective onto a direct summand?

Yes: We know that X, Y are finite complexes and can assume the same for the X_ν, Y_ν. ~~[illegible]~~ Since there is an $(f_\nu)_*$ we know

(*) $$0 \longrightarrow h(X_\nu) \longrightarrow h(Y_\nu) \longrightarrow C_\nu \longrightarrow 0$$

splits. Assume h is ℓ-primary. We must show that

$$0 \longrightarrow h(X) \otimes \mathbb{Z}_\ell \longrightarrow h(Y) \otimes \mathbb{Z}_\ell \longrightarrow (h(Y)/h(X)) \otimes \mathbb{Z}_\ell \longrightarrow 0$$

is exact and this suffices because $h(Y)$ is a finite abelian ℓ-group. But this follows by tensoring (*) with $\mathbb{Z}_\ell$ and passing to the inverse limit.

<u>Conjecture:</u> There should be a canonical trace homomorphism $f_* : h(Y) \longrightarrow h(X)$ with $f_* 1 = 1$ obtained by passage to the inverse limit from the $(f_\nu)_*$.

<u>Conjecture:</u> The condition that $h^0(pt)$ be finite should be superfluous since by direct limits we can assume $h^0(pt)$ finitely generated and then can deduce injectivity by completion and

$$h^0(X)^\wedge = \varprojlim h^0(X; \mathbb{Z}_n)$$

<u>Corollary:</u> An char. class $\theta \in h(BU_n)$ is determined by its effect on "monomial" bundles.

One might try to construct characteristic classes for vector bundles starting with one for line bundles. For an induced bundle f_*L, $f: Y \to X$ one might put

$$\varphi(f_*L) = \mathrm{Norm}_f\, \varphi(L)$$

and hope that this descends to an element of $h(BU)$. Unfortunately this formula is already false in cohomology for w_t, e.g. for a double covering $f: Y \to X$ and real line bundle

$$\begin{array}{ccccc} \mathrm{Norm}_f(1+t\,e(L)) & = & 1 + t\, f_*e(L) & + & t^2\, \mathrm{Norm}_f(e(L)) \\ & & \nparallel & & \parallel \\ w_t(f_*L) & = & 1 + t\, w_1(f_*L) & + & t^2\, e(f_*L) \end{array}$$

but $w_1(f_*L) = f_* w_1(L) + w_1(f_* 1)$.

March 2, 1970

Segal's localization for non-simply-connected spaces ?

Let S be a multiplicative system in $\mathbb{Z}$. We say that a group G is S-local if for each $s \in S$ the map $g \mapsto g^s$ is bijective as a map of G to itself as a set. Let Gp_S be the full subcategory of the category Gp of groups consisting of the S-local groups. It is clear that Gp_S is closed under inverse limits as these are computed from the underlying set. Hence modulo set theory we have

Proposition: There is a "S-localization" functor $G \mapsto S^{-1}G$ left adjoint to the inclusion $Gp_S \longrightarrow Gp$.

(Will check this if everything else goes through.)

Definition: A space X will be called S-local if $\pi_g(X,x)$ is an S-local group for each $x \in X$ and $g \geq 1$.

I want to show that there is an "S-localization" functor $X \mapsto S^{-1}X$ left adjoint to the inclusion functor $Ho_S \longrightarrow Ho$ where Ho is the homotopy category. First we consider the pointed case, where I can use simplicial groups.

Lemma: If G is a simplicial group and each G_v is S-local, then $\pi_g(G)$ is S-local for each $g \geq 0$.

Proof: One knows that $\pi_g G$ depends only on the underlying pointed simplicial set of G and that the map $g \mapsto g^{s}$ of the underlying simplicial set induces the same map on the homotopy groups. Thus the lemma is clear.

The functor $G \longmapsto S^{-1}G$ induces a functor

$$\underline{\underline{L}}S^{-1}: \text{Hosgps} \longrightarrow \text{Hosgps}_S \quad (= \text{full subcat of } G \ni \pi_q G \in \mathcal{G}ps_S)$$

by just applying it to free simplicial groups. I want to prove this is an adjoint functor and need

Conjecture: If G is a free simplicial group and $\pi_q G$ is S-local for all q, then $G \to S^{-1}G$ is a weq.

First reduction step:

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & \widetilde{S^{-1}G} \\ \downarrow & & \downarrow \\ G & \longrightarrow & S^{-1}G \\ \downarrow & & \downarrow \\ \pi_0 G & \longrightarrow & \pi_0 S^{-1}G \end{array}$$

Now $S^{-1}\pi_0 G = \pi_0 S^{-1}G$ as both are left adjoint to inclusion functor of $\mathcal{G}ps_S$ as constant simplicial groups. Thus bottom arrow is an isomorphism. The top arrow is a map of connected simplicial groups with S-local homotopy groups, hence to prove it is ~~an isomorphism~~ a weq it suffices to show the induced map

$$\begin{array}{ccc} H^q(B\widetilde{S^{-1}G}, A) & \longrightarrow & H^q(B\tilde{G}, A) \\ \wr\!\! | & & \wr\!\! | \\ H^q(BS^{-1}G, M) & \longrightarrow & H^q(BG, M) \end{array}$$

is an isomorphism where $M = \mathrm{Map}(\pi_0 G, A)$ and A is any S-local abelian group. However we have the spectral sequence map

$$
\begin{array}{ccc}
E_2^{pq} = \check{H}^p(\nu \longmapsto H^q(BG_\nu, M)) & \Longrightarrow & H^{p+q}(BG, M) \\
\uparrow & & \uparrow \\
E_2^{pq} = \check{H}^p(\nu \mapsto H^q(BS^{-1}G_\nu, M)) & \Longrightarrow & H^{p+q}(BS^{-1}G, M)
\end{array}
$$

and so we are reduced to the following

don't delete

~~Lemma: Let G be a free group and let M be an S-local abelian group on which $S^{-1}G$ acts. Then~~

$$H^q(B(S^{-1}G), M) \xrightarrow{\sim} H^q(BG, M).$$

false for $\mathbb{Z}$

~~Proof:~~ $G = \bigvee_I \mathbb{Z}$ where the wedge denotes the free product. Then by universality one sees that

$$S^{-1}G = \bigvee_I S^{-1}\mathbb{Z}$$

false because in S-groups not $\vee$ in grps.

where $S^{-1}\mathbb{Z}$ a priori might be different from $\mathbb{Z}[S^{-1}] \subset \mathbb{Q}$. However note that if $x \in H$ and H is S-local then all of the sth-roots of x, $s \in S$, commute since

$$x^{\frac{1}{s}} = \left(x^{\frac{1}{ss'}}\right)^{s'} \qquad x^{\frac{1}{s'}} = \left(x^{\frac{1}{ss'}}\right)^{s}.$$

So consequently $\mathbb{Z}[S^{-1}] = S^{-1}\mathbb{Z}$. Now cohomology adds for free products

$$H^q(B(S^{-1}G), M) = \sum_I H^q(BS^{-1}\mathbb{Z}, M) \qquad q \geq 1$$

so we are reduced to the case of $G = S^{-1}\mathbb{Z}$. ~~But everything is abelian here, so it's clear, in fact~~

$$\cancel{H^q(B\mathbb{Z}, M) = \begin{cases} 0 & q > 1 \\ M/(\sigma-1)M & q = 1 \\ M^\sigma & q = 0 \end{cases}}$$

~~and~~

$$\cancel{H^q(B(S^{-1}\mathbb{Z}), M) = \varprojlim H^q(B\mathbb{Z}, M)} \qquad (M \text{ finite?})$$

$$\cancel{= \begin{cases} 0 & q > 1 \\ \cdots & \\ \bigcap_s M^{\sigma^{1/s}} & \end{cases}}$$

~~and the point here is that $\mathbb{Z} \xrightarrow{s} \mathbb{Z}$ is finite of index s which acts invertibly on M.~~ ?

So is $M^{S^{-1}\mathbb{Z}} \xrightarrow{\sim} M^{\mathbb{Z}}$? In general this is false because you take $M = \mathrm{Map}(S^{-1}\mathbb{Z}, S^{-1}\mathbb{Z})$. Then

$$M^{S^{-1}\mathbb{Z}} = S^{-1}\mathbb{Z}$$

$$M^{\mathbb{Z}} = \mathrm{Map}(S^{-1}\mathbb{Z}/\mathbb{Z}, S^{-1}\mathbb{Z})$$

?

<u>Lemma:</u> Let G be a free group, ~~and let~~ and let Q be an S-local quotient group of G. If M is a Q-module, then

$$H^q(B(S^{-1}G), M) \xrightarrow{\sim} H^q(BG, M) \qquad q \geq 0.$$

(true $q \neq 1$
false $q = 1$

<u>Proof:</u> Problem with $q = 1$.

$$H^1(B(S^{-1}\mathbb{Z}), M) = \varprojlim_{s} M/(\sigma^{1/s}-1)M$$

if M finite

$$(1+\sigma+\cdots+\sigma^{s-1})m = (\sigma^s - 1)n$$

$$(\sigma+1)m = (\sigma^2-1)n = (\sigma+1)(\sigma-1)n$$

~~$2\pi M$~~

$$M/(\sigma-1)M \longrightarrow M^\sigma \longrightarrow M/(\sigma-1)M$$

$$m \longmapsto \underline{(\sigma+1)m}$$

$$\begin{array}{ccc} M^\sigma & & M/(\sigma-1)M \\ \downarrow & & \downarrow \\ M^{\sigma^2} & & M/(\sigma^2-1)M \end{array}$$

~~It would seem that there is trouble with the case where a free group F maps onto the S-local group $\pi_0 G$. The kernel is then free~~

The lemma on page 5 looks false for $q = 1$, but true in other dimensions, hence the whole setup looks fishy.

Note that Segal said he could prove that a localization functor in the domain of homotopy theory existed and you are trying to identify this localization functor with the map $G \longmapsto S^{-1}G$ for simplicial groups. In any case the latter functor does give a good candidate for what localized spaces should be, namely they should be homotopy equivalent to those of the form BG where G is a free S-local simplicial group. The lemma on page 1 shows that the homotopy groups of such a thing are S-local groups.

March 5, 1970. localization again

S multiplicative system in $\mathbb{Z}$ (say S = powers of ℓ a prime no. for simplicity). Then the category of S-local groups is a category of universal algebras, since the structure is given by operations

$$\begin{array}{lll} \text{mult.} & G\times G \longrightarrow G & \\ \text{identity} & e \longrightarrow G & \\ \text{inverse} & G \longrightarrow G & \\ x \mapsto x^{\frac{1}{s}} & G \longrightarrow G & s\in S \end{array}$$

subject to the usual identities and

$$\left(x^{\frac{1}{s}}\right)^{s} = x = \left(x^{s}\right)^{\frac{1}{s}},$$

which forces the map $x \mapsto x^s$ to be bijective. Consequently Gr_S is a Lawvere category.

If G is a S-local group, then there is an associated category of S-local G-modules defined as the abelian group objects over G. Such a thing is of the form $G\times_s M$ where M is a G-module such that this semi-direct product is S-local. The category of S-local G-modules will be denoted $S\mathrm{Mod}_G$ and is a full subcategory of the category of $S^{-1}\mathbb{Z}[G]$-modules. Note that if M is S-local then for $s\in S$ $s\geq 0$ we have that

$$mg \longmapsto (mg)^s = (mg)^{s-2}\, m m^g g^2 = \cdots$$
$$= m^{1+g+\cdots+g^{s-1}} g^s$$

is bijective from $M \times_s G \longrightarrow M \times_s G$. As G is S-local already this means that

$$m \longmapsto (1+g+\cdots+g^{s-1})\, m$$

is bijective, consequently the group ring for S-local G-modules is

$$\underbrace{S^{-1}\mathbb{Z}[G]}_{\substack{\text{ordinary}\\ \text{group ring}}} \ \left[(1+g+\cdots+g^{s-1})^{-1}\right]_{g\in G,\, s\in S_+}$$

<u>Questions</u>: 1.) Suppose that G is a free simplicial group such that $\pi_0 G$ is S-local and $\pi_q G$ are S-local $\pi_0 G$-modules. Then is $G \to S^{-1}G$ a weq? (This implies that the homotopy category of $\mathcal{G}p_S$ is the full subcategory of $Ho(\mathcal{G}p)$ consisting of $G \ni \pi_0 G \times_s \pi_q G$ is S-local.)

<u>Example</u>: Suppose G is connected, i.e. $\pi_0 G = 0$. Now both G and $S^{-1}G$ have S-local abelian homotopy groups hence to prove $G \to S^{-1}G$ is a weq it would suffices to show that $H^q(BS^{-1}G, M) \xrightarrow{\sim} H^q(BG, M)$ for any $S^{-1}\mathbb{Z}$-module M. ~~But this [illegible]~~ It therefore would suffice to know that for a free group G

$$(*) \qquad H^*(BS^{-1}G, M) \simeq H^*(BG, M)$$

if M is an $S^{-1}\mathbb{Z}$ modules with trivial action. Now this is clearly true in dimensions ≤ 2. It therefore suffices to show that $M \mapsto H^*(BS^{-1}G, M)$ are derived functors on the category of S-local $S^{-1}G$-modules, ~~[illegible]~~ but this seems hard.

~~[illegible]~~

A possibility for proving (*) is this. Suppose we can construct $S^{-1}G$ with G a free group by a succession of adjoining sth roots of elements so that a typical transition step is of the form of a cocartesian square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & H_1 \\ \downarrow{\scriptstyle s} & & \downarrow \\ \mathbb{Z} & \longrightarrow & H_2 \end{array}$$

By induction one can assume that H_1 is torsion-free whence the top arrow is injective and $H_2 = \mathbb{Z} *_{\mathbb{Z}} H_1$. By Serre's course H_2 is torsion-free. Also one has a long exact sequence

$$0 \to H^0(BH_2, M) \to H^0(BH_1, M) \oplus H^0(B\mathbb{Z}, M) \to H^0(B\mathbb{Z}, M) \to$$
$$\to H^1(BH_2, M) \to \cdots$$

for any H_2 module M. Take M to be an $S^{-1}\mathbb{Z}$-module with trivial action and assume by induction that

$$H^+(BH_1, M) = 0.$$

Then one has

$$\begin{array}{ccccccccc} & & & & M & \xrightarrow[\cong]{s} & M & & \\ & & & & \| & & \| & & \\ 0 \to & H^1(BH_2, M) & \to & & H^1(B\mathbb{Z}, M) & \to & H^1(B\mathbb{Z}, M) & \to & H^2(BH_2, M) \to 0 \end{array}$$

$$H^q(BH_2, M) = 0 \qquad \text{all } q > 2$$

so $H^+(BH_2, M) = 0$, continuing the induction. Observe that this argument also works to show

$$H^+(BH_1, M) = 0 \implies H^+(BH_2, M) = 0$$

provided that M is a $\mathbb{Z}[H_2]$ module such that $1 + g + \cdots + g^{s-1}$ induces a bijection on M for every $g \in H_2$ and $s \in S_+$. In more detail recall that $Z^1(\mathbb{Z}, M)$ where $\mathbb{Z}$ is generated by σ is isomorphic to M by the map $f \mapsto f(1)$ where f is a crossed homomorphism.

Thus by five lemma

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathbb{Z}, M) & \to & M & \xrightarrow{\sigma - 1} & M & \to & H^1(\mathbb{Z}, M) & \to 0 \\
 & \downarrow & & \downarrow \text{id} & & \downarrow (1+\sigma+\cdots+\sigma^{s-1}) & & \downarrow & \\
0 \to & H^0(s\mathbb{Z}, M) & \to & M & \xrightarrow{\sigma^s - 1} & M & \to & H^1(s\mathbb{Z}, M) & \to 0
\end{array}$$

the restriction maps from $\mathbb{Z}$ to $s\mathbb{Z}$ are isomorphisms.

(A disturbing feature of the above calculation is that H_2 needn't be uniquely S-divisible, e.g. if $\sigma = x^s$ with $x \in H_1$.)

What remains to make this approach good is to show that if H_1 is ~~uniquely S-divisible and if~~ S-torsion-free ($x^s = y^s \implies x = y$) and if the image of the generator of $\mathbb{Z} \to H_1$ is not an s-th root. Then H_2 is also S-torsion-free.

March 5, 1970. Higher alg. K-theory (cont.)

Let R be a ring (not necessarily commutative) I have proposed to define

$$K_i(R) = \pi_i\Big(\Omega B \coprod_{n\geq 0} BGl_n(R)\Big) \qquad i \geq 1$$

Now $\coprod_{n\geq 0} BGl_n(R)$ is a ultra-commutative H-space so has various higher classifying space if Boardman-Vogt theorem holds for it. In any case it gives rise to a trace theory on spaces with values in abelian monoids, namely

$[X, \coprod_{n\geq 0} BGl_n(R)]$ = isomorphism classes of sheaves of R-modules over X which are locally ~~trivial~~ isomorphic to $X \times R^n$ for various n.

Thus this is the monoid of R-vector bundles over X. Denote this by $m(X,R)$. Then (modulo B-V thm.) $K_i(R)$ are the coefficient groups of the universal gen. coh. theory $K_*(X,R)$ endowed with a trace compatible map $m(X,R) \longrightarrow K_0(X,R)$.

From this point of view there is another candidate namely one should take the Grothendieck group $k(X,R)$ of sheaves of R-modules over X locally isomorphic to $X \times P$ where P is a projective R-module of finite type. (In the Grothendieck group exact sequences of such things add). Note that if X is connected, then this is the same as the Grothendieck group of representations of $\pi_1(X)$ in proj. f.t. R-modules.

__Conjecture:__ A generalized cohomology theory is determined by its effect on spaces of the form $K(\pi, 1)$.

notes on the Boardman–Vogt theorem.

$$MX = \coprod_n E\Sigma_n \times_{\Sigma_n} X^n. \quad \text{associative}$$

~~form a category suspension (X).~~
form the suspension category starting from framed compact manifolds. [effective motives.]

what is a generalized cohomology theory? It is a specific functor on the motive category, satisfying an exactness axiom.

$$X \longrightarrow Y \longrightarrow Y/X$$

$$k(X) \longleftarrow k(Y) \longleftarrow k(Y/X) \quad \text{exact.}$$

$$d_g s_j = \begin{cases} s_{j-1} d_g & g<j \\ id & g=j \\ & g=j+1 \\ s_j d_{g-1} & g>j+1 \end{cases}$$

equivalently one has to formulate

$$QQB \underset{d_1}{\overset{d_0}{\rightrightarrows}} QB \xrightarrow{d_0} B \qquad s_{-1}, s_0 : QB \to QQB, \quad s_{-1} : B \to QB$$

$$d_0 s_{-1} = id$$

$$d_1 s_{-1} = s_{-1} d_0$$

$$d_0 s_0$$

$$QQB \underset{Q(\mu_B)}{\overset{\mu_{Q(B)}}{\rightrightarrows}} QB \xrightarrow{\mu_B} B \qquad \eta_{Q(B)}, Q(\eta_B) : QB \to QQB, \quad \eta_B : B \to QB$$

what about η_B compatible with trace?

$$\mu_B \mu_{Q(B)} = \mu_B Q(\mu_B)$$

$$d_0 d_1 = d_0 d_0$$

triv. $\overset{d_1}{\mu_{Q(B)}} \overset{s_0}{\eta_{Q(B)}} = id_{Q(B)}$

triv $\overset{d_1}{\mu_{Q(B)}} \overset{s_{-1}}{Q(\eta_B)} = \overset{s_{-1}}{\eta_B} \overset{d_0}{\mu_B} \cancel{id}$ X

triv. $\overset{d_0}{Q(\mu_B)} \overset{s_{-1}}{\eta_{Q(B)}} = \cancel{id}\ \eta_B \mu_B$

triv. $\overset{d_0}{Q(\mu_B)} \overset{s_0}{Q(\eta_B)} = \cancel{id}\ id$

thus if you can construct $\mu_B : QB \longrightarrow B$ $\ni$ the triple identities

$$\begin{cases} \mu_B \, \mu_{Q(B)} = \mu_B \, Q(\mu_B) \\ \mu_B \, \eta_B = id_B \end{cases}$$

hold, then you get a simplicial object by Godement, with a contraction operator

$$QQQB \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; QQB \rightrightarrows QB \underset{\eta_B}{\overset{\longleftarrow}{\longrightarrow}} B$$

So what happens is that ultimately we have an exact diagram

$$[X, B] \longrightarrow [X, QB] \underset{\eta_{Q(B)*}}{\overset{Q(\eta_B)_*}{\rightrightarrows}} [X, QQB]$$

which permits one to extend to a general coh. theory. Thus the candidate is always the kernel of this pair of maps.

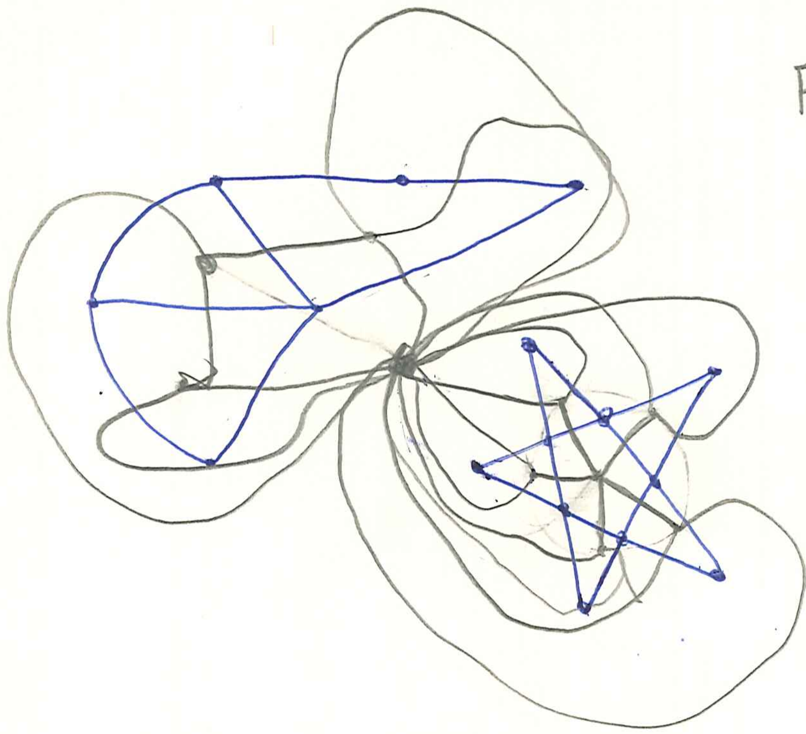
$$\{X, B\} \rightrightarrows \{X, QB\}.$$

which makes it seem that the coh. theory is independent of the actual trace structure on B. no you

you eventually need $QB \rightrightarrows QQB$ desuspension maps

$$\Omega^\infty S^{-k\infty} B \rightrightarrows \Omega^\infty S^\infty \Omega^\infty S^{+k\infty} B$$ no-

$$F_g f = \sum_{i=1}^{g} {}^{iF} f(p_i, X)$$



the power series in question is certainly

$$P(e(L)) = \cancel{e(L)\,F(\omega, e(L))} = e(L)\,e(\eta\otimes L)$$

$$X\,F(\omega, X)$$

$$\hat{\theta}\,P(e(L)) = \theta(e(L))\,\theta(F(\omega, e(L)))$$

so the question becomes

How pleasant can I make

$$\theta(X)\,\theta(F(\omega, X))$$

given a power series $\theta(X) = \sum_{j\geq 0} b_j X^{j+1}$

where $b_j(\omega) \in L[[\omega]]$ and $b_0(\omega)$ unit

$$\theta(X)\cdot\theta(F(\omega, X)) \equiv \theta(X)^2 \qquad (\mathrm{mod}\ \omega)$$

~~$b_0 X \cdot b_0$~~

$X \cdot F(\omega, X)$

$\theta(X) \equiv X$ mod ω

question about [illegible]

additive law $\quad X(\omega + X) = \omega X + X^2$

mult law $\quad X(\omega + X + \omega X) = \omega X + (1+\omega) X^2$

If you had a logarithm then taking $\theta(X) = \ell(X)$ you get

$$\ell(X)\{\underline{\ell(\omega)} + \ell(X)\} = \ell(X) \ldots \quad ?$$

---

assume $\exists$ $\boxed{\text{isomorphism}}$ of law with multiplicative one!

Then $\qquad 2\omega + \omega^2 = 0$

$$X(\omega + X + \omega X)$$

In unoriented cases you take $\theta(X) = \ell(X)$ where ℓ is the canonical logarithm of the law, whence you get

$$\ell(X)\{\ell(\omega) + \ell(X)\}$$

$X = 0 \qquad \theta(X)\,\theta(F(\omega, X))\;\theta(F(\omega^{(2)}, X))$

$\theta(X)\,\theta(F(\omega, X))$ divisible by $X(X - \omega)$

---

can you solve the equation

$$\theta(X)\theta(F(\omega, X)) = X(X - \omega)$$

$\Rightarrow F(\omega, X)\{F(\omega, X) - \omega\} = X(X - \omega)$

$$\frac{\theta(X)}{X}\,\theta(F(\omega, X)) = X - \omega$$

$\theta'(0)\theta(\omega) = -\omega$

$\theta(\omega) = -\omega$

March 7, 1970

Cohomology mod l of the group of rational points of a (non-split) torus.

Let G be a torus defined over $k = \mathbb{F}_q$, with character group

$$M = \mathrm{Hom}(G, \mathbb{G}_m).$$

Then M is a free abelian group endowed with a Frobenius automorphism F, necessarily of finite order since Galois acts continuously. Then

$$G(k) = \mathrm{Hom}_F(M, \bar{k}) = (\check{M} \otimes \bar{k})^F$$

and so if l is prime to the characteristic of k, we have

$$\begin{tikzcd}
0 \arrow[r] & {}_l G(k) \arrow[r] \arrow[d, phantom, "\cup"] & G(k) \arrow[r, "l"] \arrow[d, phantom, "\|"] & G(k) \arrow[r] \arrow[d, phantom, "\|"] & G(k)/l \arrow[r] \arrow[d, phantom, "\|"] & 0 \\
0 \arrow[r] & (\check{M} \otimes \mu_l)^F \arrow[r] & (\check{M} \otimes \bar{k})^F \arrow[r, "l"] & (\check{M} \otimes \bar{k})^F \arrow[r] & H^1(\mathbb{Z}, \check{M} \otimes \mu_l) \arrow[r] \arrow[d, phantom, "\wr|"] & 0 \\
& & & & (\check{M} \otimes \mu_l)_F &
\end{tikzcd}$$

Now since $G(k)$ is a finite abelian group there is a canon. isom.

$$H^*(BG(k)) \cong \Lambda\, (G(k)/l)^\vee \otimes S({}_l G(k))^\vee$$

at least if l is odd, which on account of what we've just shown is canonically isomorphic to

$$\Lambda (M \otimes \check{\mu}_l)^F \otimes S(M \otimes \check{\mu}_l)_F$$

Now from the spectral sequence method for computing $H^*(BG(k))$ I need

$$H^*(BG) = S(M \otimes \check{\mu}_\ell)$$

$$H^*(BG) = \Lambda(M \otimes \check{\mu}_\ell)$$

and the formula I get is

$$H^*(BG(k)) = \Lambda(M \otimes \check{\mu}_\ell)^F \otimes S(M \otimes \check{\mu}_\ell)_F$$

which checks nicely.

March 8, 1970: "Geometric Frobenius"

If X is a scheme over $\mathbb{F}_q$, then $\bar{X} = \bar{\mathbb{F}}_q \otimes_{\mathbb{F}_q} X$ has an endomorphism F, "the geometric Frobenius," defined on the ring level as the map

$$\bar{\mathbb{F}}_q \otimes_{\mathbb{F}_q} A \longrightarrow \bar{\mathbb{F}}_q \otimes_{\mathbb{F}_q} A$$

$$\lambda \otimes a \longmapsto \lambda \otimes a^q.$$

It may be defined also as the composition of the absolute Frobenius of $\bar{X}$, $\lambda \otimes a \longmapsto \lambda^q \otimes a^q$, followed by the base extension of the inverse Frobenius auto. of $\bar{\mathbb{F}}_q$, $\lambda^q \otimes a \longmapsto \lambda \otimes a$. By descent $(\bar{X}, F)$ determines X over $\mathbb{F}_q$.

Note that F is not an arbitrary endo. since it differs from the absolute Frobenius by a semi-linear automorphism. For example if T is a torus defined over $\mathbb{F}_q$, and M is the character group of T viewed as a Galois module, then the Frobenius endo. of T produces on M the map $q \cdot$(Frob. auto. of M). Thus if G is a reductive conn. alg. gp. over $\mathbb{F}_q$ with torus T again over $\mathbb{F}_q$, the F-endo. of T is q times an automorphism θ. I think from what Borel said that θ needn't preserve the root system of G.

example of a non-split torus

$T_1 \subset O_2$ is the subgroup of matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{with } a^2+b^2=1$$

This is not a split torus if $\sqrt{-1} \notin k$. For example take $k = \mathbb{R}$ or $\mathbb{F}_7$. In the latter cases the only solutions of $a^2+b^2=1$ are $(\pm 2, \pm 2)$, $(\pm 1, 0)$, $(0, \pm 1)$ so $T_1(\mathbb{F}_7)$ is of order 8. Moreover

$$\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{mod } 7.$$

showing that $T_1(\mathbb{F}_7) \cong \mathbb{Z}_8$. ~~[illegible]~~ In any case the order is wrong to be $\mathbb{F}_7^*$. When $\sqrt{-1}$ exists

$$a^2+b^2 = (a+\sqrt{-1}\,b)(a-\sqrt{-1}\,b)$$

and so the torus is isomorphic to $\mathbb{G}_m$.

March 8, 1970: Detecting $H^*(BG(\mathbb{F}_q))$.

Let G be a connected reductive alg. group defined over $\mathbb{F}_q = k$ and let l be a good prime for G different from the characteristic of $k=\mathbb{F}_q$. By "good" I mean that $H^*(G)$ has a simple system of transgressive generators and so $H^*(BG)$ is a polynomial ring. By a theorem of Borel, this is so if $H^*(G)$ is an exterior algebra, hence if G has no l-torsion.

By considering the spectral sequence

$$E_2 = H^p(BG)\otimes H^q(G) \Longrightarrow H^{p+q}_G(G^t) = H^{p+q}(BG(k))$$

we proved that at least additively

$$H^*(BG(k)) \cong H^*(BG)_F \otimes H^*(G)^F$$

where $H^*(BG)_F$ denotes the cokernel of $H^*(BG) \overset{id}{\underset{F}{\rightrightarrows}} H^*(BG)$ in the category of rings and is also

$$H^*(BG)_F = S(\text{Coker } F-id \text{ on } QH^*(BG))$$

while $H^*(G)^F$ is the kernel of $H^*(G) \overset{id}{\underset{F}{\rightrightarrows}} H^*(G)$ in the category of Hopf algebras and is also

$$H^*(G)^F = \Lambda(\text{Ker}(F-id) \text{ on } PH^*(G))$$

at least additively. (When l is odd the additively can be dropped.)

If l is odd, then $H^*(BG(k))_{red}$ is a polynomial ring, hence by my theorem, there is a unique maximal abelian l-subgroup A of $G(k)$ up to conjugacy. When $l=2$ I think it should be true still that $H^*(BG(k))$ is a complete intersection ring, since this is true after passing to associated graded rings with respect to ~~the~~ a suitable filtration. Consequently all maximal elementary abelian l-subgroups of $G(k)$ are of the same rank. (This situation occurs for $G = SO_n$).

So let A be a fixed maximal elementary abelian l-subgroup of $G(k)$, and let Z be the centralizer of A in G.

Theorem: $H^*(BG(k)) \hookrightarrow H^*(BZ(k))$. (if Z conn. e.g. if l is odd)

~~Proof~~ This will be done in a sequence of lemmas.

Lemma 1: $H^*(BG)_F \longrightarrow H^*(BA)$ is a finite free map.

Proof: If $l = 2$, then because both sides are polynomial rings of the same dimension and the map is finite since $A \hookrightarrow G$, the result follows. If l is odd, then ~~we shall show that Z is connected~~ ~~and of the same rank as~~ ~~there would be a torus T of G containing~~ by goodness of G at l we know that A is contained in a maximal torus T of G (it can be shown that T can be chosen so as to be defined over k; this follows from the fact that Z is of the same rank as G). Hence we see that the restriction map from G to A factors through T and so we have a factorization

$$H^*(BG)_F \longrightarrow S(A^*) \longrightarrow H^*(BA)$$

The first is finite and free by the argument used for $\ell = 2$ and the second is also so the lemma follows.

We note that since $A \subset G(k)$, the restriction of the G-space G^t to A is just A acting on G by the conjugation action. In this good situation the A-action is transparent (fiber tot. non. hom. to zero) so $H^*_A(G)$ is a free $H^*(BA)$ module of rank equal to the dimension of $H^*(G)$.

<u>Lemma 2:</u> $H^*_G(G^t) \longrightarrow H^*_A(G)$ is injective

<u>Proof:</u> Both are free modules of finite rank over $H^*(BG)_F$. Denoting by a bar the reduction modulo the maximal ideal of $H^*(BG)_F$ we have a diagram

$$\begin{tikzcd}
\overline{H^*_G(G^t)} \arrow[r] \arrow[dd, "\cong"] & \overline{H^*_A(G)} \arrow[d] \\
 & H^*_A(G) \otimes_{H^*(BA)} \mathbb{Z}_\ell \arrow[d, "\cong"'] \\
H^*(G)^F \arrow[r, hook] & H^*(G)
\end{tikzcd}$$

showing that the bar map is injective. Hence the result follows from

<u>Sublemma</u>: If $\varphi: M \longrightarrow N$ is a map of free graded (finitely generated) modules over a graded connected (noetherian) ring R such that $\bar{M} \longrightarrow \bar{N}$ is injective, then φ is an injection onto a direct summand.

<u>Proof</u>. (obvious from the vector bundle viewpoint). Introduce the kernel K, cokernel C, and image I of φ. Then we have exact sequences

$$0 \longrightarrow \mathrm{Tor}_1(I,k) \longrightarrow \bar{K} \longrightarrow \bar{M} \xrightarrow{u} \bar{I} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_2(C,k) \rightarrow \mathrm{Tor}_1(I,k) \rightarrow 0 \longrightarrow \mathrm{Tor}_1(C,k) \longrightarrow \bar{I} \xrightarrow{v} \bar{N} \longrightarrow \bar{C} \longrightarrow 0$$

The hypothesis implies that u is an isomorphism and v is injective. Hence C is free, I is free, $K = 0$, q.e.d.

(We see from the sublemma that the map of lemma 2 is injective onto a direct summand as $H^*(BG)_F$ - modules. The ranks are $r = \text{rank } A$, $d = \text{rank } G$

$$\text{rank } H^*_G(G^t) = 2^r \qquad (= \dim H^*(G)^F$$

$$\text{rank } H^*_A(G) = 2^d \cdot 2^r \cdot \underbrace{[S(A^*) : H^*(BG)_F]}_{\text{roughly order of "true" Weyl group}} \quad)$$

<u>Lemma 3</u>: $H^*_A(G) \longrightarrow H^*_A(Z)$ is injective.

<u>Proof</u>: Let $e \in H(BA)$ be the Euler class of the reduced

regular representation. Then the localization theorem shows that

$$(*) \qquad H^*_A(G)[e^{-1}] \xrightarrow{\sim} H^*_A(Z)[e^{-1}]$$

and as e is a non-zero divisor in $H^*_A(G)$, the lemma follows.

~~Remark: Note that $H^*_A(Z) = H^*_A \otimes H^*(Z)$ is a comm Hopf algebra over H^*_A, which becomes isomorphic after localization to $H^*_A(G)$ which is also a Hopf algebra. When l is odd $H_A(G)$ is a free commut~~

Remark: When l is odd, $H^*(G)$, hence also $H^*_A(G)$ is a free commutative (in the sense of alg. top.) algebra, hence $H^*_A(Z)[e^{-1}] = H^*_A[e^{-1}] \otimes H^*(Z) = S(A^*)[e^{-1}] \otimes \Lambda A^* \otimes H^*(Z)$ is an exterior algebra over $S(A^*)[e^{-1}]$. Thus $H^*(Z)$, which is a Hopf algebra, must be an exterior algebra. This shows that Z is connected and is good for l if G is. Everything becomes false for $SO(n)$ and $l=2$.

Proof of theorem follows from the following comm. diagram and the above lemmas.

$$\begin{tikzcd} H_G(G^t) \arrow[r] \arrow[d] & H_Z(Z^t) = H(BZ(k)) \arrow[d] & \left\{\begin{array}{l}\text{if } Z \text{ conn.}\\ \text{e.g. } l \text{ odd.}\end{array}\right. \\ H_A(G) \arrow[r, hook] & H_A(Z). \end{tikzcd}$$

qed.

<u>Examples:</u> 1.) $Gl_n(\mathbb{F}_q)$. Suppose $[\mathbb{F}_q(\mu_\ell) : \mathbb{F}_q] = d$ and let $n = dm + r$, $0 \le r < d$. There is a subgroup

$$Gl_m(\mathbb{F}_q(\mu_\ell)) \times Gl_r(\mathbb{F}_q) \subset Gl_n(\mathbb{F}_q)$$

and a maximal elementary abelian ℓ-group is

$$\mu_\ell^m \in Gl_m(\mathbb{F}_q)$$

The centralizer of this is

$$T_m(\mathbb{F}_q(\mu_\ell)) \times Gl_r(\mathbb{F}_q) = Z(\mathbb{F}_q)$$

$$\Big(\underset{\mathbb{F}_q \to \mathbb{F}_q(\mu_\ell)}{\mathrm{Norm}} T_m\Big) \times Gl_r = Z \qquad \text{connected}$$

since $r < d$ $Gl_r(\mathbb{F}_q)$ has no ℓ-torsion, so

$$H^*(BGl_n(\mathbb{F}_q)) \hookrightarrow H^*\big((\mathrm{Norm}\, T_m)(\mathbb{F}_q)\big)$$

March 9, 1970:

G algebraic group over $\overline{\mathbb{F}}_p$, say defined over $\mathbb{F}_q$ for some $q = p^a$. Then $G(\mathbb{F}_{q^b})$ makes sense.

Theorem: For any prime number $\ell \neq$ the characteristic p, we have

$$H^*(BG, \mathbb{Z}_\ell) \xrightarrow{\sim} \varprojlim_n H^*(BG(\mathbb{F}_{q^n}), \mathbb{Z}_\ell).$$

Proof: First choose $\mathbb{F}_q$ large enough so that the etale finite group $\pi_0(G)$ is constant, e.g. Galois acts trivially. Then one knows that if G^0 is the connected component that

$$0 \longrightarrow G^0(k) \longrightarrow G(k) \longrightarrow \pi_0(G) \longrightarrow 0$$

is exact ~~with~~ for any algebraic extension k of $\mathbb{F}_q$. So by comparing Hochschild-Serre spectral sequences one is reduced to the case where G is connected.

In this case we have our spectral sequence

$$E_2 = H^*(BG) \otimes H^*(G/G(k)) \Longrightarrow H(BG(k))$$

and the result we are after follows from

$$\varprojlim_n H^*(G/G(\mathbb{F}_{q^n})) = 0$$

(In fact this is the basic assertion valid even if G is not connected.)

But there is a commutative diagram

$$\begin{tikzcd}
& xG(\mathbb{F}_q) \arrow[r, mapsto] \arrow[d, mapsto] & xG(\mathbb{F}_{q^a}) \arrow[d, mapsto] & \\
G/G(\mathbb{F}_q) \arrow[rrr] \arrow[ddd] & xFx^{-1} & x\cdot F^a x^{-1} & G/G(\mathbb{F}_{q^a}) \arrow[ddd] \\
& & & \\
& & & \\
G^\circ \arrow[rrr] & & & G^\circ \\
& y \arrow[r, mapsto] & y\cdot Fy \cdots F^{a-1}y &
\end{tikzcd}$$

once one has arranged F to be trivial on $\pi_0(G)$. Suppose that we choose $\mathbb{F}_q$ large enough so that F acts trivially on $H^*(G^\circ)$, which is possible since the ~~group of~~ group $H^*(G^\circ)$ is finite. Then ~~we want to study~~ the effect of the map $y \mapsto y\cdot Fy \cdots F^{a-1}y$ on $H^*(G^\circ)$ is the same as $y \mapsto y^a$

<u>Lemma:</u> ~~Suppose that $H^*(G^\circ)$ ... The~~ Let $\psi^\ell : H^*(G^\circ) \to H^*(G^\circ)$ be the map on cohomology induced by $g \mapsto g^\ell$. Then ψ^ℓ is nilpotent on $H^+(G^\circ)$.

~~*Proof:* This is really an assertion about Hopf algebras. Let $\Lambda = H^*(G^\circ)$, and consider the let $\mathfrak{m}$ be the augmentation ideal of Λ and let~~

$$\cancel{\Lambda \supset \mathfrak{m} \supset \mathfrak{m}^{(\ell)} \supset \mathfrak{m}^{(\ell^2)} \supset \cdots\cdots}$$

~~be the Frobenius series, i.e.~~

$$\cancel{\Lambda/\mathfrak{m}^{(\ell^n)} = \mathrm{Coker}\left\{\Lambda \overset{id}{\underset{F^n}{\rightrightarrows}} \Lambda\right\}}$$

<u>Proof:</u> This is really an assertion about Hopf algebras.
~~Let $\Lambda = H^*(G^\circ)$, $\mathfrak{m} = \bar{\Lambda}$, and consider the $\mathfrak{m}$-adic filtration~~

Let $\Lambda = H_*(G^0)$ and consider the $\bar{\Lambda}$-adic filtration which is complete since Λ is connected. One knows that ψ^ℓ on Λ induces ψ^ℓ on $gr\,\Lambda$ and that $gr\,\Lambda$ is a primitively generated Hopf algebra. But $\psi^\ell x = 0$ if x is primitive, hence $\psi^\ell = 0$ on $gr\,\Lambda^+$ and ψ^ℓ is nilpotent on Λ^+, hence by duality nilpotent on $H^+(G^0)$. qed.

The lemma shows that if we take $a = \ell^\nu$ for ν large, then $H^*(G/G(\mathbb{F}_{q^a})) \longrightarrow H^*(G/G(\mathbb{F}_q))$ is zero. Thus we have the following refinement of the theorem.

<u>Theorem'</u>: Let G be an algebraic group defined over $\mathbb{F}_q$ and suppose that the Frobenius acts trivially on $H^*(G, \mathbb{Z}_\ell)$. Then

$$H^*(BG, \mathbb{Z}_\ell) \xrightarrow{\sim} \varprojlim_\nu H^*(BG(\mathbb{F}_{q^{\ell^\nu}}), \mathbb{Z}_\ell).$$

March 11, 1970: higher order K-groups

Let R be a ring (not nec. commutative). If X is a space consider sheaves ~~...~~ of R-modules over X locally isomorphic to $X \times E$ where E is a f.g. projective R-module. If X is connected (and locally connected) such a thing is a representation of $\pi_1(X, x)$ (a pro-group, recall!) in $\mathrm{Aut}_R(E)$. Let $k(X,R)$ be the associated Grothendieck group where exact sequences give rise to sums.

Properties: 1) $X \mapsto k(X,R)$ contravariant functor to Ab. $\ni f \sim g \Rightarrow f^* = g^*$.

2) If $f: Y \to X$ is a finite covering, there is an induced homomorphism (trace)

$$f_*: k(Y,R) \longrightarrow k(X,R)$$

defined by taking the trace map on sheaves. In effect if E is a sheaf of R-modules which is locally constant and projective f.g., then the same is true for $f_* E$. Moreover if $0 \to E' \to E \to E'' \to 0$ is exact, then

$$0 \to f_* E' \to f_* E \to f_* E'' \to 0$$

is exact since the fibre has no cohomology and f is proper (Leray)

Question: Suppose that E is a sheaf of R-modules which locally is a direct summand of $X \times R^n$. Is E of above type? Yes, because the projection operator at each x gives a (locally-defined) map $X \longrightarrow \mathrm{End}_R(R^n)$ which must be locally constant by local connectedness. Hence E is locally constant.

problem with basepoint: do you want $F(G) = \tilde{k}(BG)$ or $k(BG)$.

~~Conjecture~~ 3) If $X = \coprod_i X_i$, then

$$k(X,R) \xrightarrow{\sim} \prod_i k(X_i, R)$$

at least if we work with finite ~~[illegible]~~ disjoint unions.

~~Conjecture: Suppose that~~

<u>Question:</u> ~~Suppose~~ From a $\overset{k}{\text{functor}}$ on spaces satisfying 1) 2) 3) one obtains a contravariant functor F ~~on groups~~ from the category of groups to ~~the~~ Ab ~~satisfying~~ by setting $F(G) = k(BG)$. F has the properties

1)′ If θ is an inner automorphism of G, then $\theta^*: F(G) \to F(G)$ is the identity

2)′ If $H \overset{i}{\hookrightarrow} G$ is a subgroup of finite index, then there is a trace or transfer map

$$F(H) \longrightarrow F(G)$$

satisfying (i) transitivity
(ii) Mackey formula

Is it clear that F determines k? More or less it is because if $f: X \to Y$ induces an isomorphism on the category of locally constant sheaves, then $f^*: k(Y,R) \xrightarrow{\sim} k(X,R)$. This means that for connected X (to which we can reduce by 3)) we have

$$k(X,R) \xleftarrow{\sim} F(\pi_1(X,x))$$

My basic idea is that $K^*(X,R)$ should be the gen. coh. theory generated by $k(X,R)$, that is, endowed with a homomorphism $k(X,R) \longrightarrow K^*(X,R)$ compatible with traces. ~~Let's check~~ I'm now going to show that this idea is compatible with the definition of algebraic K-theory proposed by Bass.

Recall Fox's toral homotopy groups $[T_n, \Omega X]$.

$$[T_0, \Omega X] = \pi_1(X)$$
$$[T_1, \Omega X] = \pi_1(X) \times \pi_2(X)$$
$$[T_2, \Omega X] = \pi_1(X) \times \pi_2 \times \pi_2 \times \pi_3$$

(as sets; these are as groups semi-direct products)

This follows since one knows that $S(T_n^+)$ is a wedge of spheres. Now $k(T_n)$ is the Grothendieck group of the additive category whose objects are f.g. proj. R-modules E endowed with n-commuting automorphisms $\alpha_1, \ldots, \alpha_n$ and it seems pretty clear that

$$\tilde{K}(S^n) = \mathrm{Coker}\left\{ K(T^{n-1})^n \xrightarrow{\sum p_i^*} K(T^n) \right\}$$

~~i.e.~~ so the corresponding formula for $\tilde{k}(S^n)$ would say that $\tilde{k}(S^n)$ should receive the universal determinants

$$\det(E; \alpha_1, \ldots \alpha_n)$$

additive in E and which vanishes if $\alpha_j = id$ for any j.
~~(Perhaps from this you can deduce multiplicativity in α_j separately by the "square lemma" trick as well as the fact that it vanishes if two α_j are equal which should reduces to~~

~~one $\alpha_j = id$ + "composition" with a map of S^n of degree 1.~~)

Example: suppose k is an algebraically closed field. Then given a k-vector space E with commuting autos. $\alpha_1, \dots, \alpha_n$ it possesses an invariant flag, so

$$k(T_n) = \text{free abelian group generated by } (k^*)^n$$

which shows that

$$\operatorname{Coker}\{k(T_0) \longrightarrow k(T_1)\} = \mathbb{Z}[k^*]$$

is pretty far away from being k^*. So a great deal more analysis is required before we get anything resembling what should be true for $\overline{\mathbb{F}}_p$. (It's reasonable to conjecture that the diagram commutes

$$\begin{tikzcd}
(k^*)^n \arrow[r] \arrow[d] & K(T_1)^n \arrow[d] \\
k(T_n) \arrow[d] & K(T_1)^{\otimes n} \arrow[dl, "\text{external product}"] \\
K(T_n) &
\end{tikzcd}$$

and therefore that $k(T_n) \longrightarrow \tilde{K}(S^n)$ is zero for $n \geq 2$ for $k = \overline{\mathbb{F}}_p$ since this is so in dimension 2.)

Question: Perhaps $k(X) \xrightarrow{\sim} K^0(X)$ if X is a wedge of S^1's. If $G.$ is a free simplicial group are the cohomology groups $\check{H}^q(\nu \mapsto K(BG_\nu))$ reasonable?

(not unless $K^q(pt) = 0$ $q \neq 0$)

March 12, 1970. higher order K-groups:

Let R be a ring. By an R-vector bundle over a space X, R-bundle for short, I mean a sheaf of R-modules over X locally isomorphic to $X \times E$ where E is a f.g. projective R-module. Consider generalized cohomology theories h^*; better, consider contravariant functors h on the category of C^∞-manifolds to Ab endowed with Gysin homomorphisms for proper framed maps satisfying all the axioms together with a characteristic class $\{R\text{-bundles}/X\} \longrightarrow h(X)$ which is natural, compatible, with exact sequences and traces for finite coverings. It is pretty clear that a universal such animal exists, ~~[illegible]~~ that every element of $h(X)$ is represented by a pair $(\xi, Z \xrightarrow{f} X)$ where f is a proper framed map and ξ is an R-bundle over Z, but without precise knowledge of the equivalence relation we don't know that h satisfies the exactness axiom.

Modification: Require Gysin homomorphism for say complex-oriented maps. Thus if $K(?, R)$ is the above theory h with only framed Gysin maps, and $K^0(?, R) = \{X, \Gamma\}$ where Γ is a spectrum, then $MU \wedge \Gamma$ should be the universal theory with Gysin for complex-oriented maps. In effect if this theory is represented by a spectrum Δ, then Δ is an MU module and $\exists\ \Gamma \rightarrow \Delta$ hence a map $MU \wedge \Gamma \longrightarrow \Delta$.

March 13, 1970

Conjecture: If G is an l-local group, l a prime, then
$$H^q(BG, \mathbb{Z}_l) = 0, \qquad q > 0.$$

Proof for $q = 1$:
Suppose A is an l-torsion abelian group with trivial G-action. Then

$$H^1(BG, A) = \mathrm{Hom}(G_{ab}, A)$$

But $x \mapsto x^l$ is surjective on G hence also on G_{ab}, so the latter is zero.

Now an element of $H^2(BG, A)$ is represented by an extension

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] & \tilde{G} \arrow[r, "\pi"] & G \arrow[r] & 1 \\ & & & \mathbb{Z}[l^{-1}] \arrow[u, "\varphi"] \arrow[ul, dotted] & \end{tikzcd}$$

Given $x \in G$ there is a unique homomorphism φ as indicated with $\varphi(1) = x$ and since $H^+(B\mathbb{Z}[l^{-1}], A) = 0$ (here suppose A is finite & l-torsion) there is a unique dotted arrow covering φ. This shows that each element x of G has a unique lifting $\tilde{x}$ such that $\tilde{x}$ possesses a sequence of successive l-th roots, i.e.

$$\tilde{x} = \alpha_1^l, \quad \alpha_1 = \alpha_2^l, \quad \ldots$$

To show that $x \mapsto \tilde{x}$ is a homomorphism it suffices to show that $\tilde{x}\tilde{y}$ possesses a sequence of successive l-th roots. Don't even see why it has an l-th root?

March 14, 1970.

Let R be a ring and let $k(X,R)$ be the naive K functor constructed from R-vector bundles over X. We propose to define $K(X,R)$ as a representable functor on the homotopy category together with a map $\gamma: k(X,R) \longrightarrow K(X,R)$ which should be universal.

Suppose that there exists a universal map $\gamma: k(X) \longrightarrow [X,B]$ (where we drop the R to save writing). Here X runs over the homotopy category of simplicial sets. Fix a space Y and an element y of $k(Y)$. Then we have a map

$$k(X) \xrightarrow{\boxplus y} k(X \times Y) \xrightarrow{\gamma} [X\times Y, B] = [X, B^Y]$$

which is natural in X, so there is an ~~map~~ element of $[B, B^Y]$ by universality. Thus for each $y \in k(Y)$ we get an element of $[Y, B^B]$, and hence a map $k(Y) \longrightarrow [Y, B^B]$ which is obviously natural in Y. Thus finally we obtain $\mu \in [B, B^B] = [B\times B, B]$ uniquely characterized by the fact that

$$\begin{tikzcd}
k(X)\times k(Y) \arrow[r] \arrow[d] & {[X,B]\times[Y,B]} \arrow[d, phantom, "\shortparallel"] \\
 & {[X\times Y, B\times B]} \arrow[d, "\mu_*"] \\
k(X\times Y) \arrow[r] & {[X\times Y, B]}
\end{tikzcd}$$

commutes for all X, Y. Similar arguments show that μ makes $[?, B]$ into an abelian group.

However it seems difficult to show that $[?, B]$ has trace homomorphisms. (If so, then we have $MB \longrightarrow B$ satisfying the usual conditions, and the sequence

$$\begin{tikzcd}[X,B] \arrow[r, "\text{apply } M"', "\text{use } MB\to B"' below] & {[MX,B]} \arrow[l, bend right, "X\to MX"'] \arrow[r, shift left, "\text{apply } M \text{ then } M^2X\to MX"] \arrow[r, shift right, "\text{apply } M \text{ then } MB\to B"'] & {[M^2X,B]} \arrow[l, bend right, "MX \rightrightarrows M^2X"']\end{tikzcd}$$

is exact in either direction. This suggests that we might define $[X,B] \longrightarrow [MX,B]$ by

$$\begin{tikzcd}k(X) \arrow[r, "\text{trace map}"'] \arrow[d] & k(MX) \arrow[l, bend right] \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & k(M^2X) \arrow[l, bend right, shift right] \arrow[l, bend right, shift left] \arrow[d] \\ {[X,B]} \arrow[r, dashed] & {[MX,B]} \arrow[l, bend right] \arrow[r] & {[M^2X,B]} \arrow[l, shift left] \arrow[l, shift right]\end{tikzcd}$$

by proving that $\operatorname{Coker}\{[M^2X,B] \rightrightarrows [MX,B]\}$ is a direct factor of $[MX,B]$ representable and using the universal property. The other method might be to show that some variant of $[MX,B]$ is representable, e.g. if $\exists$ univ. map $[MX,B] \longrightarrow [X,C]$.)

Leaving aside this problem, consider the problem of characteristic classes for R-vector bundles with values in $H^* \otimes A$ where A is a graded $\mathbb{Z}_\ell$-algebra and $H^*(X) = H^*(X, \mathbb{Z}_\ell)$. Since the functor $X \longmapsto H^*(X) \otimes A$ is representable it follows that natural transformations from $k(X)$ to $H^*(X,A)$ are the same as natural transformations from $K(X)$ to $H^*(X,A)$, i.e. elements of $H^*(B,A) = \operatorname{Hom}(H_*(B), A)$. To assert that the characteristic class is additive ~~means that~~, ie $\varphi(x+y) = \varphi(x)\varphi(y)$ $\quad \varphi(0)=1$, is equivalent to $H_*(B) \longrightarrow A$ being a ring homomorphism. Therefore the Hopf algebra $H_*(B)$ (homology mod ℓ) can be determined from the characteristic classes without actually having B explicitly.

Example: Suppose $R = \mathbb{F}_q$, and let φ be an additive characteristic class for $\mathbb{F}_q$-vector space bundles with values in $H^*(?)\otimes A$ (coeffs. mod l). Here $A = A_0 + A_1 + \cdots\cdots$ is a graded ring and $\varphi(E) \in \sum_{i\geq 0} H^i(X)\otimes A_i$ is a unit satisfying

$$\varphi(E) = \varphi(E')\varphi(E'') \qquad \text{if} \qquad 0 \to E' \to E \to E'' \to 0.$$

Such a class is given by

$$\varphi_n \in \quad H(BGl_n(\mathbb{F}_q))_A^* \qquad \text{of degree } 0$$

such that when restricted to $\Gamma_{mn} \subset G_{m+n}$

$$(*) \qquad \operatorname{res}^{G_{m+n}}_{\Gamma_{mn}} \varphi_{m+n} = \varphi_m \otimes \varphi_n.$$

Now if $l \mid q$ one knows that the nilpotent part of the Borel subgroup contains a Sylow l-subgroup $P_n \subset G_n$. As $\operatorname{res}^{G_n}_{P_n}$ is injective and the standard bundle of BP_n has a flag with trivial quotients it follows that $\operatorname{res}^{G_n}_{P_n} \varphi_n = 1$, so $\varphi_n = 1$.

If $l \nmid q$, then we can suppose exact sequences split in calculating the possibilities for φ. This is because if P is a normal subgroup of G of order prime to l, then the Hochschild-Serre spectral sequence shows that

$$H^q(G) \xleftarrow{\sim} H^q(G/P)$$

hence the condition $(*)$ is equivalent to

$$\operatorname{res}^{G_{m+n}}_{G_m\times G_n} \varphi_{m+n} = \varphi_m \otimes \varphi_n.$$

Therefore the class φ gives a graded ring homomorphism

$$\bigoplus_n H_*(BG_n) \longrightarrow A_*$$

such that the element ξ_0 generating $H_0(BG_1)$ is a unit in A_0 (this corresponds to the fact that $\varphi(E)$ must be a unit and for this it need only have a unit for augmentation.)

So by my old calculations I know that

$$(*) \qquad H_*(B) = \mathbb{Z}_\ell[\xi_\nu, \tau_\nu, \xi_0, \xi_0^{-1}]_{\nu \geq 1} \qquad (\ell\text{-odd})$$

where ξ_ν, τ_ν form a basis for $H_*(BGl_1(\mathbb{F}_q(\mu_\ell)))^{\mathbb{Z}_d} \cong H_*(BG_d)$, $\mathbb{F}_{q^d} = \mathbb{F}_q(\mu_\ell)$. homological dim $\xi_\nu = 2d\nu$, h.dim $\tau_\nu = 2d\nu - 1$.

~~Now the next step is to construct a map from $k(X)$ to the fibre of $\psi^q - id : \mathbb{Z}\times BU \longrightarrow \mathbb{Z}\times BU$ using the Brauer theory perhaps and then check they give same characteristic classes.~~

Interpretation of $(*)$: Let φ be an additive characteristic class for $\mathbb{F}_q$ vector bundles with values in $\{H^*(?) \otimes A\}^0$. Let η be an irreducible (non-trivial) representation of $\mathbb{Z}_\ell$ in an $\mathbb{F}_q$-vector space (e.g. suppose $\overline{\mathbb{F}}_q$ chosen, then choose $\mathbb{Z}_\ell \cong \mu_\ell$ and take η to be the corresponding representation in $\mathbb{F}_q(\mu_\ell)$. Observe that there are $\frac{\ell-1}{d}$ possibilities for η.) Form the $\mathbb{F}_q$-vector bundle over $B\mathbb{Z}_\ell$, denoted $\eta_{\mathbb{Z}_\ell}$. Then there is an expansion

(by mult. by β)

$$\varphi(\eta_{\mathbb{Z}_\ell}) = \sum_{\nu\geq 0} a_\nu (\beta u)^\nu + \sum_{\nu \geq 1} b_\nu\, u(\beta u)^{\nu-1}$$

where $u \in H^1(B\mathbb{Z}_\ell)$ is the canonical element. ~~[illegible]~~ Now if any element of $Gal\left(\mathbb{F}_q(\mu_\ell)/\mathbb{F}_q\right) = \mathbb{Z}_d$ is made to act on $\mathbb{Z}_\ell$ then the resulting representation ~~of~~ η remains fixed, consequently $\varphi(\eta_{\mathbb{Z}_\ell})$ is invariant under this action, so only $(\beta u)^\nu$ and $u(\beta u)^{\nu-1}$ terms can occur with $\nu \equiv 0 \pmod d$. So reindexing we find that

$$\varphi(\eta_{\mathbb{Z}_\ell}) = \sum_{\nu \geq 0} a_\nu (\beta u)^{2d\nu} + \sum_{\nu\geq 1} b_\nu\, u(\beta u)^{2d\nu-1}$$

for $a_\nu \in A_{2d\nu}$, $b_\nu \in A_{2d\nu-1}$, $a_0 \in A_0^*$, $b_0 = 0$. The formula $(*)$ asserts that

$$\begin{aligned} Hom_{ab}\left\{k, (H\otimes A)^*\right\} &\xrightarrow{\ \cong\ } A_0^* \times \prod_{\nu\geq 1} A_{2d\nu} \times \prod_{\nu \geq 1} A_{2d\nu-1} \\ \varphi &\longmapsto \{(a_\nu, b_\nu)\} \end{aligned}$$

It would be nice to have a direct proof of this maximizing the geometry. ~~One proves that the map is injective by your general theorem~~ and the Hopf algebra theory.

March 15, 1970: $K_*(\mathbb{Z})$?

In order to compute $K_*(\mathbb{Z})$, which we have no conjectures about, we want to compute characteristic classes for $\mathbb{Z}$-vector bundles.

We concentrate on the prime l. We have good reason to suspect that we can compute $H^*(BGl_n(\mathbb{Z}))$ from the elementary abelian l-subgroups of $Gl_n(\mathbb{Z})$ up to F-isomorphism.

~~[illegible]~~ Assume that a universal space $B = BGl(\mathbb{Z})$ exists.

Given an element $x \in R_{\mathbb{Z}}(G)$ for any group G. Then we get a map $BG \longrightarrow B$ from the corresponding element of $k(BG, \mathbb{Z})$. If G is a finite group one knows that

$$\begin{tikzcd}
R_{\mathbb{Z}/l}(G) \arrow[r, "\iota_*"] \arrow[dr, bend right, "0"'] & R_{\mathbb{Z}}(G) \arrow[r, "j^*"] \arrow[d, "i^*"] & R_{\mathbb{Z}[l^{-1}]}(G) \arrow[r] \arrow[dl, dashed, "d"] & 0 \\
& R_{\mathbb{Z}/l}(G) & &
\end{tikzcd}$$

Now if G is an l-group, then $R_{\mathbb{Z}/l}(G) = \mathbb{Z}$ and by definition $\iota_* 1 = 0$ since $\iota_* 1 : [\mathbb{Z} \xrightarrow{l} \mathbb{Z}, \text{trivial action}] = [\mathbb{Z}] - [\mathbb{Z}] = 0$.
Thus if G is an l-group

$$R_{\mathbb{Z}}(G) \xrightarrow{\sim} R_{\mathbb{Z}[l^{-1}]}(G).$$

which is nice, since ~~[illegible]~~ in $Gl_n(\mathbb{Z}[l^{-1}])$ there might be a unique maximal elementary abelian l-subgroup up to conjugacy. (This ~~may~~ might be proved as follows? Suppose an elementary abelian group A acts on a free $\mathbb{Z}[l^{-1}]$ module L of rank n. ~~[illegible]~~ Let

$$L \otimes \mathbb{Q} \cong \bigoplus_i V_i$$

be the decomposition of $L \otimes \mathbb{Q}$ into eigenspaces and set $L_i = L \cap V_i$. I claim that $L = \oplus L_i$. To see this it is enough to assume $A = \mathbb{Z}_\ell$ ~~that the fields. Then there are only two irreducible reps. over Q~~. Here exactly two irreducibles over $\mathbb{Q}$. Note that the kernel of a map between free modules over Dedekind ring is a direct submodule. Hence L_1 being the kernel of a suitable polynomial in the generator a of A, i.e. $a-1$, is a direct summand of L. Taking a projection + averaging we see that $\exists$ invariant complement to L_1 which must be L_2 so $L = L_1 \oplus L_2$. (Quite generally the minimal polynomial

$$X^\ell - 1 = \prod p_i(X)$$

and $L_i = \operatorname{Ker} p_i(a)$. The p_i are comaximal since the derivative $\ell X^{\ell-1} \equiv \ell \mod (X^\ell - 1)$ is invertible). A acts in an eigenspace L_i through a cyclic quotient (break V_i into irreducibles and use that a finite subgroup of a field (A comm!) is cyclic.) Thus L_i is a module over $\mathbb{Z}[\ell^{-1}, \zeta_i]$ $\zeta_i^\ell = 1$ where a acts as ζ_i and it will be a free module provided the ideal class group of $\mathbb{Z}[\ell^{-1}, \zeta]$ is zero.

Question: Is the ideal class group of $\mathbb{Z}[\zeta, \ell^{-1}]$, ζ a primitive ℓ-th root of 1, zero? <u>NO</u>

$$K(\mathbb{Z}/\ell) \xrightarrow{\ i\ } K(\mathbb{Z}[\zeta]) \xrightarrow{\ j\ } K(\mathbb{Z}[\zeta, \ell^{-1}]) \longrightarrow 0$$

since $\prod_{i=1}^{p-1} 1 - \zeta^i = \lim_{x \to 1} \frac{x^\ell - 1}{x - 1} = \ell$, and since $\frac{1-\zeta^i}{1-\zeta}$ are units it follows that ~~invert~~ inverting ℓ in $\mathbb{Z}[\zeta]$ is the same as inverting $1-\zeta$ which generates the unique ideal over $\ell\mathbb{Z}$ in $\mathbb{Z}$. Also

$$\mathbb{Z}[\zeta]/(1-\zeta) = \mathbb{Z}/\ell \qquad e=\ell-1,\ f=1,\ g=1$$

Therefore i_*1 is seen ~~[illegible]~~ to be zero since there is an exact sequence

$$0 \longrightarrow \mathbb{Z}[\zeta] \xrightarrow{1-\zeta} \mathbb{Z}[\zeta] \longrightarrow \mathbb{Z}/\ell \rightarrow 0$$

<u>Conclusion:</u> $\quad \operatorname{Pic} \mathbb{Z}[\zeta] \xrightarrow{\sim} \operatorname{Pic} \mathbb{Z}[\zeta, \ell^{-1}]$

and therefore there are lots of interesting non-conjugate cyclic groups of order ℓ **already** in $Gl_{\ell-1}(\mathbb{Z}[\ell^{-1}])$ corresponding to the **different** orbits of $\text{Galois}(\mathbb{Q}[\zeta]/\mathbb{Q})$ on $\operatorname{Pic}(\mathbb{Z}[\zeta, \ell^{-1}])$.

March 18, 1970: Algebraic K-theory.

Let R be a ring and suppose the space BGl_R exists. Then it should be a commutative H-space whose cohomology in dimensions ≤ 2 should be closely connected to $K_i(R)$ for $0 \leq i \leq 2$. Now the only non-trivial **k-invariant** for BGl_R in dim ≤ 2 will be in $H^3(\pi_1, \pi_2)$ and stable, hence of order 2. So ignoring the prime 2 we have

$$BGl_R = \pi_0 \times K(\pi_1, 1) \times K(\pi_2, 2)$$

with cohomology given by

$$H^*(BGl_R, \mathbb{Z}_\ell) = \mathbb{Z}_\ell[\pi_0] \otimes H^*(\pi_1, \mathbb{Z}_\ell) \otimes H^*(K(\pi_2, 2))$$

ie.

$$H^0(BGl_R, \mathbb{Z}_\ell) = \mathbb{Z}_\ell^{\pi_0}$$

$$H^1(BGl_R, \mathbb{Z}_\ell) = \mathbb{Z}_\ell^{\pi_0} \hat{\otimes} \operatorname{Hom}(\pi_1, \mathbb{Z}_\ell)$$

$$H^2(BGl_R, \mathbb{Z}_\ell) = \mathbb{Z}_\ell^{\pi_0} \hat{\otimes} \operatorname{Hom}({}_\ell\pi_1, \mathbb{Z}_\ell) \hat{\otimes} \operatorname{Hom}(\pi_2, \mathbb{Z}_\ell).$$

Now let me compare these groups with the known $K_i(R)$.

$$H^0(BGl_R, \mathbb{Z}_\ell) = \operatorname{Map}(k_R, H^0(\mathbb{Z}_\ell))$$

$$\begin{tikzcd}
k_R(X) \arrow[r, "\varphi"] & H^0(X, \mathbb{Z}_\ell) \\
k_R(\pi_0(X)) \arrow[u] \arrow[r] \arrow[d, equal] & H^0(\pi_0(X), \mathbb{Z}_\ell) \arrow[u, "s"'] \\
\operatorname{Map}(\pi_0(X), k_R(pt)) &
\end{tikzcd}$$

Thus ~~[illegible]~~ φ is same

as a map $k_R(pt) \longrightarrow \mathbb{Z}_\ell$. Thus in fact for any abelian group A I find that

$$H^0(BGl_R, A) = \mathrm{Map}\,(k_R(pt), A) = \mathrm{Map}(K_0(R), A)$$

which shows that $\pi_0(BGl_R) = K_0(R)$.

Denote by $B\widetilde{Gl}_R$ the reduced theory ~~...~~ Then

$$\begin{array}{c} H^1(B\widetilde{Gl}_R, A) = \mathrm{Hom}(\pi_1 \widetilde{Gl}_R, A) \\ \| \\ \mathrm{Map}(\tilde{k}_R, H^1(\cdot, A)). \end{array}$$

Such a map is determined on $X = \bigvee_I S^1$

$$\begin{tikzcd} \tilde{k}_R(X) \arrow[r, "\varphi"] \arrow[d] & H^1(X, A) \arrow[d, hook] \\ \tilde{k}_R(\bigvee_I S^1) \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\text{naturality}"] & H^1(\bigvee_I S^1, A) \arrow[d, "\cong"] \\ \prod_I \tilde{k}_R(S^1) \arrow[r, "\prod\varphi"] & \prod_I H^1(S^1, A) \end{tikzcd}$$

Therefore such a φ is determined by a map

$$\tilde{k}_R(S^1) \xrightarrow{\varphi} A$$

and ~~which~~ by naturality ~~...~~

$$\begin{tikzcd} \tilde{k}_R(S^1) \arrow[r, "\varphi"] & A \\ \tilde{k}_R(S^1 \vee S^1) \arrow[u, "\mu"] \arrow[r, "\varphi i_1 + \varphi i_2"] & A \oplus A \arrow[u, "+"'] \end{tikzcd}$$

commutes where $\mu : S^1 \vee S^1 \longrightarrow S^1$ gives the multiplication on π_1. Thus φ associates to (P, α), α an automorphism of P, an element $\varphi(P,\alpha) \in A$ such that $\varphi(P, \alpha\beta) = \varphi(P,\alpha) + \varphi(P,\beta)$. It remains to show that φ is a group homomorphism, but this follows from

$$\begin{tikzcd} \tilde{k}_R(X) \oplus \tilde{k}_R(X) \arrow[r, "\varphi \oplus \varphi"] \arrow[d, "pr_1^* + pr_2^*"] & H^1(X,A) \oplus H^1(X,A) \arrow[d, "pr_1^* + pr_2^*"] \\ \tilde{k}_R(X \times X) \arrow[r, "\varphi"] \arrow[d, "\Delta^*"] & H^1(X \times X, A) \arrow[d, "\Delta^*"] \\ \tilde{k}_R(X) \arrow[r, "\varphi"] & H^1(X,A) \end{tikzcd}$$

the following argument. The point is that for X connected, $H^1(X \times X, A) \cong H^1(X,A) \oplus H^1(X,A)$, hence given $u, v \in \tilde{k}_R(X)$, we must have $\varphi(u \boxplus v) = \varphi(u) \boxplus \varphi(v)$, since both have same restriction to $X \vee X$. (Note $\varphi(0) = 0$ since $0 = f^* 0$ $f : X \longrightarrow pt$ & $H^1(pt, A) = 0$.) <u>Conclusion:</u> $\pi_1 BGl_R \cong K_1(R)$.